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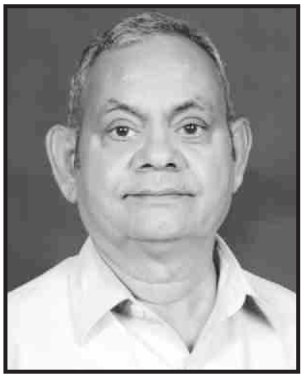
CONTENTS



Dr. T.P. Singh
Chief Editor & Professor in Maths & O.R.

FOREWORD

It gives me an immense pleasure in writing this foreword for **'Aryabhatta Journal of Mathematics & Informatics' (AJMI) Vol.13 issue 1 (Jan.-June, 2021)** published by Aryans Research & Educational Trust. The first issue of the journal was published in year 2009.Since then & till today the journal publication is regular & well in time. It gives us a great pleasure to put forward before the scholars and academicians that journal from its start is making an effort to produce good quality articles. The credit goes to its reviewer team which review sincerely and furnish valuable suggestions to improve the quality of papers.

Aryabhatta Journal of Mathematics & Informatics mainly covers areas of mathematical and statistical sciences, Operational Research, data based managerial & economical issues and information sciences. Mathematical science is an increasing component of many disciplines including bio-sciences, medical sciences, business, advanced design, climate studies etc. Being an interdisciplinary approach, it is the core of computer simulation, physical sciences, quantitative analysis of financial issues, animation, numerical weather predictions above all it is a key of key technologies of our times. Mathematical models reveal the dynamic nature of societies which presents the instability of large complex society in a predictable & uncertain phenomenon through random & fuzzy variables. The **drastic climate** change and problem **COVID-19** a danger for the existence of humanity, has posed a challenge before the mathematical community and researchers to explore such models which can represent aspects of realty and changing behavior of the virus and system in befitting situation.

AJMI VOL.13 Issue 1 is before you. I am pleased to note that research scholars, professors, executives, from different parts of country and abroad have sent their papers for this issue. The papers are relevant and focus on the futuristic trends and innovations in the related areas. We have received around **34 papers** for this issue from which on reviewer's report only **19** have been selected for publication. **DOI** no. by cross Ref. have been mentioned on each article.

1. Prof. Temur Z. Kalanov a mathematician from Tashkent, Uzbekistan proposed a critical analysis of the starting point of mathematical logic which forces us to rethink about mathematical logical theory available in science.
2. Dr. G Kavitha developed two algorithms complement to each other for finding and maintaining the shortest hyper paths in a dynamic network.
3. Prof. Shankar Lal introduced recurrent manifolds on the H-projective curvature and studied their properties with regard to other similar tensors.
4. Prof. Himanshu Pandy & Arun Kumar Rao derived the expression for Bayes estimators under different loss functions using gamma prior.
5. The Economy activities in most part of world particular in India has suffered due to localised lockdown during second wave of COVID-19, Er. Dilip Adilitya Singh in his case study explained the market size of the analytics in Indian context and elaborate the list of challenges with recommendation to tackle them.
6. Prof. Manimannan G. etal. in his paper made an attempt to evaluate of First and Second Dose of COVID-19 vacation through **K-means cluster model** in Indian environment.
7. Prof. Mallaya etal. explored the numerical evaluation of a new Quintic Spline interpolation Formula.

I would like to thank and felicitate the contributors in this issue and at the same time invite quality papers from academic and research community for Vol.13 issue 2 to be published in Nov. 2021. Comments, suggestions and feedback from discerning readers, scholars and academicians are always welcome.

DR. T.P. SINGH

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FORMAL-LOGICAL ANALYSIS OF THE STARTING POINT OF MATHEMATICAL LOGIC

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ABSTRACT

The critical analysis of the starting point of mathematical (symbolic) logic is proposed. Methodological basis of the analysis is the unity of formal logic and rational dialectics. It is shown that mathematical logic represents incorrect mathematical theory. Arguments are as follows. (1) Pure mathematics operates on mathematical quantity. Pure mathematics and mathematical (symbolic) logic ignore the correct methodological basis (i.e., criterion of truth) of science. (2) Mathematics and mathematical (symbolic) logic abolishes (deletes) essence (central point) of formal logic: concept and proposition (as a logical form of verbal expression (statement) of thought). But thought without words does not exist. If concepts and propositions as meaning contents are removed from consideration, then formal logic is destroyed. (3) Mathematical (symbolic) logic replaces proposition by the property of proposition: "truth" or "false". But the concepts "proposition" and "property of proposition" are not identical. In this case, the formal logic is destroyed. In addition, the concepts "proposition" and "property of proposition" do not represent mathematical quantities, and the concepts cannot be in mathematical formalism. (4) The starting point of mathematical (symbolic) logic represents several symbols that connect (join, unite) the words "truth" and "false" in truth tables. But words "truth" and "false" do not represent mathematical quantities, and words "truth" and "false" cannot be in mathematical formalism. The symbols are erroneously called propositional connectives. The symbols denote the words "negation", "and", "or", "if, then", "if and only if", etc. (5) The symbols of common truth-functional operations are not mathematical symbols (i.e., symbols of mathematical operations). The definition of symbols is based on set theory. But set theory is an erroneous theory. Therefore, the use of these symbols in mathematical (quantitative) expressions is a gross mistake. (6) The essences of formal logic and mathematics are different. The qualitative aspect (i.e., meaning content) is the essence of formal logic; the quantitative (i.e., numerical) aspect is the essence of mathematics. Since formal logic has no quantitative aspect, and mathematics has no qualitative aspect, the union (join, conjunction, combination, synthesis) of formal logic and mathematics is a gross methodological error.

Thus, mathematical logic is a thoughtless, absurd theory in science.

Keywords: formal logic, mathematical logic, pure mathematics, philosophy of mathematics, methodology of mathematics, history of mathematics, higher education; history of science;, dialectics, epistemology.

MSC: 03A05, 03A10, 03B05, 03B30, 03B42, 03B44, 03B80, 00A30, 00A35.

INTRODUCTION

The truth or falsity of scientific theories depends on the methodological basis used. The correct methodological basis (as the criterion of truth) of science is the unity of formal logic and rational dialectics. If a theory is based on the correct methodological basis, then the theory satisfies the criterion of truth. In this case, the theory is true. If a theory does not meet the criterion of truth, then the theory is false. The truth (validity, verity) of the theory is the property of the theory. The falsity of the theory means the absence of this property.

Mathematics in initial point and in the final analysis is quantitative science (quantitative method of cognition): the science of numbers and calculations. Pure mathematics operates on mathematical quantity and abstract numbers. Mathematical quantity has no qualitative determinacy. Therefore, abstract numbers have no names and represent the values of the mathematical quantity.

Mathematical quantity and abstract numbers have no measure (measure is the philosophical category that designates inseparable unity (connection) of the qualitative and quantitative determinacy of an object) [1-48]. Consequently, mathematical quantity and abstract numbers cannot be used to describe reality. As is shown in works [1-48], pure mathematics (including set theory) is not based on the correct methodological basis. This signifies that pure mathematics (including set theory) is not a correct science.

As is known, mathematical (symbolic) logic as a branch of modern pure mathematics represents the following: (a) the unity of pure mathematics and formal logic; (b) "a subfield of mathematics exploring the applications of formal logics to mathematics" (Wikipedia); (c) mathematical representation (formulation) of formal logic; (d) mathematical methods of research of ways of reasoning (conclusions); (e) a mathematical theory of deductive methods of reasoning. Mathematical logic (symbolic logic) as a theory was created and developed by famous mathematicians (George Boole, Augustus De Morgan, Giuseppe Peano, Ernst Zermelo, David Hilbert, Kurt Gödel, Abraham Robinson, Paul Cohen, Gottlob Frege, Charles Peirce, Bertrand Russell and others) in the 19-20 centuries [49-59]. But this historical fact is not a scientific proof of the truth of mathematical logic as a theory. To date, there are no scientific works in the world literature, devoted to the analysis of this theory within the correct methodological basis.

The purpose of this work is to analyze the starting point of mathematical logic
within the correct methodological basis: the unity of formal logic and rational dialectics. Reliable sources of methodological basis are courses in formal logic and rational dialectics.

1. The essence of formal logic

1) By definition, formal logic is the science of the laws of correct thinking. The starting point and fundamental element of formal logic is a concept. A concept is a form of thought that expresses the essential features of objects and phenomena. A concept is expressed in a word or in several words (grammatical sentences). Concepts (thoughts) cannot be expressed without words and grammatical sentences.

2) The basis of formal logic is a system of concepts. The connection of concepts forms the structure of the system. The connection of concepts is expressed by the following words: "is", "is not", "if… is…, then…", "if… is not…, then…", "consequently".

3) Proposition as a logical form of verbal expression (utterance) of thought is the essence of formal logic. The definition of proposition is the following: proposition is a statement (i.e., the act of thinking and verbal expression of thought) about the existence or non-existence of an object or phenomenon; proposition is a statement about the properties of an object or phenomenon of reality; proposition is expressed in the statement of the existence or absence of certain features of objects and phenomena. A proposition connects concepts that logically express objects. There are no true propositions that connect concepts without objects. Also, there are no true propositions that connect objects without concepts of objects (in this case, the connection between objects is not a logical connection!). Therefore, a proposition

has the following two properties: (a) the property of assertion or negation; (b) the property of truth or false. This property is expressed in the following words: “truth” or “false”.

4) The connection (combination) of propositions, which represents deriving (extracting) a new proposition from one or more propositions, is called inference. The new proposition is called a conclusion (in Latin: conclusio). Those propositions from which a new proposition is derived (extracted, follows) are called premises (in Latin: praemissae). The relation between premises and conclusion is the relation between cause and effect. Inference is based on the law of sufficient reason.

5) Inferences are divided into the following two groups: direct inferences and mediated inferences. If a conclusion (proposition) is made from only one premise (proposition), then the inference is called direct inference. If a conclusion (proposition) is made from several premises (propositions), then the inference is called mediated inference.

6) Such expressions (word combinations) as “predicate logic”, “quantificational logic”, “propositional logic”, “inferencial logic”, “logic of justification”, “logic of evidence”, “class logic”, “epistemic logic ”,“logic of truth”, “feature logic”,“action logic”, “machine logic”, “logic of reasoning”, “decision logic”, “logic of strict implication”, “feature logic”, “logic of whole”, “logic of part”, etc. do not exist (are not allowed) in formal logic. In the point of view of formal logic, these expressions are absurd.

Thus, formal logic has no quantitative aspect.

2. The essence of mathematics

Starting point of mathematics is the art of computing (calculating). The numbers represent the initial and terminal point of mathematics. “The practical application of the results of theoretical mathematical research is impossible without expressing the results in numerical form” (Russian Wikipedia). Mathematical research is based on the basic laws of formal logic: the law of identity; the law of absence of contradiction; the law of the excluded third; the law of sufficient reason.

1) Applied mathematics is used in theoretical physics. For example, the physical relationship

$$v_M = \frac{S_M}{t}$$

(where S_M is a distance traveled by the material point M ; t is time (i.e., the universal informational quantity) of the motion of the material point M)

is the mathematical definition of the speed v_M of motion of the material point M .

2) In the point of view of formal logic, physical quantities v_M and S_M express concrete concepts: the concept v_M is the concept of the speed of motion of the material point M ; the concept S_M is the concept of the distance traveled by the material point M . In the point of view of rational dialectics, physical quantities v_M and S_M have measures as the unities of qualitative and quantitative determinacy. In other words, quantities v_M and S_M have dimensions $"\frac{meter}{second}"$ and $"meter"$, respectively.

3) Physical quantities v and S also have measures and dimensions $"\frac{meter}{second}"$ and $"meter"$, respectively. In the point of view of formal logic, the quantities v and S express abstract concepts, because properties of the quantities v and S do not belong to a concrete object. If the quantities v and S did not have dimensions (i.e. properties), then the quantities v and S would be called mathematical quantities.

4) The central point of pure mathematics is the expression $y = f(x)$, where x and y are mathematical quantities; f is a symbol of the functional connection of variables x и y. The quantities x and y have no dimensions. The quantities x and y take only numerical values. Numerical values of mathematical quantities x and y represent unnamed neutral numbers (i.e., numbers without names and signs $"+"$ and $"-"$) [28, 31-38, 40]. Abstract numbers have no names and represent the values of a mathematical quantity. For example,

$$x = \frac{p\ (meter)}{q\ (meter)} = \frac{p\ (ki\log ram)}{q\ (ki\log ram)} = \frac{p\ (\sec ond)}{q\ (\sec ond)}.$$

Unnamed numbers cannot be used in practice to describe reality.

5) In the point of view of rational dialectics, mathematical quantities x and y have no the measure as the unity of qualitative and quantitative determinacy. In the point of view of formal logic, quantities and numbers without qualitative determinacy (dimension) cannot be expressed in any concepts. Really, a concept expresses the essential features of objects and phenomena. But a mathematical quantity does not express the essential features of objects, because concepts of objects and phenomena do not exist in mathematics [28, 31-38, 40]. Therefore, a mathematical quantity cannot be defined (expressed) as a concept. This means that mathematical quantities are only letter symbols for unnamed neutral numbers. Consequently, mathematical quantities cannot be considered within the framework of formal logic [1-49].

6) Mathematical expressions are constructed in the following typical way:

$$x = x,\ \ ax = ax,\ \ ax + b = ax + b,\ \ y = ax + b,\ \ldots\ \ y = f(x)$$

where a and b are the numbers; the variables x (argument) and y (function) take numerical values. The symbol f signifies set of mathematical operations on the argument x. The symbols for mathematical operations are as follows: $"+"$, $"-"$, $"\cdot"$, $":"$. The symbol $"="$ signifies the word "is". The symbol $"\neq"$ signifies the word "is not". The symbols for comparison of numbers are as follows: $"="$, $"\neq"$, $">"$, $"<"$, $"\geq"$, $"\leq"$. The double symbol $"\geq"$ signifies the words "equal-to-or-greater-than"; the double symbol $"\leq"$ signifies the words "equal-to-or-less-than". (The meanings of the words "less than" and "equal to" (also, "greater than" and "equal to") are not intersected: there is no partial coincidence (intersection) between the meanings of these terms. Therefore, the word "or" has only one meaning: separation meaning).

The punctuation marks $","$, $";"$ are symbols of enumeration in sequence of numbers. The punctuation mark $"."$ is the symbol of cessation of enumeration. The brackets $(\ )$, $[\ ]$, $\{\ \}$ are symbols of indication (designation) of order of operations. The symbols $","$, $";"$, $"."$, $(\ )$, $[\ ]$, $\{\ \}$ are separation symbols. The symbols $"\cup"$ and $"\cap"$ are not symbols of mathematical operations, because the concept "set" is not a mathematical concept ("set" is a formal-logical concept).

The symbols $"+"$, $"-"$, $"\cdot"$, $":"$, $","$, $";"$, $"."$, $(\ )$, $[\ ]$, $\{\ \}$ represent the only and complete set (exhaustive set, full set) of symbols for mathematical operations on unnamed numbers. This means that these symbols represent a single and complete set (exhaustive set, full set) of symbols for mathematical operations on mathematical quantities in the final analysis.

Thus, the essence of pure mathematics is as follows: pure mathematics abstracts away from the essential properties (qualitative determinacy) of quantities. Pure mathematics operates only with unnamed numbers

within the framework of the basic laws of formal logic (i.e., the law of identity, the law of the absence of contradiction, the law of the excluded middle, the law of sufficient reason). The unnamed numbers and the listed symbols are the essence of pure mathematics.
Formal logic operates with concepts. Concepts and propositions are the essence of formal logic. There is no connection (relationship, dependence) between unnamed numbers and concepts (propositions). Mathematical expressions are not propositions. Consequently, there is no connection (relations) between the essence of pure mathematics and the essence of formal logic. The connection (relation) between the essence of pure mathematics and the essence of formal logic would exist if these essences were identical.
Thus, pure mathematics has no qualitative aspect [1-49].

3. The essence of mathematical logic

By definition, mathematical logic is: (a) a mathematical representation (formulation) of formal logic; (b) the unity (combination) of pure mathematics and formal logic. To join (combine) pure mathematics and formal logic, one must remove concepts from formal logic. If one removes concepts from formal logic, then the following basic laws of thinking remain: (1) the law of identity, (2) the law of absence of contradiction, (3) the law of the excluded third, (4) the law of sufficient reason. But if one replaces concepts by the words “truth” and “false”, then the laws of formal logic lose meaning.
1) As is known, the starting point of mathematical logic is the following statements [49-59].
“One of the popular definitions of logic is that it is the analysis of methods of reasoning. In studying these methods, logic is interested in the form rather than the content of the argument. For example, consider the two arguments:
1. All men are mortal Socrates is a man. Hence, Socrates is mortal.
2. All cats like fish. Silly is a cat. Hence, Silly likes fish.
Both have the same form: All **A** are **B**. **S** is an **A**. Hence, **S** is a **B**. The truth or falsity of the particular premises and conclusions is of no concern to logicians. They want to know only whether the premises imply the conclusion. The systematic formalization and cataloguing of valid methods of reasoning are a main task of logicians. If the work uses mathematical techniques or if it is primarily devoted to the study of mathematical reasoning, then it may be called mathematical logic. We can narrow the domain of mathematical logic if we define its principal aim to be a precise and adequate understanding of the notion of mathematical proof.
Impeccable definitions have little value at the beginning of the study of a subject. The best way to find out what mathematical logic is about is to start doing it, and students are advised to begin reading the book even though (or especially if) they have qualms about the meaning and purpose of the subject.
Although logic is basic to all other studies, its fundamental and apparently self-evident character discouraged any deep logical investigations until the late 19th century. Then, under the impetus of the discovery of non-Euclidean geometry and the desire to provide a rigorous foundation for calculus and higher analysis, interest in logic revived. This new interest, however, was still rather unenthusiastic until, around the turn of the century, the mathematical world was shocked by the discovery of the paradoxes – that is, arguments that lead to contradictions.
Sentences may be combined in various ways to form more complicated sentences. We shall consider only truth-functional combinations, in which the truth or falsity of the new sentence is determined by the truth or falsity of its component sentences.

Negation is one of the simplest operations on sentences. Although a sentence in a natural language may be negated in many ways, we shall adopt a uniform procedure: placing a sign for negation, the symbol $\neg$, in front of the entire sentence. Thus, if **A** is a sentence, then $\neg$**A** denotes the negation of **A**. The truth-functional character of negation is made apparent in the following truth table:

A	$\neg$**A**
T	**F**
F	**T**

When **A** is true, $\neg$**A** is false; when **A** is false, $\neg$**A** is true. We use **T** and **F** to denote the truth values true and false.

Another common truth-functional operation is the conjunction: 'and'. The conjunction of sentences **A** and **B** will be designated by **A**$\wedge$**B** and has the following truth table:

A	**B**	**A**$\wedge$**B**
T	**T**	**T**
F	**T**	**F**
T	**F**	**F**
F	**F**	**F**

A$\wedge$**B** is true when and only when both **A** and **B** are true. **A** and **B** are called the conjuncts of **A**$\wedge$**B**. Note that there are four rows in the table, corresponding to the number of possible assignments of truth values to **A** and **B**.

In natural languages, there are two distinct uses of 'or': the inclusive and the exclusive. According to the inclusive usage, '**A** or **B**' means '**A** or **B** or both', whereas according to the exclusive usage, the meaning is '**A** or **B**, but not both'. We shall introduce a special sign, $\vee$, for the inclusive connective. Its truth table is as follows:

A	**B**	**A**$\vee$**B**
T	**T**	**T**
F	**T**	**T**
T	**F**	**T**
F	**F**	**F**

Thus, **A**$\vee$**B** is false when and only when both **A** and **B** are false. '**A**$\vee$**B**' is called a disjunction, with the disjuncts **A** and **B**.

Another important truth-functional operation is the conditional: 'if **A**, then **B**'. Ordinary usage is unclear here. Surely, 'if **A**, then **B**' is false when the antecedent **A** is true and the consequent **B** is false. However, in other cases, there is no well-defined truth value. For example, the following sentences would be considered neither true nor false:

1. If $1+1=2$, then Paris is the capital of France.
2. If $1+1\neq 2$, then Paris is the capital of France.
3. If $1+1\neq 2$, then Rome is the capital of France.

Their meaning is unclear, since we are accustomed to the assertion of some sort of relationship (usually causal) between the antecedent and the con-sequent. We shall make the convention that 'if **A**, then **B**' is false when and only when **A** is true and **B** is false. Thus, sentences 1-3 are assumed to be true. Let us denote 'if **A**, then **B**' by '**A** $\Rightarrow$ **B**'. An expression '**A** $\Rightarrow$ **B**' is called a conditional. Then $\Rightarrow$ has the following truth table:

A	B	A $\Rightarrow$ B
T	T	T
F	T	T
T	F	F
F	F	T

This sharpening of the meaning of 'if **A**, then **B**' involves no conflict with ordinary usage, but rather only an extension of that usage.

A justification of the truth table for $\Rightarrow$ is the fact that we wish 'if **A** and **B**, then **B**' to be true in all cases. Thus, the case in which **A** and **B** are true justifies the first line of our truth table for $\Rightarrow$, since (**A** and **B**) and **B** are both true. If **A** is false and **B** true, then (**A** and **B**) is false while **B** is true. This corresponds to the second line of the truth table. Finally, if **A** is false and **B** is false, (**A** and **B**) is false and **B** is false. This gives the fourth line of the table.

Let us denote '**A** if and only if **B**' by '**A** $\Leftrightarrow$ **B**'. Such an expression is called a biconditional. Clearly, '**A** $\Leftrightarrow$ **B**' is true when and only when **A** and **B** have the same truth value. Its truth table, therefore is:

A	B	A $\Leftrightarrow$ B
T	T	T
F	T	F
T	F	F
F	F	T

The symbols $\neg$, $\wedge$, $\vee$, $\Rightarrow$ and $\Leftrightarrow$ will be called propositional connectives. Any sentence built up by application of these connectives has a truth value that depends on the truth values of the constituent sentences. In order to make this dependence apparent, let us apply the name statement form to an expression built up from the statement letters **A**, **B**, **C**, and so on by appropriate applications of the propositional connectives" [54].

4. Critical analysis of the starting point of mathematical logic

1) As is known, the scope (volume) of general concepts is expressed in the form of a class. The logical class is a collection of objects that have common essential features (characteristics). As a consequence, these objects are covered by the general concept. One class is superior to another class if it includes another class together with other classes. The class that is superior to another is called a genus. The class that is inferior to the genus is called a species.

2) In the point of view of formal logic, the division of propositions is based on the existence of an essential feature (property) in one group of propositions and the absence of this feature (property) in another group of propositions. The proposition has the following essential feature: the property of truth or falsity. The property of falsity is the absence of the property of truth. There is the logical relation of disagreement (contradiction) between true propositions and false propositions: the feature of true negates the feature of falsity. The property of true or falsity of a proposition is determined within the framework of the correct methodological basis: the unity of formal logic and rational dialectics. The unity of formal logic and rational dialectics is the criterion of truth. The property of true or false is expressed by the following words: "truth" or "false". The properties "truth" and "false" are contradictory. Therefore, the

set of propositions can be divided into two non-overlapping classes: the "class of true propositions" and the "class of false proposition":

$$A_T = \{a_1,\ a_2, \dots, a_n, \dots\}, \quad a_n \notin B_F;$$
$$B_F = \{b_1, b_2, \dots, b_m, \dots\}, \quad b_m \notin A_T$$

where A_T is the "class of true propositions", B_F is the "class of false proposition", the element a_n is the true proposition, the element b_m is a false proposition.

The classes A_T and B_F are subordinate classes (subclasses) of the generic class C. Therefore, the volumes (scopes) of propositions are connected by the following relationship:

$$V_C = V_{A_T} + V_{B_F}.$$

This relationship corresponds to the following diagram:

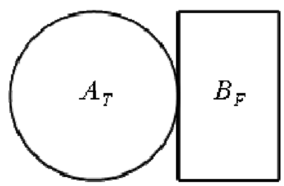

Figure 1. Graphical interpretation of the relationship $V_C = V_{A_T} + V_{B_F}$.

Hence it follows that the standard relationships

$$C = A_T \cup B_F,$$
$$C = A_T \cap B_F$$

and corresponding diagrams

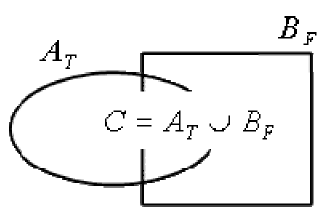

Figure 2. Graphical interpretation of the relationship $C = A_T \cup B_F$.

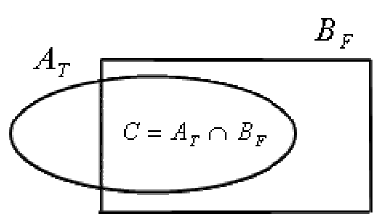

Figure 3. Graphical interpretation of the relationship $C = A_T \cap B_F$. represent formal-logical errors.

REMARK

The symbols "$\cup$" and "$\cap$" are notations of the words "or" and "and", respectively. Therefore, the symbols "$\cup$" and "$\cap$" are not symbols of mathematical (quantitative, numerical) operations, because set is not a mathematical concept. Set is a formal-logical concept. In the formal-logical point of view, there is the only one correct operation in the set theory: decomposition (partition) of set in terms of the non-overlapping (non-intersecting) sets [41, 43]. The concepts "set" and "class" are identical. Division (decomposition) of a set into the non-overlapping (non-intersecting) classes represents the inverse operation with respect to the union of the non-overlapping (non-intersecting) classes. This implies that the standard operation "$\cup$" is wrong because the standard operation "$\cap$" is wrong [41, 43]. Thus, the correct use of the word "or" is that the word "or" must have the exclusive meaning: "given proposition or other proposition, but not both". The correct use of the word "and" must have the following meaning: union of the non-intersecting (non-overlapping) classes. In this case, the symbol "$\cup$" (arises from the word "union") can be used for notation of the word "and'.

3) Proposition has two important and connected aspects: aspect of truth and aspect of content (meaning). Truth does not exist without content (meaning); content (meaning) is useless without truth. Therefore, the proposition has two important features (properties): "truth" and "content (meaning)". "Truth" is an essential feature (property) of the element a_n of the class A_T. The elements of the class A_T are independent elements. "False" is an essential feature (property) of the element b_m of the class B_F. The elements of the class B_F are independent elements. The concepts "proposition" and "property of proposition" are not identical.

(a) If one takes into account only the essential feature (property), "truth", then one comes to the following uniquely correct relationship between true propositions:

(truth) is (truth),
i.e. *(truth) = (truth).*

But mathematical logic contains standard truth tables that contradict to formal logic. For example, the following common truth-functional operations

$$(false) \lor (truth) = (truth),$$
$$(truth) \lor (false) = (truth);$$
$$(false) \land (truth) = (false),$$
$$(truth) \land (false) = (false);$$
$$(false) \Rightarrow (truth) = (truth),$$
$$(truth) \Rightarrow (false) = (false)$$

represent absurd. The absurdity is that the contradictory properties ("truth" and "false") of the elements a_n and b_m are in the left-hand sides of the common (united) relationships in mathematical logic. In addition, the symbols "$\neg$", "$\lor$", "$\land$", "$\Rightarrow$", "$\Leftrightarrow$", etc. are not symbols of mathematical (quantitative) relationships. The symbols "$\neg$", "$\lor$", "$\land$", "$\Rightarrow$", "$\Leftrightarrow$", etc. are symbols of qualitative relationships. Thus, common truth-functional operations in mathematical logic represent formal-logical errors.

(b) If one also takes into account the feature (property) "meaning content", then one can establish logical (but not quantitative) relations between the elements (propositions) of the class A_T . The property (feature) "meaning content" is not subject to mathematical (quantitative) operations. Establishment of relations between propositions is a formal-logical problem. This problem is a solved problem in formal logic.

Formal-logical solution is the following statements: (i) proposition does not exist without grammatical (verbal) form of expression of thought; (ii) proposition is the logical content of a grammatical sentence; (iii) proposition is a system of concepts (i.e., connection of concepts) defined and expressed by words and grammatical sentences. Hence, if one removes concepts, then one removes propositions from formal logic. In this case, formal logic loses its scientific meaning, and science loses its correct methodological basis.

Consequently, use of the symbols "$\neg$", "$\vee$", "$\wedge$", "$\Rightarrow$", "$\Leftrightarrow$", etc. instead of words is meaningless, useless, fruitless, unsuccessful attempt of the junction of formal logic and mathematics. The main formal-logical error in mathematical logic is that propositions are replaced by the properties: "truth" and "false".

Thus, the junction (unification) of formal logic and pure mathematics is impossible, because formal logic has no quantitative aspect, and pure mathematics has no qualitative aspect. Mathematics (as a quantitative science) cannot exist in formal logic; formal logic cannot be squeezed into mathematics.

DISCUSSION

1. Why are scientists wrong? My 40 years experience of critical analysis of the foundations of theoretical physics and mathematics [1-48] shows that the main causes are as follows:

(a) firstly, the haste and immaturity of thinking intrinsic (proper, inherent) to youth;

(b) secondly, the unwillingness and inability (inefficiency, disability) of the scientist to find the correct methodological basis and criterion of truth;

(c) thirdly, the reluctance of a scientist to admit (to acknowledge) the existence of errors in science.

2. The essence of mathematics is that mathematics is a special science that does not rely on the correct methodological basis: the unity of formal logic and rational dialectics [1-48]. The unity of formal logic and rational dialectics represents the correct criterion of truth. Therefore, mathematics does not contain the dialectical concept "measure as the unity of qualitative and quantitative determinacy (aspects)". Mathematics has the quantitative aspect but not the qualitative aspect. This means that mathematics does not contain the criterion of truth and the methodological basis [1-48]. Mathematical thinking (reasoning) ignores practice. Therefore, mathematical thinking (reasoning) is narrow, limited thinking (reasoning).

3. The essence of formal logic is that formal logic is a general science of the laws of correct thinking. Therefore, formal logic has the qualitative aspect (meaning content) but not the quantitative aspect. The unity of formal logic and rational dialectics is the correct criterion of truth and the correct methodological basis of science.

4. Junction (unification) of formal logic and mathematics is impossible as it is impossible to join philosophy (in particular, dialectics) and mathematics. The explanation is the fact that the essence of formal logic and mathematics are different. Therefore, mathematical logic is a gross methodological error. The desire of mathematicians to substantiate the junction of formal logic and mathematics is meaningless, useless, fruitless, unsuccessful effort.

CONCLUSION

A critical analysis of the starting point of mathematical (symbolic) logic within the correct methodological basis (i.e., the unity of formal logic and rational dialectics) leads to the following statements:

(1) Pure mathematics operates on mathematical quantity. Pure mathematics and mathematical (symbolic) logic ignore the correct methodological basis (i.e., criterion of truth) of science.

(2) Mathematics and mathematical (symbolic) logic abolishes (deletes) essence of formal logic: concept and proposition (as a logical form of verbal expression (statement) of thought). But thought without words does not exist. If concepts and propositions as meaning contents are removed from consideration, then formal logic is destroyed.

(3) Mathematical (symbolic) logic replaces proposition by the property of proposition: "truth" or "false". But the concepts "proposition" and "property of proposition" are not identical. In this case, the formal logic is destroyed. In addition, the concepts "proposition" and "property of proposition" do not represent mathematical quantities, and the concepts cannot be in mathematical formalism.

(4) The starting point of mathematical (symbolic) logic represents several symbols that connect (join, unite) the words "truth" and "false" in truth tables. But the words "truth" and "false" do not represent mathematical quantities, and the words "truth" and "false" cannot be in mathematical formalism. The symbols are erroneously called propositional connectives. The symbols denote the words "negation", "and", "or", "if, then", "if and only if", etc.

(5) The symbols of common truth-functional operations are not mathematical symbols (i.e., symbols of mathematical operations). The definition of symbols is based on set theory. But set theory is an erroneous theory. Therefore, the use of these symbols in mathematical (quantitative) expressions is a gross mistake.

(6) The essences of formal logic and mathematics are different. The qualitative aspect (i.e., meaning content) is the essence of formal logic; the quantitative (i.e., numerical) aspect is the essence of mathematics. Since formal logic has no quantitative aspect, and mathematics has no qualitative aspect, the union (join, conjunction, combination) of formal logic and mathematics is a gross methodological error.

Thus, mathematical logic is a thoughtless, absurd theory in science.

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SELECTING TEACHERS FOR QUALITY AWARDS THROUGH FUZZY INFERENCE SYSTEM

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ABSTRACT :
Fuzzy inference system is a combination of fuzzy logic and fuzzy set theory. In this paper we have applied fuzzy inference technique for evaluation of performance of teachers in various field. The fuzzy decision tree for evaluating staff performance have been drawn. A software package named Geodecision tree is used for computing decision trees.
Key words: Fuzzy reasoning approach, Decision tree.

INTRODUCTION

Fuzzy inference techniques are used in selecting teacher for awards and proposed an approach which is a combination of two membership functions [5]. A fuzzy expert system for select the staff's Excellence in academics, Excellence in co-curricular activity, Excellence in pedagogy, and Excellence in instructional leadership awards. Fuzzy logic techniques are used to provide practical method for staff awards with existing statistical method.

In this paper a new methodological approach using fuzzy logic reasoning has been proposed for quality awards to the teachers.

Fuzzy logic reasoning approach via Methodology:

On combining Fuzzy logic and fuzzy set theory we find a fuzzy inference system. Fuzzy inference systems are knowledge based or rule based systems that contain descriptive if then rules created from our experience. The fuzzy inference system represents the core of fuzzy logic controllers and it‘s built of rule base and data-base which constitute the knowledge base and inference engine. The fuzzy reasoning approach has found a wide application in designing of certain complex industrial and management systems which cannot be modeled precisely under various assumptions and approximations [2].

A software tool named Geo decision tree is used for computing decision trees. The algorithm makes application of two phase approach consisting of a FDT generation phase and a FDT concatenation phase in order to construct a decision tree. In the first phase, a set of FDT are generated such that there is at least one decision tree composed of the FDT in the set only. In the second phase, a subset of FDTs are selected and combined to form a decision tree. On the rectilinear plane, Geo decision remains the fastest exact algorithm for the RSMT problem but it has a limitation that on existing the obstacles in the routing plane it can't be applied.

Special awards for teacher: Selecting teachers for awards in the academic year we have proposed an approach which is a combination of membership functions. The selection of membership is based on his activities during the academic year.

excellent teacher award in academic year

- excellent teacher award in co-curricular activity
- excellent teacher award in psychology
- excellent teacher award in instructional leadership

- **The purposes of the award**

  The teaching excellence awards have several purposes:

- To recognise and reward outstanding teaching
- To recognise and reward the significant contribution made by individuals to enhancing the quality of learning and teaching in the institution
- To honor, award and convey gratitude to teachers for their excellent teaching performance
- To promote and support the highest quality teaching and learning in the institution
- To stimulate, encourage and support new members of staff in their teaching careers

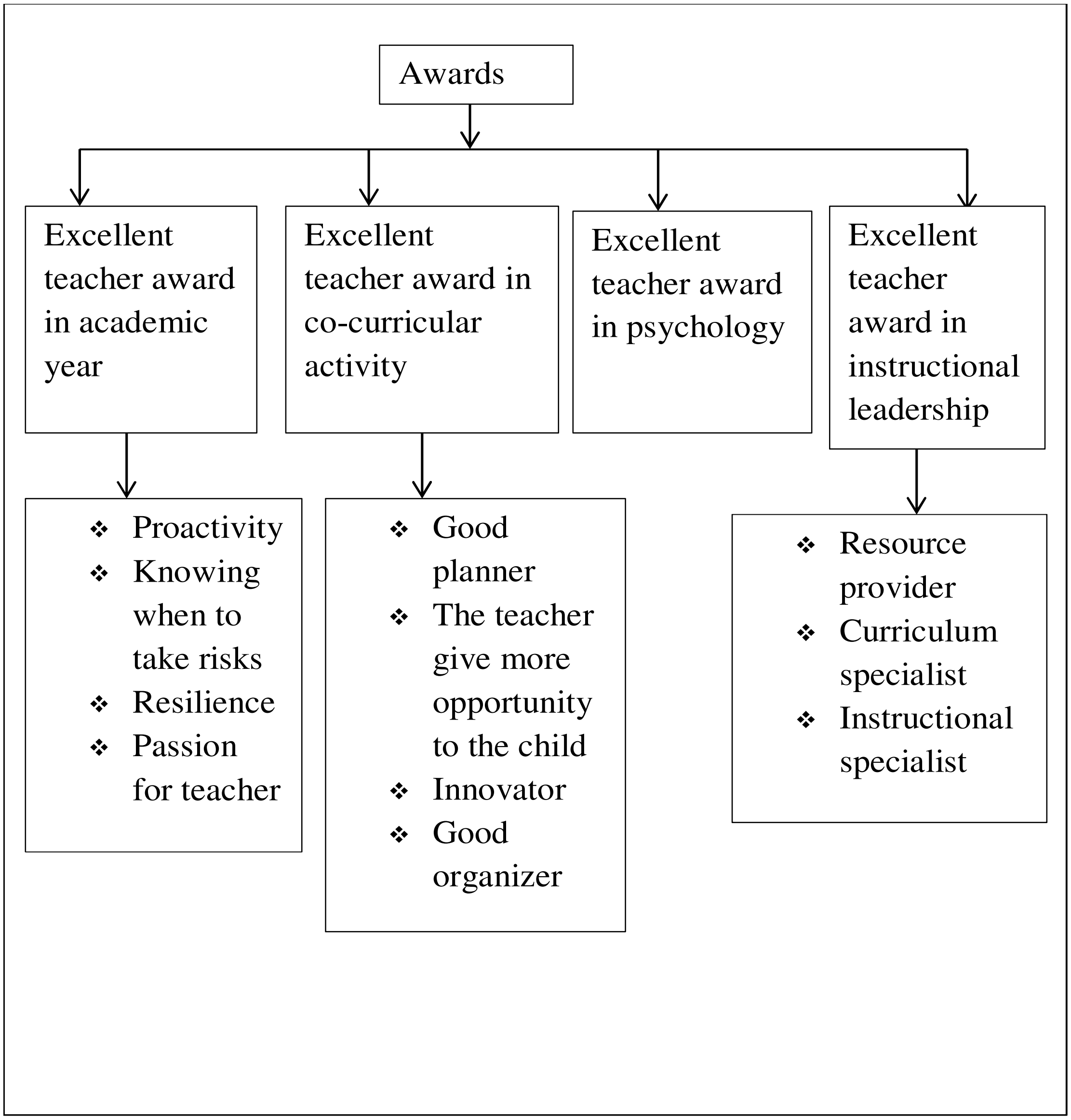

Fig 1.1Awards data

S.NO	Teachers name	Excellent teacher award in academic year	Excellent teacher award in co-curricular activity	Excellent teacher award in psychology	Excellent teacher award in instructional leadership
1	M.VIJAYA	0.8	0.8	0.7	0.9
2	C.SUJATHA	0.9	0.7	0.5	0.8
3	V.MEKALA	0.6	0.9	0.4	0.6
4	M.RAJESH	0.5	0.6	0.9	0.6

Table 1.1 Results for teachers

Examples

We performed many experiments used and draw a decision tree and the results are calculated on average rating method. Data are collected in Marudhupandiyar college Thanjavur.

Experiment 1.1

Teachers name	Excellent teacher award in academic year	Excellent teacher award in co-curricular activity	Excellent teacher award in psychology	Excellent teacher award in instructional leadership	Average rating
M.VIJAYA	0.8	0.8	0.7	0.9	0.8

Table1.2 Award in instructional leadership

Experiment 1.2

Teachers name	Excellent teacher award in academic year	Excellent teacher award in co-curricular activity	Excellent teacher award in psychology	Excellent teacher award in instructional leadership	Average rating
C.SUJATHA	0.9	0.7	0.5	0.8	0.73

Table 1.3 Award in academic year

Experiment 1.3

Teachers name	Excellent teacher award in academic year	Excellent teacher award in co-curricular activity	Excellent teacher award in psychology	Excellent teacher award in instructional leadership	Average rating
V.MEKALA	0.6	0.9	0.4	0.6	0.63

Table 1.4 Award in co-curricular activity

Experiment 1.4

Teachers name	Excellent teacher award in academic year	Excellent teacher award in co-curricular activity	Excellent teacher award in psychology	Excellent teacher award in instructional leadership	Average rating
M.RAJESH	0.5	0.6	0.9	0.6	0.65

Table 1.5 Award in psychology

S.NO	Teachers name	Excellent teacher award in academic year	Excellent teacher award in co-curricular activity	Excellent teacher award in psychology	Excellent teacher award in instructional leadership	Average rating
1	M.VIJAYA	0.8	0.8	0.7	0.9	0.8
2	C.SUJATHA	0.9	0.7	0.5	0.8	0.73
3	V.MEKALA	0.6	0.9	0.4	0.6	0.63
4	M.RAJESH	0.5	0.6	0.9	0.6	0.65

Table 1.6 Overall championship award

Theorem 1.1

For any k,m and n ,there exists a number N(k,m,n) such that the following is true.Let S be a total ordered set of size at least N(k,m,n); let $P_{1,\ldots,}P_k$ be k predicates defined on S^n. Then there exists a subset of C□S of size at least m such that each predicate P_i is order invariant on C^n.

Proof:

Let $\{x_1,\ldots.,x_r\}$ and $\{y_1,\ldots.,y_r\}$ be two r element subsets of S, indexed in increasing order. We say that$\{x_1,\ldots.,x_r\}$ is congruent to $\{y_1,\ldots.,y_r\}$ if, for each mapping σ:1…n→1…r, and each $1 \leq j \leq k$.

$$P_j(x_{\sigma(1)},\ldots,x_{\sigma(n)}) \text{ iff } p_j(y_{\sigma(1)},\ldots,y_{\sigma(n)}).$$

It is easy to see that this indeed an equivalence relation on $[S]^r$. The number G of equivalence classes of this relation is bounded by 2^{krn}. According to ramsey's theorem, for any S there is a number N=N(k,s,G) such that, if $|S| \geq N$, then S contains a subset S□ such that $|S| \geq s$ and all elements of $[S□]^k$ belong to the same congruence class.

If S is a large enough, we can repeat this process for k=1,…,n, thus building a sequence of sets $S=C_0 \supset C_1 \supset \ldots. \supset C_n=C$, such that $|C| \geq m$, and all elements of $[C_k]^k$ are congruent, k=1,…n.

Let x and y be two order equivalent tuples in C^n. Let $x_1^{□},\ldots x_k^{□}$ be the distinct components of x, indexed in increasing order, and let $y_1^{□},\ldots y_k^{□}$ be similarly defined for y. Let σ: 1…n→1…k be such that $x_i=x^{□}_{\sigma(i)}$,i=1,…,n. Since $x \equiv y$, it follows that $y_i=y^{□}_{\sigma(i)}$, i=1,…,n. Since $\{x_1^{□},\ldots.,x_k^{□}\}$ is congruent to $\{y_1^{□},\ldots.,y_k^{□}\}$ $P_j(x)$ iff $P_J(y)$, for j=1,…k.

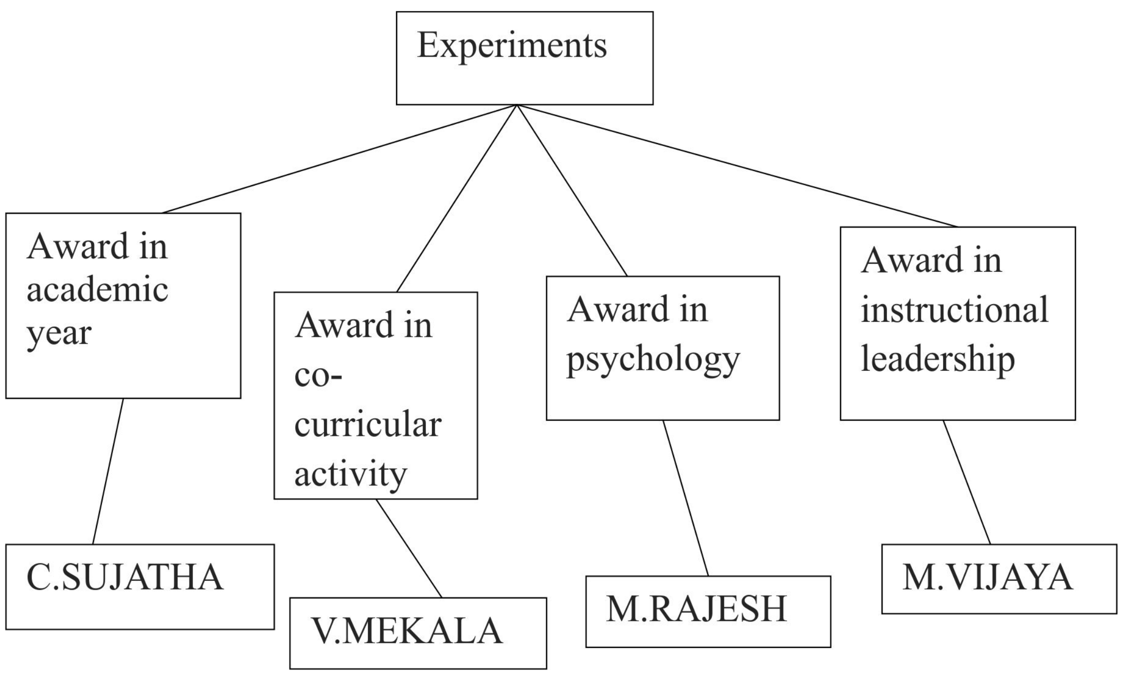

Fig 1.2 Decision tree

CONCLUSION

Fuzzy logic reasoning approach have been used to select a teacher for awards. Finally we draw the decision tree model. It is very easy to select teachers for awards without ambiguity.

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ANALYTICAL STUDY OF DETERMINISTIC INVENTORY MODEL FOR DETERIORATING ITEMS WITH CUBIC DEMAND RATE

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ABSTRACT
This paper explores a deterministic inventory mathematical model for deteriorating items with cubic demand function with respect to time. In the model, shortages are also allowed. Further, it shows that the cubic demand function is convex and gives the optimal solution. A 3-Dimensional graphical representation is used to show the convexity of this model. An illustration is also given to verify the various effects of parameters that are used to calculate the total inventory cost.
Keywords : Deterioration, Cubic demand, Shortages, Total inventory cost.

INTRODUCTION

Inventory can be explained as the stock of goods that are stored for the effective and smooth functioning of trade or business. It also helps us in the enhancement of the business. It is the key point of manufacturers, schools, farms, hospitals, and institutions of higher education. Both retailers and wholesalers need to conserve inventories of items or products. It consists of various divergent conditions that can be converted into models. These circumstances may consist of the deterministic demand, change in demand with time, deterioration, etc.

The mathematical model is distributed into two kinds, i.e., Deterministic and Stochastic models. In the Deterministic model, the demand rate is constant. Demand plays a very important part in inventory. In the classical inventory model, the demand rate was supposed to be constant, but it is not always possible.?

The assumption of constant demand rate is not appropriate for many inventory products or items like fashionable things, dairy products, electronic items, fruits, and vegetables etc.; the demand rate may be time-dependent, price dependent, and stock dependent. The products like fruits, vegetables, drugs, dairy products, etc., are having a limited lifetime. They decay according to time. Such type of items or products is known as deteriorating items. Due to the deterioration of items or products, the inventory system faces many problems.

The demand rate could also be linear in change, i.e., linearly increase or decrease in demand with time. For linear demand, we use linear polynomial as a demand function. Sometimes it should be a profound change in demand, i.e., demand is increasing so fast with relevancy time. For such increment, cubic polynomial demand function can be used. Cubic demand help to optimize the entire inventory cost and enhance the business.

Ghosh, S.K., and Chaudhuri, K.S. (2004) developed an order-level inventory model for a deteriorating item with Weibull distribution deterioration. Liang-Yuh Ouyang, Kun-Shan WU, and Mei-Chuan CHENG (2005) proposed an EOQ inventory mathematical model for deteriorating items with exponentially function as decreasing demand. Ajanta Roy(2008) developed an inventory model for deteriorating items with price dependent demand. He had also taken time dependent holding cost.C. K. Tripathy* and U. Mishra (2010) developed a listing model when the deterioration rate follows Weibull two-parameter distributions. R. Amutha and Dr. E. Chandrasekaran (2012) presented a list model for deteriorating products. In this model, demand was assumed as a linear function of time. Holding Cost takes as a linear function of time. Ravish Yadav etal (2013) shown the Volume Flexibility in Production Model with Cubic Demand Rate and Weibull Deterioration with partial backlogging. Venkateswarlu R. and Mohan R. (2014) developed a list model for deteriorating products assuming that demand rate is a quadratic function of time. This model is developed for optimizing the full Inventory Cost (TIC). Shortages are allowed and partially backlogged. Garima Sharma and Bhawna Vyas (2018) proposed a deterministic inventory model with Weibull distribution for deteriorating products. Demand rate assumed as a linear function of time within which shortages are allowed and partially backlogged. Ganesh Kumar, Sunita, and Ramesh Inaniyan (2020) proposed a listing model by using various parameters with a time-dependent demand rate. The demand rate is assumed to be a cubical polynomial of time. Recently Yadav H, Singh T.P. & Vinod Kumar (2020) explored an inventory model for Weibull rate of deteriorating items, shortages, partial backlogging with quadratic demand rate.

In this paper, the demand rate is taken as a cubical polynomial of time, and the deterioration rate varies with time. Ordering cost assumed to be constant and does not change with time. A 3-Dimensional graphical representation is used to check the convexity of this model. An illustration is also taken to verify the model. Sensitivity analysis of the optimal solution with respect to major parameters has been carried out and effects are shown.

ASSUMPTIONS AND NOTATION:

This mathematical model is described by using the following assumptions and notations

Assumptions

1. w, x, y, and z are constant.
2. The demand rate $f(t)$ at time t is assumed as $f(t) = w + xt + yt^2 + zt^3$;w, x, y, z is constants.
3. Replenishment occurs.
4. Shortages are allowed.
5. $\theta(t) = \theta t$ Is deterioration rate.

Notations

C_{SC} Shortage cost per unit per unit time.

C_{OC} Ordering cost per order.

C_{DC} Deterioration cost.

C_{HC} Holding Cost.

W The maximum inventory level for each ordering cycle.

S Shortage level for each cycle.
Q The order quantity ($Q = W + s$).
$Q(t)$ Inventory level at time t.
t_1 Time at which shortages start.
T The total length of each ordering cycle.
TC Total inventory cost over the period (0, T).

MATHEMATICAL FORMULATION:

The following graph (Figure 1) reveals the behavior of inventory at any time. This shows the optimal order quantity, which is Q, and total optimal inventory cost TC

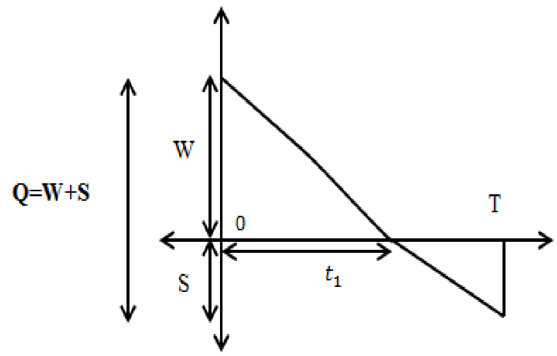

Figure 1: Inventory level (Q) vs Time

The inventory level is maximum at t=0 and replenishment is made. After this, inventory level decreases within the time period [0, t], and ultimately falls to zero at t=t_1.Further, at t_1,shortages occur during the time interval $[t_1, T]$.

Now till the shortages are allowed at interval $[0, t_1]$, the differential equation is given by:

$$\frac{dQ_1(t)}{dt} + \theta t Q_1(t) = -(w + xt + yt^2 + zt^3); 0 \le t \le t_1 \quad (1)$$

And during the interval $[t_1, T]$, the shortage occurs, so the differential equation is given by:-

$$\frac{dQ_2(t)}{dt} = -(w + xt + yt^2 + zt^3); t_1 \le t \le T \quad (2)$$

With the boundary conditions: t=0, Q(0)=W

T=t_1; Q(t_1)=0

T=T; Q(T)=S

Now, by solving the above equations (1), we get:

$$Q_1(t) = w(t_1 - t) + \frac{x}{2}(t_1^2 - t^2) + \frac{y}{3}(t_1^3 - t^3) + \frac{z}{4}(t_1^4 - t^4) + \frac{w\theta}{6}(t_1^3 - t^3) + \frac{x\theta}{8}(t_1^4 - t^4) + \frac{y\theta}{10}(t_1^5 - t^5) + \frac{z\theta}{12}(t_1^6 - t^6) - \frac{w\theta}{2}(t^2 t_1 - t^3) - \frac{x\theta}{4}(t^2 t_1^2 - t^4) - \frac{y\theta}{6}(t^2 t_1^3 - t^5) - \frac{z\theta}{8}(t^2 t_1^3 - t^5) - \frac{w\theta^2}{12}(t^2 t_1^3 - t^5) - \frac{x\theta^2}{16}(t^2 t_1^4 - t^6) - \frac{y\theta^2}{20}(t^2 t_1^5 - t^7) - \frac{z\theta^2}{24}(t^2 t_1^6 - t^8) \quad (3)$$

By solving equation (2) we get:

$$Q_2(t) = w(t_1 - t) + \frac{x}{2}(t_1^2 - t^2) + \frac{y}{3}(t_1^3 - t^3) + \frac{z}{4}(t_1^4 - t^4) \qquad (4)$$

Now, at t=0, the maximum inventory level for each cycle is given by:

Q(0)=W, t=0

$$W = Q_1(0) = wt_1 + \frac{x}{2}t_1^2 + \frac{y}{3}t_1^3 + \frac{y}{4}t_1^4 + \frac{w\theta}{6}t_1^3 + \frac{x\theta}{8}t_1^4 + \frac{y\theta}{10}t_1^5 + \frac{z\theta}{12}t_1^6$$

And at t=T, the maximum amount of cubic demand per cycle is given by:

t=T, $Q_2(t)$= -S

$$S = -w(t_1 - T) - \frac{x}{2}(t_1^2 - T^2) - \frac{y}{3}(t_1^3 - T^3) - \frac{z}{4}(t_1^4 - T^4)$$

Now, the order quantity per cycle is:

$$Q = W + S = wt_1 + \frac{x}{2}t_1^2 + \frac{y}{3}t_1^3 + \frac{y}{4}t_1^4 + \frac{w\theta}{6}t_1^3 + \frac{x\theta}{8}t_1^4 + \frac{y\theta}{10}t_1^5 + \frac{z\theta}{12}t_1^6 - w(t_1 - T) - \frac{x}{2}(t_1^2 - T^2) - \frac{y}{3}(t_1^3 - T^3) - \frac{z}{4}(t_1^4 - T^4) \qquad (5)$$

Holding cost per unit per unit time is given by:

$$\text{Holding Cost per cycle} = C_{HC}\int_0^{t_1} Q_1(t)dt$$

$$\text{Holding Cost per cycle} = C_{HC}\left(\frac{wt_1^2}{2} + \frac{xt_1^3}{3} + \left(\frac{y}{4} + \frac{w\theta}{12}\right)(t_1^4) + \left(\frac{x\theta}{15} + \frac{z}{5}\right)(t_1^5) + \left(\frac{y\theta}{18} - \frac{w\theta^2}{72}\right)(t_1^6) + \left(\frac{z\theta}{21} - \frac{x\theta^2}{84}\right)(t_1^7) - \frac{Y\theta^2 t_1^8}{96} - \frac{z\theta^2 t_1^9}{108}\right) \qquad (6)$$

Shortages cost per unit per unit time is given by:

$$\text{Shortage Cost per cycle} = (-)\, C_{SC}\int_{t_1}^{T} Q_2(t)$$

$$\text{Shortage Cost per cycle} = -C_{SC}\left[w\left(t_1 T - \frac{T^2}{2} - t_1\right) + x\left(\frac{t_1^2 T}{2} - \frac{T^3}{6} - \frac{t_1^3}{3}\right) + y\left(\frac{t_1^3 T}{3} - \frac{T^4}{12} - \frac{t_1^4}{4}\right) + z\left(\frac{t_1^4 T}{4} - \frac{T^5}{15} - \frac{t_1^5}{5}\right)\right] \quad (7)$$

Ordering cost per order is given by:

$$\text{Ordering cost per order} = C_{OC} \quad (8)$$

Now, the deteriorating cost is given by:

$$\text{Cost due to Deterioration} = C_{DC}\left[W - \int_0^{t_1} Q(t)dt\right]$$

$$\text{Cost due to Deterioration} = C_{DC}\left[\frac{w\theta t_1^3}{6} + \frac{x\theta t_1^4}{8} + \frac{y\theta\, t_1^5}{10} + \frac{z\theta\, t_1^6}{12}\right] \quad (9)$$

Therefore, the total cost per unit time per unit cycle is given by:

$$TC = \frac{1}{T}(\text{Holding Cost per cycle} + \text{Shortage Cost per cycle} + \text{Ordering Cost per cycle} + \text{Cost due to Deterioration})$$

$$TC = C_{SC}\left(\frac{wt_1^2}{2} + \frac{xt_1^3}{3} + \left(\frac{y}{4} + \frac{w\theta}{12}\right)(t_1^4) + \left(\frac{x\theta}{15} + \frac{z}{5}\right)(t_1^5) + \left(\frac{y\theta}{18} - \frac{w\theta^2}{72}\right)(t_1^6) + \left(\frac{z\theta}{21} - \frac{x\theta^2}{84}\right)(t_1^7) - \frac{Y\theta^2 t_1^8}{96} - \frac{z\theta^2 t_1^9}{108}\right) - C_{SC}\left[w\left(t_1 T - \frac{T^2}{2} - t_1\right) + x\left(\frac{t_1^2 T}{2} - \frac{T^3}{6} - \frac{t_1^3}{3}\right) + y\left(\frac{t_1^3 T}{3} - \frac{T^4}{12} - \frac{t_1^4}{4}\right) + z\left(\frac{t_1^4 T}{4} - \frac{T^5}{15} - \frac{t_1^5}{5}\right)\right] + C_{OC} + C_{DC}\left[\frac{w\theta t_1^3}{6} + \frac{x\theta t_1^4}{8} + \frac{y\theta\, t_1^5}{10} + \frac{z\theta\, t_1^6}{12}\right] \quad (10)$$

This is the essential condition to minimize the total cost of inventory.

$$\frac{d(TC)}{dT} = \frac{-C_{SC}}{T}\left[w\left(t_1 T - \frac{T^2}{2} - t_1\right) + x\left(\frac{t_1^2 T}{2} - \frac{T^3}{6} - \frac{t_1^3}{3}\right) + y\left(\frac{t_1^3 T}{3} - \frac{T^4}{12} - \frac{t_1^4}{4}\right) + z\left(\frac{t_1^4 T}{4} - \frac{T^5}{15} - \frac{t_1^5}{5}\right)\right] - \frac{1}{T^2}(C_{HC}\left(\frac{wt_1^2}{2} + \frac{xt_1^3}{3} + \left(\frac{y}{4} + \frac{w\theta}{12}\right)(t_1^4) + \left(\frac{x\theta}{15} + \frac{z}{5}\right)(t_1^5) + \left(\frac{y\theta}{18} - \frac{w\theta^2}{72}\right)(t_1^6) + \left(\frac{z\theta}{21} - \frac{x\theta^2}{84}\right)(t_1^7) - \frac{Y\theta^2 t_1^8}{96} - \frac{z\theta^2 t_1^9}{108}\right) - C_{SC}\left[w\left(t_1 T - \frac{T^2}{2} - t_1\right) + x\left(\frac{t_1^2 T}{2} - \frac{T^3}{6} - \frac{t_1^3}{3}\right) + y\left(\frac{t_1^3 T}{3} - \frac{T^4}{12} - \frac{t_1^4}{4}\right) + z\left(\frac{t_1^4 T}{4} - \frac{T^5}{15} - \frac{t_1^5}{5}\right)\right] + C_{OC} \quad (11)$$

$$\frac{d(TC)}{dt_1} = \frac{1}{T}\Big\{(C_{HC}\left(wt_1 + xt_1^2 + \left(\frac{y}{4} + \frac{w\theta}{12}\right)(t_1^3) + 5\left(\frac{x\theta}{15} + \frac{z}{5}\right)(t_1^4) + 6\left(\frac{y\theta}{18} - \frac{w\theta^2}{72}\right)(t_1^5) + 7\left(\frac{z\theta}{21} - \frac{x\theta^2}{84}\right)(t_1^6) - \frac{Y\theta^2 t_1^7}{12} - \frac{z\theta^2 t_1^8}{12}\right) - C_{SC}[w(T - t_1) + x(t_1 T - t_1^2) + y(t_1^2 T - t_1^3) + z(t_1^3 T - t_1^4)] + C_{DC}\left[\frac{w\theta t_1^2}{2} + \frac{x\theta t_1^3}{2} + \frac{y\theta\, t_1^4}{2} + \frac{z\theta\, t_1^5}{2}\right]\Big\} \quad (12)$$

We get the optimal values of t_1 and T by solving equation (11) & (12) by using MAPLE 15.

NUMERICAL ILLUSTRATION:

Now check the optimality of model with the help of numerical illustration and solve the illustration with the help of Maple 15.

To explain the model numerically, assume the following parameters of the inventory system are:

w=10, x=4, y=4, z=3, C_{HC}=4, C_{SC}=15, C_{OC}=100, C_{DC}=10, θ=0.01

Under the above-given parameters, by using Maple 15 get the optimal shortage value

t_1 =1.390326758 per unit time and the optimal length of the ordering cycle is T=1.645418489 unit time.

The total inventory cost is TC=164.7559957.

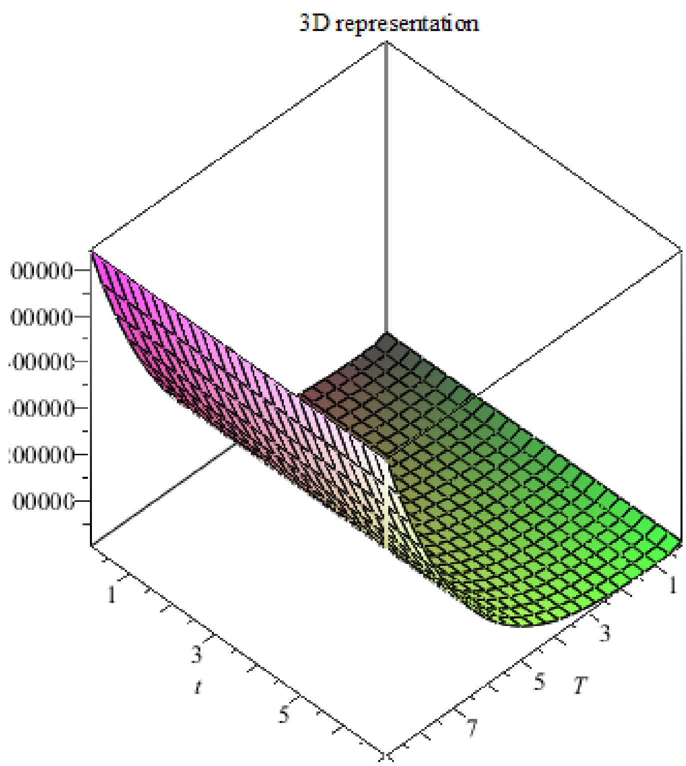

Figure 2: 3-Dimensional Graphical representation of TC of Inventory Model

SENSITIVITY ANALYSIS

Here we study the effect of changes in the model parameters such that $w, x, y, z, C_{HS}, C_{CS}, C_{OC}, C_{DC}$, $and\ \theta$.The outcome is given in the below table:

Parameter	% change	Change in			
		T	t_1	Q	TC
w	+20%	1.697087167	1.453157350	38.99668780	177.6044323
	+10%	1.671227423	1.421739503	36.16079757	171.2408318
	-10%	1.619752925	1.359044762	30.74254882	158.1526391
	-20%	1.594331847	1.328036399	28.16488024	151.4340544
x	+20%	0.9787584160	0.1235135833	14.02517649	194.7019959
	+10%	0.9829409590	0.1201833112	13.92139243	193.7798063
	-10%	0.9910385970	0.1122395164	13.69957684	191.9045320
	-20%	0.1074756238	0.1074756238	13.58018821	190.9513890
y	+20%	1.6166579650	1.3586913780	33.77408475	167.0545215
	+10%	0.9859867036	0.1233540738	13.91894291	193.3243363
	-10%	1.6605123170	1.4070662420	33.42128120	163.5542880
	-20%	0.9890154319	0.1020593534	13.59596172	191.8874648
z	+20%	0.9859486513	0.1285104754	13.93209826	193.4151291
	+10%	1.6200216850	1.3642191900	32.89682238	166.3133510
	-10%	1.6731915370	1.4188196510	38.97629230	163.08993520
	-20%	0.9881831938	0.1042543218	13.69362439	192.2765218
C_{HC}	+20%	1.562306679	1.247395801	30.12313608	173.3526512
	+10%	1.602192419	1.316958757	31.66397998	169.3423946
	-10%	1.692868107	1.468511353	35.41689607	159.4957564
	-20%	1.745684367	1.552787993	37.77243895	153.4361451
C_{SC}	+20%	1.689497717	1.490669752	35.28126987	174.7922702
	+10%	1.667257799	1.442533588	34.32592921	169.9724889
	-10%	1.624508734	1.333315502	32.54887196	159.0915236
	-20%	1.605344477	1.270531340	31.77612372	152.9133842
C_{OC}	+20%	1.691550332	1.432179886	35.34779368	176.7402864
	+10%	1.669193700	1.411999761	34.39658533	170.7895233
	-10%	1.619977623	1.366875200	32.37762862	158.6316068
	-20%	1.592543771	1.341260778	31.29569715	152.4065236
C_{DC}	+20%	1.643682900	1.387558143	33.33702159	164.8799108
	+10%	1.644549060	1.388940254	33.37270976	164.8180639
	-10%	1.646291209	1.391776860	33.44458897	164.6937046
	-20%	1.647167244	1.393113067	33.48078271	164.6311892
θ	+20%	1.643014329	1.386515883	33.50141443	164.9184147
	+10%	1.644212657	1.388416183	33.36902236	164.8361579
	-10%	1.646631918	1.392247731	33.44836821	164.6753975
	-20%	1.647853041	1.394179224	33.48843509	164.5943590

- With an increase or decrease in w, TC and Q will increase or decrease, respectively.
- With the increase in x, TC and Q decrease.
- If C_{SC} (Shortage Cost) and C_{OC} (Ordering Cost) increases, TC and Q willincrease.
- If C_{HC} (Holding Cost) increases, Q will decrease and TC willincrease.
- If θ and C_{DC} decreases, Q will decrease and TC will increase.

CONCLUSION:

The sensitivity analysis revealed that the model in case of cubic demand (a function of time) shows the deterioration rate will change with time. It shows that parameter w, C_{SC}, C_{OC} are directly proportional to TC and Q while Parameter x, C_{DC}, θ, C_{HC} show inverse relation with TC and Q. It also reveals that the total inventory cost can be obtained with this analysis. Finally, the model is verified with the help of graphical representation. The acquired results specify the stability of the model. The above model is very practical in case of constant ordering cost. This model further can be expanded or another different form of demand rate.

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3-STAGE PRODUCTIONS SCHEDULING WITH THE CONCEPT OF SET UP TIME INCLUDING ARBITRARY LAGS

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ABSTRACT
In this paper the concept of arbitrary lags (start lag and stop lag) in n-jobs, 3-machines flow shop scheduling problem where setup time are treated as separate from processing time and transportation time of jobs has been studied. The objective of the study is to propose an algorithm by which we can minimize the make-span in three stage flow shop scheduling problem. A numerical illustration is given to demonstrate the computational efficiency of proposed algorithm as a valuable analytical tool for the researchers.
Keywords: Flow Shop, Setup Time, Shipping Time, Start Lag, Stop Lag.

INTRODUCTION

Flow shop scheduling problems are one of the widest known optimization techniques. The essence of scheduling algorithm is to minimize the make span in a flow shop environment. Scheduling of operations is very difficult issues in the planning, managing of manufacturing processes. The scheduling problems depend upon the important factors like transportation time breakdown effect, total elapsed time, etc.In general, an n job- m machine scheduling problem has [(n!).(m!)] possible outcome. Such a problem does not leave any space for a pen and paper solution. However by staying in the boundaries and limiting the number of machines to 'three'the study has been conducted. Hence for 3 - stage flow shop scheduling complication with considerable set up time and arbitrary lags has been formulated and solved for the purpose of using it in the multiple organizations. The theory of shipping time is another important addition in this study. First of all in the field of scheduling theory an algorithm was introduced by Johnson [1] taking a scheduling problem in this problem n tasks are prepared on two machines. Mitten [2] treated a problem with the concept of time lags. Maggu and Das [3] established equivalent job for job – blocks theorem for 2 stage problem. The conception of shipping (transportation) time is very crucial in flow shop scheduling problem when the machines are distantly placed. Singh. T.P [4] applied the conception of shipping time in scheduling. Gupta, D. and Singh, T.P. [5] worked on nx2 production problem in which processing time are correlated with their probabilities and set up time are examined. Singh, T.P. and Gupta, D.[6] classified scheduling problem in which n tasks are prepared on 3 machines.

In it processing time of tasks and set up time of machines are correlated with probabilities including job block. Gupta, D. and Kumar, R. [7] studied the problem in which n tasks are prepared on two machines and set up time are examined separately from processing time. Gupta, D. Bala, S. and Singla, P. [8, 9] worked to reduce the lease cost of machines on nx2 open shop specially structured scheduling problem. Gupta, D and Singla, P.[10] discussed the scheduling problem with arbitrary time lags. Here we boost the study of Gupta, D. and Singla, P. [10] by taking the concept of set up time separated from processing time of jobs.

PRACTICAL SITUATION

Manufacturing industries are the backbone in the economic structure of a nation, as they contribute to increasing G.D.P. (Gross Domestic Product) and providing employment. Productivity can be maximized, if the available resources are utilized in an optimized manner. Optimized utilization of resources can only be possible if there is a proper scheduling system. In real life we always want to minimize the elapsed time in the manufacturing of products. Also during the production process a foreman takes a certain time to set up a particular machine for the processing of a particular job and that particular time is called set-up time. The practical situation may be taken in a paper mill, sugar factory and oil refinery etc. where various qualities of paper, sugar and oil are produced. The majority of scheduling research assumes set up as negligible. While this assumption adversely affects solution quality for many applications which require explicit treatment of setup, includes work to prepare the machine for processing. This includes obtaining tools, positioning work-in-process material, return tooling, cleaning up, adjusting tools and inspecting material and hence significant.

ASSUMPTIONS.

(1) No passing is allowed.
(2) Each operation once started must performed till completion.
(3) Jobs are independent to each other.
(4) No job may be processed by more than one machine at a time.

NOTATIONS.

S: Sequence of job 1, 2, 3,...., n.

M_j: Machine j, j=1, 2, 3,...., m.

A_i: Processing time of ith job on machine M_1.

B_i: Processing time of ith job on machine M_2.

C_i: Processing time of ith job on machine M_3.

S_{i1}: Set Up time of ith job on machine M_1.

S_{i2}: Set Up time of ith job on machine M_2.

S_{i3}: Set Up time of ith job on machine M_3.

D_{i1}: Start lag of ith job from machine M_1 to M_2.

E_{i1}: Stop lag of ith job from machine M_1 to M_2.

D_{i2}: Start lag of ith job from machine M_2 to M_3.

E_{i2}: Stop lag of ith job from machine M_2 to M_3.

t_i: Transportation time of ith job from M_1 machine to M_2 machine.

g_i: Transportation time of ith job from M_2 machine to M_3 machine.

t_i': Effective transportation time of ith job from M_1 machine to M_2 machine.

g_i': Effective transportation time of ith job from M_2 machine to M_3 machine.

PROBLEM FORMULATION

In this problem n - tasks are processed on three machines with processing time separated from setup time including shipping time and arbitrary time lags of tasks given below in table:

Jobs	Machine M_1		Shipping Time	Machine M_2		Shipping Time	Machine M_3		Start Lag		Stop Lag	
I.	A_i	S_{i1}	t_i	B_i	S_{i2}	g_i	C_i	S_{i3}	D_{i1}	D_{i2}	E_{i1}	E_{i2}
1.	A_1	S_{11}	t_1	B_1	S_{12}	g_1	C_1	S_{13}	D_{11}	D_{12}	E_{11}	E_{12}
2.	A_2	S_{21}	t_2	B_2	S_{22}	g_2	C_2	S_{23}	D_{21}	D_{22}	E_{21}	E_{22}
3.	A_3	S_{31}	t_3	B_3	S_{32}	g_3	C_3	S_{33}	D_{31}	D_{32}	E_{31}	E_{32}
…..	…..	…..	…….	…..	…..	…..	…..	…..	…..	…..	…..	…..
n.	A_n	S_{n1}	t_n	B_n	S_{n2}	g_n	C_n	S_{n3}	D_{n1}	D_{n2}	E_{n1}	E_{n2}

Obtained optimal sequence of jobs so as minimize the make-span.

ALGORITHM:

Step1. First of all we described effective shipping time t_i' and g_i' from machine M_1 to M_2 and from machine M_2 to M_3 respectively as follows:

$t_i' = \max(D_{i1} - A_i, E_{i1} - B_i, t_i)$

$g_i' = \max(D_{i2} - B_i, E_{i2} - C_i, g_i)$

Step2: Check the condition

either $\text{Min}\{A_i + t_i' - S_{i2}\} \geq \text{Max}\{B_i + t_i' - S_{i1}\}$

or Min{$C_i + g_i' - S_{i2}$} ≥ Max {$B_i + g_i' - S_{i3}$} or both for all i

If the conditions are satisfied then go to step 3, else the data is not in the standard form.

Step3: Introduce the two fictitious machines G and H with processing times Gi and Hi as

$G_i = A_i + B_i + \max(S_{i1}, S_{i2}) + t_i'$, $H_i = B_i + C_i - \max(S_{i2}, S_{i3}) + g_i'$

Step4: Using Johnson's procedure, obtain an optimal sequence S having minimum elapsed time.

Step5: Prepare In – Out tables for optimal sequence S and compute total elapsed time.

NUMERICAL ILLUSTRATION

Considered 5 jobs are processed on three machines with processing time separated from setup time including shipping time and also considered arbitrary lags (start lag and stop lag)

Jobs	Machine M_1		Shipping Time	Machine M_2		Shipping Time	Machine M_3		Start Lag		Stop Lag	
I.	A_i	S_{i1}	t_i	B_i	S_{i2}	g_i	C_i	S_{i3}	D_{i1}	D_{i2}	E_{i1}	E_{i2}
1.	3.2	0.6	2	0.8	0.7	2	1.2	0.6	4	2.8	2	2
2.	3.6	1.4	1	1.2	1.8	1	1.6	1.2	4.6	1.8	1.8	2
3.	2.6	1.2	2	1	1.2	2	3	1.2	2.8	3	2	4
4.	3	2.1	3	0.8	0.3	3	0.8	0.5	4	3.8	3.2	2.4
5.	1.4	2.4	4	1.2	0.6	1	1.8	0.8	3.8	2	5.2	2.5

Obtained optimal sequence of jobs so as minimize the make-span.

SOLUTION:

Step1: First of all we described effective shipping time t_i' and g_i' from machine M_1 to M_2 and from machine M_2 to M_3 respectively as follows:

$t_i' = \max(D_{i1} - A_i, E_{i1} - B_i, t_i)$ $g_i' = \max(D_{i2} - B_i, E_{i2} - C_i, g_i)$

Jobs	Machine M_1		Effective Shipping Time	Machine M_2		Effective Shipping Time	Machine M_3	
I.	A_i	S_{i1}	t_i'	B_i	S_{i2}	g_i'	C_i	S_{i3}
1.	3.2	0.6	2	0.8	0.7	2	1.2	0.6
2.	3.6	1.4	1	1.2	1.8	1	1.6	1.2
3.	2.6	1.2	2	1	1.2	2	3	1.2
4.	3	2.1	3	0.8	0.3	3	0.8	0.5
5.	1.4	2.4	4	1.2	0.6	1	1.8	0.8

Step2: Now we go through the conditions which we discussed in the algorithm step 2, and here Min{$A_i + t_i' - S_{i2}$} ≥ Max{$B_i + t_i' - S_{i1}$}condition is fulfilled. So we discover two fictitious machines G and H with

their processing time G_i and H_i as follows:

$G_i = A_i + B_i + \max(S_{i1}, S_{i2}) + t_i'$, $H_i = B_i + C_i + \max(S_{i2}, S_{i3}) + g_i'$

Jobs	Factious Machine G	Factious Machine H
I	G_i	H_i
1.	6.7	4.7
2.	7.6	5.6
3.	6.8	7.2
4.	8.9	6.1
5.	9	4.8

Step3: Now by adopting Johnson's technique, optimal sequence is S = 3,4,2,5,1

Step4:we define In-Out data for sequence S is as follows:

Jobs	Machine M_1	t_i'	Machine M_2	g_i'	Machine M_3
I.	In-Out		In – Out		In – Out
3.	0 –2.6	2	4.6- 5.6	2	7.6 – 10.6
4.	3.8 – 6.8	3	9.8 – 10.6	3	13.6 – 14.4
2.	8.9 – 12.5	1	13.5 – 14.7	1	15.7 – 17.3
5.	13.9 – 15.3	4	19.3 – 20.5	1	21.5 – 23.3
1.	17.7 – 20.9	2	22.9 – 23.7	2	25.7 –26.9

Minimum total elapsed time for the given complication is 26.9 units.

REMARK

The model presented in the section is near to real time of left communication. Our study provides a guideline to be system based on optimal continue policy. This study deals with the production scheduling problem with the leading intention to reduce the total production time of tasks. The work can be boosted by taking varied parameters like job block criteria, break-down interval, mean weightage time etc.

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FULLY DYNAMIC SHORTEST PATH ALGORITHMS FOR GENERAL HYPER-GRAPHS

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ABSTRACT :
In this paper, we consider the shortest path problem in hyper-graphs. We develop two algorithms for finding and maintaining the shortest hyper paths in a dynamic network with both weight and topological changes. These two algorithms are the first addressing the fully dynamic shortest path problem in a general hyper-graph. They complement each other by partitioning the application space based on the nature of the change dynamics and the type of the hyper-graph.
Keywords : Hyper edge, Dimension Reduction, Shortest path
MSC Code : 05C65,05C90

1. INTRODUCTION

A graph is a basic mathematical abstraction for modeling networks, in which nodes are represented by vertices and pair- wise relationships are represented by edges between vertices. A graph is thus given by a vertex set V and an edge set E consisting of cardinality-2 subsets of V. A hyper-graph is a natural extension of a graph obtained by removing the constraint on the cardinality of an edge: any non-empty subset of V can be an element (a hyper-edge) of the edge set E (see Fig 1). It thus captures group behaviors and higher-dimensional relationships in complex networks that are more than a simple union of pairwise relationships. Examples include communities and collaboration teams in social networks, document clusters in information networks, and cliques, neighborhoods, and multicast groups in communication networks.

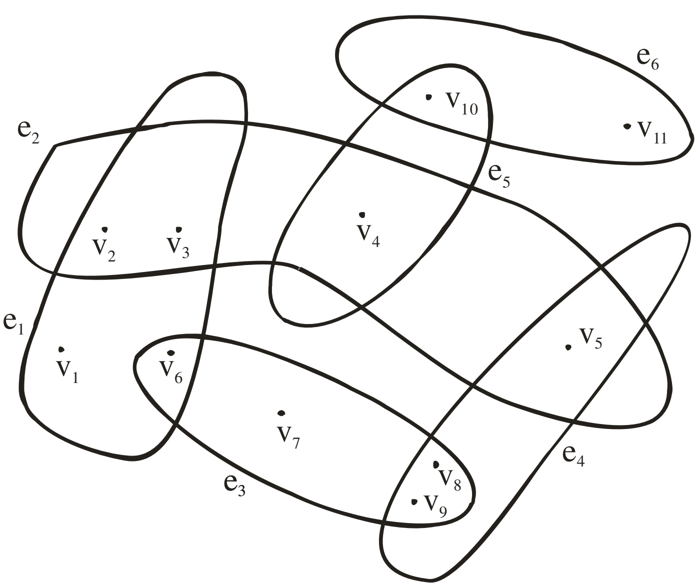

Fig.1. An example hyper-graph with 6 hyper edges: (v_1 , v_2 , v_3 , v_6), (v_2 , v_3 , v_4 , v_5), (v_6 , v_7 v_8 , v_9), (v_5 , v_8 , v_9) , (v_4 , v_{10}) and (v_{10} ,v_{11}).

While the concept of hypergraph has been around since1920's (see, for example, [1]), many well-solved algorithmic problems in graph theory remain largely open under this more general model. In this paper, we address the shortest path problem in hyper graphs.

A. Shortest Path Problem in Graphs

The shortest path problem is perhaps one of the most basic problems in graph theory. It asks for the shortest path between two vertices or from a source vertex to all the other vertices (i.e., the single-source version or the shortest path tree). Depending on whether the edge weights can be negative, the problem can be solved via Dijkstra's algorithm or Bellman-Ford algorithm [2]. This basic problem finds diverse applications in communication networks, operational research, plant and facility layout. etc., The dynamic version of the shortest path problem is to maintain the shortest path tree without recomputing from scratch during a sequence of changes to the graph. A typical change to a graph includes weight increase, weight decrease, edge insertion, and edge deletion. The last two types of changes model network topological changes, but they can be conceptually considered as special cases of weight changes by allowing weight to be infinity. Thus, if the sequence of changes contains only weight increase and edge deletion, we call it a decremental problem; if it contains only weight decrease and edge insertion, we call it an incremental problem. Otherwise, we have a fully dynamic problem. If multiple edges change simultaneously, then it is called a batch problem.

B. Shortest Path Problem in Hyper graphs

Both the static and dynamic shortest path problems have a corresponding version in hyper graphs. The static shortest hyper path problem was considered by Knuth [4] and Gallo et al., [5], in which Dijkstra's algorithm was extended to obtain the shortest hyper paths. Knuth's algorithm is for a special class of hyper graphs while Gallo's algorithm is for a general hyper graph. Ausiello et al., proposed a dynamic shortest hyperpath algorithm for directed hyper graphs, considering only the incremental problem with the weights of all hyper edges limited to a finite set of numbers [6, 7]. A dynamic algorithm for the batch problem in a special class of hypergraphs was developed in [3].

Referred to as the Hyper Edge based Dynamic Shortest Path algorithm **(HE-DSP),** the first algorithm is an extension of the dynamic Dijkstra's algorithm for graphs to hypergraphs (parallel to Gallo's extension of the static Dijkstra's algorithm to hyper graphs in [4]). The extension of the dynamic Dijkstra's algorithm to hyper graphs is more involved than that of the static Dijkstra's algorithm. This is due to the loss of the tree structure in the collection of the shortest hyperpaths from a source to all other vertices. Since the dynamic Dijkstra's algorithm relies on the tree structure to update the shortest paths after an incremental change (weight increase or edge deletion), special care needs to be given when extending it to hyper graphs.

The second algorithm is rooted in the idea of Dimension Reduction and is referred to as DR-DSP. The basic idea is to reduce the problem to finding the shortest path in the underlying graph of the hyper graph. The underlying graph of a hyper graph has the same vertex set and has an edge between two vertices if and only if there is at least one hyper edge containing these two vertices in the original hypergraph. The weight of an edge in the underlying graph is defined as the minimum weight among all hyperedges containing the two vertices of this edge. The shortest hyper path in the hyper graph can thus be obtained from the shortest path in the underlying graph by substituting each edge along the shortest path with the hyper edge that lent its weight to this edge. The correctness and advantage of this algorithm are readily seen: the definition of weight in the underlying graph captures the minimum cost offered by all hyper edges in choosing a path between two vertices, thus ensuring the correctness of the algorithm; the reduction of a hyper graph to its underlying graph removes many hyper edges from consideration when finding the shortest path, leading to efficiency and agility to dynamic changes.

HE-DSP is more efficient in hyper graphs that are densely connected through high-dimensional hyper edges and for net- work dynamics where changes often occur to hyper edges that are not on the current shortest hyperpaths. DR-DSP has lower complexity when hyper edge changes often lead to changes in the shortest hyperpaths. This is usually the case in networks where hyper edges in the shortest hyper paths are more prone to changes due to attacks, frequent use, or higher priority in maintenance and upgrade. Furthermore, DR-DSP leads to an alternative algorithm for solving the static shortest hyper path problem when the dynamic problem degenerates to the static problem. It has the same complexity as Gallo's algorithm for a general hyper graph and lower complexity for simplicial complexes (a special class of hyper graphs whose hyper edge set is closed under the subset operation). We also point out that both proposed algorithms apply to directed hyper-graphs with minor modifications in their implementation details.

A detailed time complexity analysis of these two algorithms is provided to demonstrate their performance in the worst- case change scenario. Using a random geometric hyper graph model and a real data set of a social network, we study the average performance of these two algorithms in different scenarios and demonstrate the partition of the application space between these two algorithms. In the experiment with Enron email data set, the proposed algorithms successfully identified the most important actor in this social network using the closeness centrality metric.

2. DYNAMIC SHORTEST HYPERPATH PROBLEM

We introduce some basic concepts of hyper graph [1] and Some basic properties of the shortest hyper paths define the static and the dynamic shortest hyper path problems. Some basic properties of the shortest hyper paths are established and will be used in developing the dynamic algorithms

A. Hyper-graph and Hyper path

Let V be a finite set and E a family of subsets of V. If for all elements ei ∈ E, the following conditions are satisfied : $e_i \neq \varphi$, $\cup_{e_i \in E}\ e_i = V$, then the couple $H=(V,E)$ is called a (undirected) hyper-graph. Each element $v \in V$ is called a vertex and each element $e \in E$ a hyper edge.

A weighted undirected hyper-graph is a triple $H=(V,E,w)$ with $w: E \rightarrow \{R^{+} \cup \{0\}\}$ being a nonnegative weight function defined for each hyper edge in E. In a hyper graph, a hyper path is defined as follows.

Definition 2.1 : A hyperpath between two vertices u and v is a sequence of hyper edges $\{e_0, e_1, ..., e_m\}$ such that $u \in e_0, v \in e_m$ and $e_i \cap e_{i+1} \neq \varphi$ for $i = 0, ..., m-1$. A hyper path is simple if non-adjacent hyper edges in the path are non- overlapping, i.e., $e_i \cap e_j = \varphi, \forall_j \neq i, i \pm 1$.

Let $L_e = \{e_{0,} ....e_m\}$ be a hyper path in a weighted hyper graph H . We define the weight of L_e as:

$$w(L_S) = \sum_{i=0}^{m} w(e_i) \qquad \text{..................(2.1)}$$

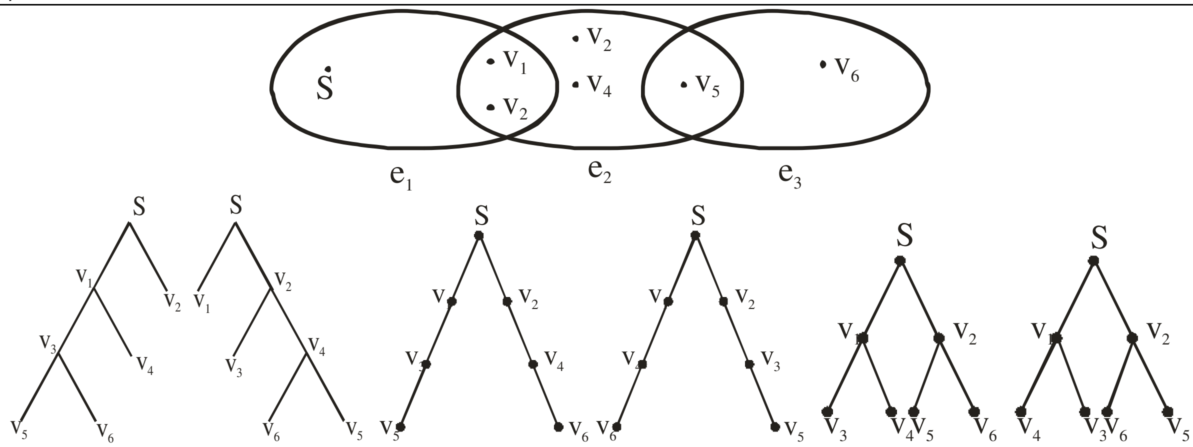

Fig.2 Hyper paths and the associated relationship trees.

B. Shortest Hyper path and Relationship Tree

Given two vertices u and v, a natural question is to find the shortest hyper path (in terms of the path weight) from u to v. Since the weight function is nonnegative, it suffices to consider only simple hyper paths. If the shortest hyper path is not simple, we can always generate a simple hyper path without increasing the weight by deleting all the hyper edges between two overlapping non-adjacent hyper edges. The dynamic shortest hyper path problem can be similarly defined for a sequence $c = \{\delta_1, \delta_2, ...\delta_l)$ of hyper edge changes. Hyper edge changes have the same six types as edge changes in a graph: weight increase, weight decrease, hyper edge insertion, and hyper edge deletion. Similarly, weight increase and hyperedge deletion will be treated together, so are weight decrease and hyper edge insertion.

Lemma 2.1 : Let $L_v = \{e_{1,} e_2, ...e_l\}$ be a shortest hyper path from $s \in e_1$ to $z \in e_l$. Then for any vertex $v \in e_i \cap e_{i+1}$, the hyperpath $L_v = \{e_{1,} e_2, ...e_l\}$ is a shortest hyper path from s to v. Furthermore, for any two vertices $u, v \in e_i \cap e_{i+1}$ (if there exist at least two vertices in $e_i \cap e_{i+1}$), D[u] = D[v].

Proof : We will prove by contradiction. Assume that $L_v = \{e_{1,} e_2, ...e_l\}$ is not a shortest hyperpath for v. Then there exists a different hyper path $L_v^{'} = \{e_1^{'}, e_2^{'}, ...e_k^{'}\}$ with $w(L_v^{'}) = w(L_V)$. Then consider the hyper path $L^{'} = \{e_1^{'}, e_2^{'}, ...e_k^{'}, e_{i+1}, e_{i+2}, ...e_l\}$, we have $w(L^{'}) = w(L)$ which contradicts the fact that L is a shortest hyper path to z. This completes the proof for the first part of the lemma. Furthermore, for any two nodes $u, v \in e_i \cap e_{i+1}$, since L_v is the shortest hyper path for both vertices, $D[v] = w(L_v) = D[u]$

…………(2.2)

Next, we introduce the concept of relationship tree that is needed in the proposed dynamic shortest hyper path algorithm HE-DSP. Since two adjacent hyper edges in a hyper path may overlap at more than one vertex, the shortest hyper paths from s to all other vertices do not generally form a tree in the original graph sense. For the development of the dynamic shortest hyper path algorithms, we introduce the concept of relationship tree to indicate the parent-child relationship along shortest hyper paths. The concept can be easily explained in the example given in Fig 2. Let $\{e_{1,} e_2 . e_3\}$ be a shortest hyper path from s to v_6. By Lemma 2.1, $\{e_1\}$ is a shortest hyper path for both v_1 and v_2. As illustrated in Fig 2, there are 6 possible relationship trees to indicate the parent-child-teacher relationship in these shortest hyper paths. The choice of the relationship tree does not affect the correctness or performance of the proposed algorithm HE-DSP.

Similar notations are used for dynamic shortest hyper path algorithms : $D[v]$ denotes the distance of a vertex v to the source s on the shortest hyper path, $P[v]$ the parent of v in the chosen relationship tree associated with the shortest hyper paths. A new notation is $E[v]$, the hyper edge containing s and $P[v]$ on the shortest hyper path (i.e., the hyper edge that leads to v from $P[v]$ on the shortest hyper path). When it is necessary to distinguish the shortest distance before and after a weight change, $d[v]$ denotes the shortest distance before the change, $d^{'}[v]$ the shortest distance after the change, and $D[v]$ the actual value stored in the data structure during the execution of the algorithm.

3. HYPER EDGE BASED DYNAMIC SHORTEST PATH ALGORITHM (HE-DSP)

We propose HE-DSP. It is an extension of the dynamic Dijkstra's algorithm to hyper graphs. The extension is more complex than Gallo's extension of the static Dijkstra's algorithm, since the dynamic Dijkstra's algorithm relies on the tree structure of the shortest paths, a structure no longer there for the shortest hyper paths.

Consider that the weight of a hyper edge eˇ decreases to w_{new} Similar to the case for graphs, we know that the vertex $x \in \tilde{e}$ with $D[x] = \min_{v \in \tilde{e}}\{D[v]\}$ will not be affected. We then check whether the other vertices in $\tilde{e}$ are affected bychecking the inequality given in (1), and put all the affected vertices into a priority queue v. The rest of the procedure is similar to that for graphs, only when we update the distance of a vertex, we check all the hyper edges that contain this vertex.

HE-DSP: Weight Decrease $(\tilde{e}, w_{new})$

Step 0 (Update the hyper graph)

$w(\breve{e}) \leftarrow w_{new}$

Step1 (Determine the affected vertices in e)

$x \leftarrow \arg\min_{v \in \breve{e}}\{D[v]\}$

for each $v \in \breve{e}$ such that $D[x] + w_{new} < D[v]$ **do**

D[v] ← D[x] + wnew ; P [v] ← x; E[v] ← eˇ

Enqueue $(Q, \langle V, D[v] \rangle)$

end

Step2 (Iteratively enqueue and update affected vertices)

while NonEmpty (Q) **do**

$\langle z, D[z] \rangle \leftarrow$ ← Dequeue(Q)

for each e $e \in E$ ∈ E s.t. $z \in e$

for each $v \in e$

if $D[v] > D[z] + w(e)$ **then**

$D[v] \leftarrow D[z] + w(e); P[v] \leftarrow z; E[v] \leftarrow e$

Enqueue or Update $(Q, \langle V, D[v] \rangle)$

end; end; end; end

Theorem 3.1 : If before the weight decrease, $D[v] = d[v], E[v]$ and $P[v]$ are correct for all $v \in V$, then after the weight decrease, $D[v] = d^{'}[v]$ D′[v] and $E[v]$ and $P[v]$ are correctly.

Proofˆ

Lemma 3.1: Let $x = \arg\min_{v\in\breve{e}}\{d[v]\}$, then $d[x] = d^{'}[x]$ and $d^{'}[x] = \min_{v\in\breve{e}}\{d^{'}[v]\}$

Proof : Proof by contradiction. Assume that $d^{'}[x] < d[x]$,then x has to use $\breve{e}$ on its new shortest hyper path. Since we consider only simple hyper paths and $x \in \breve{e}$, we have $E[x] = \breve{e}$. Therefore its parent $y = P[v]$ cannot use $\breve{e}$ on its shortest hyper path, which implies that the shortest distance to y doesnot change: $d[y] = d^{'}[y]$. Given that y is the parent of x on its new shortest hyper path, we have $d[y] = d^{'}[y] \le d^{'}[x] < d[x]$ which contradicts to the definition of x .

For the second statement, assume there exists $z \in \breve{e}$ such that $d^{'}[z] < d^{'}[x]$. Based on the definition of x and the hypothetical assumption, $d[z] \ge d[x] = d^{'}[x] > d^{'}[z]$. It thus follows that z 's shortest hyperpath changes and $E[z] = \breve{e}$ in the new shortest hyper path. Follow the same line of arguments by considering the parent of z , we arrive at the same contradiction in terms of the definition of x .

Lemma 3.2 : For any vertex v , v is enqueued into Q if and only if $d^{'}[v] < d[v]$.

Proof : Consider first that v is enqueued into Q . From the algorithm, this can only happen if there exists a neighbor z and a hyper edge $e \ni v$, z such that $D[z] + w(e) < D[v]$. We thus have $d[v] \ge D[v] > D[z] + w(e) \ge d^{'}[v]$.

We now prove the converse. Assume that $d^{'}[v] < d[v]$. Le t $p = \{e_{1,}e_{2,\dots\dots}e_i, \breve{e}, e_{i+1,\dots\dots,}e_l\}$ be v's new shortest hyperpath. There exists $u_{i+1} \in e^{ˇ} \cap e_{i+1}$ such that $d'[u_{i+1}] < d[u_{i+1}]$. In Step 1 of the algorithm, u_{i+1} is enqueued. Similarly, there exists $u_{i+2} \in e_{i+1} \cap e_{i+2}$ with $d'[u_{i+2}] < d[u_{i+2}]$. Then u_{i+2} will be enqueued in Step 2 of the algorithm when u_{i+1} is dequeued if it has not been enqueued before that. Repeating this line of argument, we conclude that there exits $u_l \in e_{l-1} \cap e_l$ with $d^{'}[u_l] < d[u_l]$ and u_l is enqueued into Q. Then v will be enqueued when u_l is dequeued if it is not enqueued already.

Lemma 3.3 : For each v dequeued from Q , D[v] = d′[v].

Proof : We first show that if u is dequeued before v, then D[u] ≤ D[v] at the instants when they are dequeued. We prove this by induction. The initial condition holds trivially. Then assume it is true for the first l dequeued vertices $z_{1,\dots\dots,}z_l$. Consider the $(l+1)$th dequeued vertex z_{l+1} . At the instant when z_l is dequeued, if $D[z_{l+1}]$ is updated based on $D[z_l]$ in Step 2, then $D[z_1] < D[z_{l+1}]$ even after the update. If, on the other hand, $D[z_{l+1}]$ is not updated at this instant, then $D[z_1] \le D[z_{l+1}]$ given that the dequeued vertex has the smallest distance.

Next, we prove the lemma by induction. From Step 1 of the algorithm, all the affected vertices v in eˇ will be dequeued first with E[v] = eˇ, P [v] = x, and D[v] = d′[x] + w(eˇ).

Based on Lemma 2, D[v] ≤ d′[u] + w(eˇ) for any u ∈ eˇ. It thus follows that the hyperpath to v through x and eˇ is the shortest one with D[v] = d′[v].Assume for $z_1, \dots, z_l$, $D[z_i] = d'[z_i]$ are satisfied for all $i = 1,\dots.l$. Consider the $(l+1)$ th dequeued vertex $z_{l+1} 6 \in \breve{e}$. Let $u = P[z_{l+1}]$ be its parent in the new shortest hyper path. Then based on the fact that distances of the dequeued vertices are monotonically increasing with the order of the dequeueing as shown at the beginning of the proof, u cannot be any vertex dequeued after z_{l+1} . Since $z_{l+1} 6 \in \breve{e}$, it is also clear that u cannot be an unaffected vertex will be unaffected, which contradicts Lemma 3). We thus have $u \in \{z_{1,\dots\dots}z_l\}$. Let $u = z_i$. Then when z_i is dequeued, $D[z_{l+1}]$ will be updated to the shortest distance $d^{'}[z_{l+1}]$ due to the induction hypothesis of $D[Z_i] = d^{'}[z_i]$. This completes the proof. .

B. Hyperedge Weight Increase

The coloring process in the graph case relies on the tree structure of the shortest paths, which is no longer present in the shortest hyper paths. Our solution is to use a relationship tree for the coloring process, and we prove the correctness of this approach regardless of the choice of the relationship tree. Consider that the weight of a hyper edge eˇ increases to wnew. First, we redefine the color of a vertex v based on the chosen relationship tree.

(i) v is colored white if d′[v] = d[v] while keeping the current P [v] and E[v].

(ii) v is colored pink if d′[v] = d[v], but only possible through a new P [v] or E[v] or both.

(iii) v is colored red if d′[v] < d[v].

With the above modified definitions of colors, the same coloring process as in the graph case can be carried out using a relationship tree. The algorithm is given below.

```
HE-DSP: Weight Increase(e?, wnew ).
Step0 (Update the hyper graph)
w(e?) ? wnew
Step1 (Determine the affected vertices in e)
for each v ? e? s.t. E[v] = e? do
Enqueue (M, hv, D[v]i)
Step2 (Coloring process)
while NonEmpty (M) **
hz, D[z]i ? Dequeue(M )
if ?  nonred q ? V   s.t. ?e ? E  with q, z ? e and
D[q] + w(e) = D[z]
 then z is pink; P [z] = q; E[z] = e;
else z is red; Enqueue(M , all z's children)
end; end
Step3.a (Initialize the distance vector for red vertices)
for each red vertex z do
if z has no nonred neighbor
then D[z] ? +?; P [z] ? Null
else
let u be the best nonred neighbor of z
E[z] ? argmine? E,e?u,z {w(e)};
D[z] ? D[u] + w(E[z]); P [z] ? u;
Enqueue(Q, hz, D[z]i)
end; end; end
Step3.b: Step2 of HE-DSP: Weight Decrease
```

The theorem below states the correctness of the algorithm

Theorem 3.2: If before the weight increase, D[v] = d[v], E[v] and P [v] are correct for all v ∈ V , then after the weight increase, D[v] = d′[v] and also E[v] and P [v] are correctly
updated.

Proof : We first show the correctness of the coloring process as given in the following lemma.

Lemma 3.4 : The coloring process correctly colors all the affected vertices.

Proof : We first state the following simple facts without proof: given a relationship tree, after the hyper edge weight increase, (i) if v is pink or white, then all its descendent in this relationship tree are white; (ii) if a v is red, then all its children in the relationship tree are either pink or red; (iii) if a v is affected, either v ∈ eˇ or P [v] is red. These facts can be directly obtained from the definition of the color. It is also easy to see that vertices are dequeued from M in a non decreasing order of their current distance D[·]. This is because each time a vertex z is dequeued from M , the possible new vertices to be enqueued into M are z's children with distances no smaller than D[z].

Then, the proof of the lemma has two parts: first we prove that all affected vertices are enqueued into M ; then we prove by induction that only affected vertices are enqueued into M and their colors are correctly identified.

We prove the first part by contradiction. Assume that there exists an affected vertex v that is not enqueued into M . It is easy to see that v 6∈ eˇ because all the affected vertices in eˇare enqueued in Step 1. Based on the third fact stated above, P [v] is red. Based on the hypothesis, P [v] is not enqueued . Continue this line of arguments, we eventually reach the root of the relationship tree and arrive at the contradiction that the source s is red.

We prove the second part by induction. It is easy to see that all the vertices initially enqueued into M are affected vertices. It remains to show that the first vertex z_l dequeued from M is colored (pink or red) correctly. To show that, we need to establish that the algorithm correctly determines whether there is an alternative shortest hyperpath to z_l with the same distance, i.e., $d[z_l] = d^{'}[z_1]$. The key here is to show that checking the currently non-red neighbors of z_l will not lead to a false alternative path. This follows from the fact that z_l has the smallest distance D[·] among all affected vertices.

Next, assume It is not difficult to see that P [·] and E[·] are correctly updated that vertices $z_{1,.......,}z_l$ dequeued from M are all affected vertices and are correctly colored. Consider the next dequeued vertex z_{l+1}. It is an affected vertex because it is either enqueued in Step 1 with E[v] = eˇ or enqueued in Step 2 with a red parent. To show that z_{l+1} will be colored correctly, we use a similar argument by showing that the currently non- red neighbors of z_{l+1} will not give a false alternative path. The latter follows from the fact that all affected vertices will be enqueued and those dequeued after z_{l+1} have distances no smaller than $D[z_{l+1}]$. This completes the induction.

4. DIMENSION REDUCTION BASE D DYNAMIC SHORTEST PATH ALGORITHM (DR-DSP)

In this section, we propose DR-DSP. When the dynamic problem degenerates to the static problem, DR-DSP leads to an alternative algorithm for solving the static shortest hyper path problem.

A. The Static Case: DR-SP

We first consider the static version of the algorithm (referred to as DR-SP), which captures the basic idea of dimension reduction.

The proposed DR-SP algorithm is based on the following theorem in which we show that for a general hyper graph H ,the weight ω(L∗) of the shortest path L_G ∗ of H is equal to the shortest path L∗of a weighted graph G derived from H . Specifically, corresponding to every hyper edge e in H , G contains a clique defined on the vertices of e.

Theorem 4.1 : Let H = (V, E, w) be a hyper graph, and G = (V, E˜) the underlying graph of H where an edge e˜ ∈ E˜ if and only if ∃e ∈ E such that e˜ ⊂ e. For each edge e˜ in G, its weight wG (e˜) is defined as wG (e˜) = min w(e)

{e∈E: e⊇e˜}

Let L^* and L^*_G be the shortest paths from u ∈ V to v ∈ V in H and G, respectively. H and G, respectively. Then we have that w(L∗) = wG (L∗).

Proof : First, for each shortest path L^*_G in G. we can obtain a corresponding hyperpath L in H with the same weight based on [2], therefore we have that

$$w_G(L_G) = w(L) \geq w(L).$$

Then it suffices to show that there exists a path L_G in G such that $w_G(L_G) \leq w(L*)$, which implies that $w_G(L*) \leq w_G(L_G) \leq w(L*)$.

Assume that $L* = \{e_0, e_1, \ldots, e_{k-1}\}$ is a shortest hyper-edge path from v_0 to v_k in H where $v_0 \in e_0$ and $v_k \in e_{k-1}$. Let $v_i \in e_{i-1} \cap e_i$ (i = 1, 2, ..., k − 1) be one of the vertices in the intersection of hyper edges e_{i-1} and e_i. Construct a path LG $= \{v_0, v_1, \ldots, v_k\}$ in the graph G. For each edge $e_i = \{v_i, v_{i+1}\}$ (i = 0, 1, ..., k − 1), since $e_i \subseteq e_i$, it follows from (2) that

$w_G(e_i) \leq w(e_i)$.

Thus, $w_G(L_G) = \sum_{I=0}^{K-1} w_G(\tilde{e}_i) \leq w(e_i) = w(L^*)$.

B. The Dynamic Case: DR-DSP

In the dynamic case, a sequence C $= \{\delta_1, \delta_2, \ldots, \delta_l\}$ of hyper edge changes in the hyper graph H results in a sequence of edge changes in the underlying graph G. For each hyper-edge change δ_i, DR-DSP first updates the underlying graph G to locate all the changed edges caused by δ_i. In the next step, DR-DSP updates the shortest path tree in the underlying graph G.

Consider first the graph update. A change to a hyper edge e only affects those edges in G that are subsets of e, i.e., a hyper edge change is localized in the underlying graph G. Furthermore, since the weight of an edge in G is the minimum weight of all hyper edges containing it, not all edges in G that are subsets of e will change weight. Based on these observations, we propose a special data structure and procedure for updating the underlying graph G without regenerating the graph from scratch using Step 1 of DR-SP.

At the initialization stage of the algorithm, a priority queue Muv for each pair of vertices (u, v) in the hypergraph is established to store the weights of all hyper edges that contain both u and v. When a change occurs to hyperedge e, all the priority queues Muv associated with the pair of vertices (u, v) that are contained in e are updated with the new weight of e. Thus, the top of these priority queues always maintain the weight for edge (u, v) in the underlying graph G for each

Let $L*^G$ and $L*$

1 for each u, v ∈ eˇ

Update(Muv , < $\check{e}$, wnew >);

wuv ←Peek(Muv);

end;

After the underlying graph G is updated, we are now facing a dynamic shortest path problem in a graph. However, since a single hyper edge change can result in multiple edge changes in G, we need to handle a batch problem. While existing batch algorithms and iterative single-change algorithms for graphs can be directly applied here, we show that the batch

Property 1: The edge changes in G caused by a hyper edge change are either all weight decreases or all weight increases.

Property 2: All changed edges in G caused by a hyper edge change belong to a clique in G.

C Hyper edge Weight Decrease

Theorem 4.1 and Property 1, there are (possibly) several edge- weight decreases in the underlying graph G. Therefore similar to HE-DSP, there is at least one unaffected node x = argminv∈$\check{e}${D[v]}. By Property 2, these affected edges are

contained in a clique derived from the changed hyper edge; therefore it is sufficient to determine the distance of every node v (other than x) in the original changed hyper edge e by checking D[x] + wnew < D[v]. And we can initialize the priority queue with those nodes whose weight decreases. After that, the procedure is similar to that in the graph case.

DR-DSP: Weight Decrease($\check{e}$, wnew).

Step0 (Update the hypergraph and G)

w($\check{e}$) ← wnew

Graph Update($\check{e}$, wnew)

Step1 of HE-DSP: Weight Decrease

Step2 of Graph: Weight Decrease

D. Hyper edge Weight Increase

If the weight of hyper edge $\check{e}$ increases to wnew , by Theorem 3 and Property 1, there are (possibly several edge-weight

DR-DSP: Weight Increase($\check{e}$, wnew).

Step0 (Update the hyper graph and G)

w($\check{e}$) ← wnew

Graph Update($\check{e}$, wnew)

Step1 of HE-DSP: Weight Increase

Step2 of Graph: Weight Increase

Step3.a of Graph: Weight Increase

Step3.b of Graph: Weight Increase.

5. APPLICATIONS

Shortest path computations on hyper graphs can be applied to communication as well as social networks. An example application in wireless communications, in particular, for multi-hop wireless networks, is in opportunistic routing schemes such as ExOR, GeRaF and MORE. In such schemes, any receiver of a packet is eligible to forward the packet. Receivers typically execute a protocol amongst themselves to decide who should forward it. This naturally leads to a hyper graph model where a node and its neighbors form a hyper edge. The cost of each hyper edge can be defined based on the cardinality of the hyper edge to capture the success rate of forwarding and the associated overhead . A shortest hyper path from the source to the destination is thus a better route than merely the traditional shortest path. And as the network topology changes, a dynamic algorithm is required to maintain the shortest hyper path.

In social networks, information (results, event reports, opinions, rumors, etc.) propagates through diverse communication means including direct links (e.g., gestures, optical, satcom, regular phone call), social media (e.g., Facebook, Twitter, blogs), mailing lists, and newsgroups. Such a network may be modeled as a hypergraph with the weight of a hyper edge reflecting the cost, credibility, and/or delay for disseminating information among all vertice of this hyper edge. In particular, the weight of a hyperedge can capture the unique effect on the information after it passes through a group of people. For instance, a result can be discussed by overlapping blog collaboration networks as it spreads, and often the discussion yields a better result than if it only spreads through individuals. The minimum cost information passing in social networks can thus be modelled as a shortest hyper path problem.

Another potential application is that of finding the most important actor in a social network. Under a graph model of social networks, the relative importance of a vertex can be measured by its between-ness and closeness centrality indices. The former is defined based on the number of shortest paths that pass through this vertex, and the latter, the total weight of the shortest paths from this vertex to all the other vertices [8]. In a social network exhibiting hyper-relationships, between-ness and closeness centrality, based on the shortest hyper paths, would be better indicators of the relative importance of each actor, we apply the proposed shortest hyper path algorithms to the Enron email data set. We propose a weight function that leads to the successful identification of the CEO of Enron as the most important actor under the closeness centrality metric. The distance of each person in the data set to the CEO along the resulting shortest hyper paths closely reflects the position of the person within the company.

6. CONCLUSION

We have presented, to our best knowledge, the first study of the fully dynamic shortest path problem in a general hyper- graph. We have developed two dynamic algorithms for finding and maintaining the shortest hyper paths. These two algorithms complement each other with each one preferred in different types of hyper graphs and network dynamics, as illustrated in the time complexity analysis and simulation experiments. We have discussed and studied via experiments over a real data set the potential applications of the dynamic shortest hyper path problem in social and communication networks.

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A STUDY OF NEURAL NETWORKS WITH GEOMETRIC MODELING

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ABSTRACT
In this paper we have studied of neural networks with geometric modeling as well as information meting out capabilities of complex hierarchical networks of plain computing units. Also, we have defined and studied artificial neural network and Geometric interpretation of Threshold Logic Unit (TLU) action through systems whose formation is only moderately stipulated. Various parameters transform the capabilities of the network.
Key words: Neural Network, Artificial, Neurons and Geometric Modeling.

1. INTRODUCTION

A neural network is an interconnected assembly of simple giving out elements, **units or nodes,** whose functionality is freely based on the animal neuron. The giving out ability of the network is stored in the inter unit connection strengths, or **weights,** obtained by a process of variation to, or knowledge from, a set of preparation patterns.

These seemingly unrelated branches of mathematics come together in a beautiful way when studying neural networks and how they operate on data manifolds. Geometric interpretations of neural networks form a minority perspective in machine learning; it is this marginal viewpoint that this study develops, showing that there exists a wealthy relationship between neural networks and Riemannian geometry. The formulate neural networks at the structural level of a Riemannian manifold. This prescription of finding and studying the minimum mathematical structure is common in physics, where for example the mathematical structure required understanding Quantum Mechanics is Hilbert spaces and for General Relativity it is Riemannian manifolds.

Research in the field of neural networks has been attracting growing awareness in modern years. While burrow [McCulloch and Walter Pitts (1943)] obtainable the initial model of artificial neurons, original and extra complicated proposals have been prepared starting every ten years. Mathematical analysis has solved a few of the mysteries posed by the latest models but has missing various questions open for upcoming investigations. Unnecessary to say, the study of neurons, their interconnections, and their role as the brain's elementary structure blocks is one of the most dynamic and important research fields in modern biology. Neural networks are frequently intended for statistical analysis and data modelling, in which their role is professed as a different to normal nonlinear regression or cluster analysis techniques (Cheng & Titterington (1994)]. Haykin (1994) gives a wide-ranging survey of various neural network techniques from an engineering viewpoint. Physicists and Mathematicians are tired to study of networks starting a concern in nonlinear dynamical systems, statistical mechanics and automata theory. It is the occupation of applied Mathematicians to determine and celebrate the properties of new systems using tools formerly working in new areas of science.. The complete Mathematical equipment for exploring these relations is developed [Amit (1989)].

To tissue this out a small, we first take a speedy seem at a few basic neurobiology. The human brain consists of an expected 1011 (100 billion) nerve cells or *neurons.* Neurons correspond via electrical signals that are small-lived impulses or "spikes" in the power of the cell wall or crust. The inter neuron relations are mediated by electro-chemical junctions called synapses, which are placed on branches of the cell referred to as dendrites. Each neuron naturally receives a lot of thousands of relations from other neurons and is therefore always getting a large number of incoming signals, which finally reach the cell body. Here, they are integrated or summed together in some way and, generally talking, if the resulting signal exceeds some doorstep then the neuron will "flames" or produce a power impulse in reply. This is then transmitted to other neurons via a branching fibred known as the ***axon*.**

2. ARTIFICIAL NEURAL NETWORKS

The word "Network" will be accustomed to mention several systems of artificial neurons. This may vary from impressive as natural as a single node to a large collection of nodes in which each one is connected to every other node in the mesh. The nodes are set in a layered structure in which each signal emanates from an input and passes via two nodes before reaching an output beyond which it is no longer changed. This nosh onward structure is only one of several obtainable and is characteristically used to place an input model into one of several module according to the ensuing model of outputs.

Artificial neural network working on usually referred to like neural network has been aggravated right from its initiation by the appreciation that the human brain computers in a wholly special technique from the straight digital computer. The brain is a vastly complex, nonlinear, and similar computer (information-processing system). It has the ability to categorize its structural constituents, known as neurons, thus as to achieve positive computations (e.g., parallel recognition, perception, and glide control) various times quicker than the top digital computer in subsistence nowadays. [Churchland and Sejnowski, (1992)].

Artificial Neural Networks be an effort on modeling the information meting out capabilities of Nervous systems. Hence, to begin with, we have to judge the necessary properties of biological Neural Networks from the opinion of information meting out. These determinations allow us to design abstract models of artificial Neural Networks, which are able to next live simulated and analyzed. Even though the models which have been planned to clarify the structure of the brain and the nervous systems of some animals are dissimilar in several respects, here is a universal consensus to the soul of the process of neural collections is control through communication [Murre, (1995)].

Now, the neurons are two types Real and Artificial. The real neurons within synaptic strengths may well, under definite conditions, be modified in order that the behaviour of each neuron can change or adapt to its exacting incentive input. Neurons are not only extremely complex but also vary significantly in the information of their structure and function. Thompson (1993) provides a good introductory text, while more comprehensive accounts are given by Kandel et al. (1991) and Kuffler et al. (1984). The artificial neurons corresponding of this is the amendment of the weight values. The conditions of information processing present are no computer programs at this time "knowledge" the network have is believed to be stored in its weights, which progress by a method of revision to motivation from a set of pattern models. The building blocks of artificial neural networks are artificial neurons.

3. GEOMETRIC INTERPRETATION OF THRESHOLD LOGIC UNIT (TLU) ACTION

If, we have a node receiving n input signals x_1, x_2,… , x_n, then these may only take on the values "0" or "1". We therefore have n weights w_1, w_2…, w_n and form the n products $w_1x_1,\ w_2x_2, \dots\dots, w_nx_n$. This will be done by simply adding them together to produce the activation a (corresponding to the axon-hillock membrane potential) so that:

$$a = w_1x_1 + w_2x_2 + \cdots \dots \dots \dots + \ w_nx_n \qquad (3.1)$$

This type of artificial neuron is known as a threshold logic unit (TLU) and was initially proposed by [McCulloch & Pitts (1943)].

Next to, writing the index symbolically (rather than numerically), we can refer to quantities commonly so that x_i , and denotes the basic or i^{th} input where it is implicit that i can be any integer between 1 and n. Similar explanation concern to the weights w_i. Use these ideas it is likely to represent (3.1) in a added compressed form:

$$a = \sum_{i=1}^{n} w_ix_i \qquad (3.2)$$

Where Σ (upper case Greek sigma) denotes summation. The expressions above and below Σ denote the upper and lower limits of the summation and tell us that the index i runs from 1 to n. Sometimes the limits are misplaced, because they have been defined elsewhere and we simply indicate the summation index (in this case *i*) by writing it below the Σ. The threshold relation for obtaining the output *y* may be written:

$$y = \begin{Bmatrix} 1\,if \quad a \geq \theta \\ 0\,if \quad a < \theta \end{Bmatrix} \tag{3.3}$$

In summing up, a TLU separates its input patterns into two categories according to its binary response ("0" or "1") to each pattern. These categories may be thought of as regions in a multidimensional space that are separated by the higher dimensional equivalent of a straight line or plane. These thoughts are now introduced little by little and in a technique that should help put to interval any concerns about higher dimensionality and multidimensional spaces.

4. NEURAL NETWORK MODELS

This paper aims to succinctly provide the necessary background for understanding the standard formulation of neural networks in machine learning in the setting of graphical models, digital signal processing and density estimation. This is by no means a complete review of neural networks, rather a short introduction to provide the reader with a working knowledge of neural networks. The two traditional paradigms of machine learning, supervised and unsupervised modeling will be reviewed here.

Supervised Models:

The objective of this part is to provide an essential working knowledge of neural networks for supervised learning. The objective of the supervised model is to learn parameters θ for a fixed model structure h_θ : $X \longrightarrow Y$ such that the output is usually interpreted as:

$P\,(Y \mid X = x) := h_\theta\,(x).$

The parameters of the model are the learned weights and biases from all the layers.

$\theta := \{(W^{(l)}, \mathrm{b}^{(l)})\}_{l=0}^{L}$

Where $(W^{(l)}, \mathrm{b}^{(l)})$ are the parameters for the softmax classification layer.

The probabilistic output usually comes from the linear softmax classifier:

$$\text{Softmax}\,(z^{(L)}) = \frac{e^{z^{(L)}}}{\sum_{k=1}^{K} e^{z_k^{(L)}}} \tag{4.1}$$

Where $z^{(L)} = W^{(L)}.\,x^{(L)}$ *and* $x^{(L)}$ is the final layer input for the softmax classifier.

Standard and Convolution feed forward Neural Networks:

A standard feed forward neural network is a multilayer perceptron [Hornik, Stinchcombe, and White (1989), Kurkova, (1992)] and is densely connected:

$$x^{(l+1)} = f\,(W^{(l)} \cdot x^{(l)} + b^{(l)}). \tag{4.2}$$

At this point, the **· and +** operations are just the standard matrix multiplication and vector addition operations of finite dimensional linear algebra, and f is some non-linear function applied element wise such as ReLu, tanh or σ **.** The universal approximator theorems [Hornik, Stinchcombe, and White (1989), Kurkova, (1992)] are for a squashing non-linearity.

The standard feed forward network is densely connected because there are no a priori restrictions on the network connections $W^{(l)}$. Convolution neural networks [LeCun, Bengio, et al. (1995) Krizhevsky, Sutskever, and Hinton (2012)] impose a substantial restriction on network connections:

$$x^{(l+1)} = f(W^{(l)} * x^{(l)} + b^{(l)}). \tag{4.3}$$

Where the $'*'$ is the standard convolution operation from digital signal processing; for example in 2-dimension:

$$(W^{(l)} * x^{(l)})_{i,j} = \sum_{j'=-J}^{J} \sum_{i'=-I}^{I} W_{i'j'}^{(l)} * x_{i-i'j-j'}^{(l)} \tag{4.4}$$

For an image $x^{(l)}$ of size $m_l \times n_l$ with c_l-channels, the dimension of the entire image is $m_l \times n_l \times c_l$. If the convolution weight matrix has c_{l+1} filters/feature detectors, all of size $u_l \times v_l$, then the matrix $W^{(l)}$ has total dimensions $u_l \times v_l \times c_l \times c_{l+1}$. For a standard feed forward network, the residual network takes the following form.

$$x^{(l+1)} = x^{(l)} + f\,(W^{(l)}.x^{(l)} + b^{(l)}) \tag{4.5}$$

5. CONCLUSION

The role of real neurons is exceedingly complex. The necessary information meting out attributes possibly will be considerable. A neuron receives input signals from numerous new neurons. Each such signal is transformed (by the synaptic mechanism) from the electrical energy point of an exploit potential into a constantly variable (graded) postsynaptic potential (PSP). Artificial neural networks possibly will be deliberation of as basic models of the networks of neurons that transpire physically in the animal brain. The biological point of view the necessary requirement for a neural network is that it should effort to confine what we judge is the necessary information meting out characteristics of the related real network.

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FLUID FLOW UNDER BOUNDARY LAYER WITH FLAT PLATE THROUGH POROUS MEDIUM

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ABSTRACT
In the present paper, we have investigated fluid flow under boundary layer with a flat plate through porous medium. The stream function, velocity components (u, v), velocity potential have also been derived.
Keywords: Boundary layer flow, complex potential, porous medium, velocity potential

INTRODUCTION

In the present research paper we have investigated, motion of boundary layer fluid flow over a flat plate through porous medium. Attempts have been made by several researchers in this direction. Walton[28], investigated the non liner instability of the thread annular flow at high Reynolds number Kader[18], investigated temperature and concentration profiles in fully turbulent boundary layer. Pozrikidis[25], explored a singularity method for unsteady linearized flow. Lynch and Holboke[23], developed normal flow boundary conditions in 3-D carnation models. Crank[13], investigated free and moving boundary problems. Merkim and. Chaudhary[24], investigated Face convection boundary layer on vertical surface driven by an exothermic surface reaction. Barichello and Siewart[5], investigated a discrete ordinates solution for poisecuille flow in a plane channel. Chatterjee[11], investigated evolution of the boundary layer over the southern ocean. Shola and Proudman[27], applied a numerical approach for free boundary fluid flow in a trough. In this study, we have investigated the stream function, velocity components (u,v), velocity potential, Complex potential of boundary layer fluid flow a flat plate through porous medium.

NOMENCLATURE

u= Velocity along the X-axis
v=velocity along the Y-axis
u_∞ =Undisturbed velocity
η=Dimension less distance parameter
Ø= Velocity potential
ψ=Stream function
W= Complex potential
υ=Kinematic velocity

FORMULATION OF THE PROBLEM

Consider a thin infinite plate submerged in steady incompressible plane parallel flow, whose undisturbed velocity is u_∞. The fluid has low viscosity, and the plate is such a way that its plane coincides with the direction of u_∞. Since the plate is of infinite length, the flow may be regarded as two dimensional, the x-axis lying along the plate parallel to du_∞ and y-axis normal to the plane, since $u_\infty$is constant then du_∞/dx=0.

Governing boundary layer equations are

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

And boundary conditions

u=0=v at y=0

u=u_∞ at y$\rightarrow\infty$

And

$$u=\frac{\partial \Psi}{\partial y},\ v=-\frac{\partial \Psi}{\partial x}$$

We take a new dimension less distance parameter

$$\eta=y\sqrt{\frac{u_\infty}{ux}} \tag{5}$$

SOLUTION OF THE PROBLEM:

Using (4) into (1), we get

$$\frac{\partial \Psi}{\partial y}\frac{\partial \Psi}{\partial x\partial y}-\frac{\partial \Psi}{\partial x}\frac{\partial \Psi}{\partial y}=\ \upsilon\frac{\partial \Psi}{\partial x\partial y} \tag{6}$$

And by (3) and (4), we get

On y=0 $\quad \frac{\partial \Psi}{\partial x}=0=\frac{\partial \Psi}{\partial y}$ (7)

On y→∞ $\quad \frac{\partial \Psi}{\partial x}=u_\infty$ (8)

Let G (h) $=\frac{u}{u_\infty}$ (9)

From (4), we have

$$\Psi=\int u dy$$

$$=\int u_\infty g\ (h)dy$$

$$=\int u_\infty g\ (h)\sqrt{\frac{xu}{u_\infty}}\,dh$$

Because

$$\eta=y\sqrt{\frac{u_\infty}{xu}}$$

$$y=\eta\sqrt{\frac{xu}{u_\infty}}$$

$$dy = \sqrt{\frac{xv}{u_\infty}}dh$$

$$\psi = \int(h)\sqrt{xvu_\infty}\ \ dh$$

$$\psi = \sqrt{xvu_\infty}\int G(h)dh$$

$$\psi = \sqrt{u_\infty xv}\ g(h) \tag{10}$$

$$\int G(h)dh = g(h)$$

$$\frac{\partial h}{\partial x} = y\sqrt{\frac{u_\infty}{v}}\ \left[-\frac{1}{2}x^{\frac{-3}{2}}\right]$$

$$\frac{\partial h}{\partial x} - \frac{1}{2x}y\sqrt{\frac{u_\infty}{vx}}$$

$$\frac{\partial h}{\partial x} = -\frac{1}{2x}\eta$$

$$\frac{\partial h}{\partial y} = \sqrt{\frac{u_\infty}{vx}}$$

Where

$$\frac{\partial \Psi}{\partial y}=\sqrt{}\mathrm{x}\upsilon u_\infty\ \frac{\partial}{\partial y}\mathrm{g(h)}$$

$$\frac{\partial \Psi}{\partial y}=\sqrt{}\mathrm{x}\upsilon u_\infty g'(\mathrm{h})\frac{\partial h}{\partial y}$$

$$\frac{\partial \Psi}{\partial y}=\sqrt{}\mathrm{x}\upsilon u_\infty \mathrm{g(h)}\sqrt{\frac{u\infty}{x\upsilon}}$$

$$\frac{\partial \Psi}{\partial y}=u_\infty g'(\mathrm{h}) \tag{11}$$

$$\frac{\partial \Psi}{\partial x}=\sqrt{\upsilon u}[g'(\mathrm{h})\frac{\partial h}{\partial x}x^{\frac{1}{2}}+\frac{1}{2}x^{\frac{-1}{2}}g(h)]$$

$$\frac{\partial \Psi}{\partial x}=\sqrt{\upsilon u_\infty}\ [x^{\frac{1}{2}}g'(\eta)(-\frac{1}{2x}\eta)+\frac{1}{2}x^{\frac{-1}{2}}g(h)]$$

$$\frac{\partial \Psi}{\partial x} = \sqrt{\upsilon u_\infty}[-\frac{1}{2\sqrt{x}}\eta g'(\eta) + \frac{1}{2\sqrt{x}}g(\eta)]$$

$$\frac{\partial \Psi}{\partial x} = \frac{1}{2}\sqrt{\frac{\upsilon \mathrm{u}_\infty}{x}}[-\mathrm{h}g'(f)+\mathrm{g}(\eta) \tag{12}$$

$$\frac{\partial^2 \Psi}{\partial x\partial y}=\frac{1}{2}\sqrt{\frac{\upsilon u_\infty}{x}}[g'(\ \eta)\frac{\partial h}{\partial y}\text{- } g'(\eta)\ \frac{\partial \eta}{\partial y}\text{- }\eta g''(\ \eta)\frac{\partial \eta}{\partial y}]$$

$$\frac{\partial^2 \Psi}{\partial x\partial y}=\frac{1}{2}\sqrt{\frac{\upsilon u_\infty}{x}}[-\eta g''(\ \eta)\frac{\partial \eta}{\partial y}]$$

$$\frac{1}{2}\sqrt{\frac{\upsilon u_\infty}{x}}[-\eta g''(\ \eta)\sqrt{\frac{u_\infty}{x v}}]$$

$$\frac{\partial^2 \psi}{\partial x\partial y}=-\frac{u_\infty \eta g''(\eta)}{2x} \tag{13}$$

$$\frac{\partial^2 \Psi}{\partial y^2}=u_\infty g''(\eta)\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 \Psi}{\partial y^2}=\mathrm{u}_\infty\sqrt{\frac{u_\infty}{xv}}g''(\ \eta) \tag{14}$$

$$\frac{\partial^3 \Psi}{\partial x^2\partial y}=\frac{-u_\infty}{2}[-x^{-2}\ \eta\ g''(\ \eta)+\frac{1}{x}\frac{\partial \eta}{\partial x}g''(\eta)]+\frac{\eta}{x}g'''(\ \eta)\frac{\partial h}{\partial x}$$

$$\frac{\partial^3 \Psi}{\partial\ x\partial y}=-\frac{u_\infty}{2}\ [\frac{-h}{x}g''(\eta) + \frac{1}{x}\left(-\frac{1}{2x}\eta\right)g''(\eta) + \frac{\eta}{x}g'''(\eta)(-\frac{1}{2x})]$$

$$\frac{\partial^3 \Psi}{\partial\ x\partial y} = \frac{-u_\infty}{2}\left[-\frac{3h}{2x^2}g''(\eta) - \frac{\eta^2}{2x^2}g'''(\eta)\right]$$

$$\frac{\partial^3\Psi}{\partial x^2\partial y}=\frac{3u_\infty h}{4x^2}g''(\eta)+\frac{u_\infty\eta^2}{4x^2}g'''(\eta) \tag{15}$$

USING (11),(12),(13),(14) and (15) into (16),we get

$$[u_\infty g'(\eta)]\left[-\frac{u_\infty\eta}{2x}g''(\eta)\right]-\frac{1}{2}\sqrt{\frac{vu_\infty}{x}}[g(\eta)-hg'(\eta)]\left[u_\infty\sqrt{\frac{u_\infty}{xv}}g''(\eta)\right]$$

$$=v\left[\frac{3\eta u_\infty}{4x^2}\ g''(\eta)+\frac{u_\infty\eta^2}{4x^2}g'''(\eta)\right]-\frac{u_\infty^2}{2x}g'(\eta)g''(\eta)-\frac{u_\infty^2}{2x}[g(\eta)-\eta g'(\eta)]g''(\eta)$$

$$=v\left[\frac{3\eta u_\infty}{4x^2}\ g''(\eta)+\frac{u_\infty\eta^2}{4x^2}g'''(\eta)\right]-\frac{u_\infty^2}{2x}g'(\eta)g''(\eta)-\frac{u_\infty^2}{2x}g(\eta)g''(\eta)+\qquad\frac{u_\infty^2\eta}{2x}g'(\eta)g''(\eta)$$

$$=v\left[\frac{3\eta u_\infty}{4x^2}\ g''(\eta)+\frac{u_\infty\eta^2}{4x^2}g'''(\eta)\right]-\frac{u_\infty^2}{2x}g'(\eta)g''(\eta)=v\left[\frac{3\eta u_\infty}{4x^2}\ g''(\eta)+\frac{u_\infty\eta^2}{4x^2}g'''(\eta)\right]-2U_\infty^2XG(\eta)g''(\eta)=$$

$$3u_\infty\eta vg''(\eta)+u_\infty\eta vg'(\eta)+u_\infty\eta^2vg'''(\eta)-2u_\infty xg''=3\eta vg''+\eta^2vg'''$$

$$3\,\eta vg''+\eta^2vg'''+2u_\infty xgg''=0 \tag{16}$$

For solving neglecting the third term of equation (16)

We get

$$v[3\eta g''(\eta)+\eta^2g'''(\eta)]=0$$

$$\eta^2g'''(\eta)+3\eta g''(\eta)=0$$

$$\eta^3g'''(\eta)+3\eta^2g''(\eta)=0 \tag{17}$$

Which is homogeneous linear diff.eq. So put $\eta=e^z$ then

$$[D(D-1)(D-2)+3D(D-1)]g(\eta)=0$$

$$[D(D^2-3d+2)+3D^2-3d]g(\eta)=0$$

$$(D^3-D)g(\eta)$$

a.e

$$m^3-m=0\qquad\qquad m=0,1,-1$$

$$C.f=C_1+C_2\eta+\frac{C_3}{\eta}$$

$$P.I=0$$

there fore

$$g(\eta)=C.f+P.I$$

$$g(\eta)=C_1+C_2\eta+\frac{C_3}{\eta} \tag{18}$$

where C_1,C_2,C_3 are constant whose value can be determine with help of initial and boundary conditions .Then by equation (10) we get…………..

$$\varphi=\sqrt{u_\infty vx}\left[C_1+C_2\eta+\frac{C_3}{\eta}\right] \tag{19}$$

there fore

$$u=\frac{\partial\psi}{\partial y}=\sqrt{u_\infty vx}\left[C_2\frac{\partial\eta}{\partial y}-\frac{C_3}{\eta^2}\frac{\partial\eta}{\partial y}\right]$$

$$=\frac{\partial\psi}{\partial y}=\sqrt{u_\infty vx}\left[C_2\frac{\partial\eta}{\partial y}-\frac{C_3}{\eta^2}\sqrt{\frac{u_\infty}{xv}}\right]$$

$$= \sqrt{u_\infty v x}\sqrt{\frac{u_\infty}{xv}}\left[C_2 - \frac{C_3}{\eta^2}\right]$$

$$u = u_\infty\left[C_2 - \frac{C_3}{\eta^2}\right] \tag{20}$$

And

$$V = -\frac{\eta\partial\psi}{\partial x}$$

$$= -\sqrt{vu_\infty}\left[\frac{1}{2}\ \mathrm{x}(\mathrm{c1} + \mathrm{C}_2\eta + \frac{C_3}{\eta}) + \sqrt{x}\ (\mathrm{C}_2\frac{\partial\eta}{\partial x} - \frac{C_3}{\eta^2}\frac{\partial\eta}{\partial x})\right]$$

$$= -\sqrt{vu_\infty}\left[\frac{1}{2\sqrt{x}}\ (\mathrm{C}_1 + \mathrm{C}_2\eta + \frac{C_3}{\eta}) + \sqrt{x}\ (\mathrm{C}_2\left(\frac{-\eta}{2x}\right) - \frac{C_3}{\eta}\left(\frac{-\eta}{2x}\right))\right]$$

$$= \sqrt{-vu_\infty}\left[\frac{1}{2\sqrt{x}}(\mathrm{C}_1 + \mathrm{C}_2\eta + \frac{C_3}{\eta} - \frac{\sqrt{x}}{2}\ \frac{\eta}{\mathrm{x}}(\mathrm{C}_2 - \frac{C_3}{\eta^2})\right]$$

$$= \sqrt{-vu_\infty}\left[\frac{1}{2\sqrt{x}}(\mathrm{C}_1 + \mathrm{C}_2\eta + \frac{C_3}{\eta}) - \frac{1}{2\sqrt{x}}(\mathrm{C}_2\eta - \frac{C_3}{\eta})\right]$$

$$= \sqrt{-vu_\infty}\left[\frac{1}{2\sqrt{x}}(\mathrm{C}_1 + \mathrm{C}_2\eta + \frac{C_3}{\eta} - \mathrm{C}_2\eta + \frac{C_3}{\eta})\right]$$

$$\mathrm{V} = -\frac{1}{2}\sqrt{\frac{u_\infty v}{x}}\left[\mathrm{C}_1 + \frac{2}{\eta}\mathrm{C}_3\right] \tag{21}$$

We have

$$\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y}$$

$$\frac{\partial\emptyset}{\partial x} = -\sqrt{u_\infty xv}\left[\mathrm{C}_2\frac{\partial\eta}{\partial y} - \frac{C_3}{\eta^2}\frac{\partial\eta}{\partial y}\right]$$

$$= -\sqrt{u_\infty xv}\left[\mathrm{C}_2\sqrt{\frac{u_\infty}{xv}} - \frac{C_3}{\eta^2}\ \sqrt{\frac{u_\infty}{xv}}\right]$$

$$= -\sqrt{u_\infty xv}\sqrt{\frac{u_\infty}{xv}}\left[\mathrm{C}_2 - \frac{C_3}{\eta^2}\right]$$

$$\frac{\partial\emptyset}{\partial x} = -u_\infty\left[\mathrm{C}_2 - \frac{C_3}{\eta^2}\right]$$

$$\emptyset = -u_\infty\left[\mathrm{C}_2\mathrm{x} - \mathrm{C}_3\int\ \frac{1}{\eta^2}dx\right]$$

$$\emptyset = -u_\infty\left[\mathrm{C}_2\mathrm{x} - \mathrm{C}_3\int\ \frac{vx}{y^2u_\infty}dx\ \right]$$

$$\emptyset = -u_\infty\left[\mathrm{C}_2\mathrm{x} - \frac{C_3v}{y^2u_\infty}\int \mathrm{x\ dx}\right]$$

$$\emptyset = -u_\infty\left[\mathrm{C}_2\mathrm{x} - \frac{C_3vx^2}{2u_\infty y^2}\right] \tag{22}$$

The complex potential w=∅+iΨ

$$\mathrm{w} = -\!-u_\infty\left[\mathrm{C}_2\mathrm{x} - \frac{C_3vx^2}{2u_\infty y^2}\right]\ + \mathrm{i}\sqrt{u_\infty xv}\ [\mathrm{C}_1 + \mathrm{C}_2\eta + \frac{C_3}{\eta}] \tag{23}$$

$$\phi = -U\left[C_2x - \frac{C_3v}{2u}\ \frac{X^2}{Y^2}\right] \tag{24}$$

CONCLUSION AND DISCUSSION

In the present study we have made an attempt to investigate, the stream function, velocity components (u,v) ,velocity potential ,and complex potential of boundary layer fluid flow over a flat plate through porous medium.

We have investigated stream function given by equation no. (19) and calculated the velocity component by equation no. (20, 21). We tried to find out the velocity potential in given by equation no. (22) and we have investigated the complex potential is given by equation no. (23), fluid flow under boundary layer with a flat plate through porous medium.

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EVOLUTION OF TSALLIS HOLOGRAPHIC DARK ENERGY MODELS : A REVIEW

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ABSTRACT
In the present review Tsallis holographic dark energy models in existing theories of gravity were discussed. The evolution of different cosmological parameters like Equation of State, Deceleration parameter, Energy density is observed. The Statefinder diagnostic pair (r,s), w_D - w'_D analysis is also discussed. The necessary remedies are suggested to the existing results to get more better results.
Keywords: Tsallis, Gravity, Dark energy, Diagnostic

1. INTRODUCTION

The surprising result that our universe is accelerating, available in cosmology nowadays. It is believed that approximate ninety five percent of our universe consists of two remarkable components known as Dark Matter (DM) and Dark Energy(DE)[33, 35, 34, 40, 19] and remaining percentage include radiation and baryonic matter. The DM which is about 25% the universe is pressureless dark matter which contribute to the velocity of galaxy clusters[1]. The DE which is about 70% of the universe is a keystone of present accelerating expansion of the universe. The dark sector of the cosmos arouses interest of physcist and cosmologist to know about the factor responsible for current accelerating expansion of the universe. The most famous General Theory of Relativity (GTR) lacks in explaining the current accelerating expansion of the universe[20, 21]. The GTR shows incompatibility with other well established theories of gravity and lacks of uniqueness. This problem can be eliminated by introducing modification in General Theory of Relativity or by source of DE having negative pressure. It is well established fact in cosmology that the early universe experienced an accelerated phase followed by radiation and matter dominated epoch before current accelerated expansion of the universe. A large number of models are available in literature to explain the present accelerated expansion of the universe[29, 36, 41]. The ΛCDM known as Λ-Cold Dark Matter model containing the cosmological constant was the main candidate to explain DE phenomenon but it suffers from two serious problems namely fine tuning and cosmic coincidence problem. The cosmological coincidence problem was observed first time by Steinhardt [7] and Zlatev et al.[9]. The fine tuning problem of the ΛCDM model is related to difference between observed value and theoretical value. The observations predict a very tiny value in comparison to the theoretical value of it which is about to 120 orders higher. Fine-tuning and coincidence problems associated with the cosmological constant have led to the search for dynamical DE models [30]. A variety of models with a family of scalar field like phantom[15, 22, 23], quintessence[3, 4, 8, 10], K-essence[12] , quintom [28, 37], Chaplygin gas model, generalized Chaplygin gas model and modified Chaplygin gas model[13, 16, 24, 25, 31, 42, 44, 45, 47,48 49, 50, 51, 52, 53,69] have been discussed. These models includes various cosmological parameter to discuss the dynamics of the universe. The Equation of State parameter(EOS) defines about accelerated and decelerated expansion of the universe. It is defined as the ratio of pressure p_D to energy density ρ_D and looks like $w_D = p_D/\rho_D$. The value of w_D is classified as follows for w = 1, it represent stiff fluid, w =1/3 shows radiation dominated phase whereas w=0 shows matter dominated phase, -1 < w < -1/3 shows DE dominanted accerlerated phase with quintessence and w = -1 shows the cosmological constant where w <-1 present phantom era. The deceleration parameter (DP) denoted by q= -1 - $\dot{H}/H^2$ is used to determine the accelerated or decelerated expansion of the universe.

The GTR shows good result at solar system scale but lacks at cosmic scale to find out origin of mysterious component of DE which has negative pressure. The first modification to Eienstein theory of gravity was Kaluza -Klein gravity[38] in which gravity was unified with electromagnetic force. Brans-Dicke(BD) theory, which was proposed by Brans and Dicke[2] in 1961 is a natural extension of Eienstein General Theory of Relativity. In this theory the dynamics of gravity were represented by a scalar field and dynamics of spacetime were represented by the metric tensor. This theory is most successful theory nowadays because of its association with string theory and extra dimensional theory. It is solely based on dimensionless argument and with the matter Lagrangian being minimally coupled. In BD theory, Newton's gravitational constant G is not presumed to be constant but is proportional to the inverse of the scalar field ϕ , which can vary from place to place and with time.

2. TSALLIS HOLOGRAPHIC DARK ENERGY MODEL

An interesting phenomenon to explain origin and nature of DE is application of holographic principle[5, 6, 17] at cosmological framework. The relation between the Ultra violet cut off the quantum field theory and vacuum energy with largest distances of the theory leads to the form of DE known as Holographic Dark Energy[11,26]. According to holographic principle, the number of degrees of freedom in a bounded system should be finite and has relations with the area of its boundary. For a system with size L and ultra-voilet (UV) cut off without decaying in to a black hole , it is required that total energy in region of size L should not exceed the mass of a black hole. Various HDE models [56, 43, 32] have been studied to observe the dynamic of universe. The area entropy relation is the key ingredient of this hypothesis assuming that black hole relates short distances cut off to the long distances cut off, results to an upper bound for the zero point energy density. The Holographic Dark Energy(HDE) faces some problems and unable to explain the timeline of a flat FRW universe. In extension to HDE models, new model of DE has been proposed using Tsallis entropy and holographic principle known as Tsallis holographic dark energy(THDE)[57, 58]. Many DE models on THDE are available in literature to search for the behavior of present universe[59, 64, 65, 72, 73, 66, 67, 55]. It is observed that using Tsallis statics to the system horizon[39, 18, 14] the Bekenstein entropy is obtained and approaches to stability of the model[60]. The energy density of HDE can be given by $\rho_D = 3c^2M_p^2$ where $c^2$is a numerical constant, and M_p^2 is reduced Planck mass and L denotes the size of current universe[27, 26]. The standard HDE depends on the entropy area relation S ~A ~L^2 of black hole with A = 4 πL^2 denotes the area of horizon.

The horizon entropy of a black hole can be modified as $S_\delta = \gamma A^\delta$[46] where γ is unknown constant and δ is non additivity parameter. A relation between the system entropy S and UV cut off was proposed by [11] as $L^{3\,3} \leq S^{3/4}$ which after combining with the relation $S_\delta = \gamma A^\delta$ gives the relation as $\Lambda^4 \leq (\gamma(4\pi)^\delta)L^{2\delta-4}$, where δ denotes the vacuum energy density . The energy density of THDE model is given by as $\rho_D = BL^{2\delta-4}$ where B is unknown parameter.

3. EVOLUTION OF TSALLIS HOLOGRAPHIC DARK ENERGY

Tsallis Holographic dark energy model was proposed [57] in standard cosmology by considering a flat Friedman-Robertson-Walker(FRW) universe without interaction between the dark sector of the universe. It is observed that the late time acceleration of the universe with Hubble radius is achieved. It is also observed that for the model parameter $\delta > 1$, the model is not stable during the cosmic evolution.

Following [57], Tsallis Holographic Dark Energy in the Brans-Dicke Cosmology was studied by[61]. The model has been discussed for non interacting and interacting case of cosmos sector in flat and non flat universe. For model parameter $\delta = 2$, the EOS parameter mimics cosmological constant in future for flat noninteracting THDE. The unstability of the model is achieved. If the interaction is added, then acceptable and proper behavior of of DP, Ω_D and EOS are not achieved , means it can meet the classical stability requirement. For non at FRW universe, the model provides the suitable description of cosmic evolution in both interacting and non interacting case .It is also observed that these description are not stable.

Another model of Tsallis holographic dark energy namely "A note on Tsallis holographic dark energy"[58] is studied in standard cosmology by using various infrared cut off including Ricci horizon, Particle horizon and Granda-Oliver(GO) cut off. It is obtained that for the Particle horizon as the IR cutoff, the late time accelerated expansion of the universe is achieved whereas the usual HDE models lacks in receiving late time acceleration .The current cosmic acceleration is achieved for interactions between the dark sector of the universe for all IR cut off .

Holographic dark energy through Tsallis entropy was studied in standard cosmology [62] with future event horizon as the IR cut off . The usual thermal history is seen means the successive order of matter and dark energy epochs observed, before resulting in dark energy dominant era.

Effects of anisotropy on sign changeable interacting Tsallis holographic dark energy [63] is investigated. The spatially homogenous Bianchi type I universe is considered with Hubble horizon, Particle horizon,GO horizon,Ricci horizon and Event horizon as IR cut off . The classical stability analysis shows that the model is unstable at the limit ($z \to -1$) . It is also mentioned that the stability may be achieved for the current universe($z \to 0$) depending on the value of model parameters like δ.

Another model of THDE named "observational constraints on interacting Tsallis holographic dark energy "is studied[69] with Hubble horizon and future horizon as IR cut off. The phenomenological non gravitational interaction between the dark sectors of the universe is considered with interaction term $Q = 3bH\rho_m$. The model is analyzed using various cosmological tests. The Pantheon Supernovae type Ia, Baryon acoustic oscillation, Cosmic Microwave Background, the local value of H_0 and Gamma ray burst data is taken for constraining the free parameters of the model as observational data. It is observed that the Hubble parameter and energy density of THDE and interacting THDE fits good with the observation.

Both interacting and non interacting THDE experience accelerating universe. The model is investigated for objective information criterion using Akaike and Bayesian Information Criterion. It is found that both interacting and non interacting THDE does not supported by observational data. The model shows instability for both interacting and non interacting THDE with Hubble horizon against the background whereas with future event horizon stability against the background in late time is observed.

Tsallis holographic dark energy in Fractal Universe[68] with Hubble Horizon as IR cut off in flat fractal universe is investigated for interacting and non interacting case with interaction term $Q = 3b^2h\rho_D$. The observational data from SNIa, Panthon data, Cosmic Microwave Background of Planck 2015, BOSS DR12, the extended Baryon Acoustic Oscillation Spectro-scopic Survey, 6df Survey , Gamma Ray Burst Data is taken for free parameters. To obtain the desired result Cosmo Hammer Paython package is applied. The DP shows accelerating expansion and remains in accelerating phase for redshift $0.6 < z_t < 0.8$,which fits good with the observation for which $0.5 < z_t < 1$. In limiting case $\beta \to 0$, the standard cosmology is recovered for DP of THDE. The evolution of density parameter Ω_D versus redshift z shows that when $(z \to \infty)$, $(\Omega_D \to 0)$ for early universe whereas when $(z \to 0)$, $\Omega_D \to 1$. The EOS parameter shows accelerated expansion at present time in absence of interaction. Also, it mimics the cosmological constant by setting coupling constant to zero. For non zero value of b^2 , the model crosses phanton divide line i.e $w_D < -1$ in future. The Statefinder diagnostic pair meets the ΛCDM point $(r, s) = (1,0)$. The value of Hubble constant is obtained as $H_0 = (68,70)$ which fits with the observation. The coupling constant is obtained as positive and small values which shows decaying DE to DM. It is suggested that in interacting case THDE can be investigated. One would like to study the dynamical system methods to point out the status of the non-linear interactions in future. The perturbation analysis compare to the gravitational lenses and the Large Scale Structure can be studied.

"Tsallis holographic dark energy in Brans-Dicke theory with logarithmic scalar field" is studied[70] with Hubble horizon as IR cut off for both interacting and non interacting case. In non interacting case the dimensionless density parameter is observed positive and decreasing function throughout the evolution and finally approaches to positive constant value for different values of model parameter δ and the steepness increases with increases in δ. The Equation of State parameter(EOS) shows matter dominant era , then adopts quintessence DE era and then turns to vacuum DE era. The model enters to an accelerated phase at an early time. It is also observed that EOS never crosses the Phantom Divide Line(PDL). The Deceleration Parameter versus redshift z is observed with smooth transition from early decelerated era to present accelerated phase .The transition redshift z_t decreases as δ increases. The transition redshift z_t from decelerated phase to an accelerated phase is observed in the interval $0.57 < z_t < 0.77$. The decreasing behavior of squared sound speed with positive sign shows stability of the model leading to unstability in late time. The w_D-w'_D plane analysis shows that expansion of the universe is more accelerating in freezing region. The interacting case with interaction term $Q = 3c^2(\rho_m + \rho_d)$ is also explore the possibility of new dynamics of the universe in the model.The density parameter has been plotted against redshift z showing decreasing behavior . However increasing the coupling constant c^2 and parameter leads the density parameter increases but effect of coupling constant ends near past and steepness is increased . The EOS parameter shows quintessence phase in starting then turns to phantom region by crossing PDL(vacuum dominated era). With increase in value of coupling constant,the EOS parameters achieve high phantom region.Increase in the value of c^2 and δ results in transition of the model from quintessence to phantom phase. The DP starts from decelerating phase then adopts accelerating phase and then finally gives q = -1. The w_D-w'_D plane is also taken in to account to discuss the dynamics of DE models. This analysis gives rise to freezing region only, means the present scenario gives consistent result with accelerated expansion of the universe. The model is seen to be stable for a short period of time and then shows instability as time increases.

Observational constraints on interacting Tsallis holographic dark energy in logarithmic Brans-Dicke theory[71] is investigated for both interacting and non interacting case with interaction $Q = 3c^2(\rho_m + \rho_d)$ for Hubble horizon as IR cut off. The dimensionless density parameter has been analyzed for non interacting and interacting case with the help of Planck's data which fits the desired requirements at present epoch. The EOS parameter also fits with the Plank's observation data. The graph of deceleration parameter for both interacting and non interacting case shows smooth transition from early decelerated era to current accelerated era of the universe. The transition red shift z_t from decelerated phase to an accelerated phase is observed in the interval $0.57 < z_t < 0.77$ which fits with observation which are given from H(z)+SN Ia for □CDM model. The model is stable for a short period of time and moves towards in-stability in later epoch for both interacting and non interacting case. The w_D-w'_D analysis corresponds to freezing region for both interacting and non interacting case. The model also meets the observation of Planck's collaboration which is given by s($--0.89 < w_D < -1.38$) and $w'_D < 1.32$.

A generalized interacting Tsallis holographic dark energy model and its thermodynamics implications[74] was investigated in standard cosmology with Hubble Horizon as IR cutoff with interaction term $Q = 3H(b_1^2\rho_m + b_2^2\rho_D)$ which is assumed to be non gravitational in nature. It is seen that the EOS of THDE is affected by Tsallis parameter δ and it can go to phantom regime or in the quintessence regime during the evolution before it asymptotically leads to cosmological constant value in future. Decelerating universe at an early epoch and accelerating universe at present is observed. It explains both the growth of structure at early time and late time acceleration of the universe. It is also observed that the Jerk parameter j is positive and approaches to ΛCDM model as $z \to -1$. The thermodynamics nature of the universe is also discussed for the model. It is observed that for Bekenstein entropy ($\delta = 1$), Generalized second law of thermodynamics is satisfied whereas for ($\delta \neq 1$), it is violated.

Study of Tsallis holographic dark energy model in the framework of Fractal cosmology [75] is studied. The evolution of fractal universe composed of Tsallis holographic dark energy and a pressure less dark matter with interaction term $Q = 3b^2H\rho_D$ is observed. The fractal function is chosen in a power-law form as, $v=a^{-\beta}$ with $\beta > 0$. The evolution of the density parameters Ω_m and Ω_D against the red shift parameter z is attained. It is seen that at high red shift, Ω_m dominates over Ω_D, while at the late time, Ω_D dominates over Ω_m. The energy transfer from THDE to DM is obtained. The flow of energy is very low at present whereas very high in early time. Although the energy transfer is from THDE to DM but in this model the growth of the universe is observed in such a manner that Ω_D dominates over Ω_m for the present universe. This gives a solution of cosmic coincidence problem. The normalized Hubble parameter h(z) for both THDE model and ΛCDM is compared with data points and observed that THDE model gives the observed value at every data point. The DP and total EOS versus redshift z is analyzed . The model shows smooth transition from early deceleration era to present acceleration era. The total EOS w_{total} attains values between -1 < w_{total} < - 1/3 < at low redshift and then remains greater than 1and approaches to zero at high redshift. It does not suffer from the problem of future singularity. THDE model shows instability against perturbation.

Cosmological Perturbation in Tsallis Holographic Dark energy Scenarios [76] with three HDE models by assuming perturbative level and observational viability is studied. The THDE with Future and Hubble Horizon as IR cutoff and standard HDE with Future event horizon as cutoff is taken into consideration. The application of linear perturbation theory is used to study the growth of matter perturbation in clustered and non clustered Holographic dark energy. A Bayesian analysis is applied using Baryon Acoustic Oscillations, Big Bang Nucleosynthesis, Cosmic Microwave Background, Cosmic Chronometer, Pantheon type Ia supernovae and redshift space distortion data to compare the model. By considering the HDE and ΛCDM, the relative deviation of matter fluctuation for present time shows minimum difference in values that the HDE behaves as cosmological constant. The clustered THDE models are greater to non clustered THDE, when impact of different parameter combination on linear growth is associated. The EOS parameter of THDE reflect a little difference from the cosmological constant with Hubble Horizon as IR cut off for the present time where it crosses the phantom barrier. For THDE with Future event horizon, there is no significant difference from standard HDE model. The new HDE and ΛCDM are taken parallel using Baysein evidence for all data combinations. It is found that the DE models studied in the present work not fit the observation in cluster scale as good as ΛCDM. This problem can be solved by taken some other experiments such J-PAS[77], DESI Aghamousa[78] and Euclid[79]

4. CONCLUSION

The present work shows Tsallis holographic dark energy models in existing theories of gravity. The literature review on Tsallis Holographic Dark Energy Models and the consequences of different cosmological parameters like Equation of State, Deceleration parameter, Energy density is observed. The Statefinder diagnostic pair (r, s), w_D-w'_D is also discussed. Recent result shows that the universe is in accelerating expansion mode described by phantom like behavior. Later results indicate that the entropy formalism plays an important role in understanding and explaining the dynamics of our universe.

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H-PROJECTIVE CURVATURE AND RICCI RECURRENT MANIFOLDS

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ABSTRACT

The theory of some H-projective curvature tensor properties inside a Kähler manifold [4]. That was developed further [5],[6], and[9]. The aim of this paper is to introduce recurrent manifolds on the H-projective Curvature and Ricci, and to research their properties with regard to other similar tensors. The H-projective curvature and n-recurrent Ricci tensor were studied. The paper is organized as follows. In section one; required details are given on recurring manifolds Curvature and Ricci. We have studied some of the curvature and Ricci tensor properties in section two and their equations are satisfied. In section three, with non-vanishing, we describe from n-recurrent to Ricci n-recurrent,, 2 forms and also define that the manifold is from n-symmetric to Ricci n-symmetric. In section four, we show that some H-projective curvature tensor theorems are in the form of n-recurrent and n-symmetric.

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KEY WORDS: Curvature and Ricci tensor; H-projective Ricci and Curvature tensor; KH-structure; con-harmonic, con-circular, H-con-harmonic Buchner curvature tensor etc.

1. INTRODUCTION

Let us consider a n-dimensional differentiable manifold of differentiability class C^{∞}. Let there is defined in M^{2n}, a C^{∞} vector valued linear function F, such that

(1.1) $F^2X = X$; for arbitrary vector field X.

(1.2) $\overline{X} \overset{def}{=} F(X)$.

Then it is said to be a recurring multiplicity, and it is said to give [8] an almost product structure. Let M^{2n} be endowed with a metric tensor g satisfying

(1.3) $`F(X,Y) \overset{def}{=} g(\overline{X},Y) = g(X,\overline{Y}) = F(Y,X)$

(1.4) $g(\overline{X},\overline{Y}) = g(X,Y)$

Then M^{2n} is called almost product Riemannian manifold [8] and $`F$ is a (0, 2) type tensor. Then the following equations hold

(1.6) $`F(\overline{X},Y) - `F(X,\overline{Y}) = 0$

(1.7) $`F(\overline{X},\overline{Y}) - F(X,Y) = 0$.

If on an Curvature and Ricci recurrent manifold M^{2n}, F is covariant constant with respect to the Riemannian connection D, i.e.,

(1.8) $(D_X F)Y = 0 = (VF)(Y,X)$,

Then M^{2n} is said to be recurrent manifold [2], [4] and [8] and V being the operator of covariant differentiation, is satisfied then M^{n} is said to be KH-structure.

2. CURVATURES AND RICCI TENSOR

Let K and Ric denote the curvature and Ricci recurrent tensor in M^{2n} and then the following equation are satisfies

(2.1) $K(X,Y,\bar{Z}) = \overline{K(X,Y,Z)}$,

(2.2) $\overline{K(X,Y,\bar{Z})} = K(X,Y,Z)$,

(2.3) $`K(X,Y,\bar{Z},T) = K(X,Y,Z,T) = -`K(\bar{X},\bar{Y},Z,T)$,

(2.4) $`K(X,Y,Z,T) \overset{def}{=} g(K(X,Y,Z),T)$

(2.5) (a) $Ric(X,Y) = g(r(Y),Z)$

(b) $Ric(Y,Z) = (C_1^1 K)(Y,Z)$,

(2.6) (a) $Ric(\bar{X},\bar{Y}) = Ric(X,Y)$

(b) $(C_1^1 r) = R$,

(2.7) $Ric(\bar{X},Y) = Ric(X,\bar{Y})$.

Then r is the self adjoins Ric map C_1^1 is the contraction with respect to first slot.

The H-projective curvature tensor in M^{2n} is given by

(2.8) $$P(X,Y,Z) = K(X,Y,Z) - \frac{1}{2(M+1)}[Ric(Y,Z) - Ric(X,Z)Y$$
$$-Ric(\bar{Y},Z)\bar{X} + Ric(\bar{X},Z)\bar{Y} - 2Ric(X,\bar{Y})\bar{Z},$$

(2.9) $`P(X,Y,Z,T) \overset{Def}{=} g(P(X,Y,Z),T)$.

Let W be the projective curvature tensor, let C be the conformal curvature tensor, let L be the con-harmonic curvature tensor, let V be the con-circular curvature tensor, let S be the H-harmonic curvature tensor and let B be the curvature tensor given by Buchner respectively (2.10)
$$W(X,Y,Z) = K(X,Y,Z) - \frac{1}{(2m-1)}[Ric(Y,Z)X - Ric(X,Z)Y]$$

(2.11) $$C(X,Y,Z) = K(X,Y,Z) - \frac{1}{2(m-1)}[Ric(Y,Z)X - Ric(X,Z)Y - g(X,Z)r(Y)$$
$$+g(Y,Z)r(X) + \frac{R}{2(2m-1)(m-1)}[g(Y,Z)X - g(X,Z)Y],$$

(2.12) $$L(X,Y,Z) = K(X,Y,Z) - \frac{1}{2(m-1)}[Ric(Y,Z)X - Ric(X,Z)Y$$
$$+g(Y,Z)r(X) - g(X,Z)r(Y)],$$

(2.13) $$V(X,Y,Z) = K(X,Y,Z) - \frac{R}{2m(2m-1)}[g(Y,Z)X - g(X,Z)Y]$$

$$(2.14)\quad S(X,Y,Z)=K(X,Y,Z)-\frac{1}{2(m+2)}[Ric(X,Z)Y-Ric(Y,Z)X$$
$$+Ric(\bar{X},Z)\bar{Y}-Ric(\bar{Y},Z)\bar{X}+2Ric(\bar{X},Y)\bar{Z}+g(X,Z)r(Y)$$
$$-g(Y,Z)r(X)+g(\bar{X},Z)r(\bar{Y})-g(\bar{Y},Z)r(\bar{X})+2g(\bar{X},Y)r(\bar{Z}),$$

$$(2.15)\quad B(X,Y,Z)=K(X,Y,Z)+\frac{1}{2(m+2)}[Ric(X,Z)Y-Ric(Y,Z)X+g(X,Z)r(Y)$$
$$-g(Y,Z)r(X)+Ric(\bar{X},Z)\bar{Y}-Ric(\bar{Y},Z)\bar{X}+g(X,Z)r(\bar{Y})-g(\bar{Y},Z)r(\bar{X})$$
$$-g(Y,Z)r(X)+2Ric(\bar{X},Y)\bar{Z}+2g(\bar{X},Y)r(\bar{Z})-\frac{R}{4(m+1)(m+2)}[g(X,Z)Y$$
$$-g(Y,Z)X+g(\bar{X},Z)\bar{Y}-g(\bar{Y},Z)\bar{X}+2g(\bar{X},Y)\bar{Z}].$$

Now, with the aid of (2.8) and (2.9), removing K from the equations (2.10), (2.11), (2.12), (2.13), (2.14) and (2.15) we have

$$(2.16)\quad P(X,Y,Z)=W(X,Y,Z)+\frac{3}{2(2m-1)(m+1)}[Ric(Y,Z)X-Ric(X,Z)Y$$
$$-\frac{1}{2(m+1)}[Ric(Y,\bar{Z})\bar{X}-Ric(X,\bar{Z})\bar{Y}-2Ric(X,\bar{Y})\bar{Z}],$$

$$(2{,}17)\quad P(X,Y,Z)=C(X,Y,Z)+\frac{1}{(m^2-1)}[Ric(Y,Z)X-Ric(X,Z)Y]$$
$$+\frac{1}{2(m+1)}[g(Y,Z)r(X)-g(X,Z)r(Y)]$$
$$-\frac{R}{2(m-1)(2m-1)}[g(Y,Z)X-g(X,Z)Y]$$
$$-\frac{1}{2(m+1)}[Ric(Y,\bar{Z})X-Ric(X,\bar{Z})\bar{Y}-2Ric(X,\bar{Y})\bar{Z},$$

$$(2.18)\quad P(X,Y,Z)=L(X,Y,Z)+\frac{1}{(m^2-1)}[Ric(Y,Z)X-Ric(X,Z)Y]$$
$$+\frac{1}{2(m+1)}[g(Y,Z)r(X)-g(X,Z)r(Y)]$$
$$-\frac{1}{2(m+1)}[Ric(\bar{Y},Z)\bar{X}-Ric(\bar{X},Z)\bar{Y}-2Ric(X,\bar{Y})\bar{Z},$$

$$(2.19)\quad P(X,Y,Z)=V(X,Y,Z)+\frac{R}{2m(2m-1)}[g(Y,Z)X-g(X,Z)Y]-\frac{1}{2(m+1)}$$
$$[Ric(Y,Z)X-Ric(X,Z)Y+Ric(Y,\bar{Z})\bar{X}-Ric(X,\bar{Z})\bar{Y}-2Ric(X,\bar{Y})\bar{Z}],$$

$$(2.20)\quad P(X,Y,Z)=S(X,Y,Z)-\frac{1}{2(m+1)(m+2)}[Ric(Y,Z)X-Ric(X,Z)Y]-\frac{(2m+3)}{2(m+1)(m+2)}$$
$$[Ric(X,Z)\bar{Y}-Ric(Y,Z)\bar{X}+2Ric(\bar{X},Y)\bar{Z}]-\frac{1}{2(m+2)}[g(X,Z)r(Y)$$
$$-g(Y,Z)r(X)+g(X,Z)r(Y)-g(Y,Z)r(X)+2g(X,Y)r(Z)]$$

(2.21) $$P(X,Y,Z)=S(X,Y,Z)-\frac{1}{2(m+1)(m+2)}[Ric(Y,Z)X-Ric(X,Z)Y]-\frac{(2m+3)}{2(m+1)(m+2)}$$
$$[Ric(\bar{Y},Z)\bar{X}-Ric(\bar{X},Z)\bar{Y}-2Ric(\bar{X},Y)\bar{Z}]-\frac{1}{2(m+2)}[g(X,Z)r(Y)$$
$$-g(Y,Z)r(X)+g(\bar{X},Z)r(\bar{Y})-g(\bar{Y},Z)r(\bar{X})+2g(\bar{X},Y)r(\bar{Z})]$$
$$+\frac{R}{4(m+1)(m+2)}[g(X,Z)Y-g(Y,Z)X+g(\bar{X},Z)\bar{Y}-g(\bar{Y},Z)\bar{X}+2g(\bar{X},Y)\bar{Z}]$$

3. RICCI RECURRENT MANIFOLD

If the manifold M^n is said to be recurrent, then

(3.1) $$(\nabla K)(X,Y,Z,U)\overset{Def}{=}A_1(U_1)K(X,Y,Z)$$

It is said to be Ricci recurrent, if

(3.2) $$\nabla Ric(Y,Z,U_1)=A_1(U_1)Ric(Y,Z),$$

This gives

(3.3) $$(\nabla_r)(Z,U_1)=A_1(U_1)r(Z),$$

(3.4) $$(\nabla R)(U_1)=A_1(U_1)R,$$

Where A_1 is a non vanishing, C^∞, 1-form. If the manifold M^n is said to be bi-recurrent [4], then

(3.5) $$(\nabla\nabla K)(X,Y,Z,U_1,U_2)=A_2(U_1,U_2)K(X,Y,Z)$$

It is said to be Ricci bi-recurrent, if

(3.6) $$(\nabla\nabla Ric)(Y,Z,U_1,U_2)=A_2(U_1,U_2)Ric(Y,Z),$$

Where A_2 is a non vanishing, C^∞ 2 form such that

(3.7) $$A_2(U_1,U_2)=(\nabla A_1)(U_1,U_2)+A_1(U_1)A_2(U_2),$$

The manifold M^n is said to be n-recurrent, if

(3.8) $$(\nabla.......\nabla\nabla K)(X,Y,Z,U_1,U_2........U_n)=A_n(U_1,U_2,......U_n)K(X,Y,Z),$$

It is said to be Ricci n-recurrent, if

(3.9) $$(\nabla.......\nabla\nabla Ric)(Y,Z,U_1,U_2........U_n)=A_n(U_1,U_2,......U_n)Ric(Y,Z)$$

This gives

(3.10) $$(\nabla.......\nabla\nabla r)(Z,U_1,U_2........U_n)=A_n(U_1,U_2,......U_n)r(Z),$$

(3.11) $$(\nabla.......\nabla\nabla R)(U_1,U_2........U_n)=A_n(U_1,U_2,......U_n)R,$$

Where A_n is non-vanishing C^∞, n form, such that

(3.12) $$A_n(U_1,U_2,......,U_n)=(\nabla A_{n-1})(U_1,U_2,......,U_n)+A_{n-1}(U_1,U_2,......,U_{n-1})A_1U_n$$

Equations (3.8), (3.9), (3.10), (3.11) and (3.12) show if the manifold is n-recurrent then n-recurrent is Ricci. Let Q, a vector valued, tri-linear function be any one of the curvature tensors W, C, L, V, S or B then M^{2n} is said to be Q n-recurrent, if

(3.13) $(\nabla.......\nabla\nabla Q)(X,Y,Z,U_1,U_2.......U_n) = A_n(U_1,U_2,......U_n)Q(X,Y,Z)$,

The manifold M^{2n} is said to be n-symmetric, if

(3.14) $(\nabla.......\nabla\nabla K)(X,Y,Z,U_1,U_2........,U_n) = 0$,

This gives

(3.15) $(\nabla.......\nabla\nabla Ric)(Y,Z,U_1,U_2........,U_n) = 0$,

Or

(3.16) $(\nabla.......\nabla\nabla r)(Z,U_1,U_2........,U_n) = 0$,

Or

(3.17) $(\nabla.......\nabla\nabla R)(U_1,U_2........,U_n) = 0$,

Equations (3.15), (3.16) and (3.17) define that the manifold is Ricci n-symmetric. The manifold M^{2n} is said to be Q n-symmetric, if

(3.18) $(\nabla.......\nabla\nabla Q)(X,Y,Z,U_1,U_2........,U_n) = 0$.

4. H-PROJECTIVE CURVATURE TENSOR

If the manifold M^{2n} is said to be H-Projective n-recurrent, if

(4.1) $(\nabla.......\nabla\nabla P)(X,Y,Z,U_1,U_2.......U_n) = A_n(U_1,U_2,......U_n)P(X,Y,Z)$

The manifold M^{2n} is said to be H-Projective n-symmetric, if

(4.2) $(\nabla.......\nabla\nabla P)(X,Y,Z,U_1,U_2,........,U_n) = 0$

Theorem 4.1. If the manifold M^{2n} be n-recurrent then it is H-projective n-recurrent for the same proportionally form A_n.

Proof. Differentiating covariant equation (2.8) with respect to $U_1,U_2,........,U_n$ successively, we get

(4.3)
$$\begin{aligned}(\nabla.......\nabla\nabla P)(X,Y,Z,U_1,U_2,........,U_n) &= (\nabla.......\nabla\nabla K)(X,Y,Z,U_1,U_2,........,U_n)\\ &-\frac{1}{2(m+1)}[(\nabla.......\nabla\nabla Ric)(Y,Z,U_1,U_2,........,U_n)X-(\nabla.......\nabla\nabla Ric)(X,Z,U_1,U_2,........,U_n)Y\\ &-(\nabla.........\nabla\nabla Ric)(\bar{Y},Z,U_1,U_2,........,U_n)\bar{X}+(\nabla........\nabla\nabla Ric)(\bar{X},Z,U_1,U_2,........,U_n)\bar{Y}\\ &-2(\nabla.......\nabla\nabla Ric)(X,\bar{Y},U_1,U_2,........,U_n)\bar{Z}]\end{aligned}$$

Since the theorem is n-recurring, using the equations (3.8), (3.9), (3.10), (3.11), (3.12) and (2.8) in (4.3), we get the theorem (4.1) hence.

Theorem 4.2. If the manifold M^{2n} be n-recurrent then it is H-projective n-recurrent and Ricci n-recurrent then it is n-recurrent for the same proportionality from A_n.

Proof. Multiplying equation (2.8) by $A_n(U_1, U_2, ...U_n)$ through, subtracting the resulting equation from (4.3) and then using (4.1) and (3.9), we get (3.8). Hence the statement.

Theorem 4.3. In the manifold M^{2n} if any two of the following conditions hold then the third also hold for the same proportionality form A_n.

(i) It is H-projective n-recurrent;

(ii) It is projective n-recurrent;

(iii) It is Ricci n-recurrent.

Proof. Differentiating covariant equation (2.16) with respect to $U_1, U_2, ...U_n$ successively, we get

(4.4) $$(\nabla.......\nabla\nabla P)(X,Y,Z,U_1,U_2,........,U_n) = (\nabla.......\nabla\nabla W)(X,Y,Z,U_1,U_2,........,U_n)$$
$$+\frac{3}{2(2m-1)((m+1)}[(\nabla.......\nabla\nabla Ric)(Y,Z,U_1,U_2,........,U_n)X$$
$$-(\nabla.......\nabla\nabla Ric)(X,Z,U_1,U_2,.....,U_n)Y - \frac{1}{2(m+1)}[(\nabla.....\nabla Ric)$$
$$(Y,\bar{Z},U_1,U_2,.....,U_n)\bar{X} - (\nabla.....\nabla\nabla Ric)(X,\bar{Z},U_1,U_2,.....,U_n)\bar{Y}$$
$$-2(\nabla.......\nabla\nabla Ric)(X,\bar{Y},U_1,U_2,.......,U_n)\bar{Z}].$$

Multiplying equation (2.16) by $A_n(U_1, U_2, ...U_n)$ throughout, subtracting the resulting equation from (4.4) and then using the fact that the manifold is H-projective n-recurrent and projective n-recurrent, we have

(4.5) $$(\nabla.......\nabla\nabla Ric)(Y,Z,U_1,U_2,........,U_n) = A_n(U_1,U_2,........,U_n)Ric(Y,Z)$$

This shows the manifold is n-recurrent to Ricci. Likewise, it can be seen that if the manifold is both H-projective n-recurrent and Ricci n-recurrent or n-recurrent projective and Ricci n-recurrent, then it is either n-recurrent projective or n-recurrent projective.

Theorem 4.4. In the manifold M^{2n} if any two of the following conditions hold then the third also hold for the same proportionality form A_n.

(i) It is H-projective n-recurrent;

(ii) It is conformal or con-harmonic or con-circular or H-con-harmonic or Bochner n-recurrent;

(iii) It is Ricci n-recurrent;

The proof follows the pattern of the proof of the above theorem (4.3).

Theorem 4.5. If the manifold M^{2n} be n-symmetric then it is H-projective n-symmetric.

Proof. Because the n-symmetric manifold, using (2.16) and (2.17) in (4.3), is Ricci n-symmetric, we get (4.2), hence the argument.

Theorem 4.6. If the manifold M^{2n} be H-projective n-symmetric and Ricci n-symmetric then it is n-symmetric.

Proof: Using equation (4.2), (2.16) and (2.17) in (4.3), we have (2.15), hence the statement.

Theorem 4.7. In the manifold M^{2n} if any two of the following conditions hold then the third conditions also hold:

(i) It is H-projective n-symmetric;

(ii) It is projective n- symmetric;

(iii) It is Ricci n- symmetric.

Proof. Replacing Q by W in (3.18) and then using the resulting equation and (4.2) in (4.4), we get the following statement.

(4.6) $(\nabla.........\nabla\nabla Ric)(Y,Z,U_1,U_2,..........,U_n)=0$,

Which are (3.15), the remaining part follows similarly.

DISCUSSIONS

This work shows that Ricci and Curvature recurrent manifold are still applicable to H-projective recurrent techniques. We studied nearly metric structure and KH-structure using advanced techniques for classifying Ricci and Curvature Tensor tasks as this is all the latest literature that the bone of the 2nd-dimensional manifolds offers. Curvature and Ricci recurrent manifold are important roles in dealing with the extended 2nd-dimensional modeling space of the celestial body because it constructs the 2nd-dimensional space and allows for more complicated Tensor structures. We can easily measure all 2n-manifold structures and spaces in H-projective form nearly n-symmetric manifold, and discuss the Buchner con-harmonic, con-circular, H-con-harmonic curvature tensor.

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SOME PRODUCTS OF PICTURE FUZZY SOFT GRAPH

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ABSTRACT
In this paper, we define picture fuzzy soft graph and some products of picture fuzzy soft graphs along with their properties.
Keywords: Picture fuzzy soft graph, Cartesian product, direct product, lexicographic product, strong product.

1. INTRODUCTION
The concept of fuzzy set theory was introduced by Zadeh [9] to solve difficulties in dealing with uncertainties. It is a generalization of the theory of crisp set. Singh [8] described picture fuzzy set. Rosenfeld [7] introduced the concept of fuzzy graph theory. The concept of fuzzy graph was introduced by Mordeson and Nair [6].
Muhammad Akram, Sairam Nawaz and Maji [4,5] introduced soft set and fuzzy soft set. Cuong and Kreinovich [3] proposed the concept of picture fuzzy set which is a modified version of fuzzy set and Intuitionistic fuzzy set. Cen Zuo [2] introduced picture fuzzy graph. [1] introduced many new concepts, including soft graphs, fuzzy soft graphs. In this paper, we defined picture fuzzy soft graph and some products of picture fuzzy soft graphs along with their properties.

2. PRELIMINARIES
Definition: 2.1
If Z is a collection of object (or element) bestowed by z. Then **fuzzy set [7,9]** A' in Z is expressed as a set of ordered pair.
$$A' = \{(z, \lambda_{A'}(z)) : z \in Z\}$$
where, $\lambda_{A'}(z)$ is called the membership function (or characteristic function) which maps Z to the closed interval [0,1].
Definition: 2.2
Let D be initial universal set, Q be a set of parameters, $\wp(D)$ be the power set of D and $K \subseteq Q$. A pair (J, K) is called **soft set [4]** over D if and only if J is a mapping of K into the set of all subsets of the set D.

Definition: 2.3

A pair (J,K) is called **fuzzy soft set** [5] over D, where J is a mapping given by $J:K \to I^D$, I^D denote the collection of all fuzzy subset of D, $K \subseteq Q$.

Definition: 2.4

Let A' be a **picture fuzzy set[2,3,8].** A' in Z defined by

$$A' = \{(z, \lambda_{A'}(z), \delta_{A'}(z), \varphi_{A'}(z)) : z \in Z\}$$

where, $\lambda_{A'}(z) \in [0,1]$, $\delta_{A'}(z) \in [0,1]$ and $\varphi_{A'}(z) \in [0,1]$ follow the condition $0 \le \lambda_{A'}(z) + \delta_{A'}(z) + \varphi_{A'}(z) \le 1$. The $\lambda_{A'}(z)$ is used to represent the positive membership degree, $\delta_{A'}(z)$ is used to represent the neutral membership degree and $\varphi_{A'}(z)$ is used to represent the negative membership degree of the element z in the set A'. For each picture fuzzy set A' in Z, the refusal membership degree is described as $\pi_{A'}(z) = 1 - (\lambda_{A'}(z) + \delta_{A'}(z) + \varphi_{A'}(z))$.

3. PICTURE FUZZY SOFT GRAPH

Definition: 3.1

A ordered pair (J,K) is called **picture fuzzy soft set** over D, where J is a mapping given by $J:K \to IP^D$, where IP^D denote the collection of all picture fuzzy subset of D, $K \subseteq Q$.

Definition: 3.2

Let $G'^{*} = (W,Y)$ be a graph, $W = \{w_1, w_2, \ldots w_n\}$ be a non-empty set, $Y \subseteq W \times W$, Q be parameter set and $K \subseteq Q$. Also let,

i. a) λ_A is a positive membership function defined on W by

$\lambda_A : K \to IP^D(W)$ ($IP^D(W)$ denote collection of all picture fuzzy subset in W)

$k \to \lambda_A(k) = \lambda_{Ak}$ (say), $k \in K$ and

$\lambda_{Ak} : W \to [0,1]$, $w_i \to \lambda_{Ak}(w_i)$

(K, λ_A) picture fuzzy soft vertex of positive membership function.

b) δ_A is a neutral membership function defined on W by

$\delta_A : K \to IP^D(W)$ ($IP^D(W)$ denote collection of all picture fuzzy subset in W)

$k \to \delta_A(k) = \delta_{Ak}$ (say), $k \in K$ and

$\delta_{Ak} : W \to [0,1]$, $w_i \to \delta_{Ak}(w_i)$

(K, δ_A) picture fuzzy soft vertex of neutral membership function.

c) φ_A is a negative membership function defined on W by

$\varphi_A : K \to IP^D(W)$ ($IP^D(W)$ denote collection of all picture fuzzy subset in W)

$k \to \varphi_A(k) = \varphi_{Ak}$ (say), $k \in K$ and

$\varphi_{Ak} : W \to [0,1]$, $w_i \to \varphi_{Ak}(w_i)$

(K,φ_A) picture fuzzy soft vertex of negative membership function.

such that $0 \le \lambda_{Ak}(w_i) + \delta_{Ak}(w_i) + \varphi_{Ak}(w_i) \le 1 \ \forall \ w_i \in W$, $k \in K$,

where A is a picture fuzzy soft set on W.

ii. a) λ_B is a positive membership function defined on Y by

$\lambda_B : K \to IP^D(W \times W)$ ($IP^D(W \times W)$ denote collection of all picture fuzzy subset in Y)

$k \to \lambda_B(k) = \lambda_{Bk}$ (say), $k \in K$ and

$\lambda_{Bk} : W \times W \to [0,1]$, $(w_i, w_j) \to \lambda_{Bk}(w_i, w_j)$

(K,λ_B) picture fuzzy soft edge of positive membership function.

b) δ_B is a neutral membership function defined on Y by

$\delta_B : K \to IP^D(W \times W)$ ($IP^D(W \times W)$ denote collection of all picture fuzzy subset in Y)

$k \to \delta_B(k) = \delta_{Be}$ (say), $k \in K$ and

$\delta_{Bk} : W \times W \to [0,1]$, $(w_i, w_j) \to \delta_{Bk}(w_i, w_j)$

(K,δ_B) picture fuzzy soft edge of neutral membership function.

c) φ_B is a negative membership function defined on Y by

$\varphi_B : K \to IP^D(W \times W)$ ($IP^D(W \times W)$ denote collection of all picture fuzzy subset in Y)

$k \to \varphi_B(k) = \varphi_{Bk}$ (say), $k \in K$ and

$\varphi_{Bk} : W \times W \to [0,1]$, $(w_i, w_j) \to \varphi_{Bk}(w_i, w_j)$

(K,φ_B) picture fuzzy soft edge of negative membership function.

where, B is a picture fuzzy soft set on Y.

Also satisfying the following condition,

$\lambda_{Bk}(w_i, w_j) \le \min(\lambda_{Ak}(w_i), \lambda_{Ak}(w_j))$, $\delta_{Bk}(w_i, w_j) \le \min(\delta_{Ak}(w_i), \delta_{Ak}(w_j))$,

$\varphi_{Bk}(w_i, w_j) \ge \max(\varphi_{Ak}(w_i), \varphi_{Ak}(w_j))$ and

$0 \le \lambda_{Bk}(w_i, w_j) + \delta_{Bk}(w_i, w_j) + \varphi_{Bk}(w_i, w_j) \le 1 \ \forall \ (W_i, W_j) \in Y$, $i, j = 1,2,..n$ and $k \in K$. Then $G'^* = (W, Y, (K,\lambda_A), (K,\delta_A), (K,\varphi_A), (K,\lambda_B), (K,\delta_B), (K,\varphi_B))$ is said to be **picture fuzzy soft graph** and this denoted by $G'^*_{K,W,Y}$.

Example: 3.2

Consider a simple graph $G'^* = (W, Y)$, where $W = \{w_1, w_2, w_3\}$ and $Y = \{(w_1, w_2), (w_2, w_3)\}$. Let $K = \{k_1, k_2, k_3\}$ be the parameter set. Then the picture fuzzy soft graph $G'^* = (W, Y, (K,\lambda_A), (K,\delta_A), (K,\varphi_A), (K,\lambda_B), (K,\delta_B), (K,\varphi_B))$ is described in the table 3.2 and figure 3.2

Table 3.2: Representation of a picture fuzzy soft graph

(a)

λ_{Ak}	w_1	w_2	w_3
k_1	0.6	0.5	0.1
k_2	0.3	0.2	0.3
k_3	0.4	0.5	0.2

(b)

δ_{Ak}	w_1	w_2	w_3
k_1	0.1	0.3	0.2
k_2	0.1	0.2	0.3
k_3	0.1	0.2	0.1

(c)

φ_{Ak}	w_1	w_2	w_3
k_1	0.1	0.2	0.4
k_2	0.3	0.3	0.4
k_3	0.1	0.3	0.7

(d)

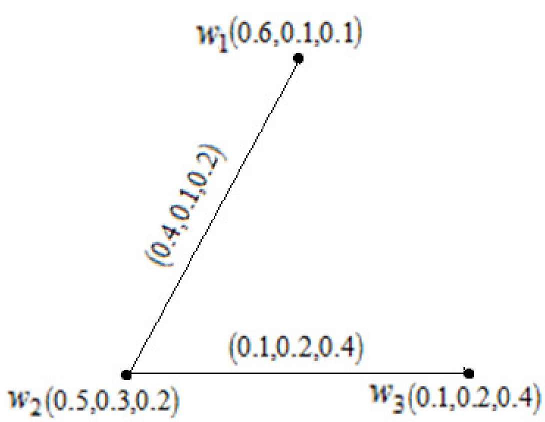
λ_{Bk}	(w_1, w_2)	(w_2, w_3)
k_1	0.4	0.1
k_2	0.2	0.2
k_3	0.4	0.2

(e)

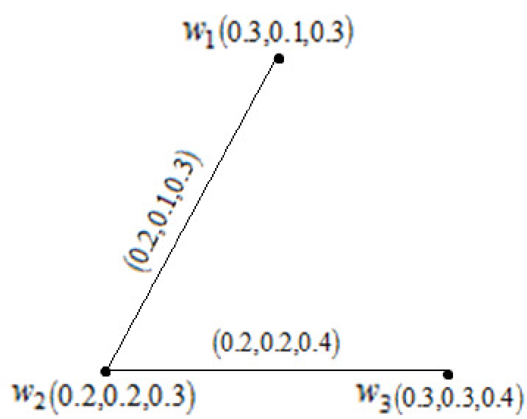
δ_{Bk}	(w_1, w_2)	(w_2, w_3)
k_1	0.1	0.2
k_2	0.1	0.2
k_3	0.1	0.1

(f)

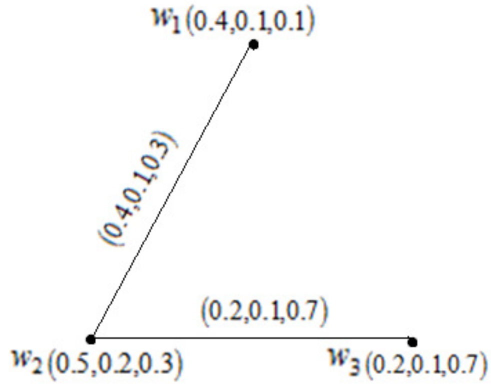
φ_{Bk}	(w_1, w_2)	(w_2, w_3)
k_1	0.2	0.4
k_2	0.3	0.4
k_3	0.3	0.7



Corresponding to the parameter (k_1)



Corresponding to theparameter (k_2)



Corresponding to the parameter (k_3)

Figure 3.2: picture fuzzy soft graph $G'^{*}_{K,W,Y}$

4. PRODUCTS OF PICTRURE FUZZY SOFT GRAPH

Definition: 4.1

The **Cartesian product** $G'^{*}_{K,W_1,Y_1} \times G'^{*}_{L,W_2,Y_2}$ of two picture fuzzy soft graph

$G'^{*}_{K,W_1,Y_1} = (W_1, Y_1, (K, \lambda_A), (K, \delta_A), (K, \varphi_A), (K, \lambda_B), (K, \delta_B), (K, \varphi_B))$ and

$G'^{*}_{L,W_2,Y_2} = (W_2, Y_2, (L, \lambda'_A), (L, \delta'_A), (L, \varphi'_A), (L, \lambda'_B), (L, \delta'_B), (L, \varphi'_B))$ is defined as (W, Y) where,

$W = W_1 \times W_2$,

$Y = \{(u, w_1), (u, w_2) : u \in W_1, (w_1, w_2) \in Y_2\} \cup \{(u_1, w), (u_2, w); w \in W_2, (u_1, u_2) \in Y_1\}$ and

$K, L \subseteq Q$ (parameter set) which satisfy the following,

i. (a) $\lambda''_{A(k,f)}(u, w) = \lambda_{Ak}(u) \wedge \lambda'_{Af}(w)$

(b) $\delta''_{A(k,f)}(u, w) = \delta_{Ak}(u) \wedge \delta'_{Af}(w)$

(c) $\varphi''_{A(k,f)}(u, w) = \varphi_{Ak}(u) \vee \varphi'_{Af}(w)$

$\forall (u, w) \in W$ and $(k, f) \in K \times L$

ii. (a) $\lambda''_{B(k,f)}((u, w_1), (u, w_2)) = \lambda_{Ak}(u) \wedge \lambda'_{Bf}(w_1, w_2)$

(b) $\delta''_{B(k,f)}((u, w_1), (u, w_2)) = \delta_{Ak}(u) \wedge \delta'_{Bf}(w_1, w_2)$

(c) $\varphi''_{B(k,f)}((u, w_1), (u, w_2)) = \varphi_{Ak}(u) \vee \varphi'_{Bf}(w_1, w_2)$

$\forall u \in W_1$, $(w_1, w_2) \in Y_2$ and $(k, f) \in K \times L$

iii. (a) $\lambda''_{B(k,f)}((u_1, w), (u_2, w)) = \lambda'_{Af}(w) \wedge \lambda_{Bk}(u_1, u_2)$

(b) $\delta''_{B(k,f)}((u_1, w), (u_2, w)) = \delta'_{Af}(w) \wedge \delta_{Bk}(u_1, u_2)$

(c) $\varphi''_{B(k,f)}((u_1, w), (u_2, w)) = \varphi'_{Af}(w) \vee \varphi_{Bk}(u_1, u_2)$

$\forall w \in W_2$, $(u_1, u_2) \in Y_1$ and $(k, f) \in K \times L$

Proposition: 4.1

If G'^{*}_{K,W_1,Y_1} and G'^{*}_{L,W_2,Y_2} are two picture fuzzy soft graph. The Cartesian product of $G'^{*}_{M,W_3,Y_3} = G'^{*}_{K,W_1,Y_1} \times G'^{*}_{L,W_2,Y_2}$ is also a picture fuzzy soft graph.

Definition: 4.2

The **direct product** $G'^{*}_{K,W_1,Y_1} * G'^{*}_{L,W_2,Y_2}$ of two picture fuzzy soft graph

$G'^{*}_{K,W_1,Y_1} = (W_1, Y_1, (K, \lambda_A), (K, \delta_A), (K, \varphi_A), (K, \lambda_B), (K, \delta_B), (K, \varphi_B))$ and

$G'^{*}_{L,W_2,Y_2} = (W_2, Y_2, (L, \lambda'_A), (L, \delta'_A), (L, \varphi'_A), (L, \lambda'_B), (L, \delta'_B), (L, \varphi'_B))$ is defined as (W, Y) where, $W = W_1 \times W_2$, $Y = \{(u_1, w_1), (u_2, w_2) : (u_1, u_2) \in Y_1, (w_1, w_2) \in Y_2\}$ and $K, L \subseteq Q$ (parameter set) which satisfy the following,

i. (a) $\lambda''_{A(k,f)}(u, w) = \lambda_{Ak}(u) \wedge \lambda'_{Af}(w)$

(b) $\delta''_{A(k,f)}(u, w) = \delta_{Ak}(u) \wedge \delta'_{Af}(w)$

(c) $\varphi''_{A(k,f)}(u, w) = \varphi_{Ak}(u) \vee \varphi'_{Af}(w)$

$\forall (u, w) \in W$ and $(k, f) \in K \times L$

ii. (a) $\lambda''_{Bk}((u_1, w_1), (u_2, w_2)) = \lambda_{Bk}(u_1, u_2) \wedge \lambda'_{Bf}(w_1, w_2)$

(b) $\delta''_{Bk}((u_1, w_1), (u_2, w_2)) = \delta_{Bk}(u_1, u_2) \wedge \delta'_{Bf}(w_1, w_2)$

(c) $\varphi''_{Bk}((u_1, w_1), (u_2, w_2)) = \varphi_{Bk}(u_1, u_2) \vee \varphi'_{Bf}(w_1, w_2)$

$\forall (u_1, u_2) \in Y_1, (w_1, w_2) \in Y_2$ and $(k, f) \in K \times L$

Proposition: 4.2

If G'^{*}_{K,W_1,Y_1} and G'^{*}_{L,W_2,Y_2} are two picture fuzzy soft graph. The direct product of $G'^{*}_{M,W_3,Y_3} = G'^{*}_{K,W_1,Y_1} * G'^{*}_{L,W_2,Y_2}$ is also a picture fuzzy soft graph.

Definition: 4.3

The **lexicographic product** $G'^{*}_{K,W_1,Y_1} \bullet G'^{*}_{L,W_2,Y_2}$ of two picture fuzzy soft graph

$G'^{*}_{K,W_1,Y_1} = (W_1, Y_1, (K, \lambda_A), (K, \delta_A), (K, \varphi_A), (K, \lambda_B), (K, \delta_B), (K, \varphi_B))$ and

$G'^{*}_{L,W_2,Y_2} = (W_2, Y_2, (L, \lambda'_A), (L, \delta'_A), (L, \varphi'_A), (L, \lambda'_B), (L, \delta'_B), (L, \varphi'_B))$ is defined as (W, Y) where,

$W = W_1 \times W_2$,

$Y = \{(u, w_1), (u, w_2) : u \in W_1, (w_1, w_2) \in Y_2\} \cup \{(u_1, w_1), (u_2, w_2) : (u_1, u_2) \in Y_1, (w_1, w_2) \in Y_2\}$ and $K, L \subseteq Q$ (parameter set) which satisfy the following,

i. (a) $\lambda''_{A(k,f)}(u, w) = \lambda_{Ak}(u) \wedge \lambda'_{Af}(w)$

(b) $\delta''_{A(k,f)}(u,w)=\delta_{Ak}(u)\wedge\delta'_{Af}(w)$

(c) $\varphi''_{A(k,f)}(u,w)=\varphi_{Ak}(u)\vee\varphi'_{Af}(w)$

$\forall(u,w)\in W$ and $(k,f)\in K\times L$

ii. (a) $\lambda''_{B(k,f)}((u,w_1),(u,w_2))=\lambda_{Ak}(u)\wedge\lambda'_{Bf}(w_1,w_2)$

(b) $\delta''_{B(k,f)}((u,w_1),(u,w_2))=\delta_{Ak}(u)\wedge\delta'_{Bf}(w_1,w_2)$

(c) $\varphi''_{B(k,f)}((u,w_1),(u,w_2))=\varphi_{Ak}(u)\vee\varphi'_{Bf}(w_1,w_2)$

$\forall u\in W_1$, $(w_1,w_2)\in Y_2$ and $(k,f)\in K\times L$

ii. (a) $\lambda''_{Bk}((u_1,w_1),(u_2,w_2))=\lambda_{Bk}(u_1,u_2)\wedge\lambda'_{Bf}(w_1,w_2)$

(b) $\delta''_{Bk}((u_1,w_1),(u_2,w_2))=\delta_{Bk}(u_1,u_2)\wedge\delta'_{Bf}(w_1,w_2)$

(c) $\varphi''_{Bk}((u_1,w_1),(u_2,w_2))=\varphi_{Bk}(u_1,u_2)\vee\varphi'_{Bf}(w_1,w_2)$

$\forall(u_1,u_2)\in Y_1,(w_1,w_2)\in Y_2$ and $(k,f)\in K\times L$

Proposition: 4.3

The lexicographic product of two picture fuzzy soft graph is a picture fuzzy soft graph.

Definition: 4.4

The **strong product** $G'^{*}_{K,W_1,Y_1}\ G'^{*}_{L,W_2,Y_2}$ of two picture fuzzy soft graph

$G'^{*}_{K,W_1,Y_1}=(W_1,Y_1,(K,\lambda_A),(K,\delta_A),(K,\varphi_A),(K,\lambda_B),(K,\delta_B),(K,\varphi_B))$ and

$G'^{*}_{L,W_2,Y_2}=(W_2,Y_2,(L,\lambda'_A),(L,\delta'_A),(L,\varphi'_A),(L,\lambda'_B),(L,\delta'_B),(L,\varphi'_B))$ is defined as (W,Y) where,

$W=W_1\times W_2$,

$Y=\{(u,w_1),(u,w_2):u\in W_1,(w_1,w_2)\in Y_2\}\cup\{(u_1,w),(u_2,w);w\in W_2,(u_1,u_2)\in Y_1\}\cup$
$\{(u_1,w_1),(u_2,w_2):(u_1,u_2)\in Y_2,(w_1,w_2)\in Y_2\}$

and $K,L\subseteq Q$ (parameter set) which satisfy the following,

i. (a) $\lambda''_{A(k,f)}(u,w)=\lambda_{Ak}(u)\wedge\lambda'_{Af}(w)$

(b) $\delta''_{A(k,f)}(u,w)=\delta_{Ak}(u)\wedge\delta'_{Af}(w)$

(c) $\varphi''_{A(k,f)}(u,w)=\varphi_{Ak}(u)\vee\varphi'_{Af}(w)$

$\forall(u,w)\in W$ and $(k,f)\in K\times L$

ii. (a) $\lambda''_{B(k,f)}((u,w_1),(u,w_2))=\lambda_{Ak}(u)\wedge\lambda'_{Bf}(w_1,w_2)$

(b) $\delta''_{B(k,f)}((u,w_1),(u,w_2))=\delta_{Ak}(u)\wedge\delta'_{Bf}(w_1,w_2)$

(c) $\varphi''_{B(k,f)}((u,w_1),(u,w_2))=\varphi_{Ak}(u)\vee\varphi'_{Bf}(w_1,w_2)$

$\forall u\in W_1$, $(w_1,w_2)\in Y_2$ and $(k,f)\in K\times L$

i. (a) $\lambda''_{B(k,f)}((u_1,w),(u_2,w))=\lambda'_{Af}(w)\wedge\lambda_{Bk}(u_1,u_2)$

(b) $\delta''_{B(k,f)}((u_1,w),(u_2,w)) = \delta'_{Af}(w) \wedge \delta_{Bk}(u_1,u_2)$

(c) $\varphi''_{B(k,f)}((u_1,w),(u_2,w)) = \varphi'_{Af}(w) \vee \varphi_{Bk}(u_1,u_2)$

$\forall\, w \in W_2, (u_1,u_2) \in Y_1$ and $(k,f) \in K \times L$

ii. (a) $\lambda''_{Bk}((u_1,w_1),(u_2,w_2)) = \lambda_{Bk}(u_1,u_2) \wedge \lambda'_{Bf}(w_1,w_2)$

(b) $\delta''_{Bk}((u_1,w_1),(u_2,w_2)) = \delta_{Bk}(u_1,u_2) \wedge \delta'_{Bf}(w_1,w_2)$

(c) $\varphi''_{Bk}((u_1,w_1),(u_2,w_2)) = \varphi_{Bk}(u_1,u_2) \vee \varphi'_{Bf}(w_1,w_2)$

$\forall (u_1,u_2) \in Y_1, (w_1,w_2) \in Y_2$ and $(k,f) \in K \times L$

Proposition: 4.4

The strong product of two picture fuzzy soft graph is a picture fuzzy soft graph.

5. CONCLUSION

In this paper, we discussed about the Cartesian product, direct product, lexicographic product, strong product of two picture fuzzy soft graph with their properties.

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TIME INDEPENDENT ANALYSIS OF BISERIAL AND PARALLEL QUEUING MODELS WITH BATCH ARRIVAL

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ABSTRACT

In the present paper we have derived the steady-state solutions for a complex network queuing model consisting of bi-serial and parallel queuing subsystems commonly connected with a single server. The customers soliciting services at bi-serial queuing subsystem arrive in batches of fixed size while at parallel queuing subsystem customer arrives one at a time. Finally after getting services at either of the subsystem the customer moves to the single server for completion of services. The mean queue length and average waiting time for customers have been calculated using generating function technique, laws of calculus and statistical tools. The numerical illustration is provided to order to validate the theoretical results.

Keywords: Bi-serial; parallel; time independent; mean queue length; batch arrival.

1. INTRODUCTION

The queuing models with various applications have been studied by many researchers and have gained tremendous significance in the present scenario due to congestion problem everywhere. Solution to queuing problems with phase type service was examined by O'Brien (1954). Suzuki (1963), Maggu (1970) and many other authors studied the waiting line problems with different parameters like bulk arrival, impatient customers, sequential arrays, and feedback queue. Madan K.C et al. (2003) examined the steady state behavior of bulk queue with multiple vacations and service breakdown. Singh T.P. et al. (2005) extended the work by introducing some additional concepts and did a remarkable work in the field of queueing theory. Kumar Vinod et al.(2006) and Gupta Deepak et al. (2007,2011) analyzed steady state behavior of the queue network models with biserial and parallel service channels linked with common channels and computed various queue characteristics. Further Mittal Meenu et al. (2018a, 2018b) studied bulk bi-serial queue model with probabilistic and fixed batch size. Gupta D et al. (2020) developed the queue model with bulk arrival at parallel queueing systems. Sachin K. et al. (2018) studied tri-cum bi-serial queuing model connected with a common Server. A stochastic queue model was designed by Singh T.P. et al. (2019) for passport office system. Mamta et al. (2019) explored the dynamical system with impatient behaviour of the customers. Impatience behaviour the customers with vacation interruption was analyzed by Bouchentouf, A. et al. (2020). Preemptive priority retrial queue with Bernoulli working vacations was examined by Rajadurai, P. (2019). Radha, J et al. (2020) investigated the retrial queue with Bernoulli-2 vacation. Bi-serial and parallel queueing system with arrival in groups was propounded by Gupta R. et al. (2020). The present paper is an extension of the work done by Gupta Deepak (2007) in the sense that batch arrival of the customers at bi-serial queuing subsystem is considered instead of individual arrival.

2. NOTATIONS

λ_1: Mean arrival rate at first bi-serial server.
λ_2: Mean arrival rate at second bi-serial server.
λ_3: Mean arrival rate at first parallel server.
λ_4: Mean arrival rate at second parallel server.
μ_1: The exponential service rate of first bi-serial server
μ_2: The exponential service rate of second bi-serial server.
μ_3: The exponential service rate of first parallel server.
μ_4: The exponential service rate of second parallel server.
μ_5: The exponential service rate of Single server.
p_{12}: Probability of customer moving from first bi-serial server to second.
p_{21}: Probability of customer moving from second bi-serial server to first.
p_{15} : Probability of customer moving from first bi-serial server to single server.
p_{25} : Probability of customer moving from second bi-serial server to single server.
b_1 : Batch size at first bi-serial server.
b_2 : Batch size at second bi-serial server.

3. THEORETICAL FRAMEWORK OF THE MODEL

The model under consideration comprises of two queuing subsystems S_B, S_P and a single server S. The constituents of S_B are two bi-serial servers S_{B1} and S_{B2} while the subsystem S_P consists of two parallel servers S_{P1} and S_{P2}. The server S is commonly connected to each of these two subsystems. The description of the model is well illustrated with the help of diagram as shown in figure 1.

3.1 Assumptions

- The customers entering into the system with mean arrival rate λ_1 and λ_2 approach to server S_{B1} and S_{B2} in batches of fixed size b_1 and b_2 respectively.
- After availing services at S_{B1}, the customer may either joins S_{B2} with probability p_{12} for further requirement of services or moves to server S with probability p_{15} for completion of services such that $p_{12} + p_{15} = 1$.
- Similarly after getting services at S_{B2} customer may either joins S_{B1} with probability p_{21}or moves to server S with probability p_{25} such that $p_{21} + p_{25} = 1$.
- Again the customer entering into the system with mean arrival rate λ_3 and λ_4 approach to server S_{P1} and S_{P2} respectively under Poisson assumptions.
- After availing services at S_{P1} and S_{P2}, the customer finally moves to the server S.
- The customer leaves the system after getting services at server S.

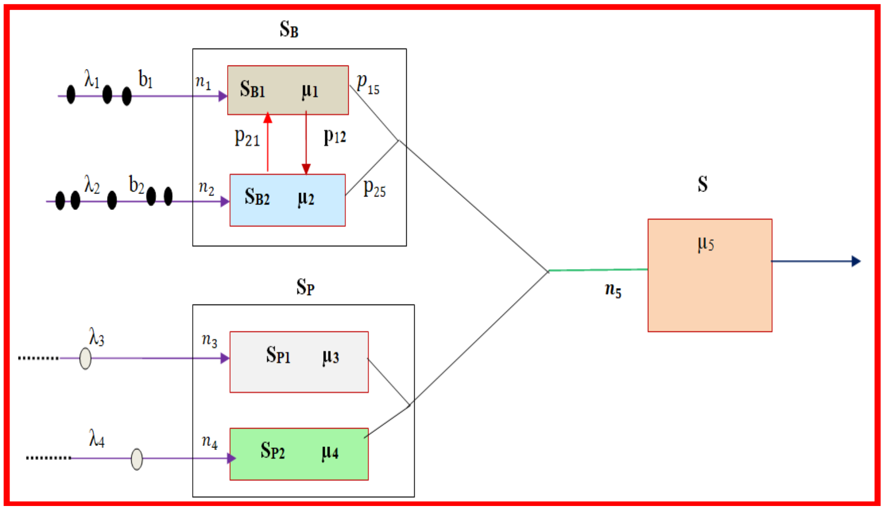

Figure 1: QUEUE NETWORK MODEL

4. PRACTICAL ASPECT

The developed model is applicable in various real world situations such as in administrative setups, office management, banking systems, handling of children parks. For example, consider the systems S_B, S_P and server S as the three different sections in a shopping complex. First is the food section, second is drink section and third is billing section, which is common for these two sections. The food section further consists of two bi-serial subsections having Indian and Chinese food. Similarly in the drink section there are two subsections having cold drink and hot drink. The customers arriving at food section either may go to take Indian food or may visit to Chinese food subsection. After taking Indian food, the customer either may have Chinese food or may directly go to billing section. Similarly the customer after having Chinese food either may have Indian food or directly may go to billing section. Again the customer arriving at drink section may either go to take cold drink or may go to take hot drink as per his interest and thereafter he will go to billing section. After paying bill the customer leaves the mall.

5. MATHEMATICAL DESCRIPTION

P_{n_1,n_2,n_3,n_4,n_5}, indicates the probability of having exactly n_1, n_2, n_3, n_4 and n_5 calling units in the queuing system, waiting in queues in front of servers $S_{B1}, S_{B2}, S_{P1}, S_{P2}$ *and server* S.

Where $n_1 > b_1, n_2 > b_2, n_3 > 0, n_4 > 0$ and $n_5 > 0$.

Differential difference equation in steady-state form is defined as follows:

- **For** $\mathbf{n_1 > b_1,\ n_2 > b_2, n_3,\ n_4,\ n_5} > 0$

$$(\lambda_1+\lambda_2+\lambda_3+\lambda_4+\mu_1+\mu_2+\mu_3+\mu_4+\mu_5)P_{n_1,n_2,n_3,n_4,n_5}$$
$$= \lambda_1 P_{n_1-b_1,n_2,n_3,n_4,n_5} + \lambda_2 P_{n_1,n_2-b_2,n_3,n_4,n_5}$$
$$+\mu_1 p_{12} P_{n_1+1,n_2-1,n_3,n_4,n_5} + \mu_1 p_{15} P_{n_1+1,n_2,n_3,n_4,n_5-1}$$
$$+\mu_2 p_{21} P_{n_1-1,n_2+1,n_3,n_4,n_5} + \mu_2 p_{25} P_{n_1,n_2+1,n_3,n_4,n_5-1} + \lambda_3 P_{n_1,n_2,n_3-1,n_4,n_5}$$
$$+\lambda_4 P_{n_1,n_2,n_3,n_4-1,n_5} + \mu_3 P_{n_1,n_2,n_3+1,n_4,n_5-1}$$
$$+\mu_4 P_{n_1,n_2,n_3,n_4+1,n_5-1} + \mu_5 P_{n_1,n_2,n_3,n_4,n_5+1} \quad (1)$$

By considering all the possible combination of different values of $\boldsymbol{n_1},\ \boldsymbol{n_2}, \boldsymbol{n_3},\ \boldsymbol{n_4}$ *and* $\boldsymbol{n_5}$,
71 more differential difference equations in steady- state are obtained.

The system of steady state equations is solved using generating function technique.

Generating function is defined as follows:

$F(X, Y, Z, R, S) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty}\sum_{n_4=0}^{\infty}\sum_{n_5=0}^{\infty} X^{n_1}Y^{n_2}Z^{n_3}R^{n_4}S^{n_5}$

Also for simplification, partial generating function is defined as:

$F_{n_2,n_3,n_4,n_5}(X) = \sum_{n_1=0}^{\infty} P_{n_1,n_2,n_3,n_4} X^{n_1}$

$F_{n_3,n_4,n_5}(X, Y) = \sum_{n_2=0}^{\infty} F_{n_2,n_3,n_4,n_5} Y^{n_2}$

$F_{n_4,n_5}(X, Y, Z) = \sum_{n_3=0}^{\infty} F_{n_3,n_4,n_5} Z^{n_3}$

$F_{n_5}(X, Y, Z, R) = \sum_{n_4=0}^{\infty} F_{n_4,n_5} R^{n_4}$

$F(X, Y, Z, R, S) = \sum_{n_5=0}^{\infty} F_{n_5} S^{n_5}$

On solving the system of steady- state equations (1-72), by applying the generating function defined as above and using the laws of calculus, we obtained the final reduced equation as:

$$F(X,Y,Z,R,S) = \left[\frac{\mu_1\left(1-\frac{p_{12}Y}{X}-\frac{p_{15}S}{X}\right)F_0(Y,Z,R,S)+\mu_2\left(1-\frac{p_{21}X}{Y}-\frac{p_{25}S}{Y}\right)F_0(X,Z,R,S)+\mu_3\left(1-\frac{S}{Z}\right)F_0(X,Y,R,S)+\mu_4\left(1-\frac{S}{R}\right)F_0(X,Y,Z,S)+\mu_5\left(1-\frac{1}{S}\right)F_0(X,Y,Z,R)}{\lambda_1(1-X^{b_1})+\lambda_2(1-Y^{b_2})+\mu_1\left(1-\frac{p_{12}Y}{X}-\frac{p_{15}S}{X}\right)+\mu_2\left(1-\frac{p_{21}X}{Y}-\frac{p_{25}S}{Y}\right)+\lambda_3(1-Z)+\lambda_4(1-R)+\mu_3\left(1-\frac{S}{Z}\right)+\mu_4\left(1-\frac{S}{R}\right)+\mu_5\left(1-\frac{1}{S}\right)}\right] \quad (2)$$

For convenience, we denote

$F_1= F_0(Y, Z, R, S)$, $F_2 = F_0(X, Z, R, S)$

$F_3 = F_0(X, Y, R, S)$, $F_4= F_0(X, Y, Z, S)$

$F_5=F_0(X, Y, Z, R)$ (3)

Using these values, equation (2) can be rewritten as:

$$F(X,Y,Z,R,S) = \left[\frac{\mu_1\left(1-\frac{p_{12}Y}{X}-\frac{p_{15}S}{X}\right)F_1+\mu_2\left(1-\frac{p_{21}X}{Y}-\frac{p_{25}S}{Y}\right)F_2+\mu_3\left(1-\frac{S}{Z}\right)F_3+\mu_4\left(1-\frac{S}{R}\right)F_4+\mu_5\left(1-\frac{1}{S}\right)F_5}{\lambda_1(1-X^{b_1})+\lambda_2(1-Y^{b_2})+\mu_1\left(1-\frac{p_{12}Y}{X}-\frac{p_{15}S}{X}\right)+\mu_2\left(1-\frac{p_{21}X}{Y}-\frac{p_{25}S}{Y}\right)+\lambda_3(1-Z)+\lambda_4(1-R)+\mu_3\left(1-\frac{S}{Z}\right)+\mu_4\left(1-\frac{S}{R}\right)+\mu_5\left(1-\frac{1}{S}\right)}\right] \quad (4)$$

At X=Y=Z=R=S=1, the above equation (4) reduces to indeterminate form $(0/0)$.

Also F (1, 1, 1, 1, 1) = 1 (Total Probability)

Applying L'Hospital Rule for indeterminate form and using the condition

$p_{12} + p_{15} = p_{21}+p_{25} = 1$, the following results are obtained:

$-\lambda_1 b_1+ \mu_1- \mu_2 p_{21} = \mu_1 F_1 - \mu_2 p_{21} F_2$ (5)

$- \lambda_2 b_2 - \mu_1 p_{12} + \mu_2 = - \mu_1 p_{12} F_1 + \mu_2 F_2$ (6)

$- \lambda_3 +\mu_3 = \mu_3 F_3$ (7)

$- \lambda_4 +\mu_4= \mu_4 F_4$ (8)

$-\mu_1 p_{15}- \mu_2 p_{25} - \mu_3- \mu_4+ \mu_5 = -\mu_1 p_{15} F_1 -\mu_2 p_{25} F_2- \mu_3 F_3-\mu_4 F_4+\mu_5 F_5$ (9)

Multiplying (6) by p_{21} & adding to (5), we get:

$$F_1 = 1 - \frac{\lambda_1 b_1+\lambda_2 b_2 p_{21}}{\mu_1(1-p_{12}p_{21})} \quad (10)$$

Again (5)* p_{12} + (6), gives

$- \lambda_1 b_1 p_{12} - \lambda_2 b_2 + \mu_2(1- p_{12}p_{21}) = \mu_2(1-p_{12}p_{21})F_2$

$$F_2 = 1 - \frac{\lambda_1 b_1 p_{12}+\lambda_2 b_2}{\mu_2(1-p_{12}p_{21})} \quad (11)$$

From (7) & (8), we obtain:

$$F_3 = 1 - \frac{\lambda_3}{\mu_3}, F_4 = 1 - \frac{\lambda_4}{\mu_4} \tag{12}$$

Substituting these values in (9), we extract:

$$F_5 = 1 - \left[\frac{(\lambda_3+\lambda_4)}{\mu_5} + \frac{p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2}{\mu_5(1-p_{12}p_{21})}\right] \tag{13}$$

The solution of the model is given as:

$$P_{n_1,n_2,n_3,n_4,n_5} = \rho_1^{n_1}\rho_2^{n_2}\rho_3^{n_3}\rho_4^{n_4}\rho_5^{n_5}(1-\rho_1)(1-\rho_2)(1-\rho_3)(1-\rho_4)(1-\rho_5)$$

Where

$$\rho_1 = 1 - F_1 = \frac{\lambda_1 b_1+\lambda_2 b_2 p_{21}}{\mu_1(1-p_{12}p_{21})},$$

$$\rho_2 = 1 - F_2 = \frac{\lambda_1 b_1 p_{12}+\lambda_2 b_2}{\mu_2(1-p_{12}p_{21})},$$

$$\rho_3 = 1 - F_3 = \frac{\lambda_3}{\mu_3},$$

$$\rho_4 = 1 - F_4 = \frac{\lambda_4}{\mu_4}$$

$$\rho_5 = 1 - F_5 = \left[\frac{(\lambda_3+\lambda_4)}{\mu_5} + \frac{p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2}{\mu_5(1-p_{12}p_{21})}\right]$$

The solution in steady state exists, if the condition

$\rho_1, \rho_2, \rho_3, \rho_4, \rho_5 < 1$ is satisfied.

6. PERFORMANCE MEASURES

6.1 Expected Number of Units In the System(Mean Queue Length) L_s

$$L_s = L_1 + L_2 + L_3 + L_4 + L_5$$

$$L_s = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \frac{\rho_3}{1-\rho_3} + \frac{\rho_4}{1-\rho_4} + \frac{\rho_5}{1-\rho_5}$$

$$L_s = \frac{\lambda_1 b_1+\lambda_2 b_2 p_{21}}{\mu_1(1-p_{12}p_{21})-(\lambda_1 b_1+\lambda_2 b_2 p_{21})} + \frac{\lambda_1 b_1 p_{12}+\lambda_2 b_2}{\mu_2(1-p_{12}p_{21})-(\lambda_1 b_1 p_{12}+\lambda_2 b_2)} + \frac{\lambda_3}{\mu_3-\lambda_3} + \frac{\lambda_4}{\mu_4-\lambda_4}$$

$$+ \frac{(\lambda_3+\lambda_4)(1-p_{12}p_{21})+p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2)}{\mu_5(1-p_{12}p_{21})-\{(\lambda_3+\lambda_4)(1-p_{12}p_{21})+p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2)\}} \tag{14}$$

6.2 Expected Waiting Time for Customers in the system

(Waiting includes service time)

$$Ws = \frac{Ls}{\lambda}, \text{ where } \lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$

$$Ws =$$

$$\frac{1}{\lambda}\left\{\frac{\lambda_1 b_1+\lambda_2 b_2 p_{21}}{\mu_1(1-p_{12}p_{21})-(\lambda_1 b_1+\lambda_2 b_2 p_{21})} + \frac{\lambda_1 b_1 p_{12}+\lambda_2 b_2}{\mu_2(1-p_{12}p_{21})-(\lambda_1 b_1 p_{12}+\lambda_2 b_2)} + \frac{\lambda_3}{\mu_3-\lambda_3} + \frac{\lambda_4}{\mu_4-\lambda_4} + \right.$$

$$\left. + \frac{(\lambda_3+\lambda_4)(1-p_{12}p_{21})+p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2)}{\mu_5(1-p_{12}p_{21})-\{(\lambda_3+\lambda_4)(1-p_{12}p_{21})+p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2)\}}\right\} \tag{15}$$

6.3 Variance of Queue Length

$$Var(n_1 + n_2 + n_3 + n_4 + n_5) = \frac{\rho_1}{[1-\rho_1]^2} + \frac{\rho_2}{[1-\rho_2]^2} + \frac{\rho_3}{[1-\rho_3]^2} + \frac{\rho_4}{[1-\rho_4]^2} + \frac{\rho_5}{[1-\rho_5]^2}$$

$$= \frac{\mu_1(1-p_{12}p_{21})(\lambda_1 b_1+\lambda_2 b_2 p_{21})}{[\mu_1(1-p_{12}p_{21})-(\lambda_1 b_1+\lambda_2 b_2 p_{21})]^2} + \frac{\mu_2(1-p_{12}p_{21})(\lambda_1 b_1 p_{12}+\lambda_2 b_2)}{[\mu_2(1-p_{12}p_{21})-(\lambda_1 b_1 p_{12}+\lambda_2 b_2)]^2} + \frac{\lambda_3\mu_3}{[\mu_3-\lambda_3]^2} + \frac{\lambda_4\mu_4}{[\mu_4-\lambda_4]^2}$$

$$+ \frac{\mu_5(1-p_{12}p_{21})\{(\lambda_3+\lambda_4)(1-p_{12}p_{21})+p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2)\}}{[\mu_5(1-p_{12}p_{21})-\{(\lambda_3+\lambda_4)(1-p_{12}p_{21})+p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2)\}]^2} \tag{16}$$

7. NUMERICAL ILLUSTRATION

We discuss the behavior of the model by considering the numeric values of various parameters as given in table 7.1

Table 1: Numeric Values

Arrival Rates	Service Rates	Batch Sizes	Associated Probabilities
$\lambda_1 = 1$ $\lambda_2 = 4$ $\lambda_3 = 2$ $\lambda_4 = 3$	$\mu_1 = 17$ $\mu_2 = 32$ $\mu_3 = 10$ $\mu_4 = 13$ $\mu_5 = 40$	$b_1 = 2$ $b_2 = 3$	$p_{12} = 0.2$ $p_{15} = 0.8$ $p_{21} = 0.4$ $p_{25} = 0.6$

The mean queue length, average waiting time and variance of queue with respect to the parametric values (given in table 1) obtained are as follows:

- **Mean queue length**

Substituting the values from table 1 in equation (14), we get:

$$L_s = \frac{\lambda_1 b_1+\lambda_2 b_2 p_{21}}{\mu_1(1-p_{12}p_{21})-(\lambda_1 b_1+\lambda_2 b_2 p_{21})} + \frac{\lambda_1 b_1 p_{12}+\lambda_2 b_2}{\mu_2(1-p_{12}p_{21})-(\lambda_1 b_1 p_{12}+\lambda_2 b_2)} + \frac{\lambda_3}{\mu_3-\lambda_3} + \frac{\lambda_4}{\mu_4-\lambda_4} +$$
$$+ \frac{(\lambda_3+\lambda_4)(1-p_{12}p_{21})+p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2)}{\mu_5(1-p_{12}p_{21})-\{(\lambda_3+\lambda_4)(1-p_{12}p_{21})+p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2)\}}$$

$$L_s = \frac{2+4\times3\times0.4}{17(1-0.2\times0.4)-(2+4\times3\times0.4)} + \frac{2\times0.2+4\times3}{32(1-0.2\times0.4)-(2\times0.2+4\times3)} + \frac{2}{10-2} + \frac{3}{13-3}$$
$$+ \frac{5(1-0.2\times0.4)+0.8(2+4\times3\times0.4)+0.6(2\times0.2+4\times3)}{40(1-0.2\times0.4)-\{5(1-0.2\times0.4)+0.8(2+4\times3\times0.4)+0.6(2\times0.2+4\times3)\}}$$

$L_s = 2.93$

- **Average waiting time of customer**

Making use of equation (15), we get:

$$W_s = \frac{Ls}{\lambda}, \text{ where } \lambda = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$$

$$W_s = \frac{1}{10}(2.93) = 0.293$$

- **Variance of queue length**

Utilizing the values from the table in equation (16), we get:

$$\mathrm{Var}(n_1 + n_2 + n_3 + n_4 + n_5) = \frac{\mu_1(1-p_{12}p_{21})(\lambda_1 b_1+\lambda_2 b_2 p_{21})}{[\mu_1(1-p_{12}p_{21})-(\lambda_1 b_1+\lambda_2 b_2 p_{21})]^2} + \frac{\mu_2(1-p_{12}p_{21})(\lambda_1 b_1 p_{12}+\lambda_2 b_2)}{[\mu_2(1-p_{12}p_{21})-(\lambda_1 b_1 p_{12}+\lambda_2 b_2)]^2} + \frac{\lambda_3\mu_3}{[\mu_3-\lambda_3]^2} + \frac{\lambda_4\mu_4}{[\mu_4-\lambda_4]^2}$$
$$+ \frac{\mu_5(1-p_{12}p_{21})\{(\lambda_3+\lambda_4)(1-p_{12}p_{21})+p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2)\}}{[\mu_5(1-p_{12}p_{21})-\{(\lambda_3+\lambda_4)(1-p_{12}p_{21})+p_{15}(\lambda_1 b_1+\lambda_2 b_2 p_{21})+p_{25}(\lambda_1 b_1 p_{12}+\lambda_2 b_2)\}]^2}$$

$$\mathrm{Var}(n_1 + n_2 + n_3 + n_4 + n_5) = \frac{268.69}{[8.84]^2} + \frac{365.05}{[17.04]^2} + \frac{20}{[8]^2} + \frac{39}{[10]^2} + \frac{643.26}{[19.32]^2}$$

$= 7.07$

8. CONCLUSION

The proposed model studies a queue network model with batch arrival consisting of biserial and parallel channels linked with a common server. The time independent behavior of the system is discussed and analyzed numerically. If individual arrival is considered at biserial queuing subsystem instead of batch arrival, the results obtained tally with that of Gupta et al. (2007), which confirms the validity of results. The performance measures obtained have managerial implications, the system analyst and industry people may redesign the system by managing the queues in an effective way and hence the congestion problem may be reduced.

9. LIMITATIONS AND FUTURE SCOPE

The model under consideration is limited to two queueing subsystems only. The research can be further extended in future by:

- Adding one more parallel or biserial queueing subsystem.
- Considering tri-cum biserial servers instead of two biserial servers.
- Introducing three or more parallel servers with bulk arrival.

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BAYES ESTIMATION UNDER DIFFERENT LOSS FUNCTION FOR EXPONENTIATED WEIBULL DISTRIBUTION

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ABSTRACT
In this paper, exponentiated Weibull distribution is considered for Bayesian analysis. The expressions for Bayes estimators of the parameter have been derived under squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions by using quasi and gamma priors.
Keywords : Bayesian method, exponentiated Weibull distribution, quasi and gamma priors, squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions.
MSC: 60E05, 62E15, 62H10, 62H12

1. INTRODUCTION

Mudholkar and srivastava [1] introduced the exponentiated Weibull distribution. According to Oluwafemi and Olakan [2], the probability density function of exponentiated Weibull distribution is given by

$$f\left(x;\theta\right)=\frac{k\theta}{\lambda^k}x^{k-1}\,exp\left[-\left(\frac{x}{\lambda}\right)^k\right]\left[1-exp\left(-\left(\frac{x}{\lambda}\right)^k\right)\right]^{\theta-1}\quad ;\, x>0. \tag{1}$$

The joint density function or likelihood function of (1) is given by

$$f\left(\underline{x};\theta\right)=\left(\frac{k\theta}{\lambda^k}\right)^n\left(\prod_{i=1}^{n}x_i^{k-1}\right)e^{-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k}\,exp\left[\left(\theta-1\right)\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]\right]. \tag{2}$$

The log likelihood function is given by

$$log\, f\left(\underline{x};\theta\right)=n\,log\left(\frac{k\theta}{\lambda^k}\right)+log\left(\prod_{i=1}^{n}x_i^{k-1}\right)-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k+\left(\theta-1\right)\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]. \tag{3}$$

Differentiating (3) with respect to θ and equating to zero, we get the maximum likelihood estimator of θ which is given as

$$\hat{\theta}=n\left(\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-1}. \tag{4}$$

2. BAYESIAN METHOD OF ESTIMATION

The Bayesian inference procedures have been developed generally under squared error loss function

$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^2. \tag{5}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_S=E\left(\theta\right). \tag{6}$$

Zellner [3], Basu and Ebrahimi [4] have recognized that the inappropriateness of using symmetric loss function. Norstrom [5] introduced precautionary loss function is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}}. \tag{7}$$

The Bayes estimator under this loss function is denoted by $\hat{\theta}_P$ and is obtained as

$$\hat{\theta}_P=\left[E\left(\theta^2\right)\right]^{1/2}. \tag{8}$$

Calabria and Pulcini [6] points out that a useful asymmetric loss function is the entropy loss

$$L\left(\delta\right)\propto\left[\delta^p-p\ log_e\left(\delta\right)-1\right]$$

where $\delta=\frac{\hat{\theta}}{\theta}$, and whose minimum occurs at $\hat{\theta}=\theta.$ Also, the loss function $L\left(\delta\right)$ has been used in Dey et al. [7] and Dey and Liu [8], in the original form having $p=1.$ Thus $L\left(\delta\right)$ can written be as

$$L\left(\delta\right)=b\left[\delta-log_e\left(\delta\right)-1\right];\ b>0. \tag{9}$$

The Bayes estimator under entropy loss function is denoted by $\hat{\theta}_E$ and is obtained by solving the following equation

$$\hat{\theta}_E=\left[E\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{10}$$

Wasan [9] proposed the K-loss function which is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}\theta}. \tag{11}$$

Under K-loss function the Bayes estimator of θ is denoted by $\hat{\theta}_K$ and is obtained as

$$\hat{\theta}_K=\left[\frac{E\left(\theta\right)}{E\left(1/\theta\right)}\right]^{\frac{1}{2}}. \tag{12}$$

Al-Bayyati [10] introduced a new loss function which is given as

$$L\left(\hat{\theta},\theta\right)=\theta^c\left(\hat{\theta}-\theta\right)^2. \tag{13}$$

Under Al-Bayyati's loss function the Bayes estimator of θ is denoted by $\hat{\theta}_{Al}$ and is obtained as

$$\hat{\theta}_{Al} = \frac{E\left(\theta^{c+1}\right)}{E\left(\theta^{c}\right)}. \tag{14}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.

(i) Quasi-prior: For the situation where we have no prior information about the parameter θ, we may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d} \ ; \theta > 0,\ d \geq 0, \tag{15}$$

where $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior.

(ii) Gamma prior: Generally, the gamma density is used as prior distribution of the parameter θ given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \ ; \theta > 0. \tag{16}$$

3. POSTERIOR DENSITY UNDER $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (2), is given by

$$f\left(\theta/\underline{x}\right) = \frac{\left(\frac{k\theta}{\lambda^k}\right)^n \left(\prod_{i=1}^n x_i^{k-1}\right) e^{-\sum_{i=1}^n \left(\frac{x_i}{\lambda}\right)^k} exp\left[(\theta-1)\sum_{i=1}^n log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]\right]\theta^{-d}}{\int_0^\infty \left(\frac{k\theta}{\lambda^k}\right)^n \left(\prod_{i=1}^n x_i^{k-1}\right) e^{-\sum_{i=1}^n \left(\frac{x_i}{\lambda}\right)^k} exp\left[(\theta-1)\sum_{i=1}^n log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]\right]\theta^{-d} d\theta}$$

$$= \frac{\theta^{n-d}\ e^{-\theta\sum_{i=1}^n log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}}}{\int_0^\infty \theta^{n-d}\ e^{-\theta\sum_{i=1}^n log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}} d\theta}$$

$$= \frac{\left(\sum_{i=1}^n log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)} \theta^{n-d}\ e^{-\theta\sum_{i=1}^n log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}} \tag{17}$$

Theorem 1. On using (17), we have

$$E\left(\theta^c\right) = \frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)}\left(\sum_{i=1}^n log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-c}. \tag{18}$$

Proof. By definition,

$$E\left(\theta^c\right) = \int \theta^c f\left(\theta/\underline{x}\right) d\theta$$

$$=\frac{\left(\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)}\int_0^{\infty}\theta^{(n-d+c)}\ e^{-\theta\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}}d\theta$$

$$=\frac{\left(\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{n-d+1}}{\Gamma(n-d+1)}\frac{\Gamma(n-d+c+1)}{\left(\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{n-d+c+1}}$$

$$=\frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)}\left(\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-c}.$$

From equation (18), for $c=1$, we have

$$E(\theta)=(n-d+1)\left(\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-1}. \tag{19}$$

From equation (18), for $c=2$, we have

$$E(\theta^2)=\left[(n-d+2)(n-d+1)\right]\left[\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right]^{-2}. \tag{20}$$

From equation (18), for $c=-1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{1}{(n-d)}\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}. \tag{21}$$

From equation (18), for $c=c+1$, we have

$$E(\theta^{c+1})=\frac{\Gamma(n-d+c+2)}{\Gamma(n-d+1)}\left(\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-(c+1)}. \tag{22}$$

4. BAYES ESTIMATORS UNDER $g_1(\theta)$

From equation (6), on using (19), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S=(n-d+1)\left(\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-1}. \tag{23}$$

From equation (8), on using (20), the Bayes estimator of θ under precautionary loss function is obtained as

$$\hat{\theta}_P = \left[(n-d+2)(n-d+1)\right]^{\frac{1}{2}} \left(\sum_{i=1}^{n} log \left[1 - exp\left(-\left(\frac{x_i}{\lambda} \right)^k \right) \right]^{-1} \right)^{-1}. \tag{24}$$

From equation (10), on using (21), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E = (n-d) \left(\sum_{i=1}^{n} log \left[1 - exp\left(-\left(\frac{x_i}{\lambda} \right)^k \right) \right]^{-1} \right)^{-1}. \tag{25}$$

From equation (12), on using (19) and (21), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K = \left[(n-d+1)(n-d)\right]^{\frac{1}{2}} \left(\sum_{i=1}^{n} log \left[1 - exp\left(-\left(\frac{x_i}{\lambda} \right)^k \right) \right]^{-1} \right)^{-1}. \tag{26}$$

From equation (14), on using (18) and (22), the Bayes estimator of θ under Al-Bayyati's loss function comes out to be

$$\hat{\theta}_{Al} = (n-d+c+1) \left(\sum_{i=1}^{n} log \left[1 - exp\left(-\left(\frac{x_i}{\lambda} \right)^k \right) \right]^{-1} \right)^{-1}. \tag{27}$$

5. POSTERIOR DENSITY UNDER $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (2), is obtained as

$$f(\theta/\underline{x}) = \frac{\left(\frac{k\theta}{\lambda^k} \right)^n \left(\prod_{i=1}^{n} x_i^{k-1} \right) e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\lambda} \right)^k} exp\left[(\theta-1) \sum_{i=1}^{n} log \left[1 - exp\left(-\left(\frac{x_i}{\lambda} \right)^k \right) \right] \right] \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}}{\int_0^\infty \left(\frac{k\theta}{\lambda^k} \right)^n \left(\prod_{i=1}^{n} x_i^{k-1} \right) e^{-\sum_{i=1}^{n} \left(\frac{x_i}{\lambda} \right)^k} exp\left[(\theta-1) \sum_{i=1}^{n} log \left[1 - exp\left(-\left(\frac{x_i}{\lambda} \right)^k \right) \right] \right] \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta}$$

$$= \frac{\theta^{n+\alpha-1} e^{-\left(\beta + \sum_{i=1}^{n} log \left[1 - exp\left(-\left(\frac{x_i}{\lambda} \right)^k \right) \right]^{-1} \right)\theta}}{\int_0^\infty \theta^{n+\alpha-1} e^{-\left(\beta + \sum_{i=1}^{n} log \left[1 - exp\left(-\left(\frac{x_i}{\lambda} \right)^k \right) \right]^{-1} \right)\theta} d\theta}$$

$$= \frac{\theta^{n+\alpha-1} e^{-\left(\beta + \sum_{i=1}^{n} log \left[1 - exp\left(-\left(\frac{x_i}{\lambda} \right)^k \right) \right]^{-1} \right)\theta}}{\Gamma(n+\alpha) \Big/ \left(\beta + \sum_{i=1}^{n} log \left[1 - exp\left(-\left(\frac{x_i}{\lambda} \right)^k \right) \right]^{-1} \right)^{n+\alpha}}$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{n+\alpha}}{\Gamma(n+\alpha)}\theta^{n+\alpha-1}\ e^{-\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)\theta} \tag{28}$$

Theorem 2. On using (28), we have

$$E\left(\theta^c\right)=\frac{\Gamma(n+\alpha+c)}{\Gamma(n+\alpha)}\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-c}. \tag{29}$$

Proof. By definition,

$$E\left(\theta^c\right)=\int\theta^c f\left(\theta/\underline{x}\right)d\theta$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{n+\alpha}}{\Gamma(n+\alpha)}\int_0^\infty\theta^{n+\alpha+c-1}\ e^{-\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)\theta}d\theta$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{n+\alpha}}{\Gamma(n+\alpha)}\frac{\Gamma(n+\alpha+c)}{\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{n+\alpha+c}}$$

$$=\frac{\Gamma(n+\alpha+c)}{\Gamma(n+\alpha)}\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-c}.$$

From equation (29), for $c=1$, we have

$$E(\theta)=(n+\alpha)\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-1}. \tag{30}$$

From equation (29), for $c=2$, we have

$$E\left(\theta^2\right)=\left[(n+\alpha+1)(n+\alpha)\right]\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-2}. \tag{31}$$

From equation (29), for $c=-1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{1}{(n+\alpha-1)}\left(\beta+\sum_{i=1}^{n}log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right). \tag{32}$$

From equation (29), for $c=c+1$, we have

$$E\left(\theta^{c+1}\right)=\frac{\Gamma(n+\alpha+c+1)}{\Gamma(n+\alpha)}\left(\beta+\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-(c+1)} . \qquad (33)s$$

6. BAYES ESTIMATORS UNDER $g_2(\theta)$

From equation (6), on using (30), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S=(n+\alpha)\left(\beta+\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-1} . \qquad (34)$$

From equation (8), on using (31), the Bayes estimator of θ under precautionary loss function is obtained as

$$\hat{\theta}_P=\left[(n+\alpha+1)(n+\alpha)\right]^{\frac{1}{2}}\left(\beta+\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-1} . \qquad (35)$$

From equation (10), on using (32), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E=(n+\alpha+1)\left(\beta+\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-1} . \qquad (36)$$

From equation (12), on using (30) and (32), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K=\left[(n+\alpha)(n+\alpha-1)\right]^{\frac{1}{2}}\left(\beta+\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-1} . \qquad (37)$$

From equation (14), on using (29) and (33), the Bayes estimator of θ under Al-Bayyati's loss function comes out to be

$$\hat{\theta}_{Al}=(n+\alpha+c)\left(\beta+\sum_{i=1}^{n} log\left[1-exp\left(-\left(\frac{x_i}{\lambda}\right)^k\right)\right]^{-1}\right)^{-1} . \qquad (38)$$

CONCLUSION

In this paper, we have obtained a number of estimators of parameter of exponentiated Weibull distribution. In equation (4) we have obtained the maximum likelihood estimator of the parameter. In equation (23), (24), (25), (26) and (27) we have obtained the Bayes estimators under different loss functions using quasi prior. In equation (34), (35), (36), (37) and (38) we have obtained the Bayes estimators under different loss functions using gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution. We therefore, recommend that the estimator's choice lies according to the value of the prior distribution which in turn depends on the situation at hand.

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AN INVENTORY MODEL FOR DETERIORATING ITEMS WITH DEMAND DEPENDENT ON TIME RAISE TO POWER m BY 3 WITH CONSTANT DETERIORATION RATE

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ABSTRACT

The paper introduces a deterministic inventory model for declining goods with a demand rate that is increased to the power m by three. The cost of keeping is believed to be constant. This listing is for a single item. The lead time is zero. The shortages are approved and are completely backlogged. The solution of a differential equation for a sale price-dependent demand rate is the focus of this paper. This model estimated the total optimal cost, which was then numerically checked. Two-dimensional and three-dimensional graphs are used to verify the model's convexity. For the graphical presentation, Maple software is used.

Keywords: Inventory model, Time Dependent, Deterioration, Shortages, Fractional Polynomial.

INTRODUCTION

Some products, such as tomatoes, fruit, biscuits, and other products, deteriorate after a certain period of time. Some researchers, while it was supposed to be a constant by others, considered demand rate a linear or quadratic function of time. The selling price or the passing of time decide the demand rate for such goods. Mandal and S. Phaujdhar [1] created an inventory model that took demand rate into account for declining goods. MohitRastogi, S.R. Singh, Prashant Kushwaha, and Shilpy Tayal [2]invented the inventory model with price-sensitive demand. With linear demand and constant deterioration, Vinod Kumar Mishra and Lal Sahab Singh formed deteriorating inventory.

K. Jalan, R. R. Giri, and K. S. Chaudhuri [3] developed EOQ model with linear demand and Weibull distribution deterioration and [4] considered linear demand with exponential decline. Kuo-Lung Hou [5] constructed a model with linear demand and constant deterioration and is applied to on-hand inventory. Y. K. Shah [6], and RAM B. MISRA [7] invented deterministic inventory with constant demand rate, and no replacement or repair is permitted within a given period.

Mingbao Cheng and Guoqing Wang [8]With a trapezoidal style demand rate and a constant deterioration rate, built an inventory model for deteriorating goods. Chaitanya Kumar Tripathy and Umakanta Mishra in [9] presented an order- level inventory system with time-dependent Weibull deterioration and a ramp type demand rate, in which supply and demand are time-dependent. S.K.Ghosh and K.S.Chaudhuri [10] developed an inventory model for deteriorating products with instantaneous supply, quadratic time-varying demand, and a two parameter Weibull distribution to reflect time deterioration. S.R. Singh and Himanshu Rathore found the rate of deterioration to be a regulated variable and demand to be linearly related to time in 2014. Later in 2106, Nita H. Shah with Urmila Chaudhari and Mrudul Y. Jani [11]For the time-varying deteriorating item under time and price-sensitive demand, developed an integrated production-inventory model with preservation technology investment. and Nita Shah in 2018 with Monika K. Naik constructed an EOQ model for the deteriorating item under full advance payment availing of discount when demand is price-sensitive. Abu Hashan Md Mashud [12] developed An EOQ deteriorating inventory model with different types of demand and fully backlogged shortages where he considers demand rate as a function of time as well as selling price.

Ajanta Roy [13] invented an inventory model for deteriorating items with price dependent demand and time-varying holding cost. R. Amutha and Dr.E .Chandrasekaran [14] developed an inventory model for deteriorating products with Weibull Distribution deterioration, partial backlogging, and time-varying demand. Recently Yadav, H.,Vinodkumar and Singh, T.P.[21] explored an inventory model for weibull rate of deteriorating items, shortages, partial backlogging with quadratic demand rate.

This paper establishes an inventory model for deteriorating goods based on a single item with a demand rate that is a function of time raised to the power m by three and a constant deterioration rate. The cost of keeping is thought to be constant, and shortages are both permitted and prohibited.

FORMULATION AND SOLUTION OF MATHEMATICAL MODE

Assumptions And Notations:

Notations:

D - Demand Rate.

C_D - Deterioration Cost.

A - The ordering cost.

C_H – Holding Cost.

Q - Order Quantity

θ - Deterioration Rate.

C - Purchase Cost per unit

L - Order Quantity

Ω - The time when inventory level reached zero.

T - The length of a cycle time.

C_S - Shortage cost per unit time.

TIC - Total inventory cost.

I(t) - Inventory level.

Assumptions:

The inventory Model uses the following assumptions:-

- The demand rate is fractionally dependent on time which is of type m by three

$D(t) = \alpha + \beta t^{\frac{m}{3}}$ and α≥0, β≥0.

- Holding cost per unit time is constant

  H(t) = C_H , where C_H ≥0.

- The Order quantity per cycle is Q.
- Shortages are allowed and are fully bloclogged.
- The lead time is zero.
- The deterioration rate is

  $\theta(t) = \gamma, 0 < \gamma < 1$

- Ordering cost per item is A
- The shortage cost per unit time is C_s
- This inventory Model deals with single item .
- Ω is the time at which inventory level reaches zero and Ω≥0.
- TIC is assumed as total inventory cost.
- I(t) is assumed as inventory level.
- C is assumed as the purchasing cost per unit.

In the period (0,Ω) inventory is positive and in the period (Ω,T) the inventory level is negative .

MATHEMATICAL FORMULATION

In the period (0,Ω)inventory level is positive and decreasing and at time t=Ω, the inventory level reaches to zero and shortage starts and inventory became negative in the period (Ω,T) .The differential equation for instantaneous inventory level at time t is as follows :-

$$\frac{dI_1(t)}{dt} + \gamma I(t) = -(\alpha + \beta t^{\frac{m}{3}}); 0 \le t \le \Omega \tag{1}$$

$$\frac{dI_2(t)}{dt} = -(\alpha + \beta t^{\frac{m}{3}}); \Omega \le t \le T \qquad (2)$$

where m is an odd natural number

with boundary conditions

$I_1(t) = I_2(t) = 0$ at t= Ω

$I_2(t) = J$ at $t = T$ and $I_1(t) = E$ at t=0

On solving equation (1) and neglecting higher powers of t , we get

$$I_1(t) = [\alpha(\Omega - t) + \frac{3\beta}{m+3}(\Omega^{\frac{m+3}{3}} - t^{\frac{m+3}{3}}) + \frac{\alpha\gamma}{2}(\Omega^2 - t^2) + \frac{3\beta\gamma}{m+6}(\Omega^{\frac{m+6}{3}} - t^{\frac{m+6}{3}})$$
$$+ \frac{\alpha\gamma^2}{6}(\Omega^3 - t^3) + \frac{n\beta\gamma^2}{m+9}(\Omega^{\frac{m+9}{3}} - t^{\frac{m+9}{3}})] \qquad (3)$$

On solving (2), we get

$$I_2(t) = \alpha(\Omega - t) + \frac{3\beta}{m+3}(\Omega^{\frac{m+3}{3}} - t^{\frac{m+3}{3}}) \qquad (4)$$

Now using condition $I_1(t) = E$ at t=0, we get

$$E = \alpha\Omega + \frac{3\beta}{m+3}\Omega^{\frac{m+3}{3}} + \frac{\alpha\gamma}{2}\Omega^2 + \frac{3\beta\gamma}{m+6}\Omega^{\frac{m+6}{3}} + \frac{\alpha\gamma^2}{6}\Omega^3 + \frac{3\beta\gamma^2}{m+9}\Omega^{\frac{m+9}{3}}$$

Now using the condition $I_2(t) = -J$ at t=T, we get

$$J = \alpha(\Omega - T) + \frac{3\beta}{m+3}(\Omega^{\frac{m+3}{3}} - T^{\frac{m+3}{3}})$$

Now the order quantity per cycle is given by

Q = E -J

$$Q = \alpha\Omega + \frac{3\beta}{m+3}\Omega^{\frac{m+3}{3}} + \frac{\alpha\gamma}{2}\Omega^2 + \frac{3\beta\gamma}{m+6}\Omega^{\frac{m+6}{3}} + \frac{\alpha\gamma^2}{6}\Omega^3 + \frac{3\beta\gamma^2}{m+9}\Omega^{\frac{m+9}{3}}$$
$$-\alpha(\Omega - T) - \frac{3\beta}{m+3}(\Omega^{\frac{m+3}{3}} - T^{\frac{m+3}{3}}) \qquad (5)$$

Now the inventory holding cost per cycle is given by:

$$IHC = C_H \int_0^\Omega I_1(t)dt$$

$$IHC = C_H[\frac{\alpha}{2}\Omega^2 + \frac{3\beta}{m+6}\Omega^{\frac{m+6}{3}} + \frac{\alpha\gamma}{3}\Omega^3 + \frac{3\beta\gamma}{m+9}\Omega^{\frac{m+9}{3}} + \frac{3\alpha\gamma^2}{8}\Omega^4 + \frac{3\beta\gamma^2}{m+12}\Omega^{\frac{m+12}{3}}] \qquad (6)$$

Total amount of inventory shortage cost per unit time during the period (Ω,T) is given by:

$$ISC = C_S \int_\Omega^T I_2(t)dt$$

$$ISC = C_S \int_\Omega^T [\alpha(\Omega - t) + \frac{3\beta}{m+3}(\Omega^{\frac{m+3}{3}} - t^{\frac{m+3}{3}})]dt$$

$$ISC = C_S[\alpha(\Omega T - \frac{T^2}{2} - \Omega^2 + \frac{\Omega^2}{2}) + \frac{3\beta}{m+3}(\Omega^{\frac{m+3}{3}} T - \frac{3}{m+6} T^{\frac{m+6}{2}} - \Omega^{\frac{m+6}{3}} + \frac{m}{6}\Omega^{\frac{m+6}{3}})$$

$$ISC = C_S[\alpha(\Omega T - \frac{T^2}{2} - \frac{\Omega^2}{2}) + \frac{3\beta}{m+3}(\Omega^{\frac{m+3}{3}} T - \frac{3}{m+6} T^{\frac{m+6}{3}} - \frac{(m+3)}{(m+6)}\Omega^{\frac{m+6}{3}})] \tag{7}$$

Now, Inventory deterioration cost per item is given by:

$$IDC = C_D[E - \int_0^{\Omega}(\alpha + \beta t^{\frac{m}{3}})dt]$$

$$IDC = C_D[\alpha\Omega + \frac{3\beta}{m+3}\Omega^{\frac{m+3}{3}} + \frac{\alpha\gamma}{2}\Omega^2 + \frac{3\beta\gamma}{m+6}\Omega^{\frac{m+6}{3}} + \frac{\alpha\gamma^2}{6}\Omega^3$$

$$+\frac{3\beta\gamma^2}{m+9}\Omega^{\frac{m+9}{3}} - \alpha\Omega - \frac{3\beta}{m+3}\Omega^{\frac{m+3}{3}}] \tag{8}$$

The inventory ordering cost per order during $(0,\Omega)$ is given by:

$$IOC = A \tag{9}$$

Now, the total cost per unit time for the cycle is given by

$$TIC = \frac{1}{T}[IHC + ISC + IDC + IOC]$$

$$TIC = \frac{1}{T}[C_H[\frac{\alpha}{2}\Omega^2 + \frac{3\beta}{m+6}\Omega^{\frac{m+6}{3}} + \frac{\alpha\gamma}{3}\Omega^3 + \frac{3\beta\gamma}{m+9}\Omega^{\frac{m+9}{3}} + \frac{3\alpha\gamma^2}{8}\Omega^4 + \frac{3\beta\gamma^2}{m+12}\Omega^{\frac{m+12}{3}}]$$

$$+C_S[\alpha(\Omega T - \frac{T^2}{2} - \frac{\Omega^2}{2}) + \frac{3\beta}{m+3}(\Omega^{\frac{m+3}{3}} T - \frac{3}{m+6} T^{\frac{m+6}{3}} - \frac{(m+3)}{(m+6)}\Omega^{\frac{m+6}{3}})]$$

$$+C_D[\alpha\Omega + \frac{3\beta}{m+3}\Omega^{\frac{m+3}{3}} + \frac{\alpha\gamma}{2}\Omega^2 + \frac{3\beta\gamma}{m+6}\Omega^{\frac{m+6}{3}} + \frac{\alpha\gamma^2}{6}\Omega^3 + \frac{3\beta\gamma^2}{m+9}\Omega^{\frac{m+9}{3}} - \alpha\Omega - \frac{3\beta}{m+3}\Omega^{\frac{m+3}{3}}] + A]$$

(10)

In order to minimize the total inventory cost , we find the optimal values of T and Ω. The optimal values of T and Ω can be obtained by equating the partial derivatives of total inventory cost with respect to T and Ω respectively to zero.

For optimal value of Ω and T we have

$$\frac{\partial(TIC)}{\partial\Omega} = 0 \tag{11}$$

and

$$\frac{\partial(TIC)}{\partial T} = 0 \tag{12}$$

The total minimum cost per unit time TIC (T, Ω) satisfy by sufficient condition

$$\frac{\partial^2(TIC)}{\partial T^2} > 0 \tag{13}$$

and

$$\frac{\partial^2(TIC)}{\partial\Omega^2} > 0 \tag{14}$$

and

$$\frac{\partial^2(TIC)}{\partial T^2} * \frac{\partial^2(TIC)}{\partial\Omega^2} - \frac{\partial^2(TIC)}{\partial\Omega\partial T} > 0 \tag{15}$$

We get the value of Ω by solving equation **(11)** and We can get the value of T by solving equation **(12)** and putting these value in equation**(10)** , we obtain the minimum cost per unit time for the values which satisfy the necessary condition**(13)** , **(14)** and**(15).**

NUMERICAL EXAMPLE

Now, we consider a numerical example to examine the optimization of the solution. We used maple 18 Mathematical software to solve example.

To explain the model numerically, the following parameters of the inventory system are: α= 14, β= 10, ϒ= 12, C_H = 28, C_S = 15, A = 500, C_D =10, n=1.

We get the optimal shortage value by the above given parameters by using Maple 18 ,Ω=0.9333011846 e^{-1} , T = 5.044843514.

Finally the total optimal cost obtained is TIC =5559894.478. The graph is as shown in figure 1.and its three dimensional picture is shown in figure 2.

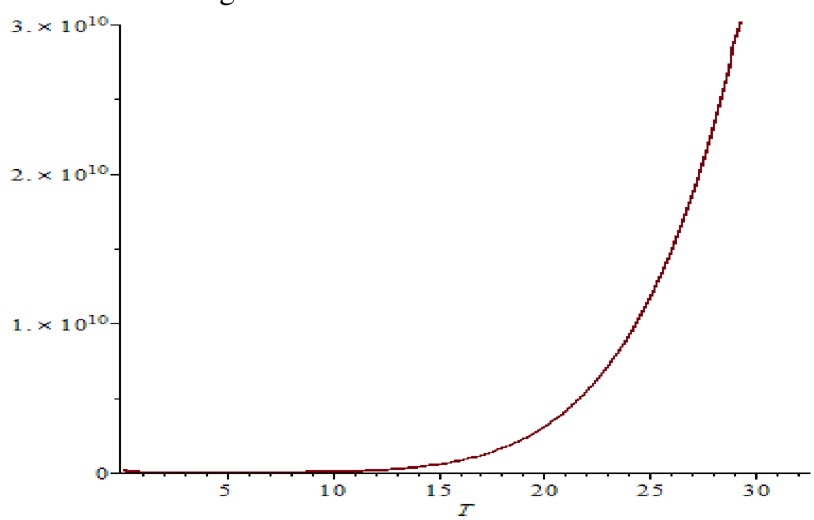

Figure1. shows total cost function verse

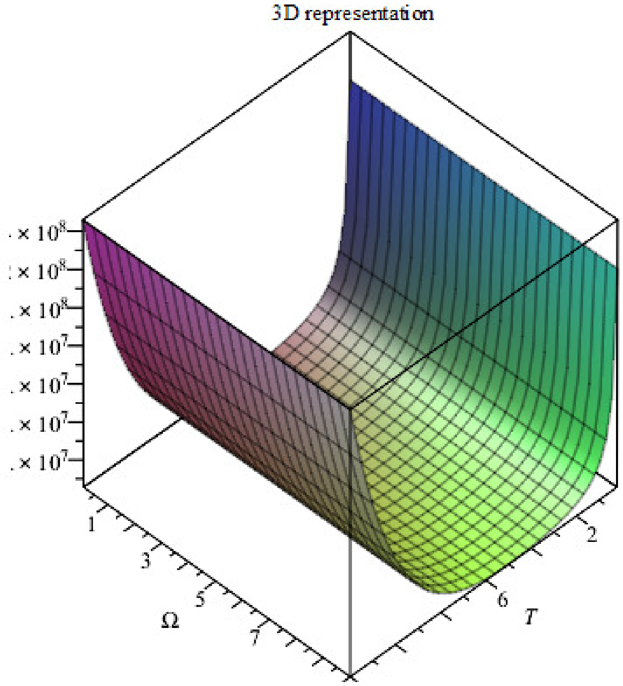

Figure 2. shows total cost function verses Ω and T

CONCLUDING REMARKS

A deterministic inventory model based on time raised to the power n by two, constant decline rate, and constant holding cost is built in this paper. The total optimal cost has been calculated for the values of Ω and T, satisfying the necessary condition. The model has been tested using numerical and graphical diagrams. This model refers to the model in which demand changes fractionally over time as commodity prices increase day by day. As a result, future cities and businesses will benefit greatly from the paradigm.

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ANALYTICS IN INDIAN ECONOMY: HOW TO OVERCOME CHALLENGES

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EXECUTIVE SUMMARY/ABSTRACT
As per market study (Analytics India Magazine and Praxis Business School study, 2019), the cumulative data analytics market size is $30 Bn. The further breakup reveals that the outsourcing is the main driver of revenue for Indian vendors accounting for $27 billion in revenue andthe domestic analytics market stands at $3.03 billion in size. The domestic market is expected to double by 2025. The Indian IT services companies such as TCS, Wipro are leading revenue generators in data analytics services. The market study further states that Bengaluru is the analytics hub with market share of 28%. Looking at the market size for outsourcing business and growth rate of domestic business for data analytics, it is apt to consider challenges for both type of businesses. Currently, India is top outsourcing destination. However, to retain the top spot and harness domestic business opportunities, India needs to tackle challenges such as availability of right skill sets, legal framework/ data privacy laws, data capturing mechanism, data availability, accuracy, and timeliness of data. This paper starts with briefly explaining the India's market size of the analytics (offshore and domestic), her current standing and then elaborates the list of challenges and finally concludes with the recommendations to tackle these challenges.

1. CURRENT CHALLENGES

Availability of enough Data

Data availability is a major challenge in our country. The driver to availability of quality data is digitalization and record of every transaction. The very first step whenever a newborn land on earth is to get it registered within 21 days of birth. However, according to a report published in 2020, still approx. 20% of births in India go unregistered leading to a parallel economy of cash surviving on street vending and on cash. According to report published in Bloomberg ,"India do not produce data on household income and expenditure, something which is common in most other countries, at even an annual frequency. Similarly, we do not have high-frequency data on the services sector of the economy – a sector that accounts for almost 60 percent of output. The list of important but missing data points on the economy is long".A vision with name SDG(sustainable Development Goals) was shared by UN with member nations with list of 17 goals and corresponding 169 indicators for measurement to address the social, economic, and environmental dimensions of sustainable development in a balanced manner. India too have embarked on the journey but currently ranks poorly at 117th place with Sweden on top as on 2020.The measurement involved in these indicators' banks primarily on percentage and data is key to such measurement. Due to lack of enough data availability, India is not able to produce metrics for majority of these indicators.

Timeliness of Data

Second challenge analytics facing w.r.t Indian economy is timeliness of the data. According to a report in Bloomberg quint1 "some high-frequency data – the current account data, which is important for the currency market, and by implication for monetary policy, is released by the RBI 2.5 to 3 months after the quarter has ended. The data for a quarter is thus released almost at the end of the following quarter, by which time it becomes stale. The industrial production data, which is an important gauge of short-term economy is released more than 40-days after the month has ended. The data for August, the month in which we currently are, will be released in mid-October. In contrast, inflation data is released with a lag of fewer than two weeks after the end of the month and is thus highly useful as it is near 'real-time'. The latest period for which the highly comprehensive data on organized manufacturing – the Annual Survey of Industries was available for the year 2015-16. There was no data yet for the year 2016-17 untilmid of 2018-19. Take the whole debate over minimum support prices or farm loan waivers. The last survey that went into detail over what farmer incomes were, what their debt levels are and of the reach and effectiveness of MSPs was in 2013 – 5 years ago. And the one prior to that was 10 years back. So, we have just two data points in the last 15 years."

Accuracy of Data

Multiple revisions in key data such as GDP, IIP, WPI inflation from government leads to confusion and questioning on veracity of data produced.According to a report in Bloomberg1" Take the latest revision to the GDP series a couple of years back. The data as per the new series was counter-intuitive to many, and questions were raised about its accuracy. The data might very well be accurate, but when honest questions are simply ignored and brushed over, this deepens suspicion. There must be a willingness on the part of bureaucrats to engage with users of data to address the to engage with users of data toaddress their concerns.

Take the survey on household expenditure that NSSO conducts or used to conduct. It gives very detailed data on household consumption across categories and regions. This is a widely used data set. However, the expenditure aggregated as per this survey is only 50 percent of the household expenditure reported by National Accounts. This clearly seems wrong prima facie. The government knows this. But beyond attributing this to methodological issues, there is no attempt to explain the difference in such a way that the data could be then used keeping its limitations in mind".

Analyzing massive, complex, and ever-changing data sets

We live in a country of diversity and varied tastes.Even with homogeneity, data insights become meaningless if there are frequent changes to behavior of customers. A data arising from a country like India is not only massive but rather changing frequently as people's transactions in day to day life are also changing with upgradation in lifestyles, social media promotions and change in phase of life (e.g. single to married, married to being parent, from being parent to grow into senior citizens,changing health etc.). Producing a stable analytics conclusion out of such data was and will remain a big challenge.

Analyzing unstructured data

Analyzing Unstructured data that is available in legacy systems and other various forms such as paper, in tacit will be challenge due to non availability of centralized system from where analytics systems can pull data to give meaningful insights. Lot of time need to be spent in data cleaning, data structuring and by the time data becomes ready to produce useful report, new set of data arises which has capability to change the meaning of the data produced earlier

Variety and Interoperability of Data

Currently the data has variety of sources such as legacy system, cloud platforms, customized platforms etc. So, to analyze such data, it is essential to have the data in single format and at the same time interoperable despite its source of origin.

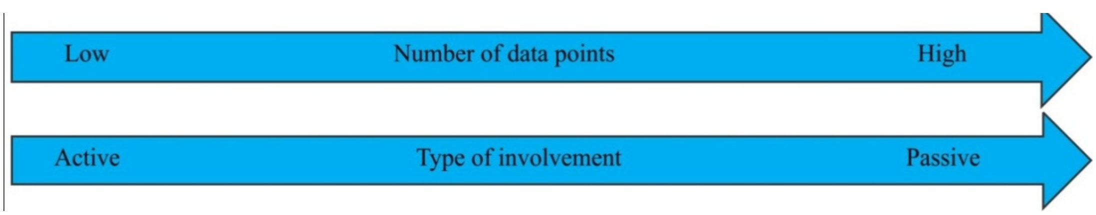
Let us take the example of healthcare sector in India where lack of interoperability is not able to generate the value for our people. The current healthcare industry has multiple closed ecosystems where information is locked. Every private healthcare provider has its own diagnostic / pathology and partnership labs to solve patient's problem. Such data resides in silos due to walled approach. And this data is hardly shared with any other healthcare providers partly to lock the patient in the system and leverage it for future use. Sometimes such data is also not made available to patient himself. The other reason could be the fear of losing the monopoly to algorithms/ machine learning tools if the data is shared. Today, the incidence rate of cancer is increasing in India. If all the private healthcare industry decides to share the cancer diagnostic data seamlessly, an algorithm can also do it. So, the interoperability lies at the heart of the problem and is preventing India to develop patient centric health care system.

However, the early passage of PDP bill (Privacy and Data Protection Bill) by Indian Parliament should address this challenge. The PDP bill is discussed in detail in one of the following sections.

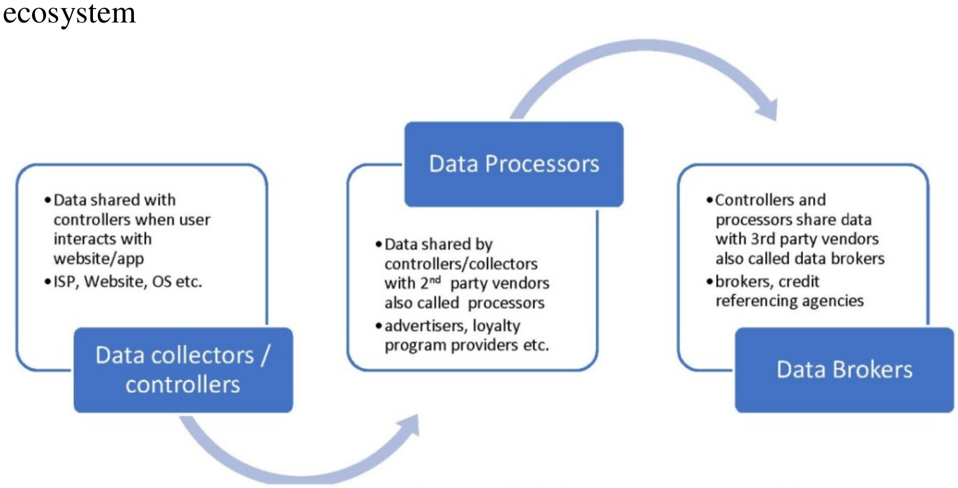
Data Collection Methods in Private Indian Enterprises

As per Deloitte research across Indian private enterprises, it is found that collecting, using, and sharing personal data is governed by complicated privacy policies often difficult to read and understand. However, there are very few enterprises who seek "explicit and transparent" consent. The lack of awareness about the personal data among many users help these enterprises to continue with such practices. Following are the data sources

Name, health details, finance, contact details etc.	Purchase activity, browsing activity	SMS, camera, microphone, location etc.
Volunteered	Collected during usage	Collected in background



Data collection ecosystem



Many private enterprises collect data without well-defined use cases and hence only 5-10% of the collected data is useful. Such wasteful practices result in unrelated data collection and inflated storage costs. Also, private enterprises frequently share personal data of consumers with data brokers and consumers are seldom aware about the number of third-party entities accessing their personal data. Such practices dent the confidence of consumers who want to share genuine information, issues, and feedback.

Legal Framework / Data Privacy Laws

Data Protection refers to the set of privacy laws, policies and procedures that aim to minimise intrusion into one's privacy caused by the collection, storage, and dissemination of personal data. Personal data generally refers to the information or data which relate to a person who can be identified from that information or data whether collected by any Government or any private organization or an agency.

According to article in Mondaq "The Constitution of India does not patently grant the fundamental right to privacy. However, the courts have read the right to privacy into the other existing fundamental rights, i.e., freedom of speech and expression under Art 19(1)(a) and right to life and personal liberty under Art 21 of the Constitution of India."

The data protection law encourages domestic and foreign businesses to enable outsource or build the competency in house in the data analytics.Currently, India does not have any formal law to tackle the data privacy issues. This is the biggest impediment at least for foreign firms to outsource to India. As the firsthand experience while working with healthcare giant Roche Diagnostics, Switzerland, the company has classified India as high-risk outsourcing destination from legal risk perspective. This as they have stated is due to absence of EU equivalent GDPR (General Data Protection Regulation).And unpredicted events such as COVID pandemic which has forced organization to go digital has in a way forced the Governments across world to take notice of data protection regulations. India is also no exception. In January 2021, India took significant steps in tech policy and data regulation. It covered sectors such as non- personal data, health data, financial data, e-commerce, and other consumer facing services. The PDP (Privacy and Data Protection) bill which GOI has drafted is a right step to regulate the data and below are some of the highlights of various privacy bills under consideration by GOI

Proposed Privacy Law: Inspired by the EU's GDPR, PDP Bill was introduced in year 2019 to address the data privacy issues which is currently under purview of Information Technology Act, 2000. The joint parliamentary committee (JPC) is still reviewing the bill and proposed some modifications, but it is expected that the bill would take considerable time to become law. The scope of bill includes non- Indian organizations, Indian organizations, personal and non-personal data.

Proposed Non-personal Data Framework: NPD (non-personal data) framework studies the issues related to NPD and to make specific suggestions to central government. This framework provides guidelines for data collection entities and separate treatment for "High Value Datasets" (data pertaining to public good and community)

DEPA for Fintech: Data Empowerment and Protection Architecture (DEPA) is policy framework released by NITI Aayog for fintech sector. This policy is aimed towards consent-based data sharing in the fintech sector. Through this framework customers will be able to share their financial data securely across banks, mutual funds, insurers, tax collectors, and pension funds. Similar frameworks are on anvil for healthcare and telecom sectors.

Data Sharing policy for Health Sector: HDM (Health Data Management) Policy inspired from PDP bill to govern data in healthcare ecosystem. It establishes the consent-based data processing and sharing. HDM policy once implemented will have impact on medical and pharmaceutical industry as it will have compliance obligation guidelines.However, HDM policy has some overlaps with PDP bill so potential conflicts are anticipated.

Repercussions due to Impending E-commerce policy: e-commerce policy which is still under works by GOI proposes sweeping changes such as sharing of source codes, algorithms, cross border data flow, anti-piracy, and other non-personal data etc. with government.

Data governance for Motor Vehicle Aggregators: Motor Vehicle Aggregator Guidelines 2020 was published in Nov 2020 to regulate the business transport aggregators. Under the guideline, data generated on aggregators app must be stored in India for minimum 3 months to maximum 24 months. The data needs to be shared with State Government as per law and data cannot be shared without consent of customer.

Required Skillsets and India's Standing

To reap the benefits of data analytics stream, it is essential to understand the interdependencies of key domains and skillsets. Broadly, the domains can be classified into three categories.

- Skills in Business Domain focus is on the practice and day-to-day running of a business. Some examples of this domain are accounting, finance, marketing, and sales etc.
- Skills in Technology Domain focus is on the creation, maintenance, and scaling of computer systems and software. Some examples of this domain are databases, computer networking etc.
- Skills in Data Science Domain focus is on capturing and utilizing the data generated within a business for decision-making and/or powering underlying products and services. The data science skills can be further subdivided into
  - ❖ Data Management comprises everything related to managing and accessing data for reporting, analysis, and model building. Sample skills: Cloud APIs, Hadoop
  - ❖ Data Visualization involves the creation and study of visual representations of data to communicate information clearly and efficiently. Sample skills: Tableau, Plotting Data
  - ❖ Machine Learning creates algorithms and statistical models that computer systems can use to perform a specific task without explicit instructions. Sample skills: Multi-Task Learning, Deep Learning
  - ❖ Math is the study of numbers and their relationships, applying these principles to models of real phenomena. Sample skills: Calculus, Linear Algebra
  - ❖ Statistical Programming is the set of programming languages and tools used to create statistical models and algorithms. Sample skills: R, Python
  - ❖ Statistics deals with all aspects of data collection, organization, analysis, interpretation, and presentation. Sample skills: Regression, AB Testing

It is pertinent to note that professionals with business, technology and data science skills will have competitive edge in the global workforce.

As per Global Skills Index Report, Coursera, India lags in data science skills particularly in data management skills. In Data Science domain, India ranks at 51 globally and at 12 among APAC countries. However, with in data management competency, India ranks at 58 globally and at 15 among APAC countries.

However, India surges ahead China in business and technology skills ranking. Within business domain, India ranks at 34 globally compared to China's rank of 45. Within Technology domain, India ranks at 40 globally compared to China's rank of 50.

	Business	Technology	Data Science
Asia Pacific			12
Global	34	40	51

Table 1: India's rank at APAC and Global level in key domain

Based on above assessment, India is currently facing challenges in Data science skills which can be attributed to brain drain reducing supply of skilled workers locally which is further exacerbated by partly poor quality of STEM education.

As per Banerjee et al (2013), for effective usage of data analysis, the simultaneous presence of all three factors mentioned in Error! Reference source not found. is essential.According to a "towards data science" report , there is shortage of more than 93000 data scientists in India as on August 2020. Further, even if some people get trained, the business acumen to process the data and deduce meaningful results doesn't lay with many people leading to shortage of experts in this field which in itself is becoming challenge to Indian government given the task of putting India ahead in Sustainable Development Goals list.Data analytics which can help government to prioritize area of exact emphasis and create policy around it so that India improves rank in SDG which is much critical for Indian economy.

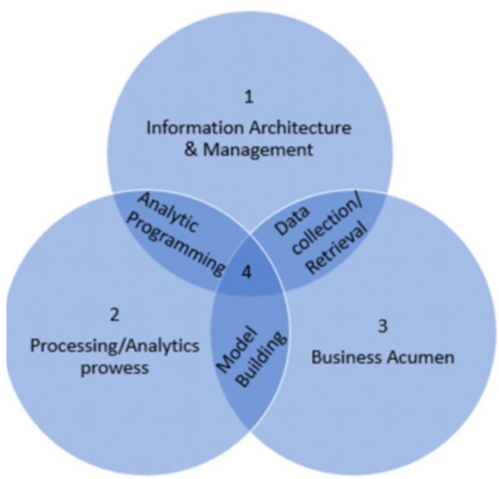


Figure 1: Analytics skill sets

RECOMMENDATION TO TACKLE CHALLENGES

Faster Adoption of 5G Network

The key benefits of 5G network are better reliability and latency. On adoption of 5G network, faster exchange of information among connected systems is expected. The ground is already set with the sudden explosion of IoT devices in the Indian market. The number of IoT devices expected to increase from 200 million unit in 2016 to 2.7 billion unit in 2020 . Such IoT devices will enable system to collect and organize data automatically. Manually performing this process is far too time-consuming. An automated system will allow employees to focus on productive/core activities rather than processing the data. A data system that collects, organizes, and automatically alerts users of trends will help solve the issue of timeliness and accuracy of data. Due to automation, it would be possible to generate intelligent business reports with satisfaction and in lesser time. With real-time reports and alerts, decision-makers can be confident they are basing any choices on complete and accurate information. So, it is essential for India to lay a roadmap to faster adoption of 5G network. However according to parliamentary panel, India is still in preparatory phase and sufficient groundwork is still under progress for launch of 5G network.

Data Centralization and Accessibility

A comprehensive and centralized system will solve problem of accessibility. Such system will save time spent accessing multiple sources and it will also allow cross-comparisons while ensuring data is complete.Decision-makers and risk managers need access to all an organization's data for insights on what is happening at any given moment, even if they are working off-site. Adatabase useful to all employee in decision making role with appropriate authorization and ability to access from anywhere will resolve accessibility issues.

Well Deliberated and Contemporary Data Privacy Law

India presently does not have any express legislation governing data protection or privacy. However, the relevant laws in India dealing with data protection are the Information Technology Act, 2000 and the (Indian) Contract Act, 18725. A codified law about data protection is likely to be introduced in India soon. The (Indian) Information Technology Act, 2000 deals with the issues relating to payment of compensation (Civil) and punishment (Criminal) in case of wrongful disclosure and misuse of personal data and violation of contractual terms in respect of personal data5. A codified law around data protection or privacy will bolster confidence in data security, authenticity and avoid misuse. We have already discussed (refer Legal Framework / Data Privacy Laws) at length on the PDP bill being deliberated in Indian parliament. With mushrooming online marketplaces, offshore and onshore data analytics in India, we feel there is an urgent need to implement well deliberated and contemporary PDP bill.

Strategies to Build and Improve Skillsets

India doesn't have good pool of experts in analytics area with business acumen required to provide meaningful information to government so that effective policies to make India a 5 trillion dollar economy by 2025 and also its ranking on SDG indicators look much better than present rank of 117. Fundamentally, the STEM education forms the base for future skillsets for data analytics professionals. The STEM education is relatively new concept in India. A robust STEM education equips individuals with critical thinking, innovation, and problem-solving skills. As per research, student develop interest in STEM fields at average age of eight. Traditionally, our education system promoted rote learning while STEM education emphasizes on Do-It-Yourself approach to promote learning. We therefore hopeful that GOI's National Education Policy will bring desired change in the education system. However, it is long term solution so in short term,collaborating with premier institutes such as IIMs/IITs on preparing sound business knowledge bended with technical knowledge possessing talent pool will help significantly in overcoming challenges.

Take Cues from Decision in "SCHREMS II" by EU Courts

The Austrian complainant, Schrems contested the privacy shield, seamless EU-US personal data transfer mechanism, in CJEU (Court of Justice European Union). The outcome of the decision (invalidating the privacy shield) had significant impact not only EU-US but also EU- other country data transfers and transferee countries will have to update the laws pertaining to data transfer mechanisms. Due to invalidation of privacy shield, the transferee must demonstrate that it is adhering to the Standard

Contractual Clauses in GDPR respecting individual rights. If transferee is found to be in violation, then the data sharing and transfer mechanism is to be immediately stopped. So, India will have to update its data privacy laws taking into consideration the Standard Contractual clauses to ensure data sharing and transfer between EU and India for data analytics work.

Role of Indian Judiciary on Data Regulation

The Kerala high court in case of Balu Gopalakrishnan Vs State of Kerala passed order on April 2020 on export of COVID 19 related data by State of Kerala to US based data analytics firm Sprinklr. In its judgement, high court directed the State of Kerala to take measures such as anonymizing the data, consent to data sharing and return of data once contractual obligation ends. This judgement will have far reaching consequences to data analytics firms in India and foreign firms. Such historic judgements should be part of formal laws which we are hopeful but if not must be included in PDP Bill.

Offshore-Onshore Model to Circumvent Offshore Analytics Issues

Currently, physical barriers for technology services companies based out of India faces challenge of not being part of key decision maker meetings happening at client location (onshore). This aspect creates information asymmetry and leads to mediocre data analytic solution. Generally, services firm nowadays employ a strategy to locate an expert resource at client location and rest of the team is located at offshore location. The onsite resource acts as proxy to client in a sense that he attends the important decision maker's meeting to get the clarity on the context and passes this information to offshore team. The expert acts more of a facilitator and communication bridge to save valuable time of client. Such offshore-onsite model can be recommended and resource rotation through such model will help in building the desired skills in the team.

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EVALUATION OF FIRST AND SECOND DOSAGE OF COVID-19 VACCINATION USING K-MEANS CLUSTERING MODEL AND VISUALIZATION OF INDIAN STATES AND UNION TERRITORIES

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ABSTRACT

Application of Orange Data mining software determines the clusters and plots the graph of vaccination data for various states and union territories. The file widget open new vaccination data set and perform k-mean++ from 2 to 9 with silhouette distance. The silhouette scores and cluster information are achieved. The three zones are visualized and the zones are labeled as green, Blue and Red: The green zone indicates that states and union territories are high vaccinated, the blue zones indicates states and union territories are Moderatelyvaccinated, and the red zone are low vaccinated states and union territories of India. The states and union territories' of Sikkim, Tripura, Ladakh and Lakshadweep have low population butfalls in high vaccinated states of first and second dose. The states and union territories of Goa, Mizoram, Delhi, Arunachal Pradesh, Chandigarh, Uttarakhand, Gujarat, Rajasthan, Kerala, Jammu and Kashmir, Dadra and Nagar Haveli, Damn and Diu, Himachal Pradesh, Chhattisgarh and Andaman Nicobar Islandshavediverse population and come in the category of low vaccinated states of first and second dose. The states and union territories of Manipur, Meghalaya, Nagaland, Odisha, West Bengal, Haryana, Karnataka, Andhra Pradesh, Maharashtra, Telengana, Jharkhand, Madhya Pradesh, Punjab, Assam, Uttar Pradesh, Tamil Nadu Puduchery and Bihar have high population and are moderately vaccinated states of first and second dose. The open source tools like Orange Data mining found useful for exploring appropriate and applicable functions in data science. Several partitions with different values of k- number of clusters or partitions are recommended to review along with cluster quality index for optimum solution. K-means can be adapted to micro-level demarcation of containment zones. The clusters formed based on COVID-19 patient's vaccination data using Data Science techniques specifically K-means will be active, unbiased, accurate, visible, economic and easy to apply.

Keywords: k-means++, Visualization, COVID-19, Vaccination of first and Second Dosage and Indian States and Union Territories.

1. INTRODUCTION

COVID-19 is a Data Science issue” (Callaghan, 2020) the comprehensive article gives various ideas and inspiration to think about the data and how it can be effectively used in current pandemic situation.Quarantine is nothing but the separation and restriction of movement or activities of persons who are not ill but who are believed to have been visible to infection, for the purpose of avoiding transmission of diseases. People are usually isolated in their homes, but they may also be quarantined in community-based accommodations. Considering the increasing volume of number of patients and limited community-based facilities, most of the people are being asked to quarantine in their homes.

The Cluster Containment Strategy would be to control the disease within a defined geographic area by early vaccinated of populations, breaking the chain of transmission and thus preventing its spread to new areas. This would include vaccinations percentage of first and second dose and total measures. Many sensitive factors are associated while defining the containment zones in states and union territories.

Manually defining these continuously changing clusters in size and location may not be feasible. The techniques from data science especially k-means can help in this situation by defining the clusters as vaccinated states and union territories. The objective of this paper is to bring neutrality and accuracy in creating the COVID-19 vaccination zones dynamically and consistently using k-means technique of Data Science. This is desirable to get continuous updating on accurate and latest micro-level isolation information of vaccination zones to plan the unbiased strategies for separating contacts of COVID-19 patients from community. The scope of this research paper is to apply K-means technique from Data Science on the collected vaccination data like first, second dose of vaccine and total population to define and visually plot the vaccinated states to visualize on the actual map.

2. REVIEW OF LITERATURE

According to Wollersheim, 2020 during the COVID-19 crisis the field of Data Science is in epicenter. Maximum of the public is interested, watching and looking forward the statistical analysis and epidemiology graphs and sharing the same in social media on a large scale. The probability from Data Science is very high. Data Science is adeveloping field consists of number of appropriate and useful tools, functions and techniques.

Singh *et.al.* 2018, suggested the cluster containment strategy for Zika virus outbreak was found effective in Rajasthan, India. It explained that how surveillance strategies are used to control the disease from spreading beyond containment zones of 3 km radius. The article gives emphasis on creating containments to prevent the outburst of disease, however it does not explain about how to make these zones quickly and accurately. In their paper (Maier &Brockmann, 2020) it is also explained about the effective containment to control COVID-19 cases in Chinaspecifically. The model which they explained in their paper captures both isolation of symptomatic infected individuals and other population isolation practices. The focus of the research is on contagion process and general effects as well as significance of the containment. Their research work implies and supports the need to define the containment zones accurately.

Manimannan G. *et.al* (2021),predicts and classifies the data of COVID-19 based on four machine learning algorithm with four major parameters namely confirmed cases, recoveries, deaths and active cases. The secondary sources of database were collected from Ministry of Health and Family Welfare Department (MHFWD), from Indian State and Union Territories up to March, 2021. Based on these background, the database classified and predicted various machine learning Algorithm, like SVM, kNN, Random Forest and Logistic Regression. Initially, k-means clustering analysis is used to perform and identified five meaningful clusters and is labeled as Very Low, Low, Moderate, High and Very High of four major parameters based on their average values. In addition the five clusters are cross validated using four machine algorithm and affected states are visualized in the table with help of prediction and probabilities. The different machine learning models cross validation and classification accuracy are 88%, 97%, 91% and 91%. The Classification of States and Union Territories were named as Very Low Affected (VLA), Low Affected (LA), Moderately Affected (MA), Highly Affected (HA) and Very Highly Affected (VHA) States and Union Territories of India by COVID-19 cases. Maharashtra is correctly classified as Very High Affected States, Delhi, Uttar Pradesh and West Bengal falls in Moderately Affected States, Assam, Bihar, Chattisgarh, Haryana, Gujarat, Madhya Pradesh, Odisha, Punjab, Rajasthan and Telangana falls in Low Affected States and Tamilnadu, Kerala, Andhra Pradesh and Karnataka forms a group of highly affected States. Remaining States and Union Territories falls in Very Low affected by Covid-19 Cases.

2.1 Application of Data Science

Data science generally has a five-stage lifecycle and is given below (Figure 1).

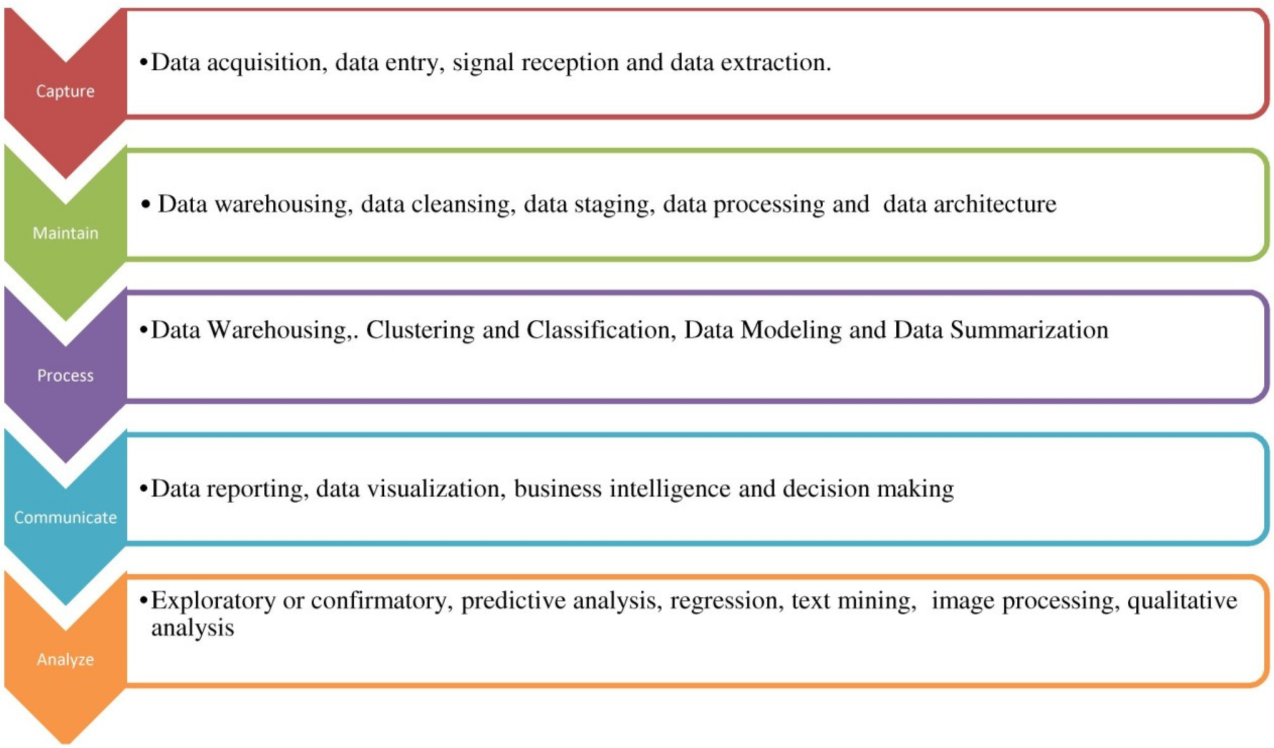

Figure 1. Five Stage of Data Science

The article (Weatherill & Burton, 2009) explains how the seismic source zones were created using k-means cluster analysis for the Aegean region. The paper describes the significance of applying k-means algorithm for hierarchical cluster analysis and was found useful for partition regions based on observed seismicity to have consistent approach to source model development.

Clustering of various Indian states and union territories is prepared which gives the Silhouette distances between each pair of the cases in Table 1.

	Pop.Census2011	itateunionterritory	Cluster	Silhouette	first_percentage	econd_percentag
1	380581.0	Andaman and ...	C1	0.638057	18.7495	1.8075
2	49577103.0	Andhra Pradesh	C2	0.688111	8.09476	1.20999
3	1383727.0	Arunachal Prad...	C1	0.629764	9.78495	2.69359
4	31205576.0	Assam	C2	0.683697	4.17573	0.943655
5	104099452.0	Bihar	C2	0.682619	4.78915	0.67443
6	1055450.0	Chandigarh	C1	0.657515	11.3222	2.47288
7	25545198.0	Chhattisgarh	C1	0.661322	17.4165	1.98895
8	343709.0	Dadra and Nag...	C1	0.556202	10.6695	1.92692
9	243247.0	Daman and Diu	C1	0.659671	14.6736	1.92849
10	16787941.0	Delhi	C1	0.675566	12.8801	2.73401
11	1458545.0	Goa	C1	0.628213	13.6004	3.16816
12	60439692.0	Gujarat	C1	0.684097	14.8778	2.41419
13	25351462.0	Haryana	C2	0.62867	10.811	1.2842
14	6864602.0	Himachal Prad...	C1	0.665122	16.7971	1.98763
15	12267032.0	Jammu and Kas...	C1	0.584099	11.3036	1.92008
16	32988134.0	Jharkhand	C2	0.695922	7.3577	1.05573
17	61095297.0	Karnataka	C2	0.637188	10.6924	1.25082
18	33406061.0	Kerala	C1	0.672189	15.5355	2.03686
19	274000.0	Ladakh	C3	0.689766	24.5201	3.89489
20	64473.0	Lakshadweep	C3	0.663558	23.4594	4.80666
21	72626809.0	Madhya Pradesh	C2	0.683027	9.04137	1.05119
22	112374333.0	Maharashtra	C2	0.667914	9.75275	1.16965
23	2570390.0	Manipur	C2	0.611044	4.00706	1.95017
24	2966889.0	Meghalaya	C2	0.664248	4.23376	1.48192
25	1097206.0	Mizoram	C1	0.654146	11.1573	2.89499
26	1978502.0	Nagaland	C2	0.663511	5.53803	1.54339
27	41974219.0	Odisha	C2	0.609048	10.272	1.47422
28	1247953.0	Puducherry	C2	0.595827	12.1546	1.11887
29	27743338.0	Punjab	C2	0.686325	7.94818	0.767903
30	68548437.0	Rajasthan	C1	0.668851	13.799	2.07999
31	610577.0	Sikkim	C3	0.674875	22.9486	3.42938
32	72147030.0	Tamil Nadu	C2	0.691738	5.67154	0.859428
33	35003674.0	Telangana	C2	0.695855	7.40264	1.05448
34	3673917.0	Tripura	C3	0.611522	20.9457	3.17378
35	199812341.0	Uttar Pradesh	C2	0.685419	4.55745	0.806766
36	10086292.0	Uttarakhand	C1	0.679639	13.1523	2.39523
37	91276115.0	West Bengal	C2	0.681095	8.3188	1.28302

Table 1. Cluster Information using K-means++ with silhouette distance

The main objective of this research paperis to explore the applications of k-means technique of Data Science influence towards defining, visualizing and maintaining the vaccination states impacted by COVID-19

3. DATA COLLECTION

The secondary source of vaccination data were collected from Kaggle.com website, up to April 30.04. 2021 based on 2011 population. The original data consist of Total population, First dose, second dose, first dose percentage, second dose percentage and cumulative percentage of Indian states and union territories. In this research paper used only three parameters and they are Total population, first dose percentage and second dose percentage.

4. METHODOLOGY

In methodology, the researcher use python based orange data mining software to identify the vaccinated states of India using k-means clustering techniques with help of silhouette distance matrix.

The widget applies of k-Means clustering algorithm to the data and give outputs as a new data set in which the cluster index is used as a class attribute. The original class attribute, if it exists, is moved to meta attributes. Silhouette distance scores of clustering results for various k are also shown in the widget (Table 2).

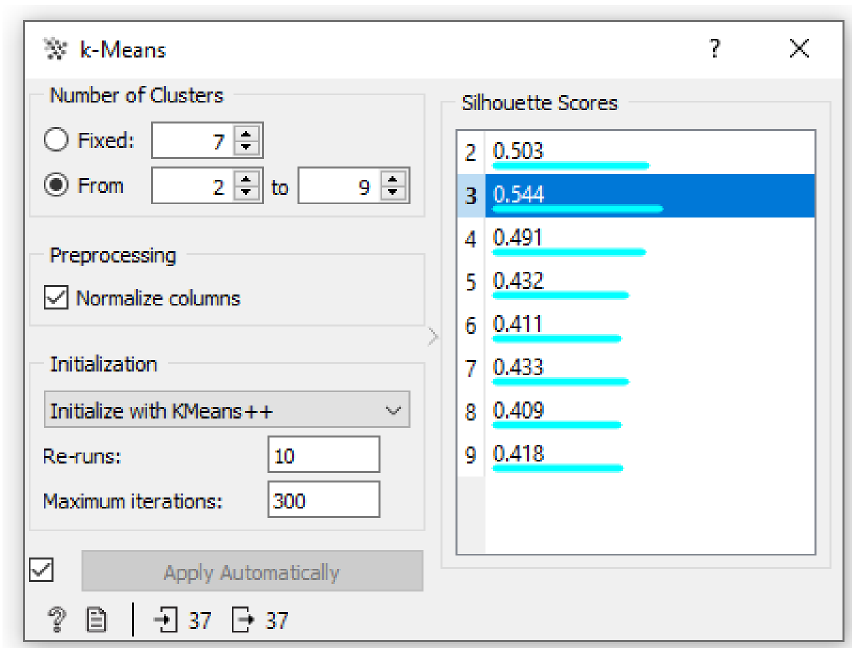

Table 2. K-means of Silhouette Scores

4.1. Proposed Algorithm:

The proposed orange data mining algorithm to execute k-means clustering step by step:
Step 1: Initially, Select the number of clusters with fixed algorithm of clusters data in specified number of clusters.
Step 2: Select the initialization method of k-means++ and first center is selected randomly, subsequent are chosen from the remaining points with probability proportioned to squared distance from the closest center.
Step 3: The algorithm is torun the maximum of iteration with each cluster, it can be set manually.
Step 4: The widget outputs a new vaccination data set with appended cluster information and select how to append cluster information and name the column.
Step 5: If *Apply Automatically* is ticked, the widget will commit changes automatically. Alternatively, click *Apply*.
Step 6: Produce a report.
Step 7: Check scores of clustering results for various k. (Figure 2)

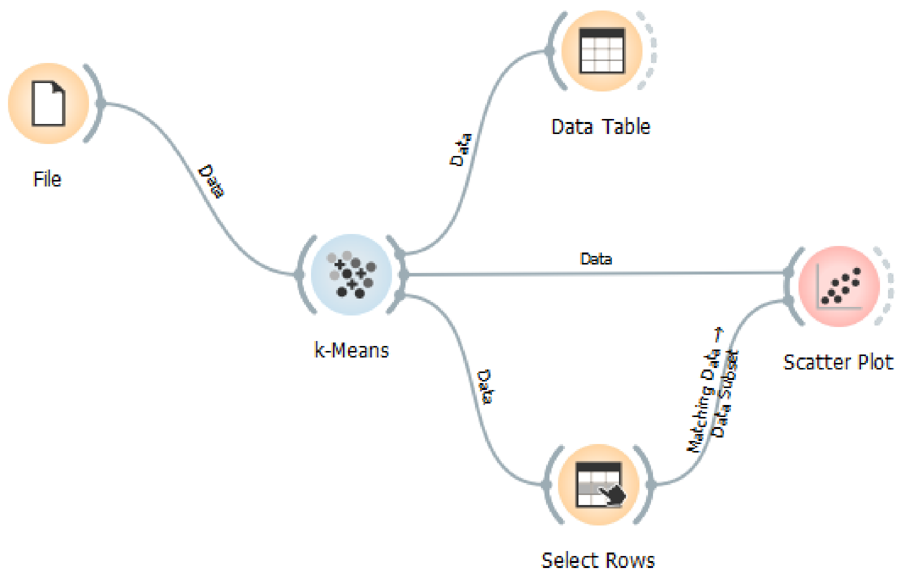

Figure 2. K-means Data Mining Work Flow Widget

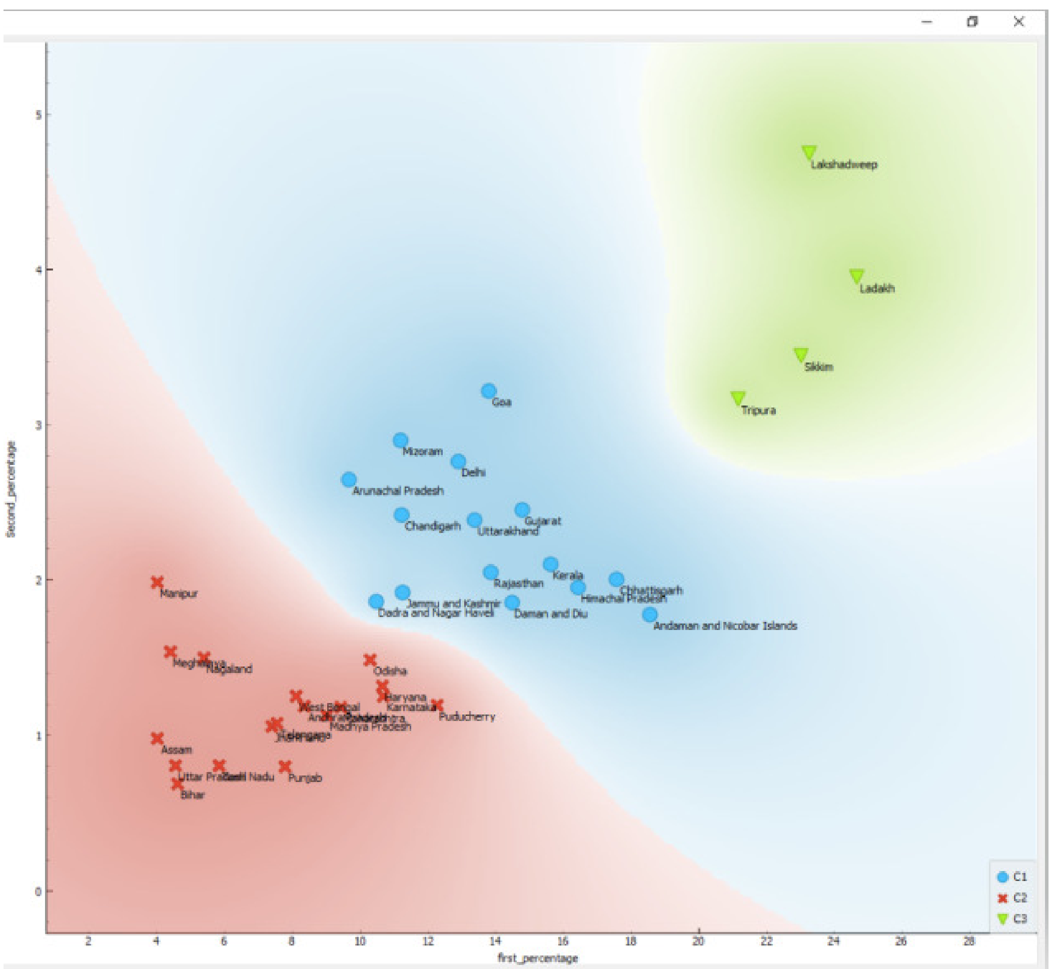

Figure 3. Data Visualization of three clusters of Indian States and Union Territories (Vaccination Data)

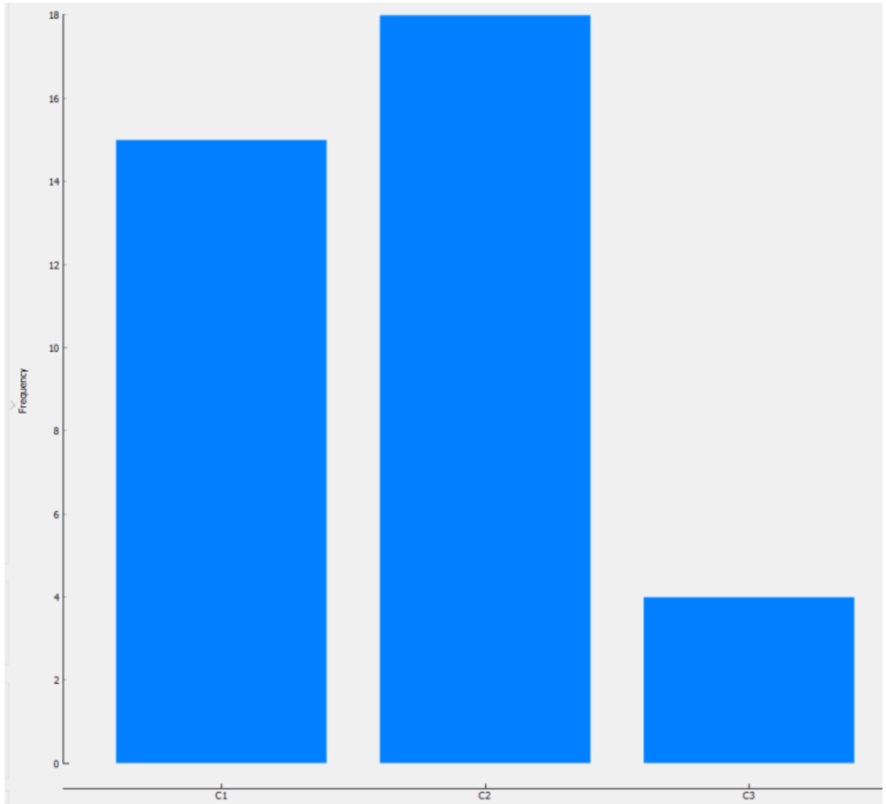

Figure 4. Data Distribution of three clusters of Indian States and Union Territories (Vaccination Data)

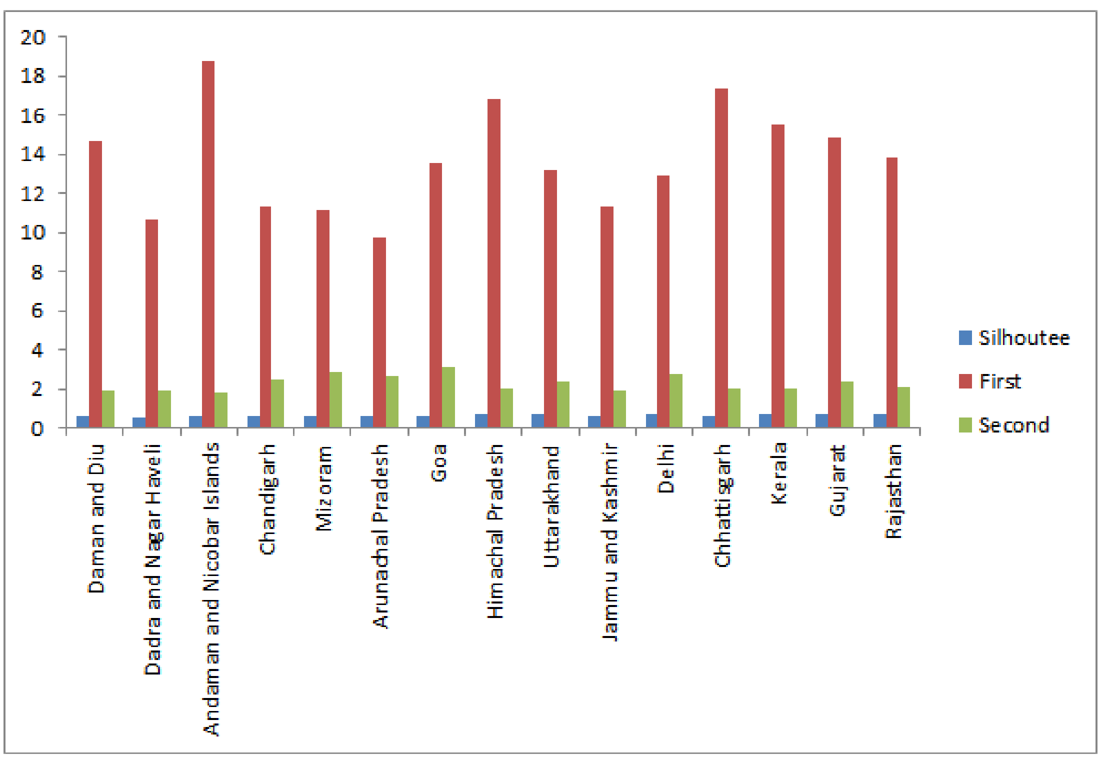

Figure 5. Data Distribution (Cluster 1) of Indian States and Union Territories (Vaccination Data)

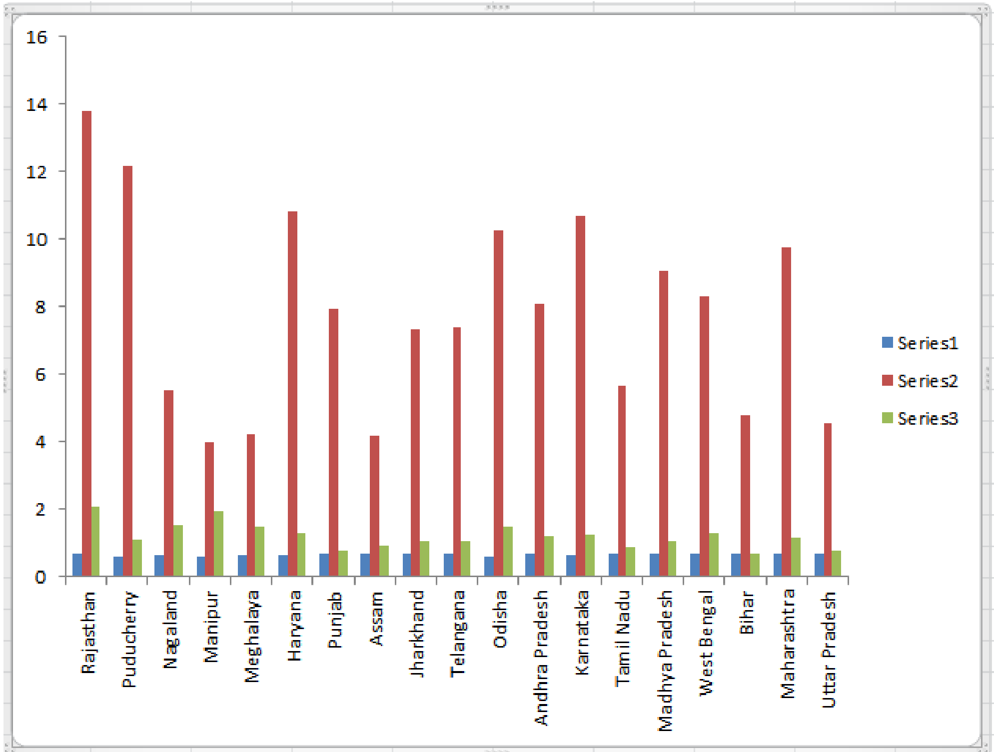

Figure 6. Data Distribution (Cluster 2) of Indian States and Union Territories (Vaccination Data)

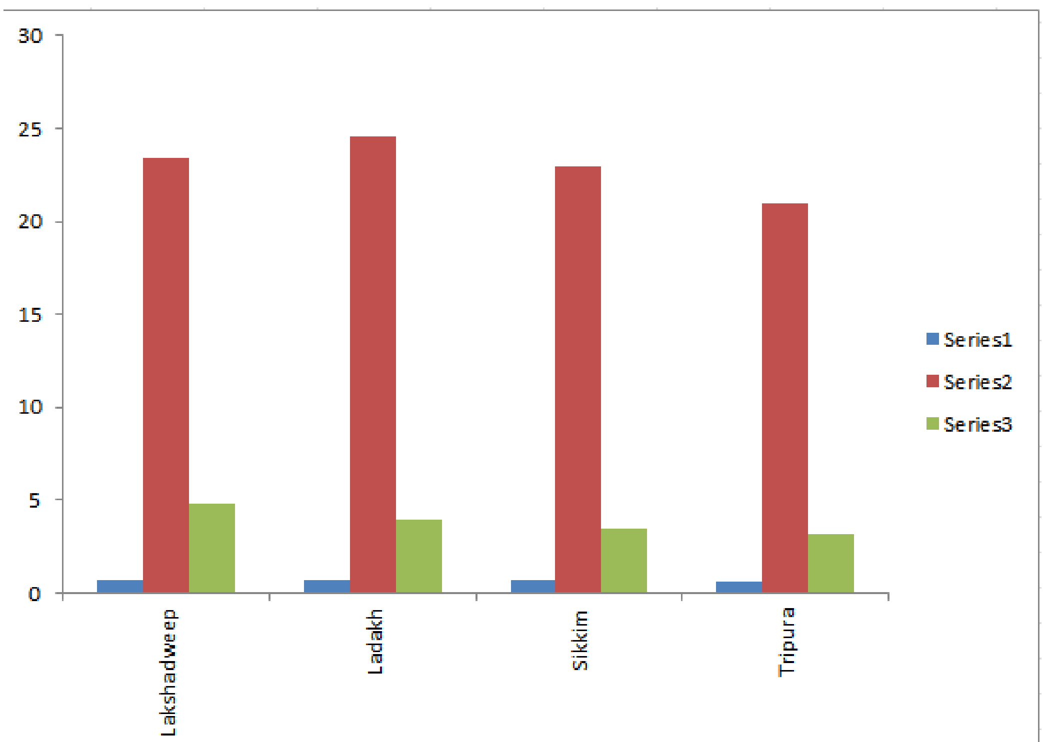

Figure 7. Data Distribution (Cluster 3) of Indian States and Union Territories (Vaccination Data)

5. DISCUSSION

Application of Orange Data mining software is to determine the clusters and visualize the graph of vaccination data for various states and union territories. The file widget opens newvaccination data set and perform k-mean++ from 2 to 9 with silhouette distance. The silhouette scores are displayed in the above Table 2. The results of Cluster information is exhibited in Table 1. The k-mean clusters achieved three cluster zones of states and union territories based on the first and second dose vaccination. The three zones are visualized in the figure 3to7. The green zone states and union territories are high vaccinated, the blue zones states and union territories are Moderate vaccinated states and union territories and the red zone are low vaccinated states and union territories of India.

The states and union territories' of Sikkim, Tripura, Ladakh and Lakshadweep have low population and falls in high vaccinated states of first and second dose. The states and union territories of Goa, Mizoram, Delhi, Arunachal Pradesh, Chandigarh, Uttarakhand, Gujarat, Rajasthan, Kerala, Jammu and Kashmir, Dadra and Nagar Haveli, Damn and Diu, Himachal Pradesh, Chhattisgarh and Andaman Nicobar Islands have adequate population and falls in Low vaccinated states of first and second dose.

The states and union territories of Manipur, Meghalaya, Nagaland, Odisha, West Bengal, Haryana, Karnataka, Andhra Pradesh, Maharashtra, Telengana, Jarghant, Madhya Pradesh, Punjab, Assam, Uttar Pradesh, Puduchery and Bihar have high population but falls in moderately vaccinated states of first and second dose.

5.1 Findings and Recommendations

The open source tools like Orange Data mining isfound useful for exploring the appropriate and applicable functions in data science. Several partitions with different values of k- number of clusters or partitions are recommended to review along with cluster quality index for optimum solution. K-means can be improved to smallseparation of suppressionstates and union territories.

6. CONCLUSION

To attainuniqueresults of vaccination assessment using k-means cluster and achieved three meaningful clusters for vaccination data set of Indian states and union territories. It is recommended that the Data Science techniques such as K-means can be adopted to define the micro-level segregation of vaccination zones and manage them effectively. The clusters formed based on COVID-19 patient’s vaccination data using Data Science techniques specifically K-means will be active, unbiased, accurate, visible, economic and easy to apply.

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In memory of Late Shri Parameswara Iyer N Vaikom

CONTINUED DIFFERENTIAL OPERATOR

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ABSTRACT :

This Paper is based on the concept of a 'Continued Differential Operator, that makes easy the process of integration contained in unpublished work 'Differentiation by a method of Continued Differentiation' by (Late) Parameswara Iyer N Vaikom, This is similar to the Differential operator D. Using this concept, explicit formulae for Ordinary Differential Equations (ODE) with constant coefficients, in the form of $f(D)y = f(x).\phi(x)$ ***; where*** $f(x)$ ***is nth degree polynomial and*** $\phi(x)$ ***is*** $\sin ax, \cos ax, \sinh ax$ ***or*** $\cosh ax$ ***could be derived.***

Keywords: Linear Differential Equations, Particular Integral, Direct formulae, Differential Operator, Continuous Differential Operator.

2010 AMS Classification: 34A30 34 A05

1. INTRODUCTION

This paper is based on the research work carried out by author's Father Late Shri Parameswara Iyer N Vaikom. Though the work was compiled in a book form in the year 1976, the manuscript remained unpublished neither as research papers in journals nor as a printed book. The work mainly deals with the direct solution of certain differential equations with constant coefficients. The concept underlying the derivation of these results is briefly discussed in this paper.

Solution of certain 'Ordinary Differential Equations (ODE)' with is made simple and handy with the introduction of the concept of "Differential Operator". Differential Operator is usually denoted as 'D' which stands for $\frac{d}{dx}$.

Differential equations are transformed to algebraic expressions by making use of this operator 'D'. For example, a standard form of an ODE becomes $f(D)y = \psi(x)$, where $f(D)$ represents the expression involving the operator 'D' depending on the differential terms; 'y' represents the dependent variable and $\psi(x)$ represents the function of independent variable, 'x'. The solution of the differential equations contains two parts i.e. complimentary function (CF) and Particular Integral (PI). Complimentary function can be easily found when the ODE is written in the above-mentioned standard form, since it depends only on the expression $f(D)$ and derivable from the ancillary equation. However, Particular Integral (PI) is problem specific and the method of solution for an equation will be varied depending on the nature of $f(D)$ and $\psi(x)$.

Considering the equations $\frac{dy}{dx}=\sin ax$ and $\frac{dy}{dx}=\cos ax$

Particular Integral

$$y_p=\int \sin ax.dx=\frac{1}{D}(\sin ax)=\frac{D}{D^2}(\sin ax)=\frac{a.\cos ax}{\left(-a^2\right)} \quad \text{.......................} \mathbf{(1)}$$

Similarly,

$$y_p=\int \cos ax.dx=\frac{1}{D}(\cos ax)=\frac{D}{D^2}(\cos ax)=\frac{-a.\sin ax}{\left(-a^2\right)} \quad \text{.......................} \mathbf{(2)}$$

This is a well-established procedure that when D^2 operates on *sinax* or *cosax* becomes. $-a^2$. Similarly, D^2 on *sinhax* or *coshax* can be found by substituting $D^2=a^2$.

This procedure helps to integrate such functions easily as it could be converted to a 'differential operation' simply by multiplying both numerator and denominator with 'D', (or the conjugate of the factors of denominator function $f(D)$ since operator 'D' behaves like any other algebraic term).

As an illustration, consider the following equations.

1. $Dy=\sin 3x$

Particular Integral could be found as follows.

$$y_p=\frac{1}{D}(\sin 3x)=\frac{D}{D^2}(\sin 3x)=\frac{3\cos 3x}{(-3^2)}=\frac{3\cos 3x}{-9}=\frac{-\cos 3x}{3}$$

2. $(D+1)y=\cos 4x$

$$y_p=\frac{1}{D+1}(\cos 4x)=\left\{\frac{(D-1)}{\left(D^2-1\right)}\right\}(\cos 4x)=\frac{-4\sin 4x-\cos 4x}{-16-1}=\frac{4\sin 4x+\cos 4x}{17}$$

However, the above advantage is lost when the right hand side (RHS) of the equation is a product of a polynomial with *sin*, *cos*, *sinh* or *cosh*.

For example when RHS is $x^3\sin x, x^2\cos x$ or any similar expression the substitution with ($-a^2$) for (D^2) is not possible. Thus while solving the equation, $Dy=f(x).\sin ax$, the shortcut of substituting $D^2=-a^2$ does not work.

The following proves the above statement.

$$\frac{1}{D}(f(x).\sin ax)=\frac{D}{D^2}(f(x)\sin ax)\neq\frac{f(x).D.(\sin ax)+\sin ax.D.(f(x)).}{-a^2}$$

Thus, even for simple integration case, for instance $\frac{dy}{dx} = x^2 \cos 4x$, it becomes necessary to carry out the conventional methods of integration by parts or Bernoulli's method. This prompted to explore whether it is possible to retain the substitution $D^2 = -a^2$ in the case of *sin* and *cos* functions (or $D^2 = a^2$ in the case of *sinhax* or *coshax* functions)) in the denominator so as to have an easy solution.

Present paper presents the possibilities of retaining the advantage of substituting $-a^2 or + a^2$ as the case may be, in the denominator so as an integration of such functions also could be conveniently done simply by differentiation.

2. CONTINUED DIFFERENTIAL OPERATOR

Consider the Integration of $\int f(x) \sin ax dx$

Using integration by parts,

$$\int f(x) \sin ax dx = f(x) \int \sin ax dx - \int \frac{d}{dx}\left(f(x)\right) \int \sin ax dx.dx$$

By using operator 'D', the equation transforms to

$$\frac{1}{D}\left(f(x)\sin ax\right) = f(x)\frac{1}{D}\left(\sin ax\right) - \frac{1}{D}\left[D\{f(x)\}\frac{1}{D}\left(\sin ax\right)\right] \qquad \text{.............................. (1)}$$

In both terms where D is operating on "*sin ax*", independently and it can be converted as $\frac{D \sin ax}{D^2}$ and equation (1) could be rewritten as follows

$$\frac{f(x)\sin ax}{D} = f(x)\frac{D}{D^2}\left(\sin ax\right) - \frac{1}{D}\left[Df(x)\frac{D}{D^2}\left(\sin ax\right)\right] = \frac{1}{-\left(a^2\right)}\left\{f(x)D\sin ax - \frac{1}{D}\{Df(x)D(\sin ax)\}\right\}$$

In the above equation 'D^2' in the denominator is substituted with ($-a^2$) as it operates only on sin ax. The second term is a similar expression compared to the original term and it could be seen that the process can be repeated. It is also to be noted that all combinations of separate differentials of $f(x)$ and $\sin ax$; the case is also similar for $\cos ax$ and when $\sin ax$ is replaced by $\sinh ax$ or $\cosh ax$ $D^2 = a^2$ could be substituted and same technique could be adopted for the evaluation of the integral.

$$\frac{1}{\left(-a^2\right)}\left\{f(x).D\sin ax - \frac{1}{D}\left(Df(x).D\sin ax\right)\right\} =$$

$$\frac{1}{\left(-a^2\right)}\left[f(x).D\sin ax - \frac{1}{\left(-a^2\right)}\left\{Df(x).D^2\sin ax - \frac{1}{D}\left(D^2\left(f(x) - D^2\left(\sin ax\right)\right)\right)\right\}\right]$$

It be seen that if $f(x)$ is a polynomial of nth degree, there will *(n+1)* terms in the Integral. Further it will be all combinations of separate differentials of $f(x)$ and $\sin ax$. The case with $\cos ax$ is similar. So also when $\sin ax$ is replaced by $\sinh ax$ or $\cosh ax$; $D^2 = a^2$ could be substituted and same technique could be adopted for the evaluation of the Integral.

The above result can be conceived in a different way by introducing a new concept of '**Continued Differential Operator'**, and let '**Δ'** represent a **"Continued Differential Operator".**

$$\frac{D\{f(x).\phi(x)\}}{D^2} = \frac{\Delta\{f(x).\phi(x)\}}{(-a^2)}$$; where ϕ(x) = $\sin ax$ (or $\cos ax$)

In the above D^2 is substituted by ($-a^2$) even though it a product of polynomial and sine function by replacing the D in the numerator by Δ, a 'Continued Differential operator'.

In this particular case when ϕ(x) = $\sin ax$ (or $\cos ax$)

$$\int x^n \sin axdx = \frac{x^n \sin ax}{D} = \frac{D}{D^2}\left[x^n \sin ax\right] = \left(\frac{\Delta}{-a^2}\right)\left[x^n \sin ax\right]$$

Then

$$\Delta\left[x^n \sin ax\right] = \left\{x^n D \sin ax - \frac{Dx^n D^2 \sin ax}{\left(-a^2\right)} + \frac{D^2 x^n D^3 \sin ax}{\left(-a^2\right)^2} + \ldots.(-1)^{n+1}\frac{D^n x^n D^{n+1}}{\left(-a^2\right)^n}\right\}$$

Here D^2 in RHS denominator is be substituted by (–a^2) since **a continuous differential operation** is carried out. Case when $\sin ax$ is replaced by $\cos ax$ is similar.

Thus

$$\Delta\left[x^n \cos ax\right] = \left\{x^n D \cos ax - \frac{Dx^n D^2 \cos ax}{-a^2} + \frac{D^2 x^n D^3 \cos ax}{\left(-a^2\right)^2} + \ldots.(-1)^{n+1}\frac{D^n x^n D^{n+1} \cos ax}{\left(-a^2\right)^n}\right\}$$

And hence

$$\int x^n \sin axdx = \frac{1}{\left(-a^2\right)}\left\{x^n D \sin ax - \frac{Dx^n D^2 \sin ax}{-a^2} + \frac{D^2 x^n D^3 \sin ax}{\left(-a^2\right)^2} + \ldots.(-1)^{n+1}\frac{D^n x^n D^{n+1} \sin ax}{\left(-a^2\right)^n}\right\}$$

$$\int x^n \cos axdx = \frac{1}{\left(-a^2\right)}\left\{x^n D \cos ax - \frac{Dx^n D^2 \cos ax}{-a^2} + \frac{D^2 x^n D^3 \cos ax}{\left(-a^2\right)^2} + \ldots.(-1)^{n+1}\frac{D^n x^n D^{n+1} \cos ax}{\left(-a^2\right)^n}\right\}$$

Case when ϕ(x) = $\sinh ax$ or $\cosh ax$;-

$$\int x^n \sinh axdx = \frac{x^n \sinh ax}{D} = \frac{D\left[x^n \sinh ax\right]}{D^2} = \frac{\Delta\left[x^n \sinh ax\right]}{a^2};$$

Where

$$\Delta\left[x^n \sinh ax\right] \equiv \left\{ x^n D \sinh ax - \frac{Dx^n D^2 \sinh ax}{a^2} + \frac{D^2 x^n D^3 \sinh ax}{\left(a^2\right)^2} + \ldots.(-1)^{n+1} \frac{D^n x^n D^{n+1} \sinh ax}{\left(a^2\right)^n} \right\}$$

Hence

$$\int x^n \sinh ax dx = \frac{1}{a^2}\left\{ x^n D \sinh ax - \frac{Dx^n D^2 S \sinh ax}{a^2} + \frac{D^2 x^n D^3 \sinh ax}{\left(a^2\right)^2} + ..(-1)^{n+1} \frac{D^n x^n D^{n+1} \sinh ax}{\left(a^2\right)^n} \right\}$$

$$\int x^n \cosh ax dx = \frac{1}{a^2}\left\{ x^n D \cosh ax - \frac{Dx^n D^2 S \cosh ax}{a^2} + \frac{D^2 x^n D^3 \cosh ax}{\left(a^2\right)^2} + ..(-1)^{n+1} \frac{D^n x^n D^{n+1} \cosh ax}{\left(a^2\right)^n} \right\}$$

An illustration by integrating $\int x^2 Sin.3xdx$ using the above concept is shown below.

$$y_p = \frac{1}{\left(-3^2\right)}\left\{ x^2 D \sin 3x + \frac{Dx^2 D^2 Sin3x}{3^2} + \frac{D^2 x^2 D^3 \sin 3x}{\left(3^2\right)^2} \right\}$$

$$= \frac{1}{9}\left\{ -3x^2 \cos 3x + 2xSin3x + \frac{2}{3}\cos 3x \right\}$$

The above concept '**Continued Differential Operator'** could be efficiently deployed for deriving direct (general) formulae for finding Particular Integrals (PI) of certain types of ODEs and is discussed in detail in the above mentioned manuscript . The types equations considered for deriving direct formulae are given below; however the formulae are not included in this paper.

Type: I

$D^m y = x^n \sin ax$ *(or cos ax, sinhax or coshax)*

Type III

$(D^2 + a^2)^m y = x^n \sin ax$ *(or cos ax, sinhax or coshax)*

Type II

$(D^2 - a^2)^m y = x^n \sin ax$ *(or cos ax, sinhax or coshax)*

Type: IV

$(D^2 \pm b^2)^m y = x^n \sin ax$ *(or cos ax, sinhax or coshax)*

3. CONCLUSION

The advantage of the proposed technique is that the 'Integrals' of a product involving trigonometric or hyperbolic with polynomial expressions could be worked out easily and directly. Moreover this can be easily extended to higher orders though this is not illustrated in this paper. The tediousness of successive Integration by parts is totally eliminated. Furthermore, the derived formulae even for the first order is very much simpler than Bernoulli's formulae which is only an extension of Integration by parts and that is limited to first order alone In conventional methods repeated application of Integration becomes necessary while solving higher orders and it becomes more and more laborious as order becomes higher and higher..

By adopting the method of continued differentiation for the types of equations cited above, explicit formulae can be derived which helps to find the Particular Integrals (PI) of each type of equation in two or three lines admirably reducing the steps, that too, for any order of the equation.

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SRGM INVOLVING IMPERFECT DEBUGGING AND TIME DEPENDENT NON-REMOVAL FAULT RATE FUNCTION

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ABSTRACT

Paper presents development of NHPP software reliability growth model in which fault detection rate follows S-shaped curve. Various existing models considered constant non-removal fault rate function. Proposed model incorporates imperfect debugging and non-removal fault rate function which depends on time. Existing models have been compared with proposed model using statistical tools R2 and MSE. Result validates model fits to given dataset better.

Keywords : NHPP, Software Reliability, Imperfect debugging, Fault Removal Rate.

1. INTRODUCTION

SRGM is the mathematical model used to estimate the reliability of software system. Software is reliable if it works properly without failure under specified design specifications. Reliability depends on debugging process which involves fault detection and removal. The foremost objective of software developer is to detect and remove maximum number of faults during testing phase so that reliability is maximized. Practically it is not possible to remove all faults completely. SRGM represents mathematical relation of number of faults detected at any point of time along with parameters which influence the reliability. A large number of models have been developed past thirty years to analyze reliability of software.

Various models have considered that faults detected are removed completely, known as perfect debugging, however it has been observed that debugging process introduce some new faults in software system, known as imperfect debugging. This means reliability depends on imperfect debugging. Goel and Okumoto[2] developed first model involving imperfect debugging. Later on Yamada[13] proposed SRGM incorporating S-shaped imperfect debugging and fault introduction rate. Ohba and Chou[6] discussed effect of imperfect debugging on reliability of software system. Pham et. al [9] developed model based on imperfect debugging and S shaped fault detection rate. Xie and Yang [12] analyzed effect of imperfect debugging on software development cost.

Fault removal efficiency is another parameter which influences the reliability of software. It is the proportion of total faults removed by time 't' . If non removed faults are present in software system then its reliability decreases. Zang et al. [16] incorporated fault removal efficiency in software reliability growth model and estimated the reliability. Jones[5] developed model involving fault removal efficiency. Kapur et. al. [4] analyzed the effect of fault removal efficiency on project and product type software in operational phase and estimated the reliability. Zhu and Pham [17] presented model involving non removal faults and imperfect debugging. They have considered constant non removal error rate function. Motivated by Zhu and Pham [17] model, in this paper we propose a model with time dependent non removal error rate function along with imperfect debugging. This paper divided into 5 sections. Section 1 is introduction. In section 2 some existing SRGM's have been discussed. Section 3 presents model development. Section 4 deals with parameter estimation and comparison of proposed model with existing model. Finally conclusion is drawn in section 5.

Notation

A Maximum number of faults.

m(t) Expected number of faults by time t.

a(t) Fault content function.

b(t) Fault detection rate per unit of time.

$r(t)$ Non-removed error rate per unit of time.

Assumptions

1.1 Failure intensity follows S-shaped curve.

1.2 New faults introduced during debugging process.

1.3 Failure detection rate is time dependent involving Non Homogeneous Poisson Process.

1.4 $r(t)$ Assumed to be time dependent function

1.5 Fault detection is learning curve phenomenon.

2. SOFTWARE RELIABILITY GROWTH MODELS

2.1 Yamada et.al. model [15]: It is S-shaped and concave model involving imperfect debugging. This model is parameter dependent with assumption $m(0) = 0$.

$m(t) = \alpha(1 + \gamma t)(\gamma t + \exp(-\gamma t) - 1)$

2.2 PNZ model [9]: It is S- shaped and concave model with imperfect debugging. $m(t)$ Is given by $m(t) = \frac{a\left[(1-\exp(-bt))\left(1-\frac{\alpha}{b}\right)+\alpha t\right]}{1+\beta\exp(-bt)}$, $m(0) = 0$

2.3 Pham [10] : It is also S-shaped and concave model incorporating dependency of parameters.

$m(t_o) = m_o \neq 0, t_o \neq 0.$

$$m(t) = m_o\left(\frac{\gamma t+1}{rt_o+1}\right)\exp\left(-\gamma(t-t_o)\right) + \alpha(\gamma t+1)\left[\gamma t - 1 + (1-\gamma t_o)\exp\left(-\gamma(t-t_o)\right)\right].$$

2.4 Pham-Chang model[1]: S-shaped and concave model involving uncertainty in operating environment. Here $m(0) = 0$.

$$m(t) = \frac{1}{1+\beta\exp(-bt)}\left[(c+a)(1-\exp(-bt)) - \frac{ab}{b-\alpha}(\exp(-\alpha t) - \exp(-bt))\right]$$

2.5 Pham-Zhu model[16]: S-shaped and concave model involving imperfect debugging and non- removed faults. $m(t_o) = m_o$

$$m(t) = \frac{\beta+\exp(bt)}{\frac{b}{L(b-c)}[\exp(bt)-\exp(ct)]+\frac{1+\beta}{m_o}\exp(ct)}$$

3. MODEL DEVELOPMENT

Let at any time t proportion of maximum faults to be detected are $\left[\frac{A-m(t)}{A}\right]$. Faults detected by time t are $b(t)m(t)\left[\frac{A-m(t)}{A}\right]$. Also $r(t)m(t)$ are non-removed faults by time t. Involving above terms, the differential equation formulated by Zhu and Pham [17] as

$$\frac{dm(t)}{dt} = b(t)m(t)\left[\frac{A-m(t)}{A}\right] - r(t)m(t) \tag{1}$$

Assuming $b(t) = \frac{b}{1+\beta e^{-bt}},\ b > 0,\ \beta > 0.$ $also$ $r(t) = \frac{r}{1+\alpha e^{-rt}},\ r > 0,\ \alpha > 0$

Where b and r are constant detection and error rate respectively. β and α are parameters of detection and error rate functions respectively.

Also $r(t) \to r$ as $t \to \infty$

Using initial condition $m(0) = a$, solution of equation (1) can be obtained as

$$m(t) = \frac{G(t)}{H(t)} \quad (2)$$

Where $G(t) = EXP\left[\int_0^t \{b(x) - r(x)\}dx\right]$

$$= EXP\left[\int_0^t \{\frac{b}{1+\beta e^{-bt}} - \frac{r}{1+\alpha e^{-rt}}\}dx\right]$$

$$= \left[\frac{(\beta+e^{bt})(\alpha+1)}{(\alpha+e^{rt})(\beta+1)}\right] \quad (3)$$

Also $H(t) = \frac{1}{A}\int_0^t EXP\left[\int_0^t \{b(y) - r(y)\}dy\right] b(x)dx + \frac{1}{a}$

$$= \frac{1}{A}\int_0^t \left[\frac{(\beta+e^{bx})(\alpha+1)}{(\alpha+e^{rx})(\beta+1)}\right]\left\{\frac{b}{1+\beta e^{-bx}}\right\} dx + \frac{1}{a}$$

$$= \frac{(\alpha+1)}{A(\beta+1)(b-c)}\left[(b-c)\alpha(e^{bt}-1) - b(e^{(b-c)t}-1)\right] + \frac{1}{a} \quad (4)$$

Using equations (3) and (4) we get expression of $m(t)$ from (2).

4. PARAMETER ESTIMATION AND MODEL COMPARISIONS

4.1 Parameter Estimation

Parameters of models are estimated by using Non Linear Least Square Method. Data set collected by Zhu and Pham [17] is used.

4.2 Model Comparisons

Proposed model has been compared with existing models using following tools of Goodness of Fit (GoF).

4.2.1 Coefficient of Determination R^2

Coefficient of Determination is also known as multiple correlation coefficient. It measures the correlation between the dependent and independent variables. Value of R^2 vary from 0 to 1. If $R^2 = 1$ then perfect fitting, $R^2 = 0$ no fitting, and R^2 close to 1 good fitting. R^2 is defined as

$$R^2 = \frac{\sum_{j=1}^{n}(Y_J - \bar{y})^2}{\sum_{j=1}^{n}(y_j - \bar{y})^2} \qquad \bar{y} = \frac{1}{n}\sum_{j=1}^{n} y_j$$

where y_j is observed cumulative faults at time j and Y_J estimated cumulative faults at time j. n is number of data points. Model fits better to given dataset if R^2 close to 1.

4.2.2 Mean of Square Error MSE

MSE measures the mean of squares of the deviation between estimated and observed data. It is defined as

$\mathrm{MSE} = \frac{1}{n}\sum_{j=1}^{n}\left(Y_J - \bar{y}\right)^2$ if MSE close to 0 then model fits data better.

Comparative analysis as follows:

Table 1. Goodness of Fit

		R^2	MSE
Model 1	Pham-Chang model[1]	0.77	0.64
Model 2	Zhu and Pham model [17]	0.89	0.53
Model 3	Proposed Model	0.93	0.58

5. CONCLUSION

In this paper we have discussed NHPP software reliability growth model. Model incorporated imperfect debugging and time dependent non-removal fault function. Failure detection rate follows S-shaped increasing/decreasing phenomenon. Two existing models have been compared with proposed model using statistical tools R^2 and MSE. For Model 3 value of R^2 is highest. However, MSE is relatively same with model 2. Result validates Model 3 fits to given dataset better.

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EVALUATION OF A NEW QUINTIC SPLINE FORMULA

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ABSTRACT

This paper deals with the numerical evaluation of a new Quintic Spline interpolation formula at certain intervening points in each of its sub-intervals. The functional values so obtained are compared with the corresponding actual values and the corresponding error examined.

Keywords: Interpolation, Spline, quintic spline, boundary conditions, Numerical Analysis

2010 AMS Classification:97N40, 97N50

1. INTRODUCTION

Spline Interpolation assumes a significant part in Numerical Analysis, Computation, Integration, Differentiation and several areas in Physical Sciences. In Interpolating problems, Spline Interpolation is frequently liked to Polynomial Interpolation. Spline functionshave been introduced by I. J. Schoenberg [4] in 1946. A spline is a function characterized by piecewise polynomials, each piece being a polynomial function in their respective subintervals. A linear spline is characterized by piecewise first degree polynomials, quadratic splines by quadratic polynomials, cubic spline by cubic polynomials [1,2, 5], quartic spline by fourth degree polynomials [8], and a quintic spline by fifth degree polynomials [6]. In the sixteenth century, thecalculation of a quintic spline was a great issue in Algebra, while cubic spline and quartic spline has been determined [3, 7].

A Spline is like an adaptable bend which comprises of a long piece of metal or other material, which might be bowed into a bend and fixed in position at various predefined points calledknots. These predefined points permit to draw a smooth bend or curve to another curve. A quintic spline being a polynomial of degree 5 has first derivative is a quartic function, second derivative cubic, third derivative quadratic and fourth derivative linear. In a quintic spline problem,the boundary conditions are applied to the function itself, and furthermore to the primary, second, third and fourth order subordinates of the spline functions, considering the referred to information values as the spline knots and at these knotsthe polynomial pieces are combined together. The continuity conditions are likewise applied at the knots.

2. NATURAL QUINTIC SPLINE INTERPOLATION FORMULA [6]

Consider the given set of data values $(x_i, y_i),\ i = 0,1,2\ \ldots\ldots n$ of the function $y = f(x)$. Let$I = [x_i, x_{i+1}]$be i^{th} subinterval of$[x_0, x_n]$ and $S_i(x)$ be the spline function defined in the interval$[x_i, x_{i+1}]$. A quintic spline is a polynomial of degree 5 and has continuous derivatives up to order 4 and the piecewise polynomials in each subinterval are of degree 5.

$$S(x) = \begin{cases} S_0(x), & x_0 < x < x_1 \\ S_1(x), & x_1 < x < x_2 \\ & . \\ & . \\ & . \\ S_n(x), & x_{n-1} < x < x_n \end{cases}$$

The boundary conditions for the quintic splines are,

a) $S_i(x_i) = y_i, = M_i,\ i = 0,1,2 \ldots\ldots n$

b) $S_i(x_{i+1}) = y_{i+1} = M_{i+1},\ i = 0,1,2 \ldots\ldots n-2$

c) $S_i(x_i),\ S_i^{'}(x_i),\ S_i^{''}(x_i),\ S_i^{'''}(x_i)\ and\ S_i^{'v}(x_i)$ are continuous.

d) $S_i^{'v}(x_0) = S_{i+1}^{'v}(x_n) = 0\ \ ie)\ M_0 = M_n = 0$

e) $S_{n-1}^{''}(x_n) = 0$

f) $S_{n-1}^{'''}(x_n) = 0$

A brief outline of the derivation of coefficients of $S_i(x)$is given below for quick reference, details of which can be had from [6].$S_i(x)$being a quintic spline, its fourth derivative $S_i^{'v}(x)$is linearso that it can be assumed as

$$S_i^{'v}(x) = \frac{1}{h_i}[(x_{i+1} - x)M_i + (x - x_i)M_{i+1}], \text{where } h_i = x_{i+1} - x_i,\ i = 0,1,2 \ldots\ldots n-1 \qquad (1)$$

which when integrated four times gives

$$S_i(x) = \frac{1}{h_i}\left[\frac{(x_{i+1}-x)^5}{120}M_i + \frac{(x-x_i)^5}{120}M_{i+1}\right] + C_i(x_{i+1}-x)(x-x_i)^2 + D_i(x_{i+1}-x)(x-x_i) + E_i(x_{i+1}-x) + F_i(x-x_i) \quad (2)$$

where $C_i,\ D_i,\ E_i and\ F_i$'s are the coefficients obtained in terms of M_i's

Application of the boundary conditions (a) and (b) in (1), gives

$$E_i = \frac{y_i}{h_i} - \frac{{h_i}^3}{120}M_i \quad (3)$$

$$F_i = \frac{y_{i+1}}{h_i} - \frac{{h_i}^3}{120}M_{i+1} \quad (4)$$

On differentiating (2) and applying the continuity condition, the following results are obtained

$$S_i'(x) = \frac{1}{h_i}\left[-\frac{(x_{i+1}-x)^4}{24}M_i + \frac{(x-x_i)^4}{24}M_{i+1}\right] + C_i(2xx_{i+1} - 2x_ix_{i+1} - 3x^2 + 4xx_i - {x_i}^2) + D_i(x_{i+1} + x_i - 2x) + E_i(-1) + F_i \quad (5)$$

$$\left[\frac{{h_i}^3}{24} - \frac{{h_{i+1}}^3}{24}\right]M_{i+1} + \frac{{h_{i+1}}^3}{120}[M_{i+2} - M_{i+1}] - \frac{{h_i}^3}{120}[M_{i+1} - M_i] + [C_{i+1}{h_{i+1}}^2 - C_i{h_i}^2] + [D_{i+1}h_{i+1} - D_ih_i] = \Delta_{i+1} - \Delta_i, \text{ where } \Delta_i = \frac{y_{i+1}-y_i}{h_i} \quad (6)$$

Now differentiating (5)and applying the continuity condition, we get the following results:

$$S_i''(x) = \frac{1}{h_i}\left[\frac{(x_{i+1}-x)^3}{6}M_i + \frac{(x-x_i)^3}{6}M_{i+1}\right] + C_i[2x_{i+1} - 6x + 4x_i] + D_i[-2] \quad (7)$$

$$D_i = D_{i+1} - \left[\frac{Z_i}{12}\right]M_{i+1} + 2[C_{i+1}h_{i+1} - C_ih_i] \text{for } i = 0,1,2 \ldots\ldots n-1 \quad (8)$$

Likewise, differentiation of (7), and application of the continuity condition gives the following :

$$S_i'''(x) = \frac{1}{h_i}\left[-\frac{(x_{i+1}-x)^2}{2}M_i + \frac{(x-x_i)^2}{2}M_{i+1}\right] - 6C_i \quad (9)$$

$$C_i = C_{i+1} + \left[\frac{h_i - h_{i+1}}{12}\right]M_{i+1} \text{for } i = 0,1,2 \ldots\ldots n-1 \quad (10)$$

Using (8) in (6) the following recurrence relation is obtained

$$h_i^3[M_i - 6M_{i+1}] + h_{i+1}^3[M_{i+2} - 6M_{i+1}] + 120C_{i+1}Z_i + 10h_i^2h_{i+1}M_{i+1} + 120[D_{i+1}h_{i+1} - D_ih_i] = 120[\Delta_{i+1} - \Delta_i] \quad (11)$$

From these the values of C_i, D_i, E_i, F_ican be computed using which $S_i(x)$can be obtained from (2).

3. ILLUSTRATION

Consider a set of data points of the function $y = ln\ (x)$ given by

{(1.2, 0.1823215568), (1.7, 0.5306282511), (2.1,0.7419373447),(2.4, 0.8754687374),(2.8, 1.029619417), (3.0, 1.098612289),(3.3, 1.193922468)}

$h_0 = 0.5,\quad h_1 = 0.4,\quad h_2 = 0.3,\quad h_3 = 0.4,\quad h_4 = 0.2,\quad h_5 = 0.3$

$\Delta_0 = 0.6966133885$ $\qquad Z_0 = -0.09$

$\Delta_1 = 0.5282727342$ $\qquad Z_1 = -0.07$

$\Delta_2 = 0.4451046421$ $\qquad Z_2 = \ \ 0.07$

$\Delta_3 = 0.3853766996$ $\qquad Z_3 = -0.12$

$\Delta_4 = 0.3449643574$ $\qquad Z_4 = \ \ 0.05$

$\Delta_5 = 0.3177005993$

$C_0 = -9.251714$ $\qquad D_0 = 0.470195682$

$C_1 = -5.0007109$ $\qquad D_1 = 0.343321$

$C_2 = 1.795985095$ $\qquad D_2 = 0.0838189905$

$C_3 = 1.262647595$ $\qquad D_3 = -0.2220442785$

$C_4 = -0.304852405$ $\qquad D_4 = -0.0304852405$

$C_5 = 0$ $\qquad D_5 = 0$

$E_0 = 0.3646431136$ $\qquad F_0 = 1.592631499$

$E_1 = 1.054506629$ $\qquad F_1 = 2.28983183$

$E_2 = 2.656635243$ $\qquad F_2 = 2.932629237$

$E_3 = 2.222805443$ $F_3 = 2.523888543$

$E_4 = 5.141827086$ $F_4 = 5.490622624$

$E_5 = 3.653809947$ $F_5 = 3.979741562$

Hence the Natural Quintic Spline$S_i(x)$ are

$S_0(x) = -8.502x^5 + 51.012x^4 - 113.177085x^3 + 108.512336x^2 - 34.487718x - 3.742,$

$x \in [1.2,1.7]$

$S_1(x) = -6.364237x^5 + 32.841016x^4 - 17.388454x^3 - 177.252048x^2 + 376.529917x -$

$226.628369,$ $x \in [1.7,2.1]$

$S_2(x) = 20.877858x^5 - 253.20098x^4 + 1224.768808x^3 - 2955.506005x^2 + 3560.056374x -$

$1712.573708,$ $x \in [2.1, 2.4]$

$S_3(x) = 3.292719x^5 - 42.179313x^4 + 216.131502x^3 - 553.741454x^2 + 709.714287x -$

$363.468389,$ $x \in [2.4, 2.8]$

$S_4(x) = -2.394488x^5 + 37.441582x^4 - 232.880505x^3 + 720.865255x^2 - 1110.961806x +$

$683.662832,$ $x \in [2.8, 3.0]$

$S_5(x) = -1.016175x^5 + 16.766882x^4 - 110.661423x^3 + 365.182696x^2 - 602.225517x +$

$397.802304,$ $x \in [3.0, 3.3]$

4. EVALUATION RESULTS

Interval	**Value of x**	**Actual value of $y = ln\ (x)$**	**Computed value**	***Error = Actual Value – Computed Value***
(1.2, 1.7)	1.5	0.4054651	0.391739	0.013726
(1.7, 2.1)	1.9	0.641853	0.633762	0.0080918
(2.1, 2.4)	2.2	0.788457	0.822591	0.0341336
(2.4, 2.8)	2.5	0.916290	0.913137	0.0031537
(2.8, 3.0)	2.9	1.064710	1.06269	0.0020207
(3.0, 3.3)	3.2	1.16315081	1.161339	0.0018118

5. CONCLUSION

The functional values computed at intervening points in each interval are seen to be close to the corresponding actual values with error as shown in the table.

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SOME FIXED POINT THEOREMS IN Ƒ- METRIC SPACES

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ABSTRACT

In this paper, the existence of fixed point for Kannan's type mapping and using other contraction mappings in a complete F-Metric space is proved.

Keywords: Banach Contraction principle, Kannan's type mapping, Complete metric space, Cauchy Sequence.
Mathematics Subject Classification: 37C25; 47H10; 54H25; 54E50.

1. INTRODUCTION

The theory of metric spaces is the general theory which is used in several branches of mathematical analysis. In recent years, many interesting generalizations of the metric space concept appeared. Czerwik [4] introduced the notion of b-metric space. Gahler [7] introduced the notion of 2-metric space. In 2018, Mohamed Jleli and Bassem Samet [8] introduced a new concept of F-metric space which is generalization of metric space. They also defined a natural topology on this space and studied their topological properties.

The popular Banach contraction principal [3] not only initiated the metric fixed point theory, but also played an important role in the development of nonlinear functional analysis and applied mathematical analysis. Due to the importance and application potential to several quantitative sciences, the generalizations of the Banach contraction principle have been investigated heavily by several authors in various distinct directions [1-12]. In this paper, we obtain fixed point theorems in Ƒ-metric spaces using Kannan's type condition for the self mapping.

Theorem 1.1.[3] (Banach contraction principal). Let (X,d) be complete metric space. Suppose T: $X \to X$ is Banach type contraction; i.e., there exists a non-negative number $\lambda < 1$ such that $d(Tx,Ty) \leq \lambda\, d(x,y)$ for all x,y in X. Then T has a unique fixed point in X.

In 1969, Kannan [9] established fixed point theorem, which is different from Banach contraction principle as follows:

Theorem 1.2.[9]: Let (X,d) be complete metric space. Suppose T: $X \to X$ is Kannan type contraction; i.e., there exists a non-negative number $\lambda < \frac{1}{2}$ such that

$d(Tx,Ty) \leq \lambda\, (d(x,Tx)+d(y,Ty)$ for all x,y in X. Then T has a unique fixed point in X.

2. F -Metric space:

In 2018, Mohamed Jleli and Bassem Samet [8] generalized the concept of metric space as F-metric space as follows:

Definition2.1.[8]: Let F be the set of functions $f:(0,+\infty)\to R$ satisfying the following conditions:

(F_1) f is non-decreasing, i.e., $0<s<t\Rightarrow f(s)\le f(t)$.

(F_2) for every sequence $\{t_n\}\subset(0,+\infty)$, we have

$\lim\limits_{n\to+\infty} t_n=0 \Leftrightarrow \lim\limits_{n\to+\infty} f(t_n)=-\infty$.

Definition2.2. [8]: Let X be a nonempty set and let $D:X\times X\to[0,+\infty)$ be a given mapping. Suppose that there exists $(f,\alpha)\in F\times[0,+\infty)$ such that

(D1) $(x,y)\in X\times X, D(x,y)=0\Leftrightarrow x=y.$

(D2) $D(x,y)=D(y,x), \text{for all}\,(x,y)\in X\times X.$

(D3) For every $(x,y)\in X\times X$, for every natural number $N\ge 2$, and for every $(u_i)_{i=1}^{N}\subset X$ with $(u_1,u_N)=(x,y)$, we have

$$D(x,y)>0\Rightarrow f(D(x,y))\le f\left(\sum_{i=1}^{N-1}D(u_i,u_{i+1})\right)+\alpha.$$

Then D is said to be an F-Metric on X, and the pair (X,D) is said to be an F-Metric space.

Every metric space on X is an F-metric space on X with $f(t)=\text{In}\,t, t>0, \text{and}\,\alpha=0$.

But F-metric space may not be metric space.

Example 2.3.[8]: Let X=N, and let $D:X\times X\to[0,+\infty)$ be the mapping defined by

$$D(x,y)=\begin{cases}(x-y)^2 & \text{if } (x,y)\in[0,3]\times[0,3],\\ |x-y| & \text{if } (x,y)\notin[0,3]\times[0,3],\end{cases}\quad \text{for all } x,y\in X$$

Then (X,D) is an F-metric space but not metric space.

Definition2.4.[8]: Let (X,D) be an F-metric space. A subset O of X is said to be F-open if for every x in O, there is some r>0 such that $B(x,r)\subset O$, where B(x,r)={y∈ X: D(x,y)<r}. And a subset C of X is F-closed if X\C is F- open. Then τF is a topology on X.

Definition 2.5. [8]: Let (X,D) be an F-metric space. Let $\{x_n\}$ be a sequence in X is F- convergent to x in X if $\{x_n\}$ is convergent to x with respect to the topology τF.

Proposition 2.6. [8]:Let (X,D) be an F-metric space. Let $\{x_n\}$ be a sequence in X, and x in X. the following statements are equivalent:

(i) $\{x_n\}$ is F-convergent to x.

(ii) $\lim\limits_{n\to\infty} D(x_n,x)=0.$

Definition2.7. [8]:Let (X,D) be an F-metric space.Let $\{x_n\}$ be a sequence in X.

(i) $\{x_n\}$ is F-Cauchy, if $\lim\limits_{n,m\to\infty} D(x_n,x_m)=0$.

(ii) (X,D) is Ƒ-complete, if every Ƒ-Cauchy sequence in X is Ƒ-convergent to certain element in X.

Mohamed Jleli and Bassem Samet proved Banach contraction principle on Ƒ-metric spaces as:

Theorem2.8. [8]: Let (X,D) be an F- complete metric space, and $T: X \to X$ be a mapping and there exists $\lambda \in (0,1)$ such that $D(Tx,Ty) \le \lambda D(x,y)$ for all x,y in X . then T has a unique fixed point in X.

In this paper, we will establish a new version of fixed point theorems for the Kannan type contraction principal on Ƒ-metric spaces.

3. MAIN RESULTS

Theorem 3.1: Let (X,D) be an F-metric space and let T and $T: X \to X$ be a mapping. Suppose that the following conditions are satisfied:

(i) (X,D) be an F-complete metric space.

(ii) There exists k>2 such that

$$D(Tx,Ty) \le \frac{D(x,Tx)+D(y,Ty)}{k}, \text{for all } x,y \in X. \tag{3.1}$$

Then T has a unique fixed point in X.

Proof: First we prove if T has fixed point, then it will be unique. Suppose T has two fixed points u and v, $u \neq v$ in X i.e., D(u,v)>0 and T(u)=u and T(v)=v.

Then from (ii), we have

$$D(u,v)=D(Tu,Tv) \le \frac{D(u,Tu)+D(v,Tv)}{k} \le 0$$

This implies u=v.

Next, Let $(f,\alpha) \in F \times [0,+\infty)$ be such that (D3) is satisfied. Let $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\varepsilon) - \alpha \tag{3.2}$$

Let x_0 in X be an arbitrary element and let define a sequence $\{x_n\} \subset X$ such that $T(x_n)=x_{n+1}$ for natural number n.

Without restriction of the generally we may assume that $D(x_0,x_1)>0$. Otherwise x_0 will be a fixed point of T.

Now, $D(x_n, x_{n+1}) = D(Tx_{n-1}, Tx_n) \le \dfrac{D(x_{n-1},Tx_{n-1})+D(x_n,Tx_n)}{k}$

$\Rightarrow kD(x_n, x_{n+1}) \le D(x_{n-1},x_n)+D(x_n,x_{n+1})$

$\Rightarrow D(x_n, x_{n+1}) \le \dfrac{1}{k-1} D(x_{n-1},x_n) \le \dfrac{1}{(k-1)^2} D(x_{n-2},x_{n-1}) \le \ldots\ldots\ldots \le \dfrac{1}{(k-1)^n} D(x_0,x_1)$

Take $\lambda = \dfrac{1}{k-1}$. Thus $\lambda<1$.

Thus we have $D(x_n, x_{n+1}) \le \lambda^n D(x_0,x_1)$

Also, $\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \le \frac{\lambda^n}{1-\lambda} D(x_0,x_1)$,for m>n.

Since, $\lim_{n\to\infty} \frac{\lambda^n}{1-\lambda} D(x_0,x_1)=0$, there exists some natural number N such that

$$0<\frac{\lambda^n}{1-\lambda} D(x_0, x_1) < \delta, \text{for all } n \ge N \tag{3.3}$$

Hence by (3.2) and (Ƒ1),we get

$$f\left(\sum_{i=n}^{m-1} D(x_i, x_{i+1}\right) \le f\left(\frac{\lambda^n}{1-\lambda} D(x_0, x_1\right) < f(\varepsilon) - \alpha. \ m,n \ge N \tag{3.4}$$

Using (D3) and (3.4), we obtain

$D(x_n,x_m)>0, m>n \ge N$

$\Rightarrow f(D(x_n, x_m) \le f\left(\sum_{i=n}^{m-1} D(x_i, x_{i+1})\right) + \alpha < f(\varepsilon)$

Which implies by (Ƒ1) that

$D(x_n,x_m)<\varepsilon$, $m>n \ge N$

This proves that $\{x_n\}$ is Ƒ-Cauchy sequence. Since (X,D) is Ƒ-complete, there exist x* in X such that $\{x_n\}$ is Ƒ-convergent to x* i.e.,

$\lim_{n\to\infty} D(x_n, x^*) = 0$.

Now, we shall prove that x* is a fixed point of T. we argue by contradiction by supposing that $D(Tx^*,x^*)>0$.

Now, $D(Tx^*,x_{n+1})=D(Tx^*,Tx_n) \le \frac{D(x^*, Tx^*) + D(x_n, Tx_n)}{k}$

$\Rightarrow k\, D(Tx^*,x_{n+1}) \le D(x^*,Tx^*)+D(x_n,x_{n+1}) \le D(x^*,Tx^*)+ \lambda^n D(x_0,x_1)$

Applying limit $n \to \infty$, we get

$k\, D(Tx^*,x^*) \le D(x^*,Tx^*)$

this implies $Tx^*=x^*$.

as a consequence x* in X is unique fixed point of T.

Theorem 3.2: Let (X,D) be an F-metric space and let and T: $X \to X$ be a mapping. Suppose that the following conditions are satisfied:

(i) (X,D) be an F-complete metric space.

(ii) There exists k>2 such that

$$D(Tx, Ty) \le \frac{D(x, Tx) + D(y, Tx)}{k}, \text{for all } x, y \in X. \tag{3.5}$$

Then T has a unique fixed point in X.

Proof: First we prove if T has fixed point, then it will be unique. Suppose T has two fixed points u and v in X i.e., T(u)=u and T(v)=v.

Then from (ii), we have

$$D(u,v)=D(Tu, Tv)\le \frac{D(u,Tu)+D(v,Tu)}{k}$$

$$\Rightarrow (k-1)D(u,v)\le 0$$

This implies u=v.

Let x_0 in X be an arbitrary element and let define a sequence $\{x_n\}\subset X$ such that $T(x_n)=x_{n+1}$ for natural number n.

Without restriction of the generally we may assume that $D(x_0,x_1)>0$. Otherwise x_0 will be a fixed point of T.

Now, $D(x_n, x_{n+1}) = D(Tx_{n-1}, Tx_n)\le \dfrac{D(x_{n-1},Tx_{n-1})+D(x_n,Tx_{n-1})}{k}$

$$\Rightarrow D(x_n, x_{n+1})\le \frac{1}{k} D(x_{n-1},x_n) \le \frac{1}{k^2}D(x_{n-2},x_{n-1}) \le \ldots\ldots\ldots\le \frac{1}{k^n}D(x_0,x_1)$$

Take $\lambda=\dfrac{1}{k}$. Thus $\lambda<1/2$.

Thus we have $D(x_n, x_{n+1}) \le \lambda^n D(x_0,x_1)$

Also, $\sum_{i=n}^{m-1}D(x_i,x_{i+1})\le \dfrac{\lambda^n}{1-\lambda} D(x_0,x_1)$,for m>n.

Since, $\lim_{n\to\infty}\dfrac{\lambda^n}{1-\lambda} D(x_0,x_1)=0$, there exists some natural number N such that

$$0<\frac{\lambda^n}{1-\lambda}D(x_0,x_1)<\delta,\text{for all}\, n\ge N \tag{3.6}$$

Hence by (3.2) and (Ƒ1),we get

$$f\left(\sum_{i=n}^{m-1}D(x_i,x_{i+1}\right)\le f\left(\frac{\lambda^n}{1-\lambda}D(x_0,x_1\right)<f(\varepsilon)-\alpha.\ m,n\ge N \tag{3.7}$$

Using (D3) and (3.7), we obtain

$D(x_n,x_m)>0, m>n\ge N$

$$\Rightarrow f(D(x_n,x_m)\le f\left(\sum_{i=n}^{m-1}D(x_i,x_{i+1})\right)+\alpha<f(\varepsilon)$$

Which implies by (Ƒ1) that

$D(x_n,x_m)<\varepsilon$, $m>n\ge N$

This proves that $\{x_n\}$ is Ƒ-Cauchy sequence. Since (X,D) is Ƒ-complete, there exist x* in X such that $\{x_n\}$ is Ƒ-convergent to x* i.e.,

$$\lim_{n\to\infty}D(x_n,x^*)=0.$$

Now, we shall prove that x* is a fixed point of T. we argue by contradiction by supposing that

$D(Tx^*,x^*)>0$

Now, $D(Tx^*,x_{n+1})=D(Tx^*,Tx_n) \le \frac{D(x^*,Tx^*)+D(x_n,Tx^*)}{k}$

$\Rightarrow k\, D(Tx^*,x_{n+1}) \le D(x^*,Tx^*)+D(x_n,Tx^*)$

Applying limit $n \to \infty$, we get

$k\, D(Tx^*,x^*) \le 2D(x^*,Tx^*)$

this implies $Tx^*=x^*$.

as a consequence x^* in X is unique fixed point of T.

Theorem 3.3: Let (X,D) be an F-metric space and let and T: $X \to X$ be a mapping. Suppose that the following conditions are satisfied:

(i) (X,D) be an F-complete metric space.

(ii) There exists k>2 such that

$$D(Tx,Ty) \le \frac{D(x,Tx)+D(y,Ty)+D(y,Tx)}{k}, \text{for all } x,y \in X. \tag{3.8}$$

Then T has a unique fixed point in X.

Proof: First we prove if T has fixed point, then it will be unique. Suppose T has two fixed points u and v in X i.e., T(u)=u and T(v)=v.

Then from (ii), we have

$D(u,v)=D(Tu,Tv) \le \frac{D(u,Tu)+D(v,Tv)+D(v,Tu)}{k} \le 0$

$\Rightarrow (k-1)\, D(u,v) \le 0$

This implies u=v.

Let x_0 in X be an arbitrary element and let define a sequence $\{x_n\} \subset X$ such that $T(x_n)=x_{n+1}$ for natural number n.

Without restriction of the generally we may assume that $D(x_0,x_1)>0$. Otherwise x_0 will be a fixed point of T.

Now, $D(x_n, x_{n+1}) = D(Tx_{n-1}, Tx_n) \le \frac{D(x_{n-1},Tx_{n-1})+D(x_n,Tx_n)+D(x_n,Tx_{n-1})}{k}$

$\Rightarrow k\, D(x_n, x_{n+1}) \le D(x_{n-1},x_n)+D(x_n,x_{n+1})$

$\Rightarrow D(x_n, x_{n+1}) \le \frac{1}{k-1} D(x_{n-1},x_n) \le \frac{1}{(k-1)^2} D(x_{n-2},x_{n-1}) \le \ldots\ldots\ldots \le \frac{1}{(k-1)^n} D(x_0,x_1)$

Take $\lambda = \frac{1}{k-1}$. Thus $\lambda<1$.

Thus we have $D(x_n, x_{n+1}) \le \lambda^n\, D(x_0,x_1)$

Also, $\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \le \frac{\lambda^n}{1-\lambda} D(x_0,x_1)$, for m>n.

Since, $\lim_{n\to\infty}\frac{\lambda^n}{1-\lambda}$ D(x_0,x_1)=0, there exists some natural number N such that

$$0<\frac{\lambda^n}{1-\lambda}D(x_0,x_1)<\delta, \text{for all } n\geq N \tag{3.9}$$

Hence by (3.2) and (Ƒ1),we get

$$f\left(\sum_{i=n}^{m-1}D(x_i,x_{i+1}\right)\leq f\left(\frac{\lambda^n}{1-\lambda}D(x_0,x_1\right)<f(\varepsilon)-\alpha, \qquad \text{for all } m,n\geq N \tag{3.10}$$

Using (D3) and (3.10), we obtain

$D(x_n,x_m)>0, m>n\geq N$

$$\Rightarrow f(D(x_n,x_m)\leq f\left(\sum_{i=n}^{m-1}D(x_i,x_{i+1})\right)+\alpha<f(\varepsilon)$$

Which implies by (Ƒ1) that

$D(x_n,x_m)<\varepsilon$, $m>n\geq N$

This proves that $\{x_n\}$ is Ƒ-Cauchy sequence. Since (X,D) is Ƒ-complete, there exist x* in X such that $\{x_n\}$ is Ƒ-convergent to x* i.e.,

$\lim_{n\to\infty}D(x_n,x^*)=0$.

Now, we shall prove that x* is a fixed point of T. we argue by contradiction by supposing that

D(Tx*, x*)>0

Now, $D(Tx^*,x_{n+1})=D(Tx^*,Tx_n)\leq\frac{D(x^*,Tx^*)+D(x_n,Tx_n)+D(x_n,Tx^*)}{k}$

$\Rightarrow k\, D(Tx^*,x_{n+1})\leq D(x^*,Tx^*)+D(x_n,x_{n+1})+D(x_n,Tx^*)\leq D(x^*,Tx^*)+\lambda^n D(x_0,x_1)+D(x_n,Tx^*)$

Applying limit $n\to\infty$, we get

$k\, D(Tx^*,x^*)\leq 2D(x^*,Tx^*)$

this implies Tx*=x*.

as a consequence x* in X is unique fixed point of T.

Theorem 3.4: Let (X,D) be an F-metric space and let and T: $X\to X$ be a mapping. Suppose that the following conditions are satisfied:

(i) (X,D) be an F-complete metric space.

(ii) There exists k>2 such that

$D(Tx,Ty)\leq\frac{1}{k}\max\{D(x,Tx)+D(y,Ty), D(x,Tx)+D(y,Tx), D(x,Tx)+D(y,Ty)+D(y,Tx)\}$. Then T has a unique fixed point in X.

Proof of the theorem is consequence of above theorems.

CONCLUSION

In this paper, we obtain fixed point theorems using Kannan's condition in F-metric space which is introduced by Mohamed Jleli and Bassem Samet and proved Banach contraction mapping.

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