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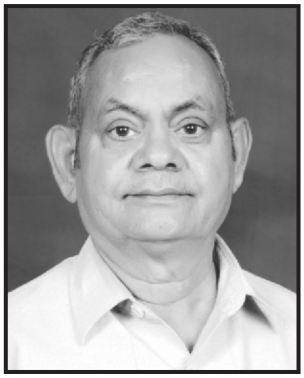
## CONTENTS



**Dr. T.P. Singh**
Chief Editor & Professor in Maths & O.R.

# FOREWORD

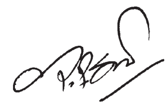
It gives me an immense pleasure in writing this foreword for **'Aryabhatta Journal of Mathematics & Informatics' (AJMI) Vol.13 issue 2 July-Dec. 2021** published by Aryans Research & Educational Trust. The first issue of the journal was published in year 2009.Since then & till today the journal publication is regular & well in time. It gives us a great pleasure to put forward before the scholars and researchers that journal from its start is making an effort to produce good quality articles. The credit goes to its reviewer team which review sincerely and furnish valuable suggestions to improve the quality of papers. The Journal covers areas of mathematical and statistical sciences, Operational Research, data based managerial ,economical issues and information sciences.

**AJMI VOL.13 Issue 2** is before you. I am pleased to note that research scholars, professors, executives, from different parts of country and abroad have sent their papers for this issue. The papers are relevant and focus on the futuristic trends and innovations in the related areas. We have received around **38 papers** for this issue from which on reviewer's report only **15** have been selected for publication. **DOI** no. by cross Ref. have been mentioned on each article.

1. Prof. Temur Z. Kalanov a mathematical philosopher from Tashkent, Uzbekistan proposed critical arguments that show scientifically impossibility of complete axiomatization of geometry within the frame work of correct methodological basis - the unity of formal logic and rational dialectics
2. Prof. Manimannan, Paranjothi, LaxmiPriyaetal.conducted a cross-sectional survey to assess prevalence of CVD risk factor among different respondent of suburban of Chennai. The results were carried out through descriptive statistics and one way analysis of Variance ( ANOVA).
3. Prof. DevanandMallayya search out the mathematical achievements made out by ancient mathematician of India Sangamagrama Madhav's and his efficiacy of end correction technique.
4. Prof. T.P.Singh & his scholar Harish explored a Buyer- Vendor fuzzy Inventory model and carried out the sensitivity analysis between crisp and fuzzy parameters.
5. In paper no.8 Prof. Madhukar expounded the ancient Indian mathematician Sankara's contribution to geometric series and made an attempt to generalize it.
6. Prof. Sandeep Khare established some new Trilateral generating relations involving I-function of one variable.
7. Dr Sapna Nagar discussed a mathematical model for a nuclear power plant for the performance measure in terms of evaluation of Reliability and Availability.
8. Dr. Arun from Jubail Industrial College Saudi Arabia applied Robust Ranking Technique in Fuzzy assignment problem for allocation of subjects in a department.
9. Prof. Anuj etal. Proposed Antenna structure helpful to minimize the potential interferences between the UWB system and the narrow band systems with reconfigurable features.
10. Dr. Smita proposed her paper related with graph theory and made an effort to show that the super division of cycle admit H-covering and also prove that it has (super) (a, d) - H-antimagic total graph.

I would like to thank and felicitate the contributors in this issue and at the same time invite quality papers from academic and research community for Vol.14 issue 1 to be published in May 2022.

Comments, suggestions and feedback from discerning readers, scholars and academicians are always welcome.



**DR. T.P. SINGH**

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# ON THE PROBLEM OF AXIOMATIZATION OF GEOMETRY

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## ABSTRACT
***An analysis of the foundations of geometry within the framework of the correct methodological basis - the unity of formal logic and rational dialectics - is proposed. The analysis leads to the following result: (1) geometry is an engineering science, but not a field of mathematics; (2) the essence of geometry is the construction of material figures (systems) and study of their properties; (3) the starting point of geometry is the following system principle: the properties of material figures (systems) determine the properties of the elements of figures; the properties of elements characterize the properties of figures (systems); (4) the axiomatization of geometry is a way of construction of the science as a set (system) of practical principles. Sets (systems) of practice principles can be complete or incomplete; (5) the book, "The Foundations of Geometry" by David Hilbert, represents a methodologically incorrect work. It does not satisfy the dialectical principle of cognition, "practice  theory practice," because practice is not the starting point and final point in Hilbert's theoretical approach (analysis). Hilbert did not understand that: (a) scientific intuition must be based on practical experience; intuition that is not based on practical experience is fantasy; (b) the correct science does not exist without definitions of concepts; the definitions of geometric concepts are the genetic (technological) definitions that shows how given material objects arise (i.e., how a person creates given material objects); (c) the theory must be constructed within the framework of the correct methodological basis: the unity of formal logic and rational dialectics. (d) the theory must satisfy the correct criterion of truth: the unity of formal logic and rational dialectics. Therefore, Hilbert cannot prove the theorem of trisection of angle and the theorem of sum of interior angles (concluded angles) of triangle on the basis of his axioms. This fact signifies that Hilbert's system of axioms is incomplete. In essence, Hilbert's work is a superficial, tautological and logically incorrect verbal description of Figures 1-52 in his work.***

***Keywords: geometry, engineering, applied mathematics, general mathematics, methodology of mathematics, philosophy of mathematics, history of mathematics, higher education, formal logic, dialectics, epistemology.***
***MSC: 00A05, 00A30, 00A35, 00A69, 03B80, 00A06, 00A69, 51A05, 51B05, 51D10, 52A38, 52C35, 53A04, 53A05.***

## 1. INTRODUCTION

As is known, science and technology are developed in an inductive way. This means that new scientific knowledge is not a consequence of old scientific knowledge. New knowledge is a guess (discovery). There is an epistemological principle (for example, in mathematics and physics), which states that the relation between old scientific knowledge and new scientific knowledge must satisfy the following condition: old knowledge must be a consequence of new knowledge. This condition would be an expression of the dialectical law of “negation of negation” if this condition had the following logical formulation: old correct knowledge should be a consequence of new correct knowledge. Correct knowledge is achieved within the framework of correct methodological basis and, therefore, satisfies the correct criterion of truth.

As is shown in my works (for example, [1-33]), the foundations of theoretical physics and mathematics do not satisfy the correct criterion of truth: the unity of formal logic and rational dialectics. The unity of formal logic and rational dialectics is the correct methodological basis of science. (In other words, the concepts "correct methodological basis" and "correct criterion of truth" are identical concepts). But scientists, as the analysis of the literature shows, ignore the correct methodological basis. The purpose of this work is to propose the arguments that show scientifically impossibility of complete axiomatization of geometry within the framework of correct methodological basis: the unity of formal logic and rational dialectics.

### 1.1 The essence of geometry

1) As is known, all material objects are bounded objects [34]. Bounded material objects have set of properties: for example, physical, chemical, geometric properties. The theoretical study of the properties of bounded material objects is carried out using applied mathematics [35]. Applied mathematics operates with quantities that represent the measure: the unity of the qualitative and quantitative determinacy of a material object. Therefore, the quantities in applied mathematics have dimensions. Unlike applied mathematics, pure mathematics operates with mathematical quantities that have only quantitative determinacy. The quantities in pure mathematics do not have dimensions: the values of mathematical quantity in pure mathematics are unnamed (abstract) numbers. Applied mathematics and pure mathematics are the science of operations with numbers.

2) Geometry studies the geometric properties of rigid (solid) and flexible bounded material bodies [35]. As is known, unbounded material bodies do not exist in reality. The geometric property of rigid (solid) and flexible bounded material bodies is manifested in the extension of material objects. The measurable (measurand) length is characterized by the dimension "meter". The dimension "meter" denotes the unity of qualitative and quantitative determinacy. As is known, a bounded material body (or part of a body) has only three extensions (three dimensions): length, width, and height. The relationship between the length, width and height of the body is called the form of the body (or the form of the part of the body). In the point of view of geometry, a form is the only essential property of a material body (or part of body). In this case, the bounded material body is called a geometric body, i.e., a geometric figure. A given geometric body (figure) can be decomposed into its component parts (fragments). Then the component parts (fragments) can be connected, combined, forming the original body. This means that the study of the form is carried out by means of geometric (material) constructions and decompositions (dissections) using applied mathematics (i.e., using real numbers) [34]. But the construction and decomposition of a form are not mathematical operations.

3) The description of the form of a bounded material body is carried out within the framework of geometric models of rigid (solid) and flexible bodies. A geometric model of a bounded material body is a system (figure) consisting of the following material elements: points, bounded lines and bounded surfaces, which are elementary models of bounded material bodies (or parts of bodies). The concepts of a

point, a bounded line and a bounded surface are as follows. A point is an elementary geometric model of a bounded body (or part of body), the three dimensions of which can be neglected (for example: material point in physics, material point drawn with a pencil or paint, the mark, rivet, the notch, the weld, welded joint, hinge, bolt, universal joint). A bounded line is an elementary geometric model of a bounded body (or part of body), two dimensions of which can be neglected (for example: material line drawn with a pencil, chalk or compasses, bar, reinforcement bar, cord, cable, string). A bounded surface is an elementary geometric model of a bounded body (or part of body), one size of which can be neglected (for example: paper, carpet, sheet, board, metal plate). Each elementary geometric model is the unity of qualitative and quantitative determinacy. Thus, the difference between the geometric models of bodies is established by comparing the qualitative and quantitative determinacy of these models. Measurement results are expressed in numbers. Therefore, comparison of sizes (i.e., numbers) is a mathematical (quantitative) operation, the result of which is indicated by the following symbols: "$=, \approx, \neq$", "$>, \geq, >>$", "$<, \leq, <<$".

4) A point, a bounded line and a bounded surface are mutually independent (free) material elements [34]. The independent existences of elements – a material point $p$, a bounded straight line $a$, and a bounded plane $\alpha$ – are defined graphically as follows (Figure 1).

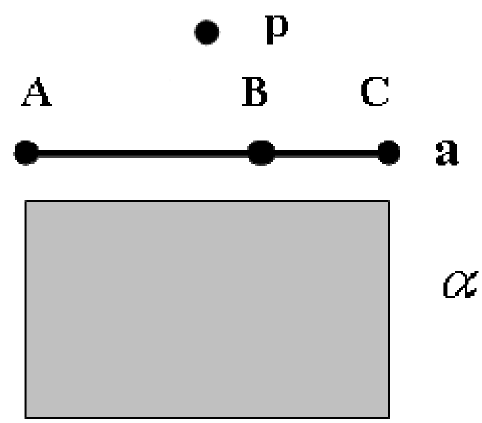


**Figure 1**. Graphic (material) definition of the following independent geometric elements: a point $p$, a bounded straight line $a$ and a bounded plane $\alpha$.

This means that the existence, position and properties (as the unity of qualitative and quantitative determinacy) of any element do not depend on the existence, position and properties of other elements. In particular, mutual movements (changes) of free elements are independent.

The independence of the existence of elements – a material point $p$, a bounded straight line $a$ and a bounded plane $\alpha$ – means that each element is not genetically determined by other elements. For example, a bounded straight line $a$ can be made (drawn on paper or board) with a pencil or a piece of chalk. But points (on paper or blackboard) or pieces of chalk do not genetically determine (define) a line. Similarly, a bounded straight line $a$ and a point $p$ not lying on $a$ do not determine (define) genetically bounded plane $\alpha$.

5) Connection between the elements puts restraints (limitations) on mutual positions and movements of the elements (Figure 2).

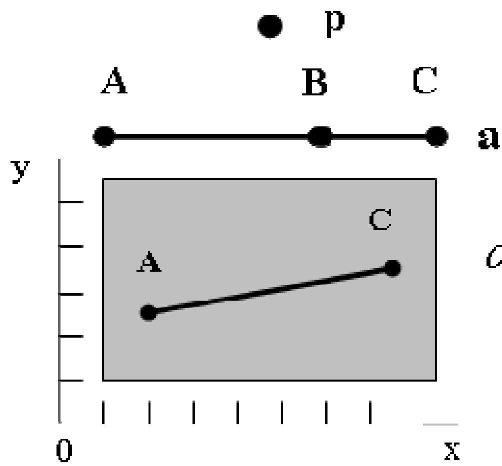


**Figure 2**. Graphical (material) definition of the following geometric elements: a point $p$, a bounded straight line $a$, a bounded plane $\alpha$. The $XOY$ coordinate system is connected with the plane $\alpha$. The $XOY$ coordinate system defines the positions of the plane $\alpha$ and the segment of the straight line $a$ lying on the plane.

For example, if any material point is fixed (i.e., if it exists) on the material plane, then a bounded material straight line can be fixed on the plane by this point [34]. If this point does not coincide with the end point of the bounded straight line, then the bounded straight line exists on the plane. "Plane + bounded straight line" is a system. The bounded straight line in the system "plane + bounded straight line" can have one degree of freedom of movement: rotation around the point of fixation. But if the bounded straight line is rigidly connected with the plane by two points (for example, nails), then the bounded straight line have no degree of freedom of movement (Figure 2).

6) All free material points are identical. If a material point $p$ is not fixed on a bounded straight line $a$ or on a bounded plane $\alpha$, then the point $p$ does not exist (does not belong) on the bounded straight line $a$ or on the bounded plane $\alpha$. In this case, the material point $p$ have no a name (designation), because a free point does not designate (does not characterize) a place. Place exists only on a bounded straight line and on a bounded surface. If a material point $p$ is fixed (belongs) on a bounded straight line $a$ or on a bounded plane $\alpha$, then the point $p$ has a name (designation), because this point names (designates, characterizes) a certain place on the bounded straight line $a$ or on the bounded plane $\alpha$. Letters $A, B, C$ denote (name, mark) identical points $p$ situated (fixed) in different (distinct) places on the bounded straight line $a$ or on the bounded plane $\alpha$. Therefore, these points $p$ have different names (designations), which characterize the different positions (places) of the points $p$. But, for example, the relationship $A = B$, expressing the identity of different (distinct) places, is a formal-logical error: the relationship $A = B$ contradicts to the formal-logical law of the lack of contradiction, $A \neq B$.

Points $A$ and $C$ belong to the bounded straight line $a$ and are called endpoints of the bounded straight line $a$ (Figures 1 and 2). The positions (places) of points on a bounded straight line are determined using a ruler or compass. A ruler or compass determines (measures) the distance between endpoints $A$ and $C$ as well as the distance of point $B$ from the endpoints of the line $a$.

7) Point $B$ is between points $A$ and $C$ (Figures 1 and 2) if the following mathematical (quantitative) relationships are satisfied:

$$d_{AB}^{(a)} < d_{AC}^{(a)}, \quad d_{BC}^{(a)} < d_{AC}^{(a)}, \quad d_{AB}^{(a)} + d_{BC}^{(a)} = d_{AC}^{(a)}$$

where $d$ is the length (distance); $d_{AB}^{(a)}$, $d_{BC}^{(a)}$ and $d_{AC}^{(a)}$ are the lengths of line segments $\overline{AB}$, $\overline{BC}$ and line $\overline{AC}$, respectively.

If the segments $\overline{AB}$ and $\overline{BC}$ form the angle $\varphi = \angle\left(\overline{BA}, \overline{BC}\right)$, $0^\circ < \varphi < 180^\circ$, then the bounded straight line $a$ is called a broken line.

If there exist (fasten, fix), for example, points $K$, $L$, $M$ on a bounded straight line $a$, then the order of arrangement (situation, disposition) of these points is determined by the following mathematical (quantitative) relationships:

$$d_{AK}^{(a)} < d_{AL}^{(a)} < d_{AM}^{(a)}$$

where $d_{AK}^{(a)}$, $d_{AL}^{(a)}$ and $d_{AM}^{(a)}$ are the lengths of lines $\overline{AK}$, $\overline{AL}$, and $\overline{AM}$, respectively.

If there exist two independent (free) bounded straight lines $a$ and $b$, then these lines can be connected using a material point $p$. In this case, the material point denotes the place of connection (intersection) of the lines and has the name $P$. Point $P$ is the vertex of the angles formed by the lines $a$ and $b$.

If bounded straight lines $a$ and $b$ are fixed by points on the bounded material plane $\alpha$, then there are two following mutual positions of the lines $a$ and $b$: the position of parallelism of the lines, and the position of non-parallelism of the lines [34]. Lines $a$ and $b$ are called parallel if these lines are equidistant lines, i.e., if the distance $d_{a,b}$ between the lines $a$ and $b$ is constant at any points of these lines: $d_{a,b} = const$. If the distance $d_{a,b}$ between the lines $a$ and $b$ is not constant at any point of these lines, then the lines are not parallel. This definition of parallel lines is a precise (exact, accurate, rigorous, strict) and correct definition based on the use of surface gage. (In other words, the genetic definition of the parallelism of lines is the following: a line $b$ is called a parallel to line $a$ if the line $b$ at any point is the equidistant line generated by surface gage).

Also, an example of equidistant curved lines is concentric circles.

If the bounded straight line $a$ is fixed on the bounded plane $\alpha$ by points $A$ and $C$, then the position of the line $a$ on the bounded plane $\alpha$ is determined by the positions of points $A$ and $C$ (Figure 2). And the positions of points $A$ and $C$ are determined by the system of rulers (i.e., the $XOY$ coordinate system) connected to the bounded plane $\alpha$. But the bounded plane $\alpha$ and the position of the bounded plane $\alpha$ are not defined by a line $a$ and a point not lying on the line $a$.

8) The connection (relation) between elements can characterize the essential properties (features) of elements.

(a) The property of a bounded straight line can be expressed as follows: any point of a bounded straight line is equidistant from the two fixed poles (explanation: fixed pole points do not belong to this bounded straight line). This property is an essential feature of a bounded straight line. Therefore, the correct definition of a bounded straight line is the following: a bounded line is called a bounded straight line if any point of the bounded line is equidistant from the two fixed poles that do not belong to this bounded line.

(b) The correct definition of a bounded plane is the following: a bounded surface is called a bounded plane if this bounded surface has the following essential property (essential feature): any bounded straight line lies (are) on this bounded surface (i.e., any point of any bounded straight line is on this bounded surface) (Figure 2).

(c) The correct definition of the three-dimensional coordinate system $XOYZ$ is the following: a system of three bounded planes is called a three-dimensional coordinate system $XOYZ$ if this system has the following constructive property (essential feature): the system consists of three intersecting material right-angled planes $XOY$, $XOZ$, and $YOZ$ with metric rulers; $O$ is the point of intersection of the planes $XOY$, $XOZ$, and $YOZ$; lines of intersections of the planes are intersecting bounded straight lines; the angles between intersecting straight lines are $90^{\circ}$. The essential (measuring, informational) property of the coordinate system $XOYZ$ is the following: if a material object is in the coordinate system $XOYZ$, then the set of positions (coordinates) of the material object in the coordinate system $XOYZ$ determines (and is called) the geometric space of this material object in the material rectangular (Cartesian) coordinate system $XOYZ$.

9) The concepts of a point, a bounded straight line, a segment of a bounded straight line and a bounded surface are geometric concepts. The definitions of geometric concepts are a genetic (technological) definitions that show how a given material objects arise (i.e., how a person creates given material objects). In particular, a point, a bounded straight line, a segment of a bounded straight line and a bounded surface drawn on paper are a material manifestation (expression) of the idea of a point, a bounded straight line, a segment of a bounded straight line, and a bounded surface. In other words, the drawn geometric objects are material objects that exist as a materialization (material manifestation) of the idea.

For example, a drawing (ornament) on a carpet is the materialization of an idea that exists in the designer's head. Complicated composition can only be created with the help of templates, because the compasses and the ruler are inadequate tools. Consequently, definitions of geometric concepts are not mathematical (quantitative, numerical) definitions. (It must be emphasized that the process of scientific thinking (abstract thinking, logical thinking) always relies on visual (material) images (i.e., on sensibly perceived material) and leads to the formulation (construction) of concepts, propositions, systems of propositions, and theories. Thinking (thought) that relies on perceivable material is called intuition).

10) Geometric elements can be concatenated (combined, connected) to each other forming a geometric system (points can be connecting elements). A geometric system can or cannot have degrees of freedom of structural movement (subject to the properties of the connecting elements [54]). (The structural movement of a given system is a change in the form of a given system without destroying this system [34]. This change carried out by man). The connection of geometric elements is manifested (reflected) in human thinking as a connection (system) of concepts.

The geometric model of a material body as a system composed (constructed) of material elements obeys to the system principle: the properties of the system determine the properties of the elements of the system; properties of elements of system characterize the properties of the system [34]. The system principle is a concretization of the laws of dialectics.

11) Definitions of geometric spaces.

(a) The set of positions (places) of a point $p$ on a bounded straight line $a$ is called the geometric space (geometric states) of a point $p$ on a bounded line $a$ .
(b) The set of positions (places) of a point $p$ on a bounded plane $\alpha$ is called the geometric space (geometric states) of a point $p$ on a bounded plane $\alpha$.
(c) The set of positions (places) of a material point $p$ in the bounded coordinate system $XOYZ$ is called the geometric space (geometric states) of a material point $p$ in the bounded coordinate system $XOYZ$.
(d) The set of positions (places) of a bounded straight line $a$ on a bounded plane $\alpha$ is called the geometric space (geometric states) of a bounded straight line $a$ on a bounded plane $\alpha$.
(e) The set of positions (places) of a bounded line $a$ in a bounded coordinate system $XOYZ$ is called the geometric space (geometric states) of a bounded line $a$ in a bounded coordinate system $XOYZ$.
(f) The set of positions (places) of a bounded plane $\alpha$ in a bounded coordinate system $XOYZ$ is called a geometric space (geometric states) of a bounded plane $\alpha$ in a bounded coordinate system $XOYZ$.
(g) Unbounded planes cannot intersect one another. Bounded planes can intersect one another and have either one common point (Figure 3) or set of common points (Figure 4):

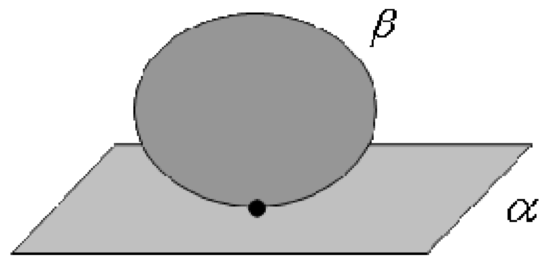


**Figure 3**. Intersection of bounded planes $\alpha$ and $\beta$ at one point.

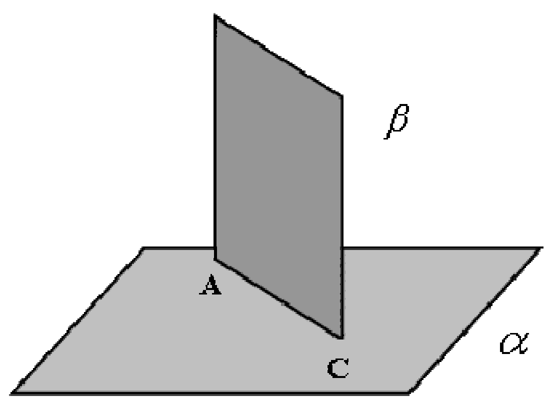


**Figure 4.** Intersection of bounded planes $\alpha$ and $\beta$ in line $AC$ containing set of points.

In this case, the set of positions of intersecting planes $\alpha$ and $\beta$ in the bounded $XOYZ$ coordinate system is called the geometric space (geometric states) of intersecting planes $\alpha$ and $\beta$ in the bounded $XOYZ$ coordinate system.

## 2. THE ESSENCE OF THE AXIOMS OF GEOMETRY

1) The definition of the axiom is as follows. "An axiom is: (a) a scientific assertion (statement) that is accepted without logical proof; (b) an obvious, convincing and true starting point of the theory" (Russian

Wikipedia). In other words, an axiom is a verbal expression of an empirical fact. In the point of view of formal logic, an axiom is a proposition (a system of concepts) based on practical experience. Therefore, an axiom is a true proposition. The explanation of the axiom represents a theorem and a theory. Examples of axioms are the following: "All men are mortal. Socrates is a man. Hence, Socrates is mortal", "Day is replaced by night (i.e., there is a cyclical change in day and night)", "Summer is replaced by winter (i.e., there is a cyclical change in seasons)".

A theorem is a statement that is based on logical proof. Usually, a theorem contains a condition and a conclusion.

2) In the point of view of formal logic, the axioms and theorems of geometry are systems of concepts. Definitions of geometric concepts are genetic (technological) definitions of concepts and systems of concepts (i.e., genetic definitions of elements and of properties of a geometric figure). But the properties of the elements do not determine the properties of the figure. The properties of a given figure can be determined if and only if one constructs the given figure as a system of certain material elements. Determination of the list of these material elements consists, first of all, in the decomposition of a given figure into a set of concrete (specific) elements. (Set of concrete (specific) elements can be created using templates (gauges) and other complex devices (gadgets), because a compass and a ruler are inadequate tools in a general case). This set of concrete (specific) elements characterizes the given figure, but not an arbitrary figure. That is why a set of axioms that does not contain a genetic definition of a given figure is always an incomplete set.

In this point of view, the starting point of geometry should be chosen as follows: (a) the axiom of the existence of geometric elements and figures: geometric elements and figures exist if they can be genetically determined as material objects; (b) the axiom of the identity of geometric figures: two geometric figures are called identical (congruent) to each other if they are copies of each other.

## 3. FORMAL LOGIC AS THE METHODOLOGICAL BASIS FOR CONSTRUCTION OF AXIOMS

1) By definition, formal logic is the science of the laws of correct thinking. The starting point and fundamental element of formal logic is a concept. A concept is a form of thought that expresses the essential features of objects and phenomena. A concept is expressed in a word or in several words (grammatical sentences). Concepts (thoughts) cannot be expressed without words and grammatical sentences. Concepts have material nature.

2) The basis of formal logic is a system of concepts. The connection of concepts forms the structure of the system. The connection of concepts is expressed by the following words: "is", "is not", "if… is…, then…", "if… is not…, then…", "consequently".

3) Proposition as a logical form of verbal expression (utterance) of thought is the essence of formal logic. The definition of proposition is the following: proposition is a statement (i.e., the act of thinking and verbal expression of thought) about the existence or non-existence of an object or phenomenon; proposition is a statement about the properties of an object or phenomenon of reality; proposition is expressed in the statement of the existence or absence of certain features of objects and phenomena. A proposition connects concepts that logically express objects. There exist no true propositions that connect concepts without objects. Also, there are no true propositions that connect objects without concepts of objects (in this case, the connection between objects is not a logical connection!). Therefore, a proposition

has the following two properties: (a) the property of assertion or negation; (b) the property of truth or false. This property is expressed in the following words: “truth” or “false”.

4) The connection (combination) of propositions which represents deriving (extracting) a new proposition from one or more propositions is called inference. The new proposition is called a conclusion (in Latin: conclusio). Those propositions from which a new proposition is derived (extracted, follows) are called premises (in Latin: praemissae). The relation between premises and conclusion is the relation between cause and effect. Inference is based on the law of sufficient reason.

5) Inferences are divided into the following two groups: direct inferences and mediated inferences. If a conclusion (proposition) is made from only one premise (proposition), then the inference is called direct inference. If a conclusion (proposition) is made from several premises (propositions), then the inference is called mediated inference.

## 4. THE ESSENCE OF DAVID HILBERT’S AXIOMS

In the conventional point of view, “the axioms are the foundation for a modern treatment of Euclidean geometry. Well-known modern axiomatizations of Euclidean geometry are those of David Hilbert, of Alfred Tarski and of George Birkhoff ”(Wikipedia). The axioms of David Hilbert, of Alfred Tarski and of George Birkhoff do not contradict to one another and, therefore, are in effect identical [36-47]. The axioms are the followings: Incidence; Order; Congruence; Parallels; Continuity.

The essence of the axioms can be understood by the example of the work of David Hilbert [56]. “Hilbert’s axioms are a set of 20 assumptions proposed by David Hilbert in 1899 in his book “Grundlagen der Geometrie” (“The Foundations of Geometry”). Hilbert’s set of axioms is constructed with six primitive notions: three primitive terms: point; line; plane; - and three primitive relations:

(a) betweenness (a ternary relation linking points);

(b) lies on (three binary relations, one linking points and straight lines, one linking points and planes, and one linking straight lines and planes);

(c) congruence (two binary relations, one linking line segments and one linking angles).

Line segments, angles, and triangles may each be defined in terms of points and straight lines, using the relations of betweenness and containment. All points, straight lines, and planes in the axioms are distinct” (Wikipedia).

The essence of the work “The Foundations of Geometry” by David Hilbert is the following [36].

“Contents.

**Fig. 1.**

II, 2. If $A$ and $C$ are two points of a straight line, then there exists at least one point $B$ lying between $A$ and $C$ and at least one point $D$ so situated that $C$ lies between $A$ and $D$. Fig. 2.

II, 3. Of any three points situated on a straight line, there is always one and only one which lies between the other two.

II, 4. Any four points $A, B, C, D$ of a straight line can always be so arranged that $B$ shall lie between $A$ and $C$ and also between $A$ and $D$, and, furthermore, that $C$ shall lie between $A$ and $D$ and also between $B$ and $D$.

Definition. We will call the system of two points $A$ and $B$, lying upon a straight line, a segment and denote it by $AB$ or $BA$. The points lying between $A$ and $B$ are called the points of the segment $AB$ or the points lying within the segment $AB$. All other points of the straight line are referred to as the points lying outside the segment $AB$. The points $A$ and $B$ are called the extremities of the segment $AB$.

II, 5. Let $A, B, C$ be three points not lying in the same straight line and let $a$ be a straight line lying in the plane $ABC$ and not passing through any of the points $A, B, C$. Then, if the straight line $a$ passes through a point of the segment $AB$, it will also pass through either a point of the segment $BC$ or a point of the segment $AC$. Axioms II, 1–4 contain statements concerning the points of a straight line only, and, hence, we will call them the linear axioms of group II. Axiom II, 5 relates to the elements of plane geometry and, consequently, shall be called the plane axiom of group II.**"** [36].

**Objections to §3:**

1) The statement "II, 1. If $A, B, C$ are points of a straight line and $B$ lies between $A$ and $C$, then $B$ lies also between $C$ and $A$" is a tautology because "$A$ and $C$" is identical with "$C$ and $A$": $\overline{AC} = \overline{CA}$. The relationship $\overline{AC} = \overline{CA}$ means that points $A$ and $C$ can be rearranged (permuted).

2) The statement "Definition. We will call the system of two points $A$ and $B$, lying upon a straight line, a segment and denote it by $AB$ or $BA$" is erroneous because the system of two points $A$ and $B$ is not a segment.

3) All statements represent the perfunctory, superficial and fallacious verbal description of Figures 1, 2, 3.

**“§4. CONSEQUENCES OF THE AXIOMS OF CONNECTION AND ORDER.**

By the aid of the four linear axioms II, 1–4, we can easily deduce the following theorems” [36].

**Objections to §4:**

All statements represent the perfunctory, superficial and fallacious verbal description of Figures 4, 5, 6, 7.

**“§5. GROUP III: AXIOM OF PARALLELS. (EUCLID’S AXIOM.)**

The introduction of this axiom simplifies greatly the fundamental principles of geometry and facilitates in no small degree its development. This axiom may be expressed as follows:

III. In a plane $\alpha$ there can be drawn through any point $A$, lying outside of a straight line $a$, one and only one straight line which does not intersect the line $a$. This straight line is called the parallel to $a$ passing through the given point $A$.

This statement of the axiom of parallels contains two assertions. The first of these is that, in the plane $a$, there is always a straight line passing through $A$ which does not intersect the given line $a$. The second states that only one such line is possible. The latter of these statements is the essential one, and it may also be expressed as follows:

Theorem 8. If two straight lines $a$, $b$ of a plane do not meet a third straight line $c$ of the same plane, then they do not meet each other.

For, if $a$, $b$ had a point $A$ in common, there would then exist in the same plane with $c$ two straight lines $a$ and $b$ each passing through the point $A$ and not meeting the straight line $c$. This condition of affairs is, however, contradictory to the second assertion contained in the axiom of parallels as originally stated. Conversely, the second part of the axiom of parallels, in its original form, follows as a consequence of theorem 8.

The axiom of parallels is a plane axiom” [36].

**Objections to §5**.

In a practical point of view, this axiom is not precise (exact, accurate) and correct. In the point of view of formal logic, the exact and correct formulation is based on the use of surface gage. The genetic definition of parallel lines is as follows: a line $b$ is called the parallel to a given line $a$ if the line $b$ passes through a point $A$ not lying upon the line $a$ and is the equidistant line generated by surface gage.

Further citation of Hilbert’s work has no sense (meaning content) because Hilbert’s work [36] represents the perfunctory, superficial and fallacious verbal description of Figures 1-52 in his work.

## 5. DISCUSSION

1. Geometry is the science of the properties of material geometric systems (figures) constructed of material geometric elements. Geometry is based on the following system principle: the properties of the system determine the properties of the elements of the system; properties of the elements of the system characterize the properties of the system. Therefore, geometry is an engineering science which uses applied mathematics. For example, a car designer first materializes an idea in the form of a drawing: the designer draws the form of a car, then he decomposes this form into its component parts and draws the component parts of the form. The material-processing robots make parts and production-line robots connect the parts into one whole (system). This means that the whole (i.e., system, form) determines the properties of the parts (i.e., elements of form). Therefore, a correct geometric theory should be based on the dialectical principle of cognition: "practice $\rightarrow$ theory $\rightarrow$ practice".

2. The system of Hilbert's axioms is incomplete because within the framework of this system it is impossible to prove the theorem of the trisection of an angle [1] and the theorem of the sum of the interior angles (concluded angles) of a triangle [23-26].

The proof of the angle trisection theorem is based on the following material operations [1]: (1) construction of a circle; (2) construction of a given central angle; (3) extraction the arc on which the central corner rests; (4) straightening of the extracted arc (i.e., converting of the arc to a straight line segment); (5) division of the straight line segment into three identical parts using a proportional compass (whole-and-half compasses); (6) designation of two marks (points) on the straight line segment; (7) transformation (bending) of the straight line segment with marks (points) into the initial arc; (8) insertion of the arc with marks (points) into the circle; (9) drawing two straight lines from the center of the circle through the marks (points) on the arc.

The proof of the theorem of the sum of the interior angles (concluded angles) of a triangle is based on the following statements [23-26]: (1) the sides of a triangle are material straight line segments; (2) the vertices of the triangle represent material universal joints; (3) universal joints allow structural (internal) movement of the triangle; (4) the structural movement of a triangle is a change in the angles and lengths of the sides of the triangle; (5) the sum of the interior angles (concluded angles) of a triangle is equal to $180^{\circ}$ in general case if some concluded angle is equal to $180^{\circ}$ in special case

3. In a practical and logical points of view, a correct system of axioms is a set of practical techniques (methods, principles) for constructing a given geometric figure. Set of practical techniques (methods, principles) for constructing given geometric figures can be complete in some cases, but can be incomplete in other cases. This is explained by the fact that practice is the supporting and developing points in the

inductive process of cognition. An inductive process – an unlimited process – obeys to the dialectical principle of cognition: “practice $\rightarrow$ theory $\rightarrow$ practice”.

4. The book, “Grundlagen der Geometrie” (“The Foundations of Geometry”). by David Hilbert, is a methodologically wrong work. It does not satisfy the dialectical principle of cognition, “practice $\rightarrow$ theory $\rightarrow$ practice,” because practice is not the starting point and final point of Hilbert’s theoretical approach (analysis). Hilbert did not understand that: (a) scientific intuition must be based on practical experience; intuition that is not based on practical experience is fantasy; (b) the definitions of geometric concepts are the genetic (technological) definitions that show how given material objects arise (i.e., how a person creates given material objects); (c) the theory must be constructed within the framework of the correct methodological basis: the unity of formal logic and rational dialectics. (d) the theory must satisfy the correct criterion of truth: the unity of formal logic and rational dialectics. Therefore, Hilbert cannot prove the theorem of trisection of angle and the theorem of sum of interior angles (concluded angles) of triangle on the basis of his axioms. This fact signifies that Hilbert’s system of axioms is incomplete. In essence, Hilbert’s work is a superficial, tautological and logically incorrect verbal description of Figures 1-52 in his work.

## CONCLUSION

The analysis of the foundations of geometry within the framework of the correct methodological basis – the unity of formal logic and rational dialectics – leads to the following result:

1) Geometry is an engineering science, but not a field of mathematics.

2) The essence of geometry is the construction of material figures (systems) and study of their properties.

3) The starting point of geometry is the following system principle: the properties of material figures (systems) determine the properties of the elements of figures; properties of elements characterize the properties of figures (systems).

4) Axiomatization of geometry is a way of construction of the science as a set (system) of practical principles. Sets (systems) of practice principles can be complete or incomplete.

5) The book, “The Foundations of Geometry” by David Hilbert, represents a methodologically incorrect work. It does not satisfy the dialectical principle of cognition, “practice $\rightarrow$ theory $\rightarrow$ practice”, because practice is not the starting point and final point in Hilbert’s theoretical approach (analysis). Hilbert did not understand that: (a) scientific intuition must be based on practical experience; intuition that is not based on practical experience is fantasy; (b) the correct science does not exist without definitions of concepts; the definitions of geometric concepts are the genetic (technological) definitions that shows how

given material objects arise (i.e., how a person creates given material objects); (c) the theory must be constructed within the framework of the correct methodological basis: the unity of formal logic and rational dialectics; (d) the theory must satisfy the correct criterion of truth: the unity of formal logic and rational dialectics. Therefore, Hilbert cannot prove the theorem of trisection of angle and the theorem of sum of interior angles (concluded angles) of triangle on the basis of his axioms. This fact signifies that Hilbert's system of axioms is incomplete. In essence, Hilbert's work is a superficial, tautological and logically incorrect verbal description of Figures 1-52 in his work.

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# PREVALENCE OF CARDIO VASCULAR DISEASE RISK FACTORS IN SUBURBAN OF CHENNAI, SOUTH INDIA : A COMMUNITY ASSESSMENT

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## ABSTRACT

***In a cross-sectional survey, 240 subjects aged ≥ 20 years were studied from June 2015 to March 2016 in Pammal Suburban Chennai, a south Indian city. Demographic history, anthropometry and blood pressure were assessed. The descriptive statistics and one way ANOVA analysis was carried out to show the sub urban of Pammal, South Chennai, India activities towards scientific production in the field of CVD patient's significance during this period. In this study population 53.8% are female, 46.2% are male with mean age (in years) of (56.48±13.503); 63.3% are found withS1S2 (+) Cardiovascular System (Heart Beat Sound). The age group of above 50 years (57.1%) is more prone to first stage of heart attack; 30.8% got cured at initial phase itself through artificial respiratory Oxygen therapy. There is significant association between Gender and age at first heart attack (p<0.05). Mean and Standard deviation of Length of stay in Hospital (L) of 240 subjects is (8.96±16.701) days; for female it is (11.6±21.520) and for male (5.9±7.038) days. Using One way ANOVA, it is found that Duration of stay (p<0.05), Temperature (p<0.05) and Pulse Rate (p<0.05) is significant with respect to Gender. Age (p>0.05), Blood Pressure (p>0.05), Respiratory Rate (p>0.05) has no significant with respect to Gender and identifies pulse rate as significant contributing factor to the model. The length of stay in hospital with cardiovascular disease is more among females when compared to male.***

***Keyword : Pammal Suburban, Greater Chennai, Cardiovascular Disease (CVD), Descriptive Statistics and One Way ANOVA.***

## 1. INTRODUCTION

Pammal is a suburban city in greater Chennai, town in Chengalpet district of Tamil Nadu, India. Leather and tannery factories are present in and around Pammalsuburban which has labour intensive and employs many people. Pammal Sambandha Mudaliar, the father of modern Tamil theatre, was born in Pammal which is very close to Chennai International Airport.

According to 2011 census, Pammal had population of 75,870 with a sex-ratio of 998 females for every 1,000 males, much above the national average of 929. A total of 8,264 were under the age of six, constituting 4,223 males and 4,041 females. Scheduled Castes and Scheduled Tribes accounted for 19.28% and 0.16% of the population respectively. The average literacy of the town was 81.13%, compared to the national average of 72.99%.The suburban had a total of 18812 households. There were a total of 29,090 workers, comprising 105 cultivators, 154 main agricultural labourers, 370 in house hold industries, 24,304 other workers, 4,157 marginal workers, 47 marginal cultivators, 63 marginal agricultural labourers, 127 marginal workers in household industries and 3,920 other marginal workers.

## 2. REVIEW OF LITERATURE

Arnab Ghosh, *et.al*.(2005 and 2009),study aimed to investigate age and sex variation in the prevalence of CVD risk factors in people of Asian Indian origin. The present community-based cross-sectional study was conducted between February 2007 and April 2010. A total of 682 (302 males and 380 females) participants aged 25–85 years took part in the study. Descriptive statistics, such as the mean and standard deviation (SD) were undertaken separately for each age group. Analysis of variance (ANOVA) was used to compare the 4 groups. Percentile distribution of BMI, WHR, TC, LDL, and TG: HDL ratio in the study population was also undertaken. Finally, Chi-square test was computed to compare the prevalence of high TG, high TC, high BP, and high BMI across the age groups.Age irrespective of sex modulates Cardio Vascular Disease risk factors and warranted prevention as early as middle age.The prevalence of CVD risk factors, such as high BMI, high TC, high TG, and high BP as per age was also evident in our study suggesting that with age factors of metabolic syndrome, dyslipidemia is accumulated among them which in turn may lead them into increasing risk of CHD and it is noteworthy to mention that CHD in Asian Indians.

G.Subramanian *et.al.*estimation of prevalence of risk factors of cardiovascular disease is important to design preventive programmers. This cross-sectional study was carried out in a port area population to assess the prevalence of traditional cardiovascular risk factors. The study included 500 participants (males/females-120/380). This cross sectional study was conducted during the period of February-November 2011 in rural population in Nellore district, with an estimated adult population 5,000. Cluster sampling technique was used for data collection. A total of 500 subjects were screened. Mean and SD were calculated for continuous variables and proportions were calculated for the categorical variables. Differences in the mean values of the subject characteristics in two groups were analyzed using student's t test and differences in the distribution of risk factors in two groups were assessed using chi square tests. A total of 500 subjects participated in the study. The mean age of the participants was 42.59± 13.63 yrs. There were 120 males and 380 females. 83.3 % (100/120) of males were smokers and alcoholic. The prevalence of prehypertension and its association with hypercholesterolemia and hypertension in this moderately physically active population indicates an urgent need for targeted interventions to reduce the cardiovascular risk.

R.Walia *et.al.* studies conducted to assess the prevalence of cardiovascular (CV) risk factors among different regions of the country show variation in risk factors in different age groups and urban and rural population. Undertook this study to determine the prevalence of cardiovascular risk factors among urban adults in a north Indian city. In a cross-sectional survey, 2227 subjects aged ≥ 20 years were studied from April 2008 to June 2009 in Urban Chandigarh, a north Indian city. Demographic history, anthropometry and blood pressure were assessed. Fasting and 2h capillary plasma glucose after 75 g glucose load, HDL-C and triglycerides were estimated. Multivariate logistic regression analysis is used in the study. The most prevalent cardiovascular risk factors in the age group of 20-29 years were sedentary lifestyle (63%), while from fourth decade and onwards, it was overweight/obesity (59-85%). The second most common prevalent cardiovascular risk factor in the age group of 20-29 year was overweight/obesity, in 30-49 year sedentary lifestyle, in 50-69 year hypertension and in subjects' ≥70 years, it was hyper triglyceridemia. The prevalence of CV risk factors significantly increased with age irrespective of gender and prevalence of low HDL-C was significantly more common in women as compared to men. A total of 2368 subjects aged ≥20 yr, were approached based on multistage cluster randomized sampling in different sectors of urban, Chandigarh. Of them, 123 were non-respondents. However, these subjects were similar to the study subjects, in terms of age, gender and body mass index (BMI). Of the remaining 2245 subjects, 18 subjects were excluded since HDL-C/TG levels were not available.

The occurrence of cardiovascular disease (CVD) is rising worldwide and it accounts for 17 per cent of the total mortality by Pagidipati NJ, Gaziano TA (2013). This escalation in the prevalence of CVD has been attributed to the paradigm shift in life style counting the fluctuations in dietary pattern particularly more consumption of refined carbohydrates and saturated fats, and physical inertia associated with progressive economic growth and urbanization.

## 3. MATERIAL AND METHODS

The primary data source collected through survey method, the respondents from Pammal, suburban of Greater Chennai, India. The targeted population size is 250 with 20 attributes, unfortunately we achieved the sample size of 240, those which are collected for the first time and thus happen to be original in character. Primary data was collected by framing questionnaires and using simple random sampling methods. The questionnaire contained questions which are both open-ended and close-ended. Open-ended questions yielded more insightful information, whereas close-ended questions were relatively simple to tabulate and analyze information. This survey has been conducted by using a case sheet questionnaire. Questions were asked to the respondents of the sample and their responses were coded and marked in data sheets. The questionnaire consists 20 questions with many small sections.

### *3.1 One Way ANOVA*

In Statistics, One-Way Analysis of Variance (abbreviated one-way ANOVA) is a technique used to compare means of three or more samples using F-distribution. This technique can be used only for numerical data. The ANOVA tests the null hypothesis that samples in two or more groups are drawn from populations with the same mean values. To do this, two estimates are made of the population variance. These estimates rely on various assumptions. The ANOVA produces an F-statistic, the ratio of the variance calculated among the means to the variance within the samples. If the group means are drawn from populations with the same mean values, the variance between the group means should be lower than the variance of the samples (R .A, Johnson and D.W. Wichern, 2021).

### *3.2 Statistical Model*

$y_{ij} = \mu_i + \alpha_i + \epsilon_{ij}$

Where $\mu_i$ is the general mean, $\alpha_i$ is the treatment and $\epsilon_{ij}$ is the error term.

### *3.3 Assumption*

- All the observations ($y_{ij}$'s) are independent and $y_{ij} \sim N(\mu_i, \sigma_e^2)$.
- Different effects are additive in nature.
- $\epsilon_{ij}$areindependent and identically distributed (iid) normal with mean $\mu_i$ and standard deviation $\sigma_e^2$.

**ONE WAY ANOVA TABLE**

| Sources of Variation | Sum of Squares | Degrees of Freedom | Mean Sum of Squares | Variance of Ratio |
|---|---|---|---|---|
| Treatment | $S_t^2$ | K-1 | $S_t^2 = S_t^2/(K-1)$ | F=$S_t^2/S_E^2$~F(K-1,N-K) |
| Error | $S_E^2$ | N-K | $S_E^2 = S_E^2/(N-K)$ | |
| Total | $S_T^2$ | N-1 | | |

## 4. RESULT AND DISCUSSION

The general descriptive statistics of the study are presented with discussions and are given below.

**Table 1: Demographic Features and Cardiovascular Disease**

| ATTRIBUTES | CODING | SIZE(N) | PERCENTAGE |
|---|---|---|---|
| Gender | Female-0 | 129 | 53.8 |
| | Male-1 | 111 | 46.2 |
| Date of Surgery | No-0 | 238 | 99.2 |
| | Yes-1 | 2 | 0.8 |
| Present Breathing Illness | 1 Hour | 34 | 14.2 |
| | 2 Hour | 127 | 52.9 |
| | 1 Day | 29 | 12.1 |
| | 2 Day | 50 | 20.8 |

The above table 1, shows the socio-economic and physical characteristics of female and male subjects in this study. Out of which 99.2% are subjected to non- surgery, 0.8% are subjected to surgery. Among the respondents individuals present breathing difficulty stayed for 2 Hours (52.9%), 1 Hours (14.2%), 1 Day (12.1%) and 2 Days (20.8%) among the patients history.

The following Figure 1. Visualized the respondents in frequency and percentage format, 128 Male respondents nearing 44 percentage of the total population. The Female respondents are 111 with 46 percentage of the population to participate in this survey.

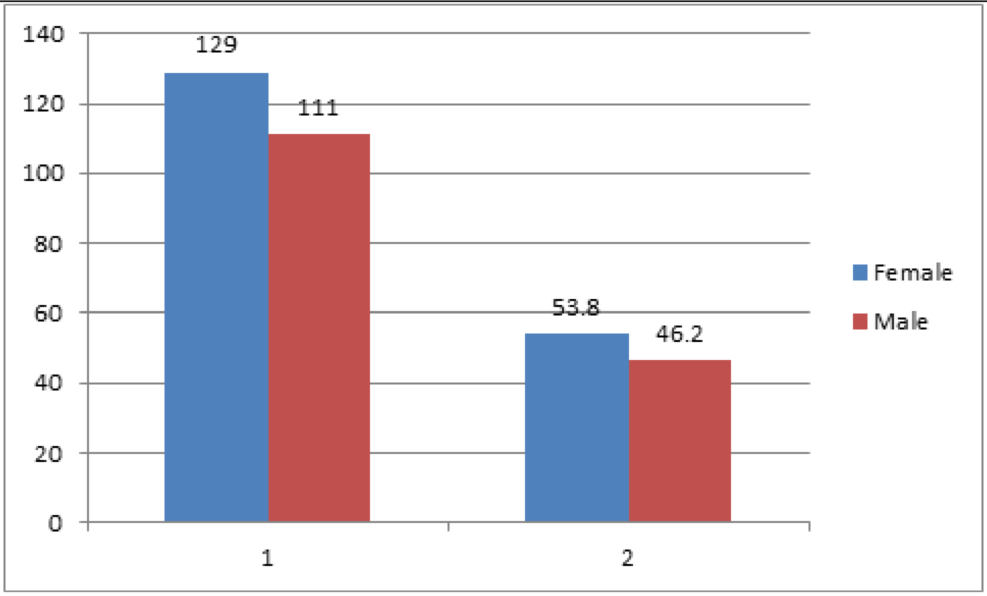


**Figure 1. Total Cardiovascular Disease Population of Male and Female**

**Table 2. Cardio Vascular System, Physical Active and after 12 months more physically Active of the Respondents**

| ATTRIBUTES | CODING | SIZE(N) | PERCENTAGE |
|---|---|---|---|
| Cardiovascular System | S1S2(+)-0 | 152 | 63.3 |
| | S1S2(++)-1 | 26 | 10.8 |
| | S1(+)S2(+)-2 | 29 | 12.1 |
| | S1S2-3 | 33 | 13.8 |
| Physically Active | No-0 | 147 | 61.2 |
| | Yes-1 | 93 | 38.3 |
| After 12 months more physically active | No-0 | 42 | 17.5 |
| | Yes-1 | 198 | 82.5 |

The above table 2, indicates that in Cardiovascular System the respondents with heart beat sound varies S1S2(+), S1S2(++), S1(+)S2(+) and S1S2 with( 63.3%, 10.8%, 12.1% and 13.8%) respectively. At time of identifying the disease their Physical Activity was 61.2%, after taking treatment for a span of 12 months and their Physical Activity had an inclination of 82.5%.

**Table 3.First Heart Attack and Treatment of Cardiovascular Disease**

| ATTRIBUTES | CODING | SIZE(N) | PERCENTAGE |
|---|---|---|---|
| First Heart Attack | Above 50 Age | 137 | 57.1 |
| | After 40 Age | 58 | 24.5 |
| | Before 40 Age | 2 | 0.8 |
| | Before 35 Age | 39 | 16.2 |
| | Below 35 Age | 2 | 0.8 |
| | Below 25 Age | 2 | 0.8 |
| Treatment Given | AR Through Oxygen Therapy-0 | 74 | 30.8 |
| | Cardiopulmary-1 | 41 | 17.1 |
| | Ventilation by ambu bag-2 | 56 | 23.3 |
| | Cardiac massages-3 | 50 | 20.8 |
| | Atovas-4 | 19 | 7.9 |

Table 3, shows that respondents of age group are above 50 years is more prone to first stage of heart attack. At initial phase itself through Artificial Respiratoy Oxygen Therapy 30.8% of respondents get cured, and then through Ventilation by ambu bag 23.3%, through Cardiac massages 20.8%, through Cardio-pulmary 17.1% and through Atovas 7.9% respectively.

**Table 4. Mean and Standard Deviation of Age and Duration of Stay**

| ATTRIBUTES | | Mean and S.E Mean |
|---|---|---|
| Age | Male | 55.76±12.881 |
| | Female | 56.48±13.503 |
| Duration of stay | Male | 21.520±7.038 |
| | Female | 11.60±21.520 |

**Table 5. Descriptive Statistics for Cardiovascular Disease using one way ANOVA**

| | | N | Mean | Std. Deviation | Std. Error | 95% CI for Mean | | Minimum | Maximum |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | L B | UB | | |
| Age | Female | 129 | 56.48 | 13.503 | 1.189 | 54.13 | 58.83 | 20 | 89 |
| | Male | 111 | 55.76 | 12.881 | 1.223 | 53.33 | 58.18 | 30 | 96 |
| | Total | 240 | 56.15 | 13.197 | .852 | 54.47 | 57.82 | 20 | 96 |
| Duration of stay | Female | 129 | 11.60 | 21.520 | 1.895 | 7.85 | 15.35 | 1 | 156 |
| | Male | 111 | 5.90 | 7.038 | .668 | 4.58 | 7.22 | 1 | 36 |
| | Total | 240 | 8.96 | 16.701 | 1.078 | 6.84 | 11.09 | 1 | 156 |
| Blood Pressure | Female | 129 | .40 | .538 | .047 | .31 | .50 | 0 | 2 |
| | Male | 111 | .45 | .599 | .057 | .34 | .56 | 0 | 2 |
| | Total | 240 | .42 | .566 | .037 | .35 | .50 | 0 | 2 |
| Temperature | Female | 129 | 98.59 | 1.271 | .112 | 98.37 | 98.81 | 92 | 101 |
| | Male | 111 | 99.22 | 2.756 | .262 | 98.70 | 99.74 | 92 | 110 |
| | Total | 240 | 98.88 | 2.112 | .136 | 98.61 | 99.15 | 92 | 110 |
| Pulse Rate | Female | 129 | 103.03 | 15.690 | 1.381 | 100.30 | 105.76 | 60 | 138 |
| | Male | 111 | 96.72 | 11.852 | 1.125 | 94.49 | 98.95 | 65 | 132 |
| | Total | 240 | 100.11 | 14.368 | .927 | 98.29 | 101.94 | 60 | 138 |
| Respiratory Rate | Female | 129 | 26.06 | 5.313 | .468 | 25.14 | 26.99 | 16 | 39 |
| | Male | 111 | 26.72 | 5.000 | .475 | 25.78 | 27.66 | 16 | 39 |
| | Total | 240 | 26.37 | 5.171 | .334 | 25.71 | 27.02 | 16 | 39 |

Table 5 indicates the results of Mean and Standard Error Mean of Age and Duration of Stay in hospital. The Male respondent mean and Standard Error is 55.76±12.881 and that of Female is 56.48±13.503. The staying duration period of male and female is 21.520±7.038 and 11.60±21.520. The summary Statistics of Age, Duration of Stay, Blood Pressure, Temperature, Pulse Rate and Respiratory Rate are clearly shown in Table 5.

$\mathbf{H_0}$=There is significant difference in the mean Age, Duration of stay, Blood Pressure,Temperature, Pulse Rate, Respiratory Rate between male and female.

$\mathbf{H_1}$ =There is no significant difference in the mean Age, Duration of stay, Blood Pressure, Temperature, Pulse Rate, Respiratory Rate between male and female.

**Table 6. One Way ANOVA for Various Cardiovascular Disease**

| Parameters | Source of Variation | Sum of Squares | Degrees of Freedom | Mean Sum Square | F – calculated Value | Significant p- Value |
|---|---|---|---|---|---|---|
| Age | Between Groups | 31.262 | 1 | 31.262 | .179 | .673 |
| | Within Groups | 41590.634 | 238 | 174.751 | | |
| | Total | 41621.896 | 239 | | | |
| Duration of Stay | Between Groups | 1935.714 | 1 | 1935.714 | 7.118 | .008 |
| | Within Groups | 64726.949 | 238 | 271.962 | | |
| | Total | 66662.662 | 239 | | | |
| Blood Pressure | Between Groups | .134 | 1 | .134 | .416 | .520 |
| | Within Groups | 76.516 | 238 | .321 | | |
| | Total | 76.650 | 239 | | | |
| Temperature | Between Groups | 23.808 | 1 | 23.808 | 5.438 | .021 |
| | Within Groups | 1041.965 | 238 | 4.378 | | |
| | Total | 1065.773 | 239 | | | |
| Pulse Rate | Between Groups | 2375.744 | 1 | 2375.744 | 12.040 | .001 |
| | Within Groups | 46962.218 | 238 | 197.320 | | |
| | Total | 49337.962 | 239 | | | |
| Respiratory Rate | Between Groups | 25.887 | 1 | 25.887 | .968 | .326 |
| | Within Groups | 6363.846 | 238 | 26.739 | | |
| | Total | 6389.733 | 239 | | | |

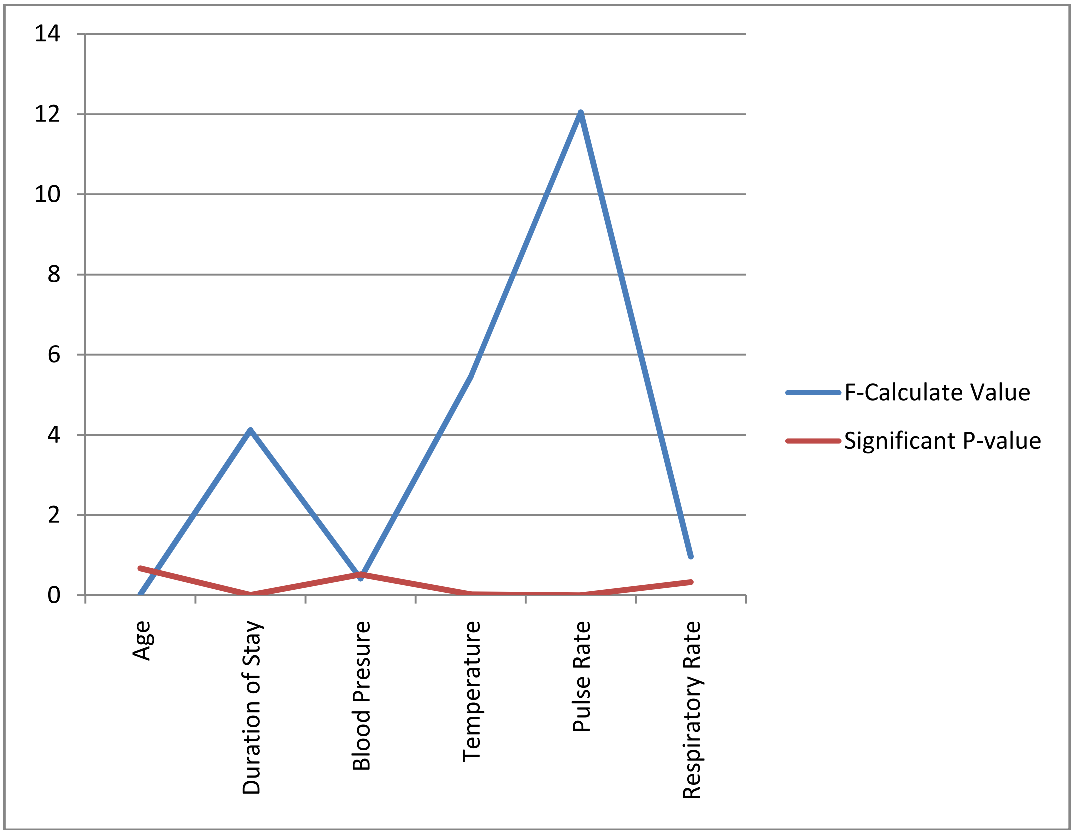


**Figure 2. Significant and not Significant of Cardiovascular Disease**

The significant and insignificant values are visualized in Figure 2. The significant difference in duration of stay (p=.008), Temperature (p=.021) and Pulse Rate (p=.001) with respect to Gender.Rest of the parameters are on insignificant difference with Age (p=.673), Blood Pressure (p=.520), Respiratory Rate (p=.326) and Cardiovascular Systemwith respect to Gender. The probability value, F-Calculated Value, Mean Square, Degrees of Freedom, Source of variation and various medical parameters of one way ANOVA results are presented in Table 6.

## 4. FINDINGS AND CONCLUSION

Studies conducted to assess the prevalence of Cardiovascular Disease (CVD) risk factors among different respondents of Pammal Suburban of Greater Chennai indicate variation in risk factors of different age groups and Cardiovascular Disease population. This study aims to determine the prevalence of cardiovascular risk factors among suburban adults in Pammal, Greater Chennai of Indian city.

In a cross-sectional survey, 240 subjects aged ≥ 20 years were studied from June 2015 to March 2016 in Pammal Suburban Chennai, a south Indian city. Demographic history, anthropometry and blood pressure were assessed. The descriptive statistics and one way ANOVA analysis was carried out to show the sub urban of Pammal, Greater Chennai, India activities towards scientific production in the field of CVD patient's significance during this period.

In this study population 53.8% are female, 46.2% are male with mean age(in years) of (56.48±13.503); 63.3% are found with S1S2(+) Cardiovascular System (Heart Beat Sound). The age group of above 50 years (57.1%) is more prone to first stage of heart attack; 30.8% got cured at initial phase itself through artificial respiratory Oxygen therapy. There is significant association between Gender and age at first heart attack ($p<0.05$). Mean and Standard deviation of Length of stay in Hospital (L) of 240 subjects is (8.96±16.701) days; for female it is (11.6±21.520) and for male (5.9±7.038) days.

Using One way ANOVA, it is found that Duration of stay ($p<0.05$), Temperature ($p<0.05$) and Pulse Rate ($p<0.05$) is significant with respect to Gender. Age ($p>0.05$), Blood Pressure ($p>0.05$), Respiratory Rate ($p>0.05$)is not significant with respect to Gender and identifies pulse rate as significant contributing factor to the model. The length of stay in hospital with cardiovascular disease is more among femaleswhen compared to male.

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**Conflict Interest**: The authors declare no competing Interest.

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# INVENTORY MODEL FOR DETERIORATING ITEMS HAVING DEMAND RATE AS A POLYNOMIAL FUNCTION OF SELLING PRICE OF DEGREE FOUR WITH VARIABLE HOLDING COST

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## ABSTRACT
***The paper contains an inventory model for deterministic inventory for deteriorating products; a particular type of polynomial of selling price dependent on degree four is taken and a demand rate. Both the deterioration rate and holding cost are assumed to be a linear function of time. The inventory involves a single item and involves zero lead time. The shortages are enabled and fully backlogged. With the help of differential equations mathematical modeling of the paper has been framed. The optimal value of parameters is evaluated, and the optimal cost price is calculated in the model. The graphs are drawn by using different parameters for the convexity of the model. Maple software is used for numerical evaluation and graphical presentation.***
***Keywords: Inventory Control, Selling Price Demand, Polynomial Function, Holding Cost.***

## 1. INTRODUCTION:

Inventory management for businesses is essentially business management in order to make a profit. It effectively controls each of the business's constituents in order to operate the business in an orderly manner. Several studies have resulted in the creation of a business model. The models include public demand for the commodity, its decaying pace, inventory keeping costs, and certain other components such as ordering costs.

In some models, the demand rate is a vector that is affected by time or the product's sale price. Several products with high retail prices are used by the majority of business class citizens, while the general class tends to purchase only the small and necessary amount of that good.

People from both the business and popular classes often use products with lower sale prices. As a result, some products sell well if they are in high demand, such as sugar. Sugar is a commodity that is needed in every household and by citizens of all social classes. Sugar is often sold at a fair price. As a result, the sugar industry will easily benefit.

Ajanta Roy(2008)[1] devised solutions for the inventory model in which demand is a linear function of sale price, storing cost is time-dependent, and depreciation rate is time-variable. Timothy H. Burwell, Dinesh S. Dave, Kathy E. Fitzpatrick, and Melvin R. Roy (1997)[2] worked with incorporated quantity and international discounts, which aided in decision-making, and they believed the demand rate was influenced by price.

T. Chakrabarty, B. C. Giri, and K. S. Chaudhuri (1997)[3] focused on immediate supply with demand that was time-dependent. T. Chakrabarti, and K.S. Chaudhuri (1997)[4]focused on inventory renewal issues with a finite time frame and a linear demand rate that was time dependent. In each period, they found equivalent renewal phases and shortages.

KUN-JEN CHUNG and PIN-SHOU TING (1993) [5]based their solutions on deterministic demand with positive linear drift and a constant rate of deterioration per unit time. Upendra Dave and L. K. Patel (1981) [6] focused on an inventory model with a time-dependent demand rate and a persistent

depreciation rate. A finite planning period was embodied in them. Okitsugu Fujiwara and U. L. J .S .R Perera (1993) [7] worked on EOQ models for uninterrupted deterioration rates and fragile products. B. C. GIRI and K. S. CHAUDHURI (1997) [8]used two forensic models to solve renewal issues. Demand rate, keeping cost, deterioration rate, ordering cost, and shortage cost were all thought to be time dependent. M. GOH (1994) [9]scoured the endless horizon in search of a single object. He found a system that was both deterministic and constant in nature. He believed that the demand rate was influenced by inventory levels.
MoncerHariga(1996)[10] used a fixed-horizon inventory lot-sizing model with a general constant time-varying demand rate and optimum inventory lot-sizing. MONCER A. HARIGA and LakdereBenkherouf(1994) [11]presented optimal and forensic inventory renewal strategies. The demand rate was assumed to differ exponentially, while the rate of depreciation was assumed to be stable.. A.K. Jalan, R. R. Giri, and K.S. Chaudhuri (1996) [12] for decaying articles, used an inventory model.They took into account acute supply constraints as well as linearly increasing demand patterns.Yadav, H., and Singh, T.P. (2019) [13] invented model with variable demand rate, Weibull distribution deterioration rate and time dependent holding cost. Yadav, H. Vinod Kumar and Singh, T.P.(2020) [14] invented model with quadratic demand rate and partial backlogging and with Weibull distribution for deterioration rate.
Kusum and Vinod Kumar [15] constructed a model with fractional polynomial of type m by 3 as a demand rate with other parameters constant. Suman and Vinod Kumar [16] constructed a model with cubic polynomial as a demand rate, linear deterioration rate and other parameters constant.
In this paper, a deterministic inventory model is developed with demand as a polynomial of degree four, deterioration rate linearly dependent on time, and linear holding cost.

## ASSUMPTIONS AND NOTATIONS

### Assumptions:

The assumptions used in the model are listed as follows:-

- The demand rate is selling price dependent as a polynomial function of degree five

$D(s)=\alpha s^4+\beta s^2+\gamma$ , where s is the selling price per unit time
and $\alpha\geq 0,\beta\geq 0,\gamma\leq 0.$

- The lead time is zero.
- The deterioration rate, $\theta(t)=\lambda t, 0<\lambda<1$
- Holding cost per unit time constant, $H(t)=q+rt, where, q, r\geq 0$.
- The Order quantity per cycle is L.
- Shortages are allowed and are fully bloclogged.
- The cost of an item is C.
- The shortage cost per unit time is $C_s$.
- The deterioration cost per unit time is $C_D$.
- This inventory Model deals with single item.
- $\upsilon$ is the time at which inventory level reaches zero, $\upsilon\geq 0$

In interval $(0,\upsilon)$ inventory is positive and, in the interval, $(\upsilon,T)$ the inventory level is negative.

### Notations:

- D - Demand Rate.
- A - The ordering cost.
- $\theta$ -Deterioration Rate.
- H(t) - Holding Cost.
- C -Purchase Cost per unit
- L - Order Quantity

- $\upsilon$ - The time when inventory level reached zero.
- T - The length of a cycle time.

## MATHEMATICAL MODEL

Period $(0,\upsilon)$ includes inventory with positive level and decreasing and at time $t=\upsilon$, the inventory level approaches to zero and shortage starts and inventory became negative in the interval $(\upsilon,T)$.

Differential equation for inventory level at time $t$ is as follows: -

$$\frac{dI(t)}{dt}+\lambda tI(t)=-(\alpha s^4+\beta s^2+\gamma);0\le t\le\upsilon \tag{1}$$

$$\frac{dI(t)}{dt}=-(\alpha s^4+\beta s^2+\gamma);\upsilon\le t\le T \tag{2}$$

having conditions $I(t)=0$ at $t=\upsilon$

On solving equation (1) and (2) and neglecting higher powers of t , we get

$$I(t)=(\alpha s^4+\beta s^2+\gamma)(\upsilon-t)+\lambda\left(\frac{\upsilon^3}{6}+\frac{t^3}{3}-\frac{\upsilon t^2}{2}\right)+\lambda^2\left(\frac{\upsilon^5}{40}-\frac{t^5}{15}-\frac{t^2\upsilon^3}{12}+\frac{\upsilon t^4}{8}\right);0\le t\le\upsilon$$

$$I(t)=-(\alpha s^4+\beta s^2+\gamma)(t-\upsilon);\upsilon\le t\le T$$

The stock loss due to deterioration is given as:

$$D'=(\alpha s^4+\beta s^2+\gamma)\int_0^{\upsilon}e^{\frac{\lambda t^2}{2}}dt\ -(\alpha s^4+\beta s^2+\gamma)\int_0^{\upsilon}dt=(\alpha s^4+\beta s^2+\gamma)\left(\frac{\lambda\upsilon^3}{6}+\frac{\lambda^2\upsilon^5}{40}\right)$$

$$L=D'+\int_0^T(\alpha s^4+\beta s^2+\gamma)dt$$

$$L=(\alpha s^4+\beta s^2+\gamma)\left(\frac{\lambda\upsilon^3}{6}+\frac{\lambda^2\upsilon^5}{40}\right)+(\alpha s^4+\beta s^2+\gamma)T \tag{3}$$

Holding Cost is given as:

$$H(t)=\int_0^{\upsilon}(q+rt)e^{\frac{-\lambda t^2}{2}}\left[\int_t^{\upsilon}(\alpha s^4+\beta s^2+\gamma)e^{\frac{\lambda u^2}{2}}du\right]dt$$

$$=(\alpha s^4+\beta s^2+\gamma)\int_0^{\upsilon}(q+rt)(1-\frac{\lambda t^2}{2}+\frac{\lambda^2t^4}{8})\cdot\left[\int_t^{\upsilon}(1+\frac{\lambda u^2}{2}+\frac{\lambda^2u^4}{8})du\right]dt$$

$$H(t)=(\alpha s^4+\beta s^2+\gamma)\times q\times\left[\begin{array}{l}\frac{\upsilon^2}{2}+\frac{\lambda\upsilon^4}{12}+\frac{\lambda^2\upsilon^6}{90}\\ +r(\alpha s^4+\beta s^2+\gamma)\left(\frac{\upsilon^3}{6}-\frac{13\lambda\upsilon^5}{120}-\frac{9\lambda^2\upsilon^7}{560}\right)\end{array}\right] \tag{4}$$

The shortages during the cycle is given as:

$$S=-\int_{\upsilon}^{T}(-(\alpha s^4+\beta s^2+\gamma)(t-\upsilon))dt$$

$$S=\frac{1}{2}(\alpha s^4+\beta s^2+\gamma)(T-\upsilon)^2 \tag{5}$$

Total cost per unit time is as follows:

$$P=\frac{1}{T}[A+C_DL+H+C_sS]$$

$$P(T,s)=\frac{1}{T}\begin{bmatrix} A+C_D(\alpha s^4+\beta s^2+\gamma)\left(T+\frac{\gamma\upsilon^3}{6}+\frac{\gamma^2\upsilon^5}{40}\right) \\ +(\alpha s^4+\beta s^2+\gamma)\times q\times\left[\frac{\upsilon^2}{2}+\frac{\lambda\upsilon^4}{12}+\frac{\lambda^2\upsilon^6}{90}+r(\alpha s^4+\beta s^2+\gamma)\left(\frac{\upsilon^3}{6}-\frac{13\lambda\upsilon^5}{120}-\frac{9\lambda^2\upsilon^7}{560}\right)\right] \\ +\frac{C_s}{2}(\alpha s^5+\beta s^3+\gamma)(T-\upsilon)^2 \end{bmatrix} \quad (6)$$

For optimal value of s and T, we have

$$\frac{\partial P}{\partial s}=0 \quad (7)$$

and

$$\frac{\partial P}{\partial T}=0 \quad (8)$$

The total minimum cost per unit time P(T,s) satisfy by sufficient condition

$$\frac{\partial^2 P}{\partial T^2}>0 \quad (9)$$

and

$$\frac{\partial^2 P}{\partial s^2}>0 \quad (10)$$

and

$$\frac{\partial^2 P}{\partial T^2}*\frac{\partial^2 P}{\partial s^2}-\frac{\partial^2 P}{\partial s\partial T}>0 \quad (11)$$

We get the value ofs and T by solving equation (7) and (8) and putting these values in equation (6), we obtain the minimum cost per unit time for the values which satisfy the necessary condition (9), (10) and (11).

| PARAMETERS | Example 1 | Example 2 | PARAMETERS | Example 1 | Example 2 |
|---|---|---|---|---|---|
| A | 99999 | 99998 | $\lambda$ | 1.5 | 1.5 |
| Γ | -9999999 | -9999999 | $\upsilon$ | 2 | 2 |
| B | 999 | 998 | Optimal Value of T | 21.14708567 | 21.10829601 |
| A(ordering cost) | 9878 | 9878 | Optimal Value of s | 3.161495772 | 3.161504462 |
| $C_S$ | 109 | 109 | Total optimal cost per unit time | 461.6535114 | 459.6938028 |
| $C_D$ | 800.001 | 798 | - | - | - |
| Q | 4 | 4 | - | - | - |
| R | 0.05 | 0.05 | | | |

The above illustration is graphically presented as follows:

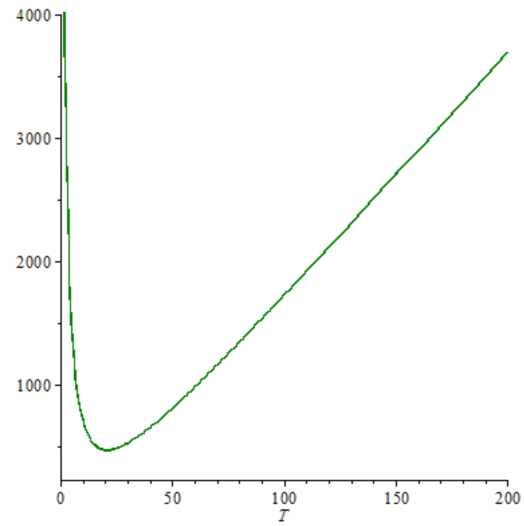


**Figure.1-shows total cost function verses T**

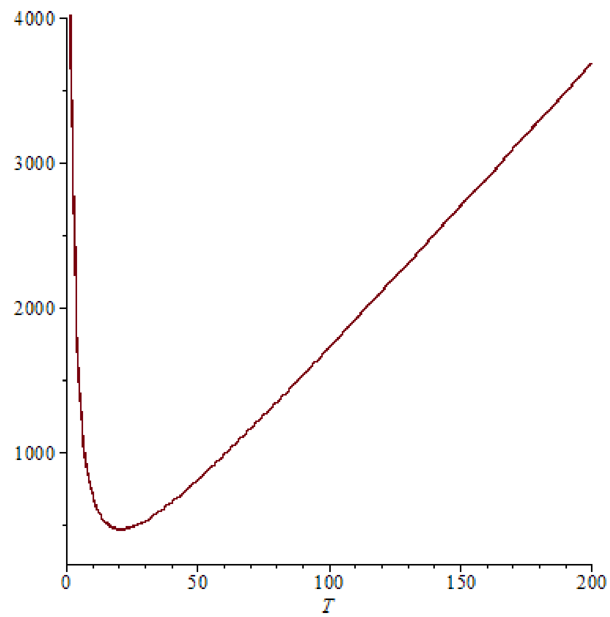


**Figure.2-shows total cost function verses T**

## CONCLUSION

In this paper, a deterministic inventory model developed with the demand rate depends on selling price raise to power four, linear decline rate, and variable holding cost. The total optimal cost has been calculated for the values of s and T, which satisfies the necessary condition. Numerical and graphical illustrations have been used to validate the model. As the price of products rises day by day, this model fits in well with the model where the demand rate is determined by the even powers of the selling price. As a result, the model will be highly advantageous to future societies and enterprises.

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# AN INTERESTING END-CORRECTION TECHNIQUE FOR ALTERNATING SERIES ESTIMATION

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## ABSTRACT

***Dealing with the problem of incommensurability of circumference and diameter, the 14th century Kerala mathematician Sangamagrāma Mādhava broke the barriers of finite and landed into the highly fertile area of infinite. He discovered infinite series expressions for circumference in terms of diameter and made startling discoveries in the field of infinite series and series approximations. An innovative and interesting end-correction technique devised by Mādhava for approximating the Mādhava circumference-diameter series is found applicable to various other alternating series and a couple of these are discussed in this paper along with the Mādhava series.***

***Keywords : Catalan, Circumference, End-correction,Mādhava, SankaraVāriyar, Sthaulya***

## 1. INTRODUCTION

In the field of astronomy and mathematics several scholars belonging to the Kerala region are found to have played a crucial role. With the emergence of SaṅgamagrāmaMādhava(1340–1425 AD) there was a huge up surge in mathematical activities in this region. Several outstanding achievements are found to have been made by Mādhava and his successors. Some of these are now available to us through various works written by scholars belonging to Mādhava school such as *Tantrasaṅgraha* of Nīlakan ṭha Somayāji (1444–1545 AD), *Kriyākramakarī*and *Yuktidīpika*of ŚaṅkaraVāriyar (1500–1560 AD), *Yuktibhāṣā* of Jyeṣṭhadeva (1500–1610 AD), *Karanapaddhati* of Putumana Somayāji (1660–1740 AD) and *Sadratnamāla* of Śaṅkara Varman (1800–1838 AD).

In order to evaluate the circumference of a circle from its diameter with desired degree of accuracy Mādhavabroke the barriers of finite and open eda new passage to the so far unknown field of infinite. Floating innovative concepts of infinitesimals and continuous summations, Mādhava discovered infinite series for circumference in terms of diameter and made startling contributions in the highly fertile area of infinite series. Mādhava discovered power series expressions for Indian trigonometric functions also. His disciples and later scholars belonging to the Mādhava school travelled far and wide along the new passage to infinity thrown open by Mādhava and made their own contributions in the field of infinite series and series approximations. The *Yuktibhāṣā*[1,2] and the *Kriyākramakarī* contains a detailed analysis of Mādhava series and series approximations [3].

## MĀDHAVA'SEND-CORRECTION TECHNIQUE IN MODERN TERMS

The Mādhava's series for circumference $S$ of a circle with diameter $d$is $S = 4d - \frac{4d}{3} + \frac{4d}{5} - \cdots$. While commenting on the *Līlāvatī* rule for determination of circumference of a circle, Śaṅkara refers to Mādhava and gives a detailed analysis of series approximation using end-corrections [3,4]. Citing

Mādhava's enunciation, Śaṅkara give $S_n = 4d - \frac{4d}{3} + \frac{4d}{5} - \cdots + (-1)^{n-1}\frac{4d}{2n-1} + (-1)^n 4d\left\{\frac{(2n/2)}{(2n)^2+1}\right\}$ which is a truncated Mādhavaseries with a compensation attached to it in order to amend the loss of neglected terms. The Mādhava series converges notoriously slowly and hence a very large number of terms are needed to get even a very small degree of accuracy. So to stop computing after a specific number of terms and get the result with desired accuracy, Mādhava applied an end-correction (*antya-saṁskāra*) to the truncated series. To the $n^{\text{th}}$ partial sum$S_n' = 4d\left\{1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1}\frac{1}{2n-1}\right\}$an end correction $(-1)^n 4dE_n$ is applied to get the $n^{\text{th}}$ *corrected* partial sum$S_n = S_n' + (-1)^n 4dE_n$ where$E_n$ is the end-correction factor or *guṇa* (being the multiplier of $(-1)^n 4d$). The corresponding relation for $(n+1)^{th}$ corrected partial sum is $S_{n+1} = S_{n+1}' + (-1)^{n+1}4dE_{n+1}$. Hence, the difference is

$\Delta S_n = S_{n+1} - S_n = (-1)^n 4d\left\{\frac{1}{2n+1} - E_{\,n+1} - E_n\right\} = (-1)^n 4dr_n$where $r_n = \frac{1}{2n+1} - E_{n+1} - E_n$

This corrected partial sum difference $\Delta S_n$is called *sthaulya* (inaccuracy) and$r_n = \frac{1}{2n+1} - E_{n+1} - E_n$ is the inaccuracy factor which should be decreased for increasing the accuracy. For *ideal* end-correction $r_n = 0$. If $E_n$ and $E_{n+1}$are both assumed to be equal to $\frac{1}{2}\left(\frac{1}{2n+1}\right)$ then $r_n = 0$. But it cannot be assumed so becauseif $E_n = \frac{1}{2}\left(\frac{1}{2n+1}\right)$ then obviously$E_{n+1} = \frac{1}{2}\left(\frac{1}{2n+3}\right)$ which are different for small and finite values of *n*. So starting from this initial value $E_n^{(0)} = \frac{1}{2}\left(\frac{1}{2n+1}\right)$a first approximation$E_n^{(1)}$ is derived by adding a suitable term to the denominator of $E_n^{(0)}$in such a way that the corresponding *sthaulya* factor $r_n^{(1)} = \frac{1}{2n+1} - E_{n+1}^{(1)} - E_n^{(1)} = \frac{N^{(1)}}{D^{(1)}}$is as small as desired. Thus taking $E_n^{(1)} = \frac{1}{2(2n+1)+A}$ Mādhava shows that $A = -2$ is the best fit value to make the corresponding *sthaulya* $r_n^{(1)}$ as small as possible. Thus the first approximation to the end-correction factor $E_n^{(1)} = \frac{1}{4n}$. Hoping to get a better one, Mādhava proceeds further to find the second approximation $E_n^{(2)}$ from the first by adding a best fit fraction of the denominator of $E_n^{(1)}$ to itself which makes *sthaulya* factor $r_n^{(2)} = \frac{1}{2n+1} - E_{n+1}^{(2)} - E_n^{(2)}$as small as possible for any *n*. Taking $E_n^{(2)} = \frac{1}{4n+\frac{A}{4n}}$ the best fit value of *A* is obtained as 4. Thus the second approximation to end-correction factor is $E_n^{(2)} = \frac{1}{4n+\frac{4}{4n}} = \frac{(2n/2)}{(2n)^2+1}$ as given by Śaṅkara in the *Kriyākramakarī* quoting Mādhava**.** Proceeding further, the third approximation is $E_n^{(3)} = \frac{n^2+1}{n(4n^2+5)} = \frac{n^2+1}{n[4(n^2+1)+1]}$.

Mādhava'send-correction technique is applicable to various other alternating series. Two such cases are given below. Excellent improvement in accuracy can be attained using only a very few terms.

**Alternating Harmonic Series**

The Alternating Harmonic Series $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ is conditionally convergent with sum $log2$. Mādhava's technique can be appliedby proceeding as stated above. In this case $\Delta S_n = (-1)^n\left\{\frac{1}{n+1} - E_{n+1} - E_n\right\}$ so that the *sthaulya* factor is $r_n = \frac{1}{n+1} - E_{n+1} - E_n$. Starting from the initial end-correction $E_n^{(0)} = \frac{1}{2(n+1)}$ the first three successive approximations are $E_n^{(1)} = \frac{1}{2n+1}$, $E_n^{(2)} = \frac{2n+1}{(2n+1)^2+1}$, $E_n^{(3)} = \frac{(2n+1)^2+4}{(2n+1)[(2n+1)^2+5]}$

**Catalan's Series**

The infinite series $\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$ convergesto a constant called Catalan's constant 0.91596559417721901505… which is named so in honor of the great Belgiam mathematician Eugène

Charles Catalan (1814-1894 AD) who discovered infinite series for its computation. Applying Mādhava's technique we get $\Delta S_n = (-1)^n \left\{\frac{1}{(2n+1)^2} - E_{n+1} - E_n\right\}$so that $r_n = \frac{1}{(2n+1)^2} - E_{n+1} - E_n$. Staring from the initial value $E_n^{(0)} = \frac{1}{2(2n+1)^2}$,the first threeapproximations in this case are$E_n^{(1)} = \frac{1}{\rho}$, $E_n^{(2)} = \frac{\alpha}{\rho\alpha-64}$, $E_n^{(3)} = \frac{\alpha\beta-1024}{\rho(\alpha\beta-1024)-64\beta}$, where $\rho = 8n^2 + 6, \alpha = \rho + 32, \beta = \alpha + 64$

**EFFICACY OF MĀDHAVA'S END-CORRECTION TECHNIQUE**

The number of terms *n* required to get an accuracy up to 8, 9,15,20 decimal places using the successively refined end-corrections for each of the above three series are given in the following tables. The efficacy of Mādhava's end-correction technique can be seen clearly in all the three cases.

**Table 1- Mādhava's circumference-diameter series**

| With end-correction | *No. of terms n required for an accuracy of* | | | |
|---|---|---|---|---|
| | 8 *dp* | 9 *dp* | 15 *dp* | 20 *dp* |
| $E_n^{(1)} = \frac{1}{4n}$ | 500 | 1077 | 107722 | $5*10^6$ |
| $E_n^{(2)} = \frac{n}{4n^2+1}$ | 42 | 66 | 1046 | 10456 |
| $E_n^{(3)} = \frac{n^2+1}{n(4n^2+5)}$ | 16 | 22 | 161 | 834 |
| Without any end-correction | $5*10^8$ | $5*10^9$ | $5*10^{15}$ | $5*10^{20}$ |

**Table 2- Alternating Harmonic Series**

| With end-correction | *No. of terms n required for an accuracy of* | | | |
|---|---|---|---|---|
| | 8 *dp* | 9 *dp* | 15*dp* | 20 *dp* |
| $E_n^{(1)} = \frac{1}{\rho}$, $\rho = 2n+1$ | 629 | 1357 | 135720 | 6299605 |
| $E_n^{(2)} = \frac{\rho}{\rho^2+1}$ | 47 | 75 | 1201 | 12011 |
| $E_n^{(2)} = \frac{\rho^2+4}{\rho[\rho^2+5]}$ | 17 | 24 | 177 | 921 |
| Without any end-correction | $10^9$ | $10^{10}$ | $10^{16}$ | $10^{21}$ |

**Table 3- Catalan's Series**

| With end-correction | *No. of terms n required for an accuracy of* | | | |
|---|---|---|---|---|
| | *8 dp* | *9 dp* | *15 dp* | *20 dp* |
| $E_n^{(1)} = \frac{1}{\rho}, \rho = 8n^2 + 6$ | 25 | 37 | 368 | 2510 |
| $E_n^{(2)} = \frac{\alpha}{\rho\alpha-64}, \alpha = \rho + 32$ | 9 | 11 | 46 | 145 |
| $E_n^{(3)} = \frac{\alpha\beta-1024}{\rho(\alpha\beta-1024)-64\beta}, \beta = \alpha + 64$ | 6 | 8 | 21 | 48 |
| Without any end-correction | 15811 | 50000 | 50000000 | 15811388301 |

## CONCLUSION

Mādhava's discovery of infinite series and his ingenious end-correction techniques are important landmarks in series estimation. They can be applied to other alternating series and can be used for reducing considerably the number of terms to get a desired accuracy.

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*In memory of Late Shri Parameswara Iyer Vaikom*

# DIRECT FORMULAE FOR FINDING PARTICULAR INTEGRAL OF CERTAIN ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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## ABSTRACT

***This paper deals with simple and direct formulae for finding particular integrals of certain differential equations with constant coefficients. These results are contained in an unpublished work 'Differentiation by a method of Continued Differentiation' by author's Father (Late) Parameswara Iyer N Vaikom. Solution of equations of the form $f(D)y = f(x).\phi(x)$ ; where $f(x)$ is $n^{th}$ degree polynomial and $\phi(x)$ is $\sin ax, \cos ax, e^{x}, \sinh ax$ or $\cosh ax$ are dealt with.***

***Key words: Linear Differential Equations, Particular Integral, Differential Operator, Continuous Differential Equations Numerator coefficients, Denominator Coefficients, Direct formulae. 2010 AMS Classification: 34A30 34 A05***

## 1. INTRODUCTION

This paper is based on the research work carried out by author's Father Late Sri Parameswara Iyer N Vaikom. Though the work was compiled in the year 1976, the manuscript remained unpublished either as research papers in journals or as a printed book.

This paper is intended for bringing the attention of Mathematicians, teachers and students to the research work, which deals with explicit and simple formulae for finding solution of certain types of differential equations.

The 'Ordinary Linear Differential Equations with Constant Coefficients' have varied application in the field of Engineering and Physics. Hence in the curriculum of Mathematics, the study of these equations and its solutions have given prominence. By using the proposed formulae, the solutions of such types can be written down in two or three steps in an elegant form for any order of the equation. The currently used methods for finding Particular Integrals (PI) are cumbersome and its complexity increases as the order of the equation becomes higher and higher.

In the solution of Ordinary Linear Differential Equations with constant coefficients, the well-established method of solution is that, to find the complete solution in two parts namely the Complimentary Function and the Particular Integral. It is well known that 'the linear differential operators with constant coefficients formally obey the laws which are valid for polynomials', (Sokolnikoff, 1941). When the given equation is written using the operator D in the standard form as $\{f(D)\}y = \psi(x)$, where $\psi(x)$ is a function of dependent variable x and y is independent variable, the complimentary Function (CF) can be easily found as it depends only on the expression $f(D)$ forming the ancillary equation directly from it. . On the other hand, the Particular Integral (PI) is definite for a given equation depends on the function $\psi(x)$ and its evolution is not easy and method to be adopted depends on the nature of $\psi(x)$ as well as $f(D)$. The paper deals with the direct formulae for finding the Particular Integral (PI) of following equations which have wide use.

**Type: I**

$D^m y = x^n \sin ax$ ***(or cos ax,)***

**Type II**

$(D^2 + a^2)^m y = x^n \sin ax$ ***(or cos ax,)***

**Type III**

$(D^2 - a^2)^m y = x^n \sin ax$ ***(or cos ax,)***

**Type: IV**

$(D^2 \pm b^2)^m y = x^n \sin ax$ ***(or cos ax,) where*** $b \neq a$

Though it may appear that above four types can be brought under one general form as given below. $(D^2 \pm b^2)^m y = x^n \sin ax$ *(or cos ax,)*.It found impossible to work out a common formulae since $(D^2 + b^2)$ in denominator makes it to an indeterminate when $D^2 = -a^2$ and $b^2 = a^2$ .Hence equations were considered separately under four types corresponding to four cases as depicted below.

*Case: 1 when* $b^2 = 0$, *Case II when* $b^2 = -a^2$ *Case III when* $b^2 = a^2$ *and case IV when* $b \neq a$

As stated above, by using the proposed formulae, the particular integral can be easily and directly written down, considerably reducing the paper work, Examples to demonstrate the simplicity and applicability of the formulae are also included. The formulae completely eliminate the successive applications of rule of integration by parts or the use of Bernoulli's method to type-I equation and transformation of the last three types by the substitution of exponential equivalences for sinax or cosax and For simplicity and for easy understanding of the formulae are presented by replacing $f(x)$ with. $x^n$ .

## 2. TYPE I EQUATION :

$$D^m y = x^n \sin ax \quad \textbf{\textit{(or cos ax,)}}$$

Here the differential equation $D^m y = f(x).\psi(x)$ represents $m^{\text{th}}$ Integral of $f(x).\psi(x)$ where $f(x)$ is $n^{\text{th}}$ degree polynomial and $\psi(x)$ is $\sin ax \; or \; \cos ax$ . For easy understanding let us consider the typical equation $D^m y = x^n \sin ax$. As stated above the proposed formulae deals only with the PI as CF can be easily written down using conventional methods.

**Proposition: 1 If Differential Equation of Type: 1, then PI ( $y_p$ ) is given by**

$$y_p = \frac{1}{(-a^2)^m}\sum_{r=0}^{r=n}\frac{K_r}{(a^2)^r}.D^r x^n.D^{m+r}\sin ax \qquad \textbf{(A)}$$

**Where numerator coefficient** $K_r = {}^{m+r-1}C_r$

The formula can be further simplified by considering the even and odd orders separately.

$$y_p = \begin{cases} \dfrac{1}{(-a^2)^{\frac{m+1}{2}}}\displaystyle\sum_{r=0}^{r=n}\frac{K_r}{(a^2)^r}.D^r x^n.D^{r+1}\sin ax ............(A_1)..when..m..is..odd \\ \\ \dfrac{1}{(-a^2)^{\frac{m}{2}}}\displaystyle\sum_{r=0}^{r=n}\frac{K_r}{(a^2)^r}D^r x^n D^r \sin ax .......................(A_2) when..m..is..even \end{cases}$$

**Where numerator coefficient** $K_r = {}^{m+r-1}C_r$

The above results are arrived using the concept '**Continued Differential Operator', (**Rajasekhar, 2021). A deductive proof is also provided in the MS, (Iyer, 1976) which is also not included in this paper.

The advantage of the proposed formulae and the easiness with which the 'Integrals' can be worked out for equations of any order are well explicit from the examples illustrated below., wherein the tediousness of successive Integration by parts is totally eliminated. Furthermore, the derived formulae even for the first order is very much simpler than Bernoulli's formulae which is only an extension of Integration by parts and that is limited to first order alone .

Following examples illustrates the applicability of the above formulae and its easiness in producing the results. In conventional methods repeated application of Integration becomes necessary while solving higher orders and it becomes laborious as order becomes higher and higher. The proposed formulae irrespective of the order of the equation produce the results in second or third line which are illustrated below.

**2.1 Illustration (1) of Preposition :**

Applying the proposed formula for finding the PI ( $y_p$ ) on $D^7 y = x^2 \sin .3x$**.**

$$y_p = \frac{1}{(-a^2)^{\frac{m+1}{2}}}\sum_{r=0}^{r=n}.\frac{K_r}{(a^2)^r}.D^r x^n.D^{r+1}\sin ax$$, using ($A_1$) since 'm' is odd

$$= \frac{1}{(-9)^{\frac{7+1}{2}}}\left\{K_0x^2D\sin 3x + \frac{K_1Dx^2D^2\sin .3x}{3^2} + \frac{K_2D^2x^2D^3\sin 3x}{(3^2)^2}\right\}$$

$$= \frac{1}{6561}\left\{3x^2\cos 3x - 14x\sin 3x - \frac{56}{3}\cos 3x\right\} \because K_0 = 1, K_1 = m = 7 \;\&\; K_2 = \frac{m(m+1)}{2} = 28$$

**2.2 Illustration (2) of Proposition : 1**

Applying the proposed formula for finding the PI ( $y_p$ ) on $D^{1000}y = x\cos x$

$$y_p = \frac{1}{(-a^2)^{\frac{m}{2}}}\sum_{r=0}^{r=n}\frac{K_r}{(a^2)^r}.D^r x^n.D^r \sin ax \text{ using } (A_2) \text{ since 'm' is even.}$$

$$= \frac{1}{(-1)^{500}}\left\{K_0x\cos x + \frac{K_1Dx.D\cos x}{(-1)^1}\right\} = x\cos x + 1000\sin x \quad \because K_0 = 1 \;\&\; K_1 = 1000$$

**3. TYPE II EQUATION :**

$$(D^2 + a^2)^m y = x^n \sin ax\,(or\ cos\ ax)$$

**Proposition: 2 If Differential Equation of Type: I1, then PI ( $y_p$ ) is given by**

$$y_p = \sum_{r=0}^{r=n}\frac{K_r}{(2)^{m+r}}.\frac{D^{(r-m)}x^n.D^{(r-m)}\sin ax}{(a^2)^r} \qquad \textbf{(B)}$$

**Where numerator coefficient** $K_r = {}^{m+r-1}C_r$

On scrutiny of the above formula, it can be seen that there will be a total of *(m+n+1)* terms, of which last *'m'* terms will merge with the complimentary function and hence first *(n+1)* terms need be considered in PI. Also it can be seen that first *m* terms will be 'integrated term' and next one term will be the 'function term' and last *n* terms will be differentiated terms. This can be noticed clearly in the Illustration 3.2.

The proposed formula completely eliminates the conventional method of substituting the trigonometric function with exponential equivalences. Applicability of the proposed formula is illustrated below.

**3.1 Illustration (1) of Preposition : 2**

Applying the formula for solving the differential equation $(D^2 + 1)^4 y = 1920x\cos x$ .

Given $(D^2 + 1)^4 y = 1920x\cos x$

Using (B)

$$y_p = \sum_{r=0}^{r=n} \frac{K_r}{(2)^{m+r}} \cdot \frac{D^{(r-m)} x^n . D^{(r-m)} \sin ax}{(a^2)^r} \quad \textbf{and} \quad K_r = {}^{m+r-1}C_r$$

$$= 1920\left\{ \frac{K_0}{2^4} . D^{(0-4)}(x) . D^{(0-4)}(\cos x) + \frac{K_1}{2^5} . D^{(1-4)}(x) D^{(1-4)}(\cos x) \right\}$$

$$= 1920\left\{ \frac{K_0}{2^4} \cdot \frac{1}{D^4}(x) . \frac{1}{D^4}(\cos x) + \frac{K_1}{2^5} \cdot \frac{1}{D^3}(x) \frac{1}{D^3}(\cos x) \right\}$$

$$= x^5 \cos x - 10x^4 \sin x \qquad \lfloor \because K_0 = 1, K_1 = 4 \rfloor$$

### 3.2 Illustration (2) of Preposition : 2

Applying the formula for solving the differential equation $(D^2+1)^4 y = x^3 \sin x$.

Given $(D^2+1)y = x^3 \sin x$

Using (B)

$$y_p = \sum_{r=0}^{r=n} \frac{K_r}{(2)^{m+r}} \cdot \frac{D^{(r-m)} x^n . D^{(r-m)} \sin ax}{(a^2)^r} \quad \textbf{and} \quad K_r = {}^{m+r-1}C_r$$

$$= \left\{ \frac{K_1}{2^1} \cdot \frac{1}{D}(x^3) . \frac{1}{D}(\sin x) + \frac{K_2}{2^2} . D^0(x^3) D^0(\sin x) + \frac{K_3}{2^3} D^1(x^3) D^1(\sin x) + \frac{K_4}{2^4} D^2(x^3) D^2(\sin x) \right\}$$

$\because K_1 = K_2 = K_3 = K_4 = 1$

$$y_p = -\frac{x^4 \cos x}{8} + \frac{x^3 \sin x}{4} + \frac{3x^2 \cos x}{8} - \frac{3x \sin x}{8}$$

## 4. TYPE III EQUATION :

$(D^2 - a^2)^m y = x^n \sin ax$ *(or cos ax,)*

**Proposition: 3 If Differential Equation of Type: I1I, then PI ( $y_p$ ) is given by**

$$y_p = \frac{1}{(-2a^2)^m} \sum_{r=0}^{r=n} \frac{K_r}{C_r} \frac{D^r x^n . D^r \sin ax}{(a^2)^r} \qquad \textbf{(C)}$$

Where

**Numerator coefficients**

$K_0 = 1$ *For any order m*

$$K_r = \begin{cases} \left[K_{(r-1)} - K_{(r-2)}\right]_m + \left[K_r\right]_{(m-1)} \text{.......} when..r..is..odd \\ \\ \left[2K_{(r-1)} - K_{(r-2)}\right]_m + \left[K_r\right]_{(m-1)} \text{.......} when..r..is..even \end{cases}$$

**Denominator coefficients**

$C_r = 1 \quad when\ r = 0$

$$C_r = \begin{cases} = 2^{\frac{r-1}{2}} ......when..r..is..odd \\ = 2^{\frac{r}{2}} .........when..r..is..even \end{cases}$$

From the above relation it can be seen that Numerator coefficients are solely dependent on the order of the equation and formulae for evaluation of numerator coefficients in terms of *m* can be arrive as shown below.

$K_0 = 1$

$K_1 = m$

$K_2 = m^2$

$$K_3 = \frac{m(m^2 - 1)}{3}$$

$$K_4 = \frac{m(m+1)(m^2 - m - 3)}{6}$$

Proceeding further coefficients can be evaluated for any order and any degree. For easiness of computation values are tabulated below. (Refer Table: 1)

**TABLE: 1 Numerator Coefficients**

| $m$ | $K_0$ | $K_1$ | $K_2$ | $K_3$ | $K_4$ | $K_5$ | $K_6$ | $K_7$ | $K_8$ | $K_9$ | $K_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 1 | 1 | 1 | 0 | -1 | -1 | -1 | 0 | 1 | 1 | 1 |
| **2** | 1 | 2 | 4 | 2 | -1 | -4 | -8 | -4 | 1 | 6 | 12 |
| **3** | 1 | 3 | 9 | 8 | 6 | -6 | -26 | -24 | -21 | 9 | 51 |
| **4** | 1 | 4 | 16 | 20 | 30 | 4 | -48 | -76 | -125 | -40 | 96 |
| **5** | 1 | 5 | 25 | 40 | 85 | 49 | -35 | -160 | -410 | -290 | -74 |
| **6** | 1 | 6 | 36 | 70 | 189 | 168 | 112 | -216 | -954 | -1028 | -1176 |
| **7** | 1 | 7 | 49 | 112 | 364 | 420 | 588 | -48 | -1638 | -2618 | -4774 |
| **8** | 1 | 8 | 64 | 168 | 636 | 888 | 1728 | 792 | -1782 | -5192 | -13376 |
| **9** | 1 | 9 | 81 | 240 | 1035 | 1683 | 4059 | 3168 | 495 | -7865 | -29601 |
| **10** | 1 | 10 | 100 | 330 | 1595 | 2948 | 8360 | 8580 | 9295 | -7150 | -53196 |

On the scrutinizing the above table, for the first order equations it can be seen that the numerator coefficient for every fourth term is zero.

**4.1 Illustration (1) of Preposition : 3**

Applying the proposed formula for finding the PI ( $y_p$ ) on the same equation $(D^2-1)y = x^2\cos x$

$$y_p = \frac{1}{\left(-2a^2\right)^m}\sum_{r=0}^{r=n}\frac{K_r}{C_r}\frac{D^r x^n . D^r \sin ax}{(a^2)^r} \qquad \text{using (C)}$$

$$=\left(\frac{1}{-2}\right)\left\{x^2\cos x + Dx^2 D\cos x + \frac{D^2x^2D^2\cos x}{2}\right\}$$

$$=\frac{1}{-2}\left\{x^2\cos x - 2x\sin x + -\cos x\right\}$$

$\because K_1 = K_2 = K_3 = 1$ and $C_1 = 1$, $C_2 = 2$

**4.2 Illustration (2) of Preposition : 3**

Applying the formula for solving the differential equation $(D^2-1)y = x^3\sin x$

$$y_p = \frac{1}{\left(-2a^2\right)^m}\sum_{r=0}^{r=n}\frac{K_r}{C_r}\frac{D^r x^n . D^r \sin ax}{(a^2)^r} \quad \text{............} \quad \text{using (C)}$$

$$=\left(\frac{1}{-2}\right)\left\{x^3\sin x + \frac{K_1}{C_1}Dx^3D\sin x + \frac{K_2}{C_2}\frac{D^2x^3D^2\sin x}{2} + 0\right\}$$

$$=-\frac{1}{2}\left\{x^3\sin x + 3x^2\cos x - 3x\sin x\right\}$$

$\because K_1 = K_2 = K_3 = 1$ and $C_1 = 1$, $C_2 = 2$

**4.3 Illustration (3) of Preposition : 3**

Applying the formula for solving the differential equation $(D^2-1)^{10}y = x\sin x$

$$y_p = \frac{1}{\left(-2a^2\right)^m}\sum_{r=0}^{r=n}\frac{K_r}{C_r}\frac{D^r x^n . D^r \sin ax}{(a^2)^r} \qquad \text{using } \textbf{(C)}$$

$$=\frac{1}{(-2)^{10}}\left\{x\sin x + \frac{K_1}{C_1}DxD\sin x\right\}$$

$$=\frac{1}{2^{10}}\left\{x\sin x + 10\cos x\right\}$$

$K_1 = m = 10; C_1 = 1$

It is quite cumbersome to solve the equations of higher orders like the above using conventional method but applying proposed formula **(C),** the solution can be easily obtainable.

## 5. TYPE IV EQUATION :

$$(D^2 \pm b^2)^m y = x^n \sin ax \,(or\ cos\ ax,)\ where\ \ b \neq a$$

**Proposition: 4 If Differential Equation of Type: IV, then PI ( $y_p$ ) is given by**

$$y_p = \frac{1}{(\theta)^m} \sum_{r=0}^{r=n} \frac{K_r}{C_r} \frac{D^r x^n . D^r \sin ax}{(\theta)^r} \qquad \textbf{(D)}$$

Where $\theta = \left(-a^2 + b^2\right)$ ; $K_r$ and $C_r$ are numerator and denominator coefficients respectively.

**Numerator coefficients**

$K_0 = 1$ *For any order m*

$$K_r = \begin{cases} \left[-2K_{(r-1)} + \theta . K_{(r-2)}\right]_m + \left[K_r\right]_{(m-1)} .............when..r..is..odd \\ \\ \left[-2a^2 K_{(r-1)} + \theta . K_{(r-2)}\right]_m + \left[K_r\right]_{(m-1)} .......when..r..is..even \end{cases}$$

**Denominator coefficients**

$C_r = 1 \quad when\ \ r = 0$

$$C_r = \begin{cases} \left(a^2\right)^{\frac{r-1}{2}} ......when..r..is..odd \\ \left(a^2\right)^{\frac{r}{2}} ..........when..r..is..even \end{cases}$$

The numerator coefficients can be evaluated from the above relation but deriving general expression as given below for the same found to be difficult as order increases since it involves '*a*' and $\theta$ also apart from '*m*'. However by generating a table the coefficients can be easily evaluated.

$K_1 = -2m$ ; $K_2 = m\left\{2a^2(m+1) + \theta\right\}$ and so on.

**5.1 Illustration (1) of Preposition : 4**

Applying the formula for solving the equation, $(D^2 - 1)^4 y = x^2 \cos 2x$

$$y_p = \frac{1}{(\theta)^4}\left\{x^2 \cos 2x + \frac{K_1}{C_1}\frac{Dx^2 D \cos 2x}{\theta} + \frac{K_2}{C_2}\frac{D^2 x^2 D^2 \cos 2x}{\theta^2}\right\} \text{ using } \textbf{(D)}$$

$$= \frac{1}{5^4}\left\{x^2 \cos 2x - \frac{32}{5}(x \sin 2x) - \frac{56}{5}(\cos 2x)\right\}$$

$$\because \theta = \left(-a^2 + b^2\right) = -4 - 1 = -5\ \ K_1 = -8 ; K_2 = 4(8 \times 5 - 5) = 140 ;\ C_1 = 1 ; C_2 = a^2 = 4$$

**5.2 Illustration (2) of Preposition : 4**

Applying the proposed formula for solving the equation, $\left(D^2 + 10\right)^5 y = x^3 \sin 3x$

$$y_p = \frac{1}{(-9+10)^5}\left\{x^2 \sin 3x + \frac{K_1}{C_1} Dx^2 D \sin 3x + \frac{K_2}{C_2} D^2 x^2 D^2 \sin 3x\right\} \qquad \text{using } \textbf{(D)}$$

$$= \left\{x^2 \sin 3x - 60x \cos 3x - 1090 \sin 3x\right\}$$

$$\because \theta = \left(-a^2 + b^2\right) = -9 + 10 = 1\ ;\ \ K_1 = 10 ; K_2 = 5(108 + 1) = 545 ;\ C_1 = 1 ; C_2 = 9$$

## 6. CONCLUSION

6.1 Proposed formulae gives direct, easy and explicit solution for the equations considered. Moreover with the formulae proposed the Particular Integrals (PI) can be written down in an elegant manner in the descending degree of the algebraic terms.

6.2 On examination , it could be seen that proposed formulae are similar to finite series expansions containing ‘ $n$ ’ terms where ‘ $n$ ’ is the degree of the algebraic expression with each term containing either as $D^r x^n D^r \sin ax$ or $D^r x^n D^{r+1} \sin ax$ depending up on the equation considered. The cases where $\sin ax$ is replaced by $\cos ax, \sinh ax, \cosh ax$ are also same.

6.3 The deductive proof for each of the formulae are given in the MS, (Iyer, 1976). These are not included in this paper.

6.4 The method by which these results were arrived are also depicted in detail in the above mentioned MS which are also not included in this paper, (Rajasekhar, 2021).

6.5 The work contains interesting study on the nature and properties of the numerator and denominator coefficients of the proposed formulae

6.6 As stated earlier the proposed formulae for the solution equations when *g(x) = sin ax* or *cos ax* can be extended with modifications for the cases when *g(x) =* $e^{ax}$ *g(x) = cosh ax* and *g(x) = sinh ax.*

6.7 The proposed formulae can be effectively used for developing computer programmes for getting closed form (explicit) solutions for the differential equations instead of numerical solutions.

6.8 Similarly the solution of equations when RHS is $(D \pm a)^m y = x^n \sin ax$ (or cos ax) can also be brought under Type-III -Equation, since

$$y_p = \frac{1}{(D \pm a)^m}\left(x^n \sin ax\right) = \left\{\frac{(D \mp a)^m}{(D^2 - a^2)^m}\right\}\left(x^n \sin ax\right)$$

$$y_p = \frac{1}{(D \pm a)^m}\left(x^n \cos ax\right) = \left\{\frac{(D \mp a)^m}{(D^2 - a^2)^m}\right\}\left(x^n \cos ax\right)$$

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# ANALYSIS OF HETEROGENEOUS FEEDBACK QUEUE SYSTEM WITH AT MOST ONE REVISIT

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## ABSTRACT
***In this paper a queuing model has been developed for a system in which a service channel commonly linked with two service channels which are heterogeneous. A customer is allowed to revisit the service channels one time and the probability of second visit has been taken different than the probability for the first visit. The arrival and departure process follows poisson distribution at each service channel. The main objective of the paper is to analyze the behaviour of the heterogeneous feedback queue system when probability of second visit has been changed and when customers arrive at different arrival rates at different service channels.***
***Keywords: Queuing, Heterogeneous, Feedback, Revisit,***

## 1. INTRODUCTION

On studying the literature of multi-server queue models it has been observed that the main assumption that the servers are of identical in nature has been taken in account while this situation is not realistic and exist only when the process is highly mechanized. In case of human servers, the people cannot work at the same rate as they vary in their efficiency. Hence, in real world situation the system is heterogeneous in nature. And in feedback queuing models a customer/ group of customers may revisit to any of the server, if either they get some defected item or they are unsatisfied with the service due to some reason. Many researchers put their efforts in describing the feedback queuing model. In their queuing model some researchers take the same probability of leaving the server as that was of leaving the server of first time. Others take different probability of leaving the server second time as that of first time. The researchers who contributed in such work include Surinder Kumar & Gulshan Taneja[6] discussed a homogenous feedback queuing model with three servers and different probability in second visit as that taken in first visit. Singh T.P and Kusum did a lot of research in developing in feedback queue models. Dr. Kusum[8] discussed a network of queue system which consists two heterogeneous service channels and both commonly linked with a single service channel. The limitations of the model are

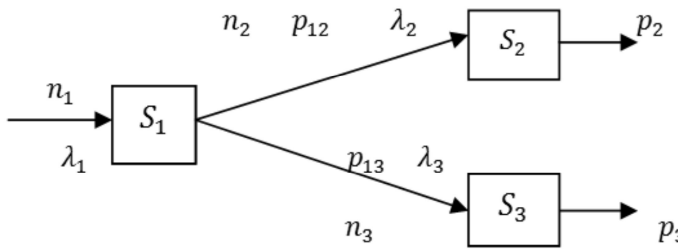
(i) On serving the customers in second time the probability of leaving the servers has not been changed.

(ii) The number of revisit of a customer has not been specified in the model.

In present paper we try to remove both the limitation of the above said paper. An effort has been made by us to discuss the behaviour of the model when customer revisit the system one time and leaves the system with changed probability. Also the customers arrive at different service channels with different arrival rate. The differential difference equation has been formed and we obtained mean queue length by using generating function technique. Numerical illustration has taken to check the validity of the result.

## 2. MODEL DISCRIPTION

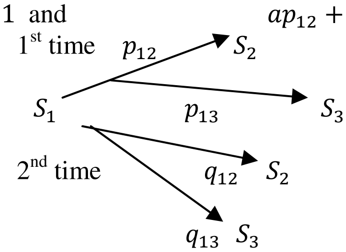
The model consists three service channels $S_1, S_2, S_3$.Service channels $S_2$ and $S_3$ are heterogeneous which are commonly connected to service channel $S_1$.Initially the customer arrives at service channel $S_1$ and after being served the customer may move to either $S_2$ or $S_3$ and if customer is unsatisfied then he can revisit the server maximum one time.



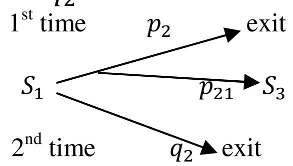

**NOTATIONS:**

| Service channels | $S_1$ | | $S_2$ | | $S_3$ | |
|---|---|---|---|---|---|---|
| Arrival rate | $\lambda_1$ | | $\lambda_2$ | | $\lambda_3$ | |
| Service rate | $\mu_1$ | | $\mu_2$ | | $\mu_3$ | |
| No. of customers | $n_1$ | | $n_2$ | | $n_3$ | |
| Probability of customers to move from one server to another | 1st time | $S_1 \rightarrow S_2$ $p_{12}$<br>$S_1 \rightarrow S_3$ $p_{13}$ | 1st time | $S_1 \rightarrow exit$ $p_2$<br>$S_2 \rightarrow S_1$ $P_{21}$ | 1st time | $S_1 \rightarrow exit$ $p_3$<br>$S_3 \rightarrow S_1$ $P_{31}$ |
| | 2nd time | $S_1 \rightarrow S_2$ $q_{12}$<br>$S_1 \rightarrow S_3$ $q_{13}$ | 2nd time | $S_2 \rightarrow exit$ $q_2$ | 2nd time | $S_3 \rightarrow exit$ $q_3$ |
| Probability of customers leaving the server | 1st time | a | 1st time | b | 1st time | c |
| | 2nd time | $a'$ | 2nd time | $b'$ | 2nd time | $c'$ |

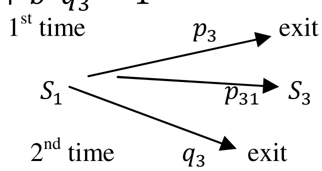
Possible states of leaving the server $S_1$: Firstly the customers will come at service channel $S_1$ with poisson arrival rate $\lambda_1$ and after being served the customer will leave the server with probability a and he goes to either server $S_2$ or server $S_3$ with probabilities $p_{12}$ and $p_{13}$ such that $p_{12} + p_{13} = 1$ .If the customer revisit ,then after getting service at service channel $S_1$,the customer will leave the server with probability $a'$ and he will again go to either server $S_2$ or server $S_3$ for service with probability $q_{12}$ and $q_{13}$ such that $q_{12} + q_{13} = 1$ and $ap_{12} + ap_{13} + a' q_{12} + a' q_{13} = 1$



Possible states of leaving the server $S_2$: The customers will arrive at server $S_2$ with arrival rate $\lambda_2$. After getting the service from server $S_2$ the customer leaves the server with probability b and either he may exit with probability $p_2$ or goes back with probability $p_{21}$ for service such that $p_2 + p_{21} = 1$ and after revisit he will leave the server with probability $b'$ and exit the system with probability $q_2$ such that, $bp_2 + bp_{21} + b'\, q_2 = 1$




Possible states of leaving the server $S_3$: At server $S_3$ the customers will arrive with arrival rate $\lambda_3$. After getting the service from server $S_3$ the customer leaves the server with probability c and either he may exit with probability $p_3$ or goes back to $S_1$ with probability $p_{31}$ for service such that $p_3 + p_{31} = 1$ and after revisit he will leave the server with probability $c'$ and exit the system with probability $q_3$ such that, $cp_3 + cp_{31} + b'\, q_3 = 1$




## FORMULATION OF DIFFERENTIAL DIFFERENCE EQUATIONS

Let $P_{n_1,n_2,n_3}$ (t) denotes the probability of $n_1$ , $n_2$, $n_3$ customers waiting for service at the queues in front of the servers $S_1, S_2, S_3$ respectively where $n_1$ , $n_2$, $n_3 \geq 0$ .The differential difference equations in steady state are as follows:

$$(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2 + \mu_3)P_{n_1,n_2,n_3} = \lambda_1 P_{n_1-1,n_2,n_3} + \lambda_2 P_{n_1,n_2-1,n_3} + \lambda_3 P_{n_1,n_2,n_3-1} + \mu_1(ap_{12} + a'\, q_{12})P_{n_1+1,n_2-1,n_3} + \mu_1(ap_{13} + a'\, q_{13})P_{n_1+1,n_2,n_3-1} + \mu_2(bp_{21})P_{n_1-1,n_2+1,n_3} + \mu_2(bp_2 + b'\, q_2)P_{n_1,n_2+1,n_3} + \mu_3(cp_{31})P_{n_1-1,n_2,n_3+1} + \mu_3(cp_3 + c'\, q_3)P_{n_1,n_2,n_3+1}, \quad n_1, n_2, n_3 \geq 0 \tag{1}$$

For $n_1 = 0$ , $n_2$, $n_3$ >0

$$(\lambda_1 + \lambda_2 + \lambda_3 + \mu_2 + \mu_3)P_{0,n_2,n_3} = \lambda_2 P_{0,n_2-1,n_3} + \lambda_3 P_{0,n_2,n_3-1} + \mu_1(ap_{12} + a'\, q_{12})P_{1,n_2-1,n_3} + \mu_1(ap_{13} + a'\, q_{13})P_{1,n_2,n_3-1} + \mu_2(bp_2 + b'\, q_2)P_{0,n_2+1,n_3} + \mu_3(cp_3 + c'\, q_3)P_{0,n_2,n_3+1}, \tag{2}$$

For $n_2 = 0$ , $n_1$, $n_3$ >0

$$(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_3)P_{n_1,0,n_3} = \lambda_1 P_{n_1-1,0,n_3} + \lambda_3 P_{n_1,0,n_3-1} + \mu_1(ap_{13} + a'\, q_{13})P_{n_1+1,0,n_3-1} + \mu_2(bp_{21})P_{n_1-1,1,n_3} + \mu_2(bp_2 + a'\, q_2)P_{n_1,1,n_3} + \mu_3(cp_{31})P_{n_1-1,0,n_3+1} + \mu_3(cp_3 + c'\, q_3)P_{n_1,0,n_3+1}, \tag{3}$$

For $n_3 = 0$ , $n_1$, $n_2$ >0

$$(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1 + \mu_2)P_{n_1,n_2,0} = \lambda_1 P_{n_1-1,n_2,0} + \lambda_2 P_{n_1,n_2-1,0} + \mu_1(ap_{12} + a'\, q_{12})P_{n_1+1,n_2-1,0} + \mu_2(bp_{21})P_{n_1-1,n_2+1,0} + \mu_2(bp_2 + b'\, q_2)P_{n_1,n_2+1,0} + \mu_3(cp_{31})P_{n_1-1,n_2,1} + \mu_3(cp_3 + c'\, q_3)P_{n_1,n_2,1}, \tag{4}$$

For $n_1 = n_2 = 0$, $n_3$ >0

$(\lambda_1 + \lambda_2 + \lambda_3 + \mu_3)P_{0,0,n_3} = \lambda_3 P_{0,0,n_3-1} + \mu_1(ap_{13} + a' q_{13})P_{1,0,,n_3-1} + \mu_2(bp_2 + b' q_2)P_{0,1,n_3} + \mu_3(cp_3 + c' q_3)P_{0,0,n_3+1}$ , (5)

For $n_1 = n_3 = 0, n_2 > 0$

$(\lambda_1 + \lambda_2 + \lambda_3 + \mu_3)P_{0,n_2,0} = \lambda_2 P_{0,n_2-1,0} + \mu_1(ap_{12} + a' q_{12})P_{1,n_2-1,0} + \mu_2(bp_2 + b' q_2)P_{0,n_2+1,0} + \mu_3(cp_3 + c' q_3)P_{0,n_2,1}$ , (6)

For $n_2 = n_3 = 0, n_1 > 0$

$(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1)P_{n_1,0,0} = \lambda_1 P_{n_1-1,0,0} + \mu_2(bp_{21})P_{n_1-1,1,0} + \mu_2(bp_2 + b' q_2)P_{n_1,1,0} + \mu_3(cp_{31})P_{n_1-1,0,1} + \mu_3(cp_3 + c' q_3)P_{n_1,0,1}$ , (7)

For $n_1 = n_2 = n_3 = 0$

$$(\lambda_1 + \lambda_2 + \lambda_3)P_{0,0,0} = \mu_2(bp_2 + b' q_2)P_{0,1,0} + \mu_3(cp_3 + c' q_3)P_{0,0,1} \tag{8}$$

## SOLUTION PROCESS

For solution of steady state equations from (1) to (8),let us define Generating function and partial generating functions as follows:

$$F(X,Y,Z)=\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} P_{n_1,n_2,n_3}(t)\, X^{n_1}Y^{n_2}Z^{n_3} \tag{9}$$

Where $|X| = |Y| = |Z| = 1$

Also, $$F_{n_2,n_3,}(X) = \sum_{n_1=0}^{\infty} P_{n_1,n_2,n_3}(t)\, X^{n_1} \tag{10}$$

, $$F_{n_3}(X,Y) = \sum_{n_2=0}^{\infty} F_{n_2,n_3}(X)\, Y^{n_2} \tag{11}$$

$$F(X,Y,Z) =\sum_{n_3=0}^{\infty} F_{n_3}(X,Y)\, Z^{n_3} \tag{12}$$

Solving equations (1) to (8) , with equations (9) to (12) and following the steps of Singh T.P & Kusum(2011) we derive the final equation as:

$$F(X,Y,Z) = \frac{\mu_1\left[1-\frac{(ap_{12}+a' q_{12})}{X}Y - \frac{(ap_{13}+a' q_{13})}{X}Z\right]F_1+\mu_2\left[1-\frac{(bp_{21})}{Y}X - \frac{(bp_2+b' q_2)}{Y}\right]F_2+\mu_3\left[1-\frac{(cp_{31})}{Z}X - \frac{(cp_3+c' q_3)}{Z}\right]F_3}{\lambda_1(1-X)+\lambda_2(1-Y)+\lambda_3(1-Z)+\mu_1\left[1-\frac{(ap_{12}+a' q_{12})}{X}Y - \frac{(ap_{13}+a' q_{13})}{X}Z\right]+\mu_2\left[1-\frac{(bp_{21})}{Y}X - \frac{(bp_2+b' q_2)}{Y}\right]+\mu_3\left[1-\frac{(cp_{31})}{Z}X - \frac{(cp_3+c' q_3)}{Z}\right]} \tag{13}$$

Assuming $F_0(Y,Z) = F_1$, $F_0(X,Z) = F_2$, $F_0(X,Y) = F_3$

At X=Y=Z=1,F(X,Y,Z) = 1 and the right hand side if equation (13) reduces to $\frac{0}{0}$ indeterminate form. Considering X→1 & Y=Z=1 and applying L' Hospital rule we get,

$$-\lambda_1+\mu_1+\mu_2(-bp_{21})+\mu_3(-cp_{31}) = \mu_1 F_1 + \mu_2(-bp_{21})F_2 + \mu_3(-cp_{31})F_3 \tag{14}$$

Taking Y→1 & X=Z=1 and using L'Hospital rule ,from (13) we get,

$$-\lambda_2-\mu_1(ap_{12} + a' q_{12})+\mu_2 = -\mu_1(ap_{12} + a' q_{12})F_1+\mu_2 F_2 \tag{15}$$

Taking Z→1 & X=Y=1 and using L'Hospital rule ,from (13) we get,

$$-\lambda_3-\mu_1(ap_{13} + a' q_{13})+\mu_3 = -\mu_1(ap_{13} + a' q_{13})F_1+\mu_3 F_3 \tag{16}$$

On solving equations (14),(15) and(16), we get

$$F_1 = 1 - \frac{(\lambda_1 + \lambda_2 bp_{21} + \lambda_3 cp_{31})}{\mu_1(1-(ap_{12}+a' q_{12})bp_{21} - (ap_{13}+a' q_{13})cp_{31})} \tag{17}$$

$$F_2 = 1 - \frac{\lambda_1(ap_{12}+a' q_{12}) + \lambda_2(1-(ap_{13}+a' q_{13})cp_{31}) + \lambda_3 cp_{31}(ap_{12}+a' q_{12})}{\mu_2(1-(ap_{12}+a' q_{12})bp_{21} - (ap_{13}+a' q_{13})cp_{31})} \tag{18}$$

$$F_1 = 1 - \frac{\lambda_1(ap_{13}+a' q_{13}) + \lambda_2\left((ap_{13}+a' q_{13})bp_{21}\right) + \lambda_3(1-(ap_{12}+a' q_{12})bp_{21}}{\mu_3(1-(ap_{12}+a' q_{12})bp_{21} - (ap_{13}+a' q_{13})cp_{31})} \tag{19}$$

The solution of differential difference equations in steady state is given by,

$P_{n_1,n_2,n_3} = {\rho_1}^{n_1}{\rho_2}^{n_2}{\rho_3}^{n_3}(1-\rho_1)(1-\rho_2)(1-\rho_3)$, where

$$\rho_1 = \frac{(\lambda_1 + \lambda_2 bp_{21} + \lambda_3 cp_{31})}{\mu_1(1-(ap_{12}+a' q_{12})bp_{21} - (ap_{13}+a' q_{13})cp_{31})} \tag{20}$$

$$\rho_2 = \frac{\lambda_1(ap_{12}+a' q_{12}) + \lambda_2(1-(ap_{13}+a' q_{13})cp_{31}) + \lambda_3 cp_{31}(ap_{12}+a' q_{12})}{\mu_2(1-(ap_{12}+a' q_{12})bp_{21} - (ap_{13}+a' q_{13})cp_{31})} \tag{21}$$

$$\rho_3 = \frac{\lambda_1(ap_{13}+a' q_{13}) + \lambda_2\left((ap_{13}+a' q_{13})bp_{21}\right) + \lambda_3(1-(ap_{12}+a' q_{12})bp_{21}}{\mu_3(1-(ap_{12}+a' q_{12})bp_{21} - (ap_{13}+a' q_{13})cp_{31})} \tag{22}$$

Solution of the model exists if, $\rho_{1,}\rho_2, \rho_3 < 1$

Expected queue length for the system is, L = $L_{Q_1} + L_{Q_2} + L_{Q_3}$

Now let, F(X,Y,Z) = $\frac{f}{g}$ where f = $\mu_1\left[1 - \frac{(ap_{12}+a' q_{12})}{X}Y \quad - \frac{(ap_{13}+a' q_{13})}{X}Z\right]F_1 + \mu_2\left[1 - \frac{(bp_{21})}{Y}X \quad - \frac{(bp_2+b' q_2)}{Y}\right]F_2 +$
$\mu_3\left[1 - \frac{(cp_{31})}{Z}X \quad - \frac{(cp_3+c' q_3)}{Z}\right]F_3$

g $=\lambda_1(1-X) + \lambda_2(1-Y) + \lambda_3(1-Z) + \mu_1\left[1 - \frac{(ap_{12}+a' q_{12})}{X}Y \quad - \frac{(ap_{13}+a' q_{13})}{X}Z\right] + \mu_2\left[1 - \frac{(bp_{21})}{Y}X \quad - \frac{(bp_2+b' q_2)}{Y}\right] +$
$\mu_3\left[1 - \frac{(cp_{31})}{Z}X \quad - \frac{(cp_3+c' q_3)}{Z}\right]$

and using $L_{Q_1} = \frac{\left(\frac{\partial g}{\partial X}\right)\left(\frac{\partial^2 f}{\partial X^2}\right) - \left(\frac{\partial f}{\partial X}\right)\left(\frac{\partial^2 g}{\partial X^2}\right)}{2\left(\frac{\partial g}{\partial X}\right)^2}$

$$L_{Q_2} = \frac{\left(\frac{\partial g}{\partial Y}\right)\left(\frac{\partial^2 f}{\partial Y^2}\right) - \left(\frac{\partial f}{\partial Y}\right)\left(\frac{\partial^2 g}{\partial Y^2}\right)}{2\left(\frac{\partial g}{\partial Y}\right)^2} \qquad L_{Q_3} = \frac{\left(\frac{\partial g}{\partial Z}\right)\left(\frac{\partial^2 f}{\partial Z^2}\right) - \left(\frac{\partial f}{\partial Z}\right)\left(\frac{\partial^2 g}{\partial Z^2}\right)}{2\left(\frac{\partial g}{\partial Z}\right)^2}$$

We get,

$$L_{Q_1} = \frac{(\lambda_1 + \lambda_2 bp_{21} + \lambda_3 cp_{31})}{\left(1-(ap_{12}+a' q_{12})bp_{21} - (ap_{13}+a' q_{13})cp_{31}\right)\left(-\lambda_1 + \mu_1 + \mu_2(-bp_{21}) + \mu_3(-cp_{31})\right)} \tag{23}$$

$$L_{Q_2} = \frac{\lambda_1(ap_{12}+a' q_{12}) + \lambda_2(1-(ap_{13}+a' q_{13})cp_{31}) + \lambda_3 cp_{31}(ap_{12}+a' q_{12})}{(-\lambda_2 - \mu_1(ap_{12}+a' q_{12}) + \mu_2)(1-(ap_{12}+a' q_{12})bp_{21} - (ap_{13}+a' q_{13})cp_{31})} \tag{24}$$

$$L_{Q_3} = \frac{\lambda_1(ap_{13}+a' q_{13}) + \lambda_2\left((ap_{13}+a' q_{13})bp_{21}\right) + \lambda_3(1-(ap_{12}+a' q_{12})bp_{21}}{(-\lambda_3 - \mu_1(ap_{13}+a' q_{13}) + \mu_3)(1-(ap_{12}+a' q_{12})bp_{21} - (ap_{13}+a' q_{13})cp_{31})} \tag{25}$$

Hence mean queue length=
L=$\frac{1}{(1-(ap_{12}+a' q_{12})bp_{21} - (ap_{13}+a' q_{13})cp_{31})}\left[\frac{(\lambda_1 + \lambda_2 bp_{21} + \lambda_3 cp_{31})}{(-\lambda_1 + \mu_1 + \mu_2(-bp_{21}) + \mu_3(-cp_{31}))} +\right.$

$$\frac{\lambda_1(ap_{12}+a' q_{12})+\lambda_2(1-(ap_{13}+a' q_{13})cp_{31})+\lambda_3 cp_{31}(ap_{12}+a' q_{12})}{(-\lambda_2-\mu_1(ap_{12}+a' q_{12})+\mu_2)}+\frac{\lambda_1(ap_{13}+a' q_{13})+\lambda_2((ap_{13}+a' q_{13})bp_{21})+\lambda_3(1-(ap_{12}+a' q_{12})bp_{21}}{(-\lambda_3-\mu_1(ap_{13}+a' q_{13})+\mu_3)}\Bigg]$$

(26)

Average waiting time

$$=\frac{L}{\lambda}=\frac{1}{\lambda(1-(ap_{12}+a' q_{12})bp_{21}-(ap_{13}+a' q_{13})cp_{31})}\Bigg[\frac{(\lambda_1+\lambda_2 bp_{21}+\lambda_3 cp_{31})}{\left(-\lambda_1+\mu_1+\mu_2(-bp_{21})+\mu_3(-cp_{31})\right)}+$$
$$\frac{\lambda_1(ap_{12}+a' q_{12})+\lambda_2(1-(ap_{13}+a' q_{13})cp_{31})+\lambda_3 cp_{31}(ap_{12}+a' q_{12})}{(-\lambda_2-\mu_1(ap_{12}+a' q_{12})+\mu_2)}+\frac{\lambda_1(ap_{13}+a' q_{13})+\lambda_2((ap_{13}+a' q_{13})bp_{21})+\lambda_3(1-(ap_{12}+a' q_{12})bp_{21}}{(-\lambda_3-\mu_1(ap_{13}+a' q_{13})+\mu_3)}\Bigg]$$ ,where $\lambda = \lambda_1+\lambda_2+\lambda_3$

## BEHAVIOR ANALYSIS OF THE MODEL

Now we will discuss about the behaviour analysis of $L_{Q_1}, L_{Q_2}, L_{Q_3}$ and L(mean queue length) in three different ways;

(I) Behaviour analysis of $L_{Q_1}, L_{Q_2}, L_{Q_3}$ and L(mean queue length) with respect to poisson arrival rate $\lambda_1$, as shown in the table:

| Service rate | $\mu_1 = 11$ | $\mu_2 = 12$ | $\mu_3 = 13$ | Arrival rate $\lambda_2$=1, $\lambda_3$=2 |
|---|---|---|---|---|
| Probability | 1st time | 2nd time | | |
| | a =0.4 , $p_{12}$=0.4, $p_{13}$=0.6 | $a'$ =0.6 , $q_{12}$ =0.3 , $q_{13}$=0.7 | | |
| | b =0.7 , $p_{21}$=0.2 | $b'$=0.3 | | |
| | c =0.8, $p_{31}$=0.1 | $c'$=0.2 | | |
| Arrival rate ($\lambda_1$) | $L_{Q_1}$ | $L_{Q_2}$ | $L_{Q_3}$ | L |
| 2 | 0.4071 | 0.2574 | 0.9859 | 1.6504 |
| 2.5 | 0.5385 | 0.2835 | 1.0840 | 1.906 |
| 3 | 0.6947 | 0.3095 | 1.1821 | 2.1863 |
| 3.5 | 0.8837 | 0.3355 | 1.2801 | 2.4993 |
| 4 | 1.1167 | 0.3616 | 1.3783 | 2.8566 |
| 4.5 | 1.4116 | 0.3876 | 1.4763 | 3.2755 |
| 5 | 1.7961 | 0.4136 | 1.5744 | 3.7841 |
| 5.5 | 2.3191 | 0.4397 | 1.6725 | 4.4313 |
| 6 | 3.0715 | 0.4657 | 1.7706 | 5.3078 |

Graphical representation of $L_{Q_1}, L_{Q_2}, L_{Q_3}$ and L(mean queue length) with respect to poisson arrival rate $\lambda_1$, as shown below in figures 1 to figure 4.

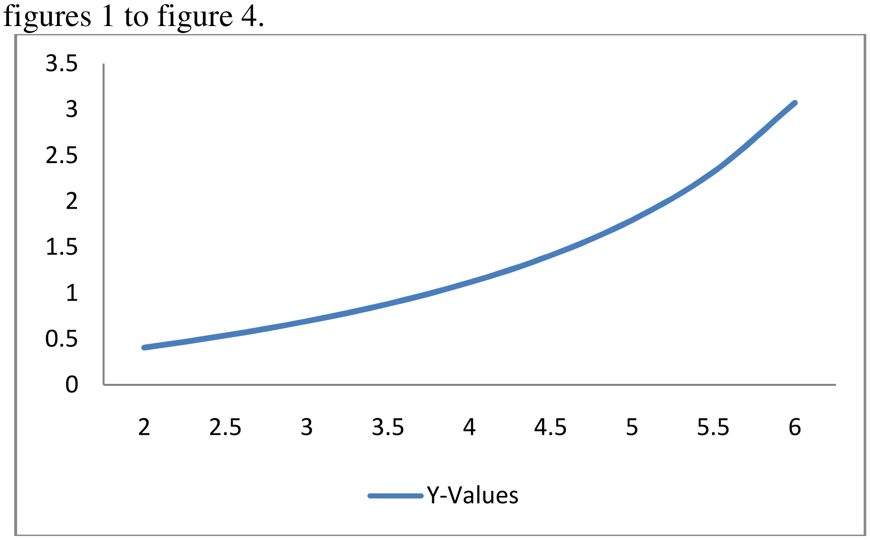



**Figure 1.** $L_{Q_1}$ **vs** $\lambda_1$**(x-axis represents different values of** $\lambda_1$**and y-axis of** $L_{Q_1}$**)**

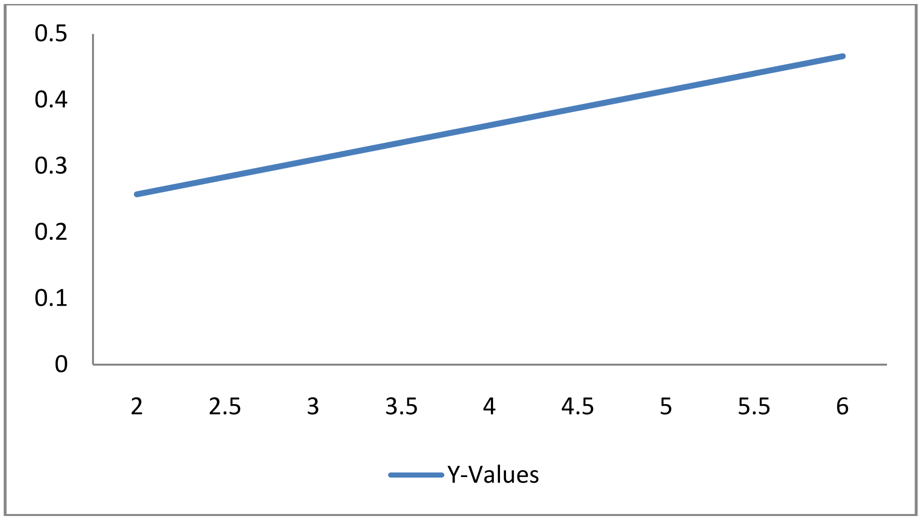



**Figure 2.** $L_{Q_2}$ vs $\lambda_1$(x-axis represents different values of $\lambda_1$and y-axis of $L_{Q_2}$)

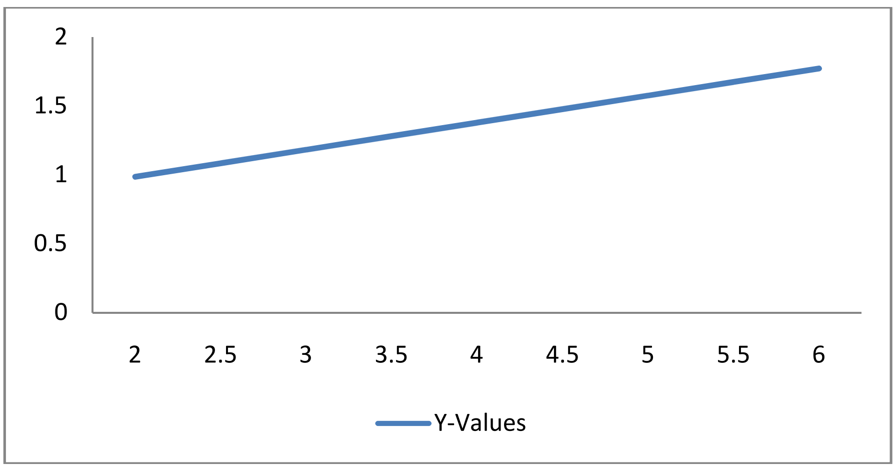



**Figure 3.** $L_{Q_3}$ vs $\lambda_1$(x-axis represents different values of $\lambda_1$and y-axis of $L_{Q_3}$)

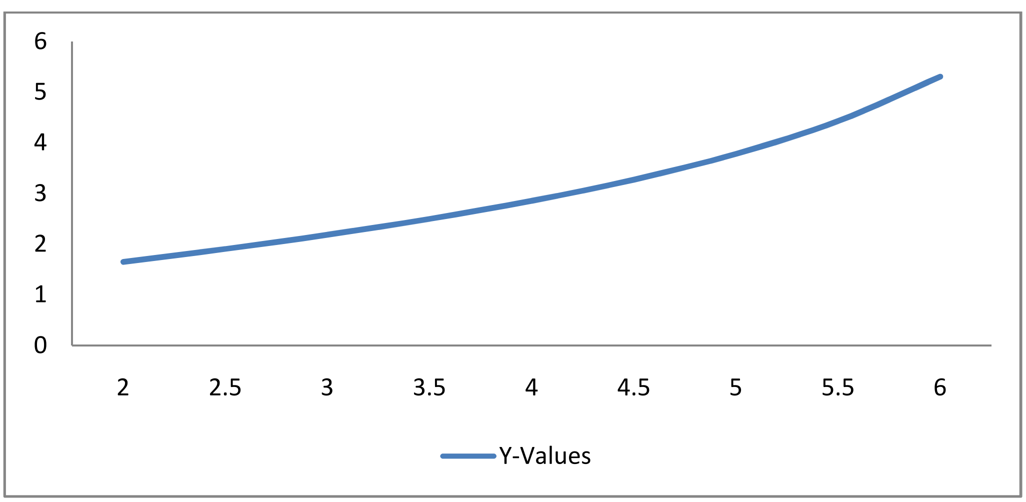



**Figure 4.** L vs $\lambda_1$(x-axis represents different values of $\lambda_1$and y-axis of L)

(II) Behaviour analysis of $L_{Q_1}, L_{Q_2}, L_{Q_3}$ and L(mean queue length) with respect to service rate $\mu_1$, as shown in the table:

| | | | | |
|---|---|---|---|---|
| Arrival rate | $\lambda_1 = 2$ | $\lambda_2 = 1$ | $\lambda_3 = 2$ | Service rate $\mu_2$= 12, $\mu_3$= 13 |
| Probability | 1[st] time | 2[nd] time | | |
| | a=0.4 , $p_{12}$=0.4, $p_{13}$=0.6 | $a'$ =0.6 , $q_{12}$ =0.3 , $q_{13}$=0.7 | | |
| | b =0.7 , $p_{21}$=0.2 | $b'$=0.3 | | |
| | c =0.8, $p_{31}$=0.1 | $c'$=0.2 | | |
| Service rate ($\mu_1$) | $L_{Q_1}$ | $L_{Q_2}$ | $L_{Q_3}$ | L |
| 11 | 0.4071 | 0.2574 | 0.9859 | 1.6504 |
| 11.5 | 0.3770 | 0.2636 | 1.0813 | 1.7219 |
| 12 | 0.3511 | 0.2701 | 1.1972 | 1.8184 |
| 12.5 | 0.3286 | 0.2769 | 1.3408 | 1.9463 |
| 13 | 0.3087 | 0.2840 | 1.5237 | 2.1164 |
| 13.5 | 0.2911 | 0.2916 | 1.7643 | 2.347 |
| 14 | 0.2755 | 0.2997 | 2.0951 | 2.6703 |
| 14.5 | 0.2614 | 0.3079 | 2.5786 | 3.1479 |
| 15 | 0.3487 | 0.3168 | 3.3521 | 4.0176 |

Graphical representation of $L_{Q_1}, L_{Q_2}, L_{Q_3}$ and L(mean queue length) with respect to service rate $\mu_1$, as shown below in figures 5 to figure 8.

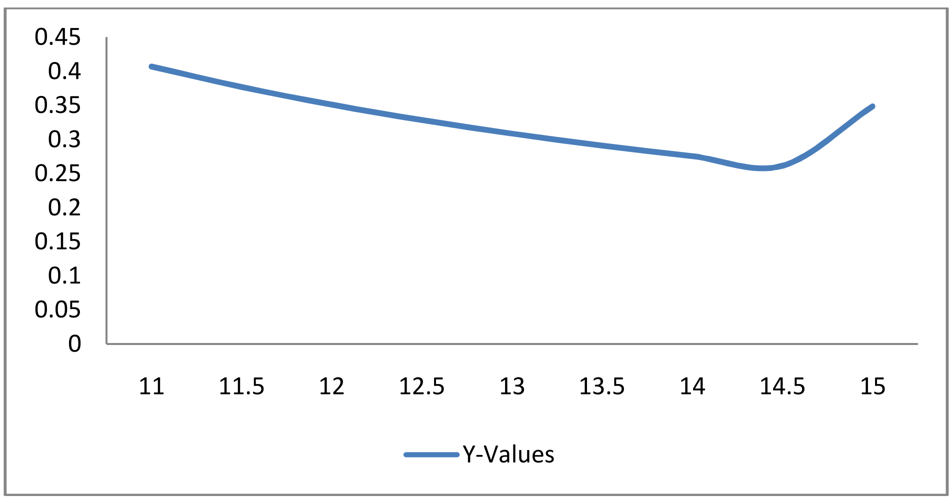



**Figure 5.** $L_{Q_1}$ **vs** $\mu_1$**(x-axis represents different values of** $\mu_1$**and y-axis of** $L_{Q_1}$)

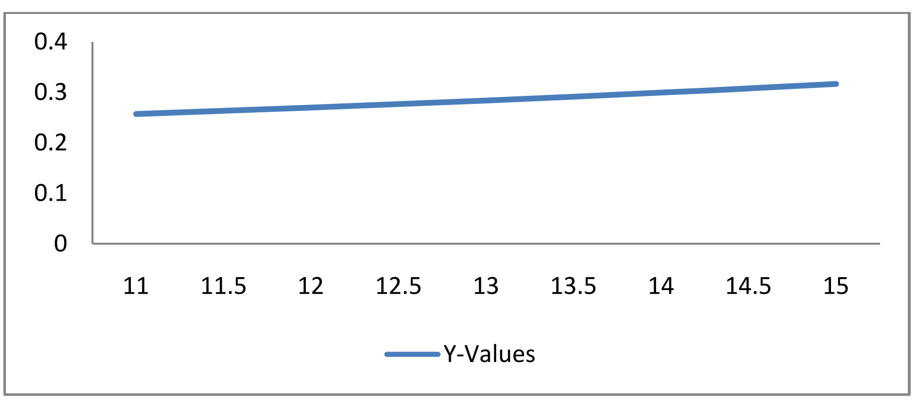



**Figure 6.** $L_{Q_2}$ **vs** $\mu_1$**(x-axis represents different values of** $\mu_1$**and y-axis of** $L_{Q_2}$)

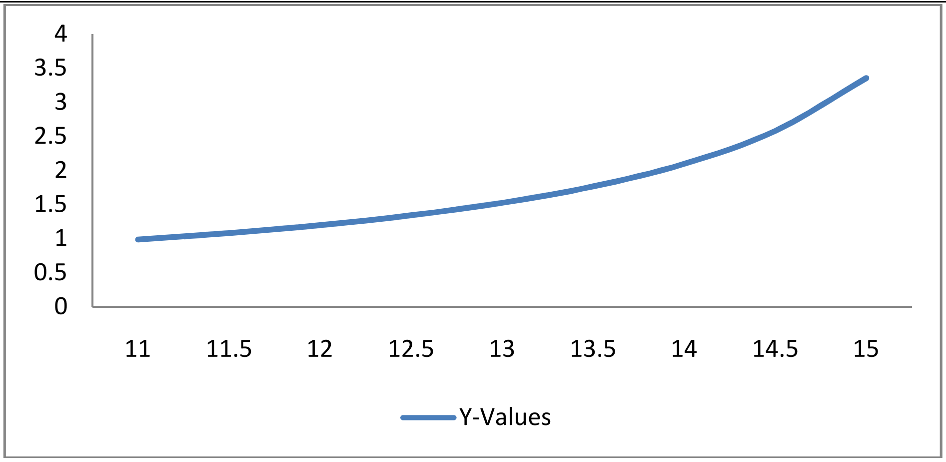



**Figure 7.** $L_{Q_3}$ **vs** $\mu_1$**(x-axis represents different values of** $\mu_1$ **and y-axis of** $L_{Q_3}$**)**

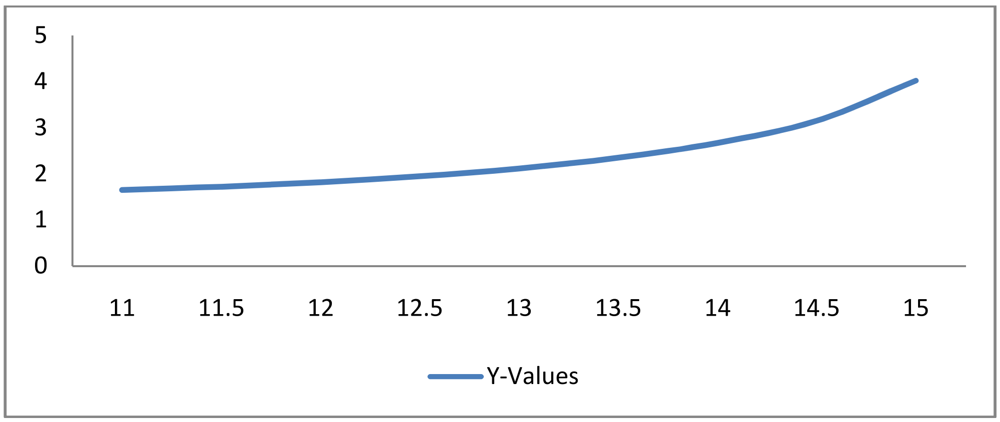



**Figure 8. L vs** $\mu_1$**(x-axis represents different values of** $\mu_1$ **and y-axis of L)**

(III) Behaviour analysis of $L_{Q_1}, L_{Q_2}, L_{Q_3}$ and L(mean queue length) with respect to increasing values of a, as shown in the table:

| Service rate | $\mu_1 = 11$ | $\mu_2 = 12$ | $\mu_3 = 13$ | Arrival rate $\lambda_2$=1, $\lambda_3$=2 |
|---|---|---|---|---|
| Probability | 1st time | 2nd time | | |
| | $p_{12}$=0.4, $p_{13}$=0.6 | $q_{12}$=0.3 , $q_{13}$=0.7 | | |
| | b =0.7 , $p_{21}$=0.2 | $b'$=0.3 | | |
| | c =0.8, $p_{31}$=0.1 | $c'$=0.2 | | |
| Probability of customers leaving first time the server $S_1$ (a) | $L_{Q_1}$ | $L_{Q_2}$ | $L_{Q_3}$ | L |
| a=0.4, $a'$=0.6 | 0.4071 | 0.2574 | 0.9859 | 1.6504 |
| a=0.6, $a'$=0.4 | 0.4076 | 0.2729 | 0.9188 | 1.5993 |
| a=0.8, $a'$=0.2 | 0.4094 | 0.3073 | 0.8598 | 1.5765 |

## RESULT AND DISCUSSION

(I) From table (1) and figure 1 to figure 4, we observe that length of $L_{Q_1}$ is increasing very fast as compared to $L_{Q_2}$ & $L_{Q_3}$ as the increase in arrival rate $\lambda_1$. Length of the entire system is also increasing with the increase of arrival rate at first server also when $\lambda_1$=5.5 there is quick increase in length of entire system.

(II) From table (2) and figure 5 to figure 8, it is observed that the length of $L_{Q_1}$ is decreasing as service rate $\mu_1$ at server $S_1$ is increasing, but suddenly beyond $\mu_1$= 15, there is an increase in $L_{Q_1}$. But there is no significant change in the length of $L_{Q_2}, L_{Q_3}$ , and the length of the entire system is increasing but in a slow speed.

(III) From table (3) it is clearly seen that with the increase in a, the lengths of $L_{Q_1}$ and $L_{Q_2}$ are increasing very slowly but the length of $L_{Q_3}$ is decreasing and the total length of the system is also decreasing.

## VALIDATION STUDY

Particular case: If we take $a'$=$b'$=$c'$= 0, a=b=c=1,$q_{12} = q_{13} = 0$, then the present model reduces to the model studied by Kusum[8]

## CONCLUSION

In the present paper, a heterogeneous feedback queue model is established in which the facility of revisit is given to the customers. Length of queues, mean queue length and average waiting are calculated with the help of probability generating function. Behaviour analysis of the model is studied with the help of numerical values and graphical representation. Validity of model is checked by taking particular case. The model can be helpful in many practical situations like communication networks, offices, shopping malls etc because facility of revisit is necessary for customer's satisfaction.

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# BUYER - VENDOR FUZZY INVENTORY MODEL HAVING TIME VARYING HOLDING COST

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## ABSTRACT

***In the present study, a Fuzzy inventory model having time dependent holding cost and changeable lead time over a time horizon is proposed. Latest Technology brought a drastic change in demand and supply chain management due which stimedependent holding cost is being necessary to be considered. We explore the Cost inventory system to incorporate two parameters. The two parameters i.e. Lead time and cycle time has been fuzzified using triangular fuzzy parameters. Graded mean integration method, Signed distance method and Centroid method are used to defuzzify the total cost. The main objective of the paper is to minimize the total cost per unit time in an inventory system.***

***Key Words: Time dependent holding cost, lead time, Graded mean integration method, Signed distance method , Centroid method***

## 1. INTRODUCTION

In recent years, Supply Chain Management has many new direction, such as EOQ with random supplier capacity, discount for vendor's and coordination of vendor buyer in inventory ordering, pricing etc. In past inventory problems, the buyer and vendor models were devised separately, but due to change in supply chain management system integrated buyer vendor models became the realistic and plays pivot role. In today cut throat market competition, individuals and independent entities cannot survive in business for long run. It is better to work together to work more efficiently. In traditional EOQ models, buyer's view, fixed demand rate and zero lead time are the main factors, which are far away from reality. But when the vendor has difficulty in supplying items, the buyer can't replenish his inventory. When the vendor holds a monopolistic status, he will control items prices as well as lead time in order to obtain maximum profit. Due to rapid change in consumer's choice or his priority or due to the research with high-tech products and for supply chain coordination, it is necessary to integrate buyer vendor inventory models with exponential demand rate and time varying holding cost. Fuzzy inventory modeling is the closest approach towards reality. Due to deterioration, inventory system faces the problem of shortages and loss of goodwill or loss of profit. Longer the waiting time leading to a larger fraction of lost sales and yield less profit. Hence, the factor of lead time as decision variable becomes significant to be considered.

In 1991, Liao and Shyu proposed a inventory model to determine the length of lead time that minimizes the total relevant cost. Ben-Dayu and Raouf [1994] , Hariga and Ben-Dayu [1999] worked on models that considers both lead time and order quantity as decision variables. Yao and Chang[2003] developed the EOQ Models with fuzzy parameters. Hsiao J.M. and Lin. C. [2005] constructed a buyer-vendor EOQ model with changeable lead-time. Ritha.W etal. [2013] proposed an vendor buyer Inventory model with fuzzy parameters. Saha.S and Chakrabarti [2016] explained a buyer-vendor EOQ model with changeable lead-time in supply chain.

Recently a stochastic inventory model with variable demand, weibull distributed deterioration rate has been developed by Harish ,Vinod kumar & T. P. Singh ( 2019,2020 ) in order to find optimal value of production cycle time minimizing the stock level & total average cost over a time horizon . This model is an extended work and makes an attempt to fuzzify the said model with triangular fuzzy parameters it is because fuzzy uncertainty justify the real time situation in more effective manner between supply & demand in order to smooth running of various operations of the system. The optimal policy for fuzzy inventory cost of the said model is defuzzified by using Centroid , signed distance and graded mean integration method. The sensitivity analysis has been carried out between crisp parameters and fuzzy parameters as well.

In the present communication, the uncertainty is described by three parameters via triangular fuzzy parameters. Following an introductory part rest of the paper is organized as follows. In section 2, some definitions and properties about fuzzy sets related to this work are given. In section 3, the notations and assumptions are described in brief for developing the model. Section 4 gives the mathematical formulations of the model. In section 5, the model has been Fuzzified and then defuzzified with the help of Centroid Method, Signed distance Method and Graded Mean Integration Representation method . Section 6 illustrates the developed models on a numerical example and sensitive analysis of the optimal solution has been carried out. Finally, conclusions are given in last section..

## 2. PRELIMINARIES

**Definition 2.1 Triangular Membership Function:** A Triangular membership function is specified by three parameters ( a,b,c) as follows:

$$\mu_A(x) = \begin{cases} 0 & if\ x < a \\ \frac{x-a}{b-a} & if\ a \le x \le b \\ \frac{c-x}{c-b} & if\ b \le x \le c \\ 0 & if\ x > c \end{cases}$$

The parameters ( a,b,c) with $a \le b \le c$ determined the x coordinate with three corners of triangular membership function.

**Definition 2.2**

If Â = (a,b,c) is a triangular fuzzy number then the graded mean integration representation of Â is defined as

$$P(\hat{A}) = \frac{a+4b+c}{6}$$

**Definition 2.3**

If Â = (a, b, c) is a triangular fuzzy number then the signed distance of Â is defined as

$$P(\hat{A}) = \frac{a+2b+c}{4}$$

**Definition 2.4**

The Centroid for a triangular fuzzy number Â = (*a*, *b*, *c)* is defined as

$$P(\hat{A}) = \frac{a+b+c}{3}$$

## 3. ASSUMPTIONS AND NOTATIONS

To develop the model following assumptions and notations have been considered :

1) T is the length of vendor order cycle time ;
2) $O_1$ be Ordering Cost of vendor;
3) $O_2$ be Ordering Cost of buyer ;
4) $C_1(t) = a\,t^n$ is the inventory holding cost per unit time of vendor where $n \ge 1$ and a are constants ;
5) $C_2(t) = f+gt$ is the inventory holding cost per unit time of buyer where $f,g>0$ are constants ;
6) $C_3$ is vendor shortage cost per unit;
7) $t_1$ is vendor lead time and $t_1 < T$.
8) C(t) is the total inventory cost ;

9) $\bar{G}(\ddot{t})$ is the total inventory cost by Graded Mean Integration Method (Triangular) ;

10) $\dot{S}(\ddot{t})$ is the total inventory cost by Signed Distance Method (Triangular);

11) Ḉ $(\ddot{t})$ is the total inventory cost by Centroid Method (Triangular) ;

12) The demand rate $D_1(t) = \int_0^t D(t)\ dt$ where $D(t) = \alpha e^{\beta t}$ , where $0 \le \beta \le 1$, $t > 0$ and $\alpha > 0$.

## 4. MATHEMATICAL DESCRIPTION OF THE MODEL

Assuming an amount S (S>0) as an initial inventory, which can meet the demand for the time $T - t_1$. Let I(t) be on hand inventory at any time t. Then

$$I(t) = S - \int_0^t D(t)\ dt = S + \frac{\alpha}{\beta}(1 - e^{\beta t}) \qquad ....(1)$$

Also $I(T - t_1) = 0$

$$\text{Therefore, } S = \frac{\alpha}{\beta}(e^{\beta(T-t_1)} - 1) \qquad ....(2)$$

$$\text{Vendor Shortage cost} = \int_{T-t_1}^{T} \{S + \frac{\alpha}{\beta}\left(1 - e^{\beta t}\right)\}dt \qquad ....(3)$$

$$\text{Vendor Holding Cost} = C_3 \int_0^{T-t_1} at^n\{S + \frac{\alpha}{\beta}\left(1 - e^{\beta t}\right)\}\ dt \qquad ....(4)$$

Then, the vendor total average cost be

$$C_v = \frac{1}{T}[O_1 + \int_0^{T-t_1} at^n\{S + \frac{\alpha}{\beta}\left(1 - e^{\beta t}\right)\}\ dt - C_3 \int_{T-t_1}^{T}\{S + \frac{\alpha}{\beta}\left(1 - e^{\beta t}\right)\}\ dt] \qquad ....(5)$$

$$C_v = \frac{1}{T}\{O_1 + a\alpha\left[\frac{(T-t_1)^{n+2}}{(n+1)(n+2)} + \beta\frac{(T-t_1)^{n+3}}{(n+1)(n+3)}\right] - \alpha C_3\left[\frac{T^2}{2} - Tt_1 - \beta\frac{T^3}{6} + \left(\frac{T\beta-1}{2}\right)(T - t_1)^2 - \frac{\beta}{3}(T - t_1)^{n+2}\right]\} \qquad ....(6)$$

$$\text{Buyer Holding Cost} = \int_0^{T-t_1}(f + gt)[S + \frac{\alpha}{\beta}\left(1 - e^{\beta t}\right)]\ dt$$

$$= \alpha f\ \{\ (T - t_1)^2\ + \frac{\alpha}{\beta^2}\} + g\alpha\ [(T - t_1)^2 + \frac{(T-t_1)^3}{3}] \qquad ....(7)$$

Buyer Total Average Cost = $C_b$

$$= \frac{1}{T}\{O_2 + \alpha[\frac{f}{2}(T - t_1)^2\ +(\frac{2f\beta+g\alpha}{6})(T - t_1)^3 + \frac{3\alpha\beta g}{8T}(T - t_1)^4]\} \qquad ....(8)$$

Average total cost per unit time $C(t_1, T)$

$= \frac{1}{T}[$ Vendor Total cost per unit time + Buyer Total cost per unit time ]

$$= \frac{1}{T}\left\{O_1 + a\alpha\left[\frac{(T-t_1)^{n+2}}{(n+1)(n+2)} + \beta\frac{(T-t_1)^{n+3}}{(n+1)(n+3)}\right] - \alpha C_3\left[\frac{T^2}{2} - Tt_1 - \beta\frac{T^3}{6} + \left(\frac{T\beta-1}{2}\right)(T - t_1)^2 - \frac{\beta}{3}(T - t_1)^{n+2}\right]\right\} + \frac{1}{T}\{O_2 + \alpha[\frac{f}{2}(T - t_1)^2\ +(\frac{2f\beta+g\alpha}{6})(T - t_1)^3 + \frac{3\alpha\beta g}{8T}(T - t_1)^4]\} \qquad ....(9)$$

For minimum $C(t_1, T)$ ,the necessary condition are

$\frac{\partial C(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial C(t_1,T)}{\partial T} = 0$ ....(10) these equation can be solved simultaneously by using Mat Lab, we get the values of T and $t_1$.

The sufficient condition for minimization of $C(t_{1,}T)$ is it must be convex function for T >0 , $t_1 > 0$ . i.e

$$\begin{vmatrix} \frac{\partial^2 \ C(t_1,T)}{\partial T^2} & \frac{\partial^2 \ C(t_1,T)}{\partial T \partial t_1} \\ \frac{\partial^2 \ C(t_1,T)}{\partial t_1 \partial T} & \frac{\partial^2 \ C(t_1,T)}{\partial t_1^2} \end{vmatrix} > 0$$

## 5. FUZZIFICATION AND DEFUZZIFICATION OF THE MODEL

let us describe cycle time as fuzzy parameter Ť and lead time ẗ

the total cost function with fuzzy cycle time be

$$C(\ddot{t}) = \frac{1}{T}\left\{\left\{O_1 + a\alpha\left[\frac{(T_i-t_i)^{n+2}}{(n+1)(n+2)} + \beta\frac{(T_i-t_i)^{n+3}}{(n+1)(n+3)}\right] - \alpha C_3\left[\frac{T_i{}^2}{2} - T_i t_i - \beta\frac{T_i{}^3}{6} + \left(\frac{T_i\beta-1}{2}\right)(T_i-t_i)^2 - \frac{\beta}{3}(T_i-t_i)^{n+2}\right]\right\} + \{O_2 + \alpha[\frac{f}{2}(T_i-t_i)^2 + (\frac{2f\beta+g\alpha}{6})(T_i-t_i)^3 + \frac{3\alpha\beta g}{8T}(T_i-t_i)^4]\}\right\} \quad .....(11)$$

For triangular fuzzy parameters i = 1,2,3. and C(ẗ) = ($A_1$,$A_2$,$A_3$), the values of $A_1$,$A_2$,$A_3$ be obtained by putting i = 1,2,3. in C(ẗ) respectively.

### 5.1 DEFUZZIFICATION BY GRADED MEAN RERESENTATION METHOD

Total cost is given by $\bar{G}(\ddot{t}) = \frac{1}{6}[A_1+4A_2+A_3]$

$$\bar{G}(\ddot{t}) = \sum_{i=1}^{3} x_i[\frac{1}{6T}\left\{\left\{O_1 + a\alpha\left[\frac{(T_i-t_i)^{n+2}}{(n+1)(n+2)} + \beta\frac{(T_i-t_i)^{n+3}}{(n+1)(n+3)}\right] - \alpha C_3\left[\frac{T_i{}^2}{2} - T_i t_i - \beta\frac{T_i{}^3}{6} + \left(\frac{T_i\beta-1}{2}\right)(T_i-t_i)^2 - \frac{\beta}{3}(T_i-t_i)^{n+2}\right]\right\} + \{O_2 + \alpha[\frac{f}{2}(T_i-t_i)^2 + (\frac{2f\beta+g\alpha}{6})(T_i-t_i)^3 + \frac{3\alpha\beta g}{8T}(T_i-t_i)^4]\}\right\}]$$

.....(12)

Where $x_i$ = 1, 4, 1 for i=1,2,3 respectively.

For minimum $\bar{G}(t_{1,}T)$ ,the necessary condition are

$\frac{\partial \bar{G}(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial \bar{G}(t_1,T)}{\partial T} = 0$ these equation can be solved simultaneously by using Mat Lab, we get the values of T and $t_1$.

The sufficient condition for minimization of $C(t_{1,}T)$ is it must be convex function for T >0 , $t_1 > 0$ . i.e

$$\begin{vmatrix} \frac{\partial^2 \ \bar{G}(t_1,T)}{\partial T^2} & \frac{\partial^2 \ \bar{G}(t_1,T)}{\partial T \partial t_1} \\ \frac{\partial^2 \ \bar{G}(t_1,T)}{\partial t_1 \partial T} & \frac{\partial^2 \ \bar{G}(t_1,T)}{\partial t_1^2} \end{vmatrix} > 0$$

## 5.2 SIGNED DISTANCE METHOD

Total cost is given by $\dot{S}(\ddot{t}) = \frac{1}{4}$ [$A_1+2A_2+A_3$ ],

$$\dot{S}(\ddot{t}) = \frac{A}{T} + \sum_{i=1}^{3} Y_i \left[ \frac{1}{4T} \left\{ \left\{ O_1 + a\alpha \left[ \frac{(T_i-t_i)^{n+2}}{(n+1)(n+2)} + \beta \frac{(T_i-t_i)^{n+3}}{(n+1)(n+3)} \right] - \alpha C_3 \left[ \frac{T_i^2}{2} - T_i t_i - \beta \frac{T_i^3}{6} + \left( \frac{T_i\beta - 1}{2} \right)(T_i - t_i)^2 - \frac{\beta}{3}(T_i - t_i)^{n+2} \right] \right\} + \left\{ O_2 + \alpha\left[ \frac{f}{2}(T_i - t_i)^2 + \left( \frac{2f\beta + g\alpha}{6} \right)(T_i - t_i)^3 + \frac{3\alpha\beta g}{8T}(T_i - t_i)^4 \right] \right\} \right\} \right] \quad .......(13)$$

Where $Y_i = 1, 2, 1$ for i=1,2,3 respectively.

For minimum $\dot{S}(t_1, T)$ ,the necessary condition are $\frac{\partial \dot{S}(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial \dot{S}(t_1,T)}{\partial T} = 0$ these equation can be solved simultaneously by using Mat Lab, we get the values of T and $t_1$.

The sufficient condition for minimization of $C(t_1, T)$ is it must be convex function for T >0 , $t_1 > 0$ . i.e

$$\begin{vmatrix} \frac{\partial^2 \dot{S}(t_1,T)}{\partial T^2} & \frac{\partial^2 \dot{S}(t_1,T)}{\partial T \partial t_1} \\ \frac{\partial^2 \dot{S}(t_1,T)}{\partial t_1 \partial T} & \frac{\partial^2 \dot{S}(t_1,T)}{\partial t_1^2} \end{vmatrix} > 0$$

## 5.3 CENTROID METHOD

Total cost is given by $\acute{Ç}(\ddot{t}) = \frac{1}{3}$ [$A_1+A_2+A_3$ ]

$$\acute{Ç}(\ddot{t}) = \sum_{i=1}^{3} \frac{1}{3T} \left\{ \left\{ O_1 + a\alpha \left[ \frac{(T_i-t_i)^{n+2}}{(n+1)(n+2)} + \beta \frac{(T_i-t_i)^{n+3}}{(n+1)(n+3)} \right] - \alpha C_3 \left[ \frac{T_i^2}{2} - T_i t_i - \beta \frac{T_i^3}{6} + \left( \frac{T_i\beta - 1}{2} \right)(T_i - t_i)^2 - \frac{\beta}{3}(T_i - t_i)^{n+2} \right] \right\} + \left\{ O_2 + \alpha\left[ \frac{f}{2}(T_i - t_i)^2 + \left( \frac{2f\beta + g\alpha}{6} \right)(T_i - t_i)^3 + \frac{3\alpha\beta g}{8T}(T_i - t_i)^4 \right] \right\} \right\} \quad ........(14)$$

For minimum $\acute{Ç}(t_1, T)$ ,the necessary condition are $\frac{\partial \acute{Ç}(t_1,T)}{\partial t_1} = 0$ and $\frac{\partial \acute{Ç}(t_1,T)}{\partial T} = 0$ these equation can be solved simultaneously by using Mat Lab, we get the values of T and $t_1$.

The sufficient condition for minimization of $C(t_{1,}T)$ is it must be convex function for T >0 , $t_1 > 0$ . i.e

$$\begin{vmatrix} \frac{\partial^2\ \acute{C}(t_1,T)}{\partial T^2} & \frac{\partial^2\ \acute{C}(t_1,T)}{\partial T\partial t_1} \\ \frac{\partial^2\ \acute{C}(t_1,T)}{\partial t_1\partial T} & \frac{\partial^2\ \acute{C}(t_1,T)}{\partial t_1^2} \end{vmatrix} > 0$$

## 6. NUMERICAL ILLUSTRATION AND SENSTIVITY ANALYSIS

To illustrate the model we consider following numerical values of the parameters.

$O_1$= 1000, $O_2$= 1200, a = 2, α = 40, β =0.5, f= 0.2, g=0.1, $C_3$ = 8.

we obtain for the crisp model Total Cost = 56493.79. and cycle time t= 0.67 years

For triangular fuzzy model Ť= ( 1.50,1.70,1.80), ẗ = (0.35,0.39,0.45) We Obtain following results;

| parameters | Method | T (Cycle Time ) | lead time | vendor Optimal cost | Buyer Optimal Cost | Total Optimal Cost |
|---|---|---|---|---|---|---|
| Crisp | Crisp | 1.750 | 0.410 | 613.5341 | 783.5878 | 1397.122 |
| Triangular | Centroid Method | 1.667 | 0.397 | 637.8205 | 805.1098 | 1442.93 |
| | Signed Distance Method | 1.675 | 0.395 | 635.0199 | 803.4654 | 1438.485 |
| | Graded Mean Integration Method | 1.683 | 0.393 | 632.2648 | 801.8855 | 1434.15 |

## 7. CONCLUSION

An Inventory model in which exponential demand rate with time varying holding cost and lead time as decision variable in supply chain process is a very uncertain situation has been developed. Model has fuzzified and then defuzzified with the help of Graded Mean Integration Method, Signed Distance Method and by Centroid Method . Graded Mean Method provides minimum total cost. The proposed model deals some realistic features likely to be associated with some kind of inventory. The model finds its application in automobile business etc.

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# GEOMETRIC SERIES GEOMETRY: ŚANKARA'S CONTRIBUTION

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## ABSTRACT
***Ancient and medieval Indian mathematicians were fond of validating arithmetic and algebraic truths using geometrical diagrams. Their geometrical imaginations prompted them to view even the progressive series through geometric figures called średhīksetras. Geometrical exposition of various arithmetic series can be had from several works, but geometric demonstration of geometric series is a special feature of the Līlāvatī commentary Kriyākramakarī by Śankara and Nārāyana. Śankara has described an interesting geometrical demonstration for summation of a geometric series with common ratio 4. Śankara's demonstration is expounded in modern terms and an attempt is made to generalize it.***

***Key words: Līlāvatī, Kriyākramakarī, Śankara, Nārāyana, citighana, średhī***

## 1. INTRODUCTION

*Līlāvatī* of Bhāskarācārya is a basic mathematical treatise dealing with mathematics essentially required for study and practice of astronomy (*graha gaṇita*) and that for domestic applications (*gṛaha gaṇita*). It gained popularity among scholars ever since its composition in 1150 CE because of the simplicity of presentation of various topics and practical utility of the contents. Consequently a large score of commentaries in Sanskrit and various regional languages were written on it. A short list of 41 of them is included in the source book '*Mathematics in Ancient and Medieval India*'[12]. Among the numerous standard commentaries on the treatise, the commentary *Kriyākramakarī* [4]by Śaṅkara and Nārāyaṇa of 16th century CE stands out. It is found to be one of the most excellent and informative commentary. Lot of valuable contributions in the field up to the 16th century CE are borne to us through this commentary. The portion of the commentary up to the verse 199 of the *Līlāvatī* was written by Śaṅkara Vāryar and the remaining part by Nārāyaṇa (1540 – 1610 CE) at the age of 18 after the demise of Śaṅkara [5].

Śaṅkara Vāryar (1500 – 1560 CE) hailed from the scholarly family of the *Tṛkkāṭiri* Vāryars at *Tṛkkuṭaveli* (*Śrīhutāśa* in Sanskrit) near Ottapālam in South Malabār area of the Kerala region. He was professionally a functionary of the *Śrīhutāśa* temple and was an illustrious student of erudite scholars namely Nīlakaṇṭha Somayāji, Dāmodara, and Citrabhānu belonging to the brilliant lineage from Saṅgamagrāma Mādhava. Śaṅkara Vāryar was patronized by Nārāyaṇa Āzhvānceri Thamprākkal who was a contemporary scholar, promoter of astronomical studies and religious head of Nampūtiris. Apart from the major part of the commentary *Kriyākramakarī*, he has written an astronomical text *Karaṇasāra* (apparently anonymous but ascribed to Śaṅkara) along with an elaborate commentary on it namely *Karaṇasārakriyākrama* in Malayalam, three commentaries on the *Tantrasaṅgraha* of Nīlakaṇṭha Somayāji, two of which are detailed commentaries namely *Yuktidīpika* and *Kriyākalāpa*and the other is a short one namely *Laghuvivṛti*. He has also written an elaborate commentary entitled *Laghuvivṛti* on the astronomical work *Pañcabodha*. In order to take up the task of writing the commentary *Yuktidīpika* on the *Tantrasaṅgraha*, Śaṅkara left the work *Kriyākramakarī* unfinished after an excellent exposition on the verse 199 of the *Līlāvatī* and the onerous task of completion of the remaining work was taken up by Mahiṣamaṅgalam Nārāyaṇa of Puruvana (Peruvana) village under the insistence of his father and great scholar Mahiṣamaṅgalam Śaṅkaran Nampūtiri[6]

The *Kriyākramakarī* edited by K.V. Sarma consists of three sections in about 500 pages of close print in small type font spanning over a variety of topics ranging from basic to advanced arithmetic, various kinds of determinations, geometrical discussions relating to plane figures and solids, trigonometric functions and their computations, interpolation techniques, gnomon shadow problems (problems on heights and distances), permutations and combinations, permutation of digits, summation of syllables in poetic meters, analysis of indeterminate equations, derivation and analysis of infinite series for evaluating circumference, arcs and chords, rational approximation of infinite series, analysis of remainder terms and error functions in series approximation, extraction of rapidly convergent series from the error functions generated from the remainder terms of slowly convergent series, arithmetic and geometric series, discrete and continuous summations, and repeated summations leading to the concept of integration, repeated integration and integration by parts.

An instructive style of exposition is generally followed in the commentary and a lot of relevant information has been provided by the commentators from several other works such as those of Śrīdhara, Govindasvāmin, Udayadivākara, Jayadeva, Saṅgamagrāma Mādhava,Citrabhānu and also from various other unidentified sources. Some of the treatises from where they are cited are not available to us now and the modern world is highly indebted to the commentators for bequeathing to us some of their valuable enunciations through such citations. Moreover, the commentators have enhanced the quality of the commentary by including own contributions in the field. Taking into account the background of the learners, the commentators have provided different types of proofs and illustrations to establish various important enunciations and the underlying principles and concepts involved therein are explained from their fundamentals. Analytical proofs, numerical illustrations, and geometrical demonstrations are provided for validating various arithmetic and algebraic truths in order to enhance the quality of the commentary and make it beneficial to all kinds of learners. Numerous *saṅgraha slokas*(summary verses) are given at the end of each exposition in order to serve the dual purpose of recapitulating the quintessence of the elaborate expositions given therein and for updating the knowledge in the field. These *saṅgraha ślokas* provide a lot of information on the advances in the field and they form a rich source of several important methods, enunciations and their rationale, illustrations and derivations including special geometrical demonstrations for validating various abstract concepts involved there in. These visual demonstrations help the learner to view the underlying concepts clearly and get convinced at once. Such geometrical techniques have immense value from the historical point of view as well as from the pedagogical aspects. Indian scholars have ingeniously yoked arithmetic and algebra to geometry and their geometrical imaginations prompted them to treat even progressive series geometrically. The full bloom of series geometry can be seen in the *Kriyākramakarī* where the commentator has treated geometric series also in a geometrical background using rectangular strips.

## SERIES DIAGRAMS (ŚREDHĪKSETRA)

Āryabhaṭa I's geometrical vision of series is evident from his geometrical terminology for series summations such as *citi* (pile) and *upaciti* (sub-pile) for sums of arithmetic series of natural numbers, *citighana*(solid pile) for sum of sums of natural numbers, *varga citighana* for sum of squares of natural numbers, *ghana citighana* for sum of cubes of natural numbers [9]. Bhāskara I in his commentary on the *Āryabhaṭīya* has given detailed illustrations revealing the geometrical imaginations of Āryabhaṭa I on progressive series. Moreover Bhāskara I has included the topic on series in the geometry section with the following remark[10]:

*gaṇitam dviprakāram – rāśi gaṇitam kṣetra gaṇitam / anupātakuṭṭākārādayo gaṇitaviśeṣāḥr āśigaṇite'bhihitāḥ, śreḍhīcchāyādayaḥ kṣetragaṇite / tadevam rāśyāśritam kṣetrāśritam vā aśeṣam gaṇitam /*

Mathematics is of two types: mathematics of numbers (*rāśi gaṇitam*:arithmetic) and mathematics of figures (*kṣetra gaṇitam*:geometry). Mathematical topics like proportion and indeterminate analysis are included in the mathematics of numbers; series, gnomon-shadow and the like in the mathematics of figures. Entire mathematics is this manner either number based mathematics (arithmetic) or figure based mathematics (geometry).

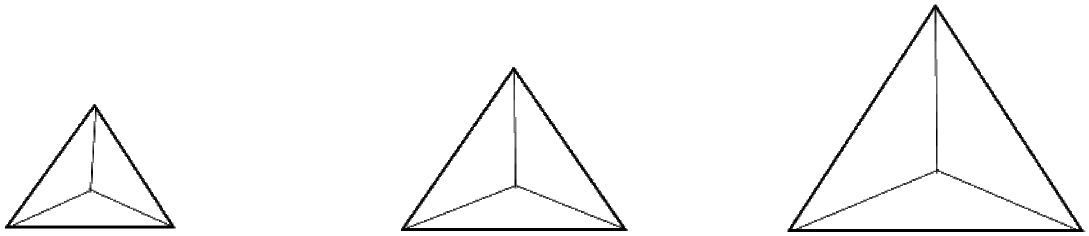
In the section on series mathematics (*śreḍhī gaṇitam*) Bhāskara we represent arithmetic series using pyramidal piles (of balls /bricks of unit size in layers) using3 illustrations[11]

*i*) *pañcānāmaṣṭānāmcaturdāśānāmca yāḥ kramāccitayaḥ /*

*gacchastarāstrikoṇā* [*rūpavidhānam ca*] *me vācyam //*

There are(three pyramidal) piles (of balls) respectively having 5, 8, and 14 layers which are triangular. Tell me the number of units (in each of them)

Number of balls in the pile with 5 layers is $\sum_{r=1}^{5} S_r = \frac{5\times(5+1)\times(5+2)}{6} = 35$ where $S_r = \sum_{i=1}^{r} i$

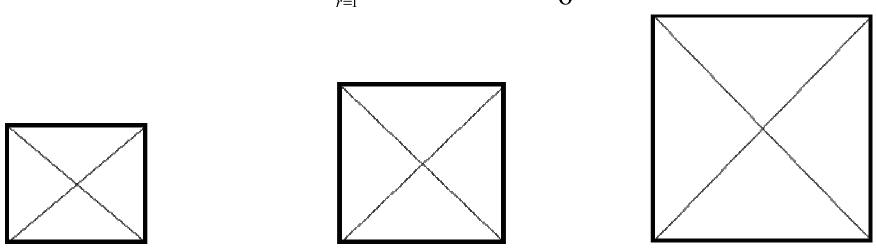


5 layers, 35 balls 8 layers, 120 balls 14 layers, 560 balls

*ii) saptānāmaṣṭānām saptadāśānām caturbhujāscitayaḥ/*
*ekavidhānam vācyam padastarāstā hi vargākhyāḥ//*

There are (three pyramidal) piles on square base having 7, 8, and 17 layers which are also squares. Say the number of units (in each of them)

Number of bricks in the pile with 7 layers is $\sum_{r=1}^{7} r^2 = \frac{7\times(7+1)\times(2\times7+1)}{6} = 140$

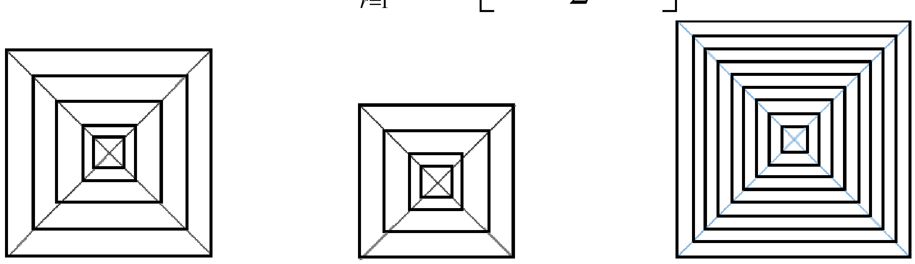


With 7 layers, 140 balls or bricks With 8 layers, 204 balls or bricks With 17 layers, 1785 balls or bricks

*iii) caturaśraghanāścitayaḥ pañcacaturnavastarāvinirdeśyāḥ/*
*ekāvaghaṭitāstāḥ samacaturaśreṣṭakāḥ kramaśaḥ//*

There are (three pyramidal) piles having 5, 4, and 9 cuboidal layers made of cuboidal bricks with one brick in the topmost layer. (Tell me the number of bricks in each of them)

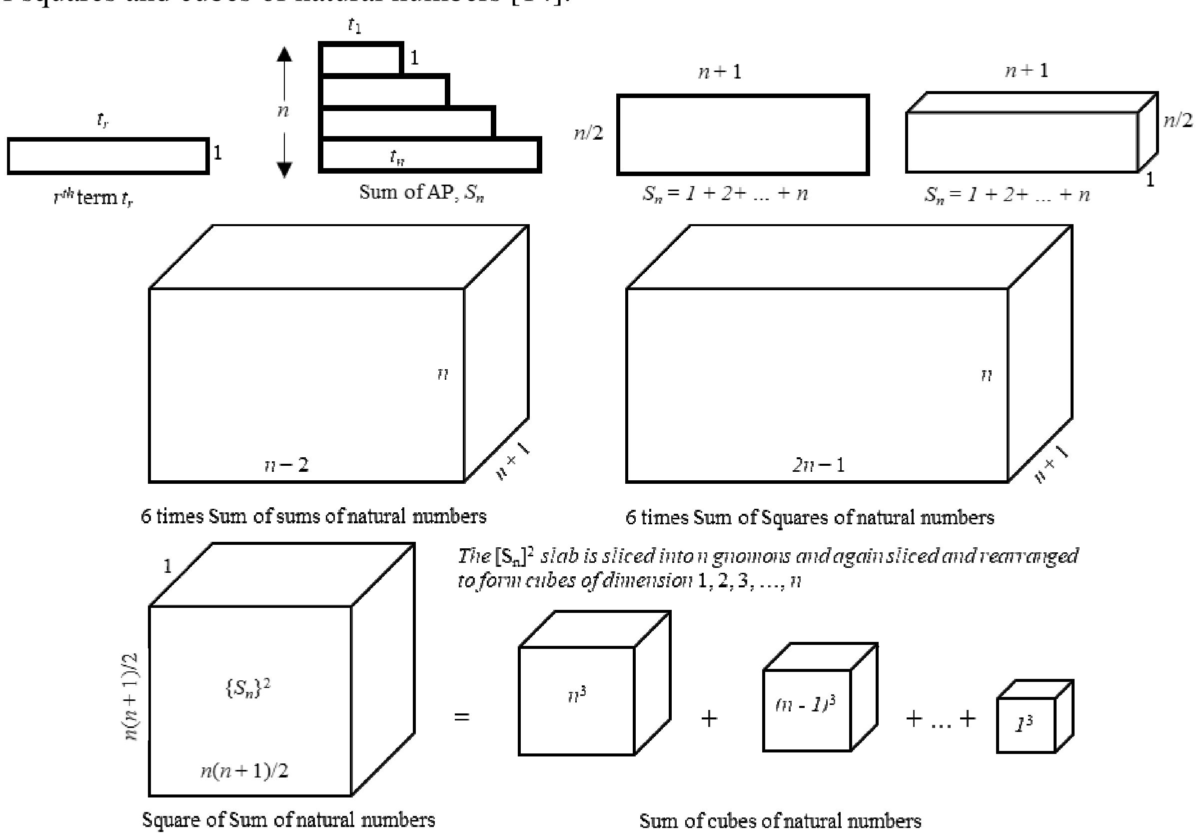
Number of bricks in the pile with 5 layers is $\sum_{r=1}^{5} r^3 = \left[\frac{5\times(5+1)}{2}\right]^2 = 225$



With 5 layers, 225 bricks With 4 layers, 100 bricks With 9 layers, 2025 bricks

Through these illustrations Bhāskara I has demonstrated Āryabhaṭa's geometrical vision on series summations termed as *citighana*, *varga citighana*, and *ghana citighana*. Śrīdharain his *Pāṭīgaṇita* [3] has given a detailed discussion on geometry of series using geometrical diagrams called‘*śreḍhīkssetra*s.

Śrīdhara's series figures are in the form of isosceles trapezia with altitude as much as the number of terms $n$ of the series, base as much as $a-\frac{d}{2}+nd$ and top $a-\frac{d}{2}$ so that its area represents the sum of $n$ terms of the series with first term $a$ and common difference $d$. Series figures given by Nārāyaṇa Paṇḍita in the *Gaṇitakaumudi* [2] are also in the form of isosceles trapezia with base $a-\frac{d}{2}$ and top $a-\frac{d}{2}+nd$. The type of *śreḍhīkṣetra* used by Kerala scholars belonging to Mādhava school is different from those of others. Nīlakaṇṭha Somayāji and Śaṅkara Vāryar have discussed series geometrically using two dimensional rectangular strips and three dimensional rectangular slabs. Nīlakaṇṭha in his *Āryabhaṭīya bhāṣya* [1] and Śaṅkara in his *Kriyākramakarī* [7] have represented a term of an arithmetic series usinga rectangular strip of unit breadth and length as much as the number of units in the term so that the area of the strip represents the term. The series sum is represented by successively piling up the strips representing the terms of the series. The area of the whole pile is equal to the sum of the series. They have also used rectangular slabs and blocks constructed from rectangular strips to represent sums, sum of sums, sum of squares and cubes of natural numbers [14].



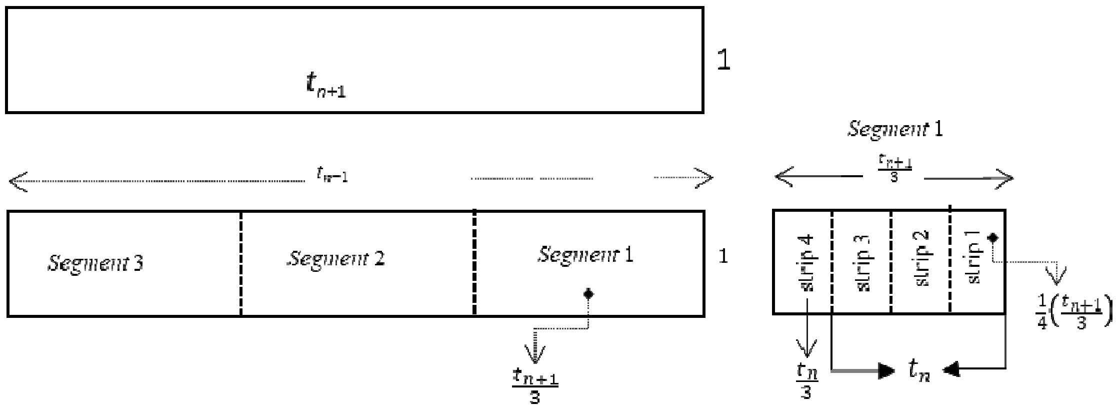

## GEOMETRIC SERIES DIAGRAMS (GUNOTTARAŚREDHĪKSETRA)

Apart from the geometrical treatment of arithmetic series, Śaṅkara has given an interesting geometrical representation of geometric series using rectangular strips in the *Kriyākramakarī* which is not found in other works. In $5\frac{1}{2}$ verses of the set of *saṁgraha slokas* given at the end of his discussion on geometric series (*guṇottaraśreḍhī*) Śaṅkara beautifully describes the summation of a geometric series with common ratio 4 as follows [8].

*caturguṇottare gacche phalamūrdhvapadasthitam /*
*tato guṇottarakṣetratrayam kuryād yathocitam //*
*tatrāntyam phalamūrdhvasthaphalasya caturaṁśakaḥ /*
*tatastaccaturaṁśānām trayeṇāntyaphalatrayam //*
*śiṣṭam tasya turīyoṁśastasyāpi caturaṁśakaiḥ /*
*kuryāt tribhirupāntyottham phalam turyo'tra śiṣyate //*
*tasyāpi ca caturthāṁśaistribhiḥ kalpyamadhaḥ phalam /*
*kṛteṣu tattadādhasthyaphaleṣu svordhvaturyataḥ //*
*kṛtsnam phalam syāt triguṇam turyo'ṁśo'nte'vaśiṣyate /*
*rūpādau śiṣyate rūpam tato vyekam tribhirharet //*
*anādau na punastyājyam rūpam sūkṣmam phalam yataḥ //*

There, from the sum of terms of a geometric series with common ratio four, make the series figure representing the 'raised term' (the term just after the last term, *ūrdhvapadam*) of the geometric series into three in a proper manner. Here the last term (*āntyaphalam*) in the sum is one-fourth of the raised term. Then three-quarter portions with one of this three will be the last term. One quarter-portion is left-over, that would also be made four parts. Three of them will be penultimate term (*upāntimam*) and one-quarter is left over. Three of four parts of this reminder should also be considered as the term just below (*adhaḥ phalam*, pre-penultimate). On proceeding like this the terms below are obtained from the one-fourth of the terms just above. Continue this process till the first term and the part left-over will be one-third of the first term. The raised term should not be discarded because the difference between the raised term and the first term divided by three gives the actual sum.

For the sake of simplicity Śaṅkara has given the geometrical demonstration for a geometric series with common ratio 4. The sum $S_n = t_1 + t_2 + t_3 + ... + t_n$ is derived from the geometric series figure [13](*guṇottaraśreḍhīkṣetram*) representing the *raised term* $t_{n+1}$obtained from the last term $t_{\text{n}}$ by multiplying it by common ratio. This raised term $t_{n+1}$ is represented by rectangular strip with length as much as the number of units in $t_{n+1}$ and breadth 1so that the strip area represents $t_{n+1}$. The series figure is divided into 3 segments so that each segment represents $\frac{t_{n+1}}{3}$. Each segment is then subdivided into 4 equal parts so that each part represents $\frac{1}{4}\left(\frac{t_{n+1}}{3}\right)$. Hence 3 parts in each segment will be $\frac{t_{n+1}}{4} = t_n$ and the remaining part in each segment being equivalent to one of the parts in $t_n$ will represent $\frac{t_n}{3}$..



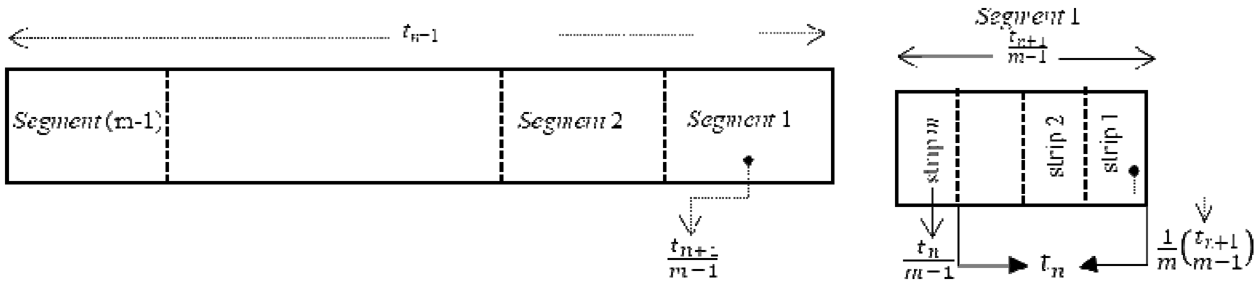

Thus the four parts in each segment will be $t_n + \frac{t_n}{3}$. Hence one segment gives $\frac{t_{n+1}}{3} = t_n + \frac{t_n}{3}$ (Taking all the 3 segments we get $t_{n+1} = 3t_n + t_n$) Now dividing the remaining part representing $\frac{t_n}{3}$ in each segment into 4 parts and grouping three of them together with one remainder part gives $\frac{t_n}{3} = t_{n-1} + \frac{t_{n-1}}{3}$. Proceeding like this we get $\frac{t_2}{3} = t_1 + \frac{t_1}{3}$. Combining all these we get $\frac{t_{n+1}}{3} = t_n + t_{n-1} + \cdots + t_1 + \frac{t_1}{3}$

Thus the rectangular strip representing raised term gives $t_{n+1} = 3t_n + 3t_{n-1} + 3t_{n-21} + \ldots + 3t_1 + t_1 = 3S_n + t_1$. Hence $S_n = \frac{t_{n+1} - t_1}{3}$.

An attempt is made here to extend Śaṅkara's demonstration to four different cases as follows.

**Case 1:** $r = m$, a positive integer

As suggested by Śaṅkara, divide the rectangular strip representing $t_{n+1}$ into $m - 1$ equal segments so that each segment represents $\frac{t_{n+1}}{m-1}$. Now subdivide each segment into $m$ equal strips so that each strip represents $\left\{\frac{1}{m}\left(\frac{t_{n+1}}{m-1}\right)\right\}$ and so $m$ - 1strips in one segment taken together will represent $(m-1)\left\{\frac{1}{m}\left(\frac{t_{n+1}}{m-1}\right)\right\} = \frac{t_{n+1}}{m} = \frac{t_{n+1}}{r} = t_n$. The remaining strip in each segment represents $\frac{t_n}{m-1}$ since it is equal to any one of the strips from the other set of $m - 1$strips representing $t_n$. Thus each segment representing $\frac{t_{n+1}}{m-1}$ is the join of $(m - 1)$ strips representing $t_n$and one remainder strip representing $\frac{t_n}{m-1}$. Thus one segment represents $\frac{t_{n+1}}{m-1} = t_n + \frac{t_n}{m-1}$ [so all the $(m - 1)$ segments taken together will represent $t_{n+1} = (m-1)t_n + t_n$].




Now subdivide the remainder strip representing $\dfrac{t_n}{m-1}$ in *each* of the ($m$ – 1) segments into $m$ equal parts so that($m$ – 1) of them *in each* will represent the preceeding term $t_{n-1}$ and the remainder *in each* will represent $\dfrac{t_{n-1}}{m-1}$ thereby giving $\dfrac{t_n}{m-1} = t_{n-1} + \dfrac{t_{n-1}}{m-1}$ [or all the $m$ – 1segments taken together will represent $t_n = (m-1)t_{n-1} + t_{n-1}$. Thus $t_{n+1} = (m-1)t_n + (m-1)t_{n-1} + t_{n-1}$

Proceeding like this, $t_{n+1} = (m-1)t_n + (m-1)t_{n-1} + (m-1)t_{n-2} + \ldots + (m-1)t_1 + t_1$

i.e; $t_{n+1} = (m-1)[t_n + t_{n-1} + t_{n-2} + \ldots + t_1] + t_1 = (m-1)S_n + t_1$. Thus $S_n = \dfrac{t_{n+1} - t_1}{m-1}$

**Case 2**: $r$ is a negative integer, say $r = -m$ where $m$is a positive integer

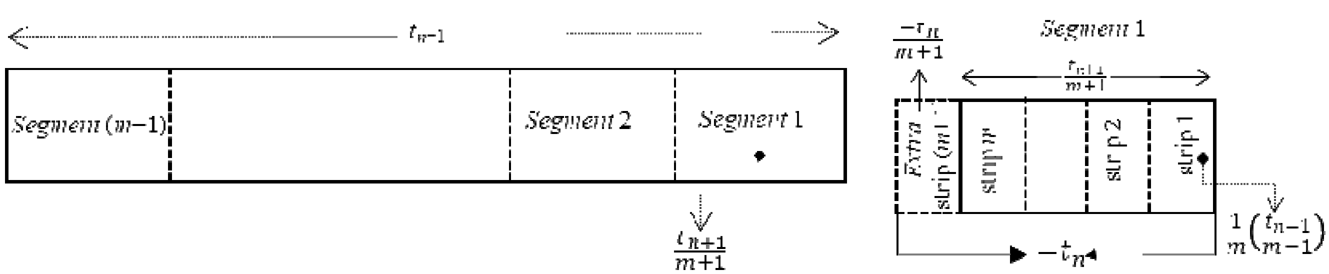
In this case divide the rectangular strip representing $t_{n+1}$ into $m$ + 1 equal segments so that each segment represents $\dfrac{t_{n+1}}{m+1}$

Now subdivide each of the segments into $m$ equal strips so that each strip represents $\left\{\dfrac{1}{m}\left(\dfrac{t_{n+1}}{m+1}\right)\right\}$

Create*one extra strip* of same size for each segment and add it to the $m$ strips so that theresulting $m$ + 1 strips taken together represents $(m+1)\left\{\dfrac{1}{m}\left(\dfrac{t_{n+1}}{m+1}\right)\right\} = \dfrac{t_{n+1}}{m} = \dfrac{t_{n+1}}{-r} = -t_n$ and each *extra strip*added toeachof the segment represents $\dfrac{-t_n}{m+1}$ since it is equivalent to any one of the $m$ + 1 strips representing– $t_n$. Thus each segment representing $\dfrac{t_{n+1}}{m+1}$together with the extra strip representing$\left(\dfrac{-t_n}{m+1}\right)$will represent– $t_n$.Thus

$$\frac{t_{n+1}}{m+1} + \frac{-t_n}{m+1} = -t_n$$

$\dfrac{t_{n+1}}{m+1} = -t_n + \dfrac{t_n}{m+1}$ [so that all the (m + 1) segments taken together gives $t_{n+1} = -(m+1)t_n + t_n$ ].



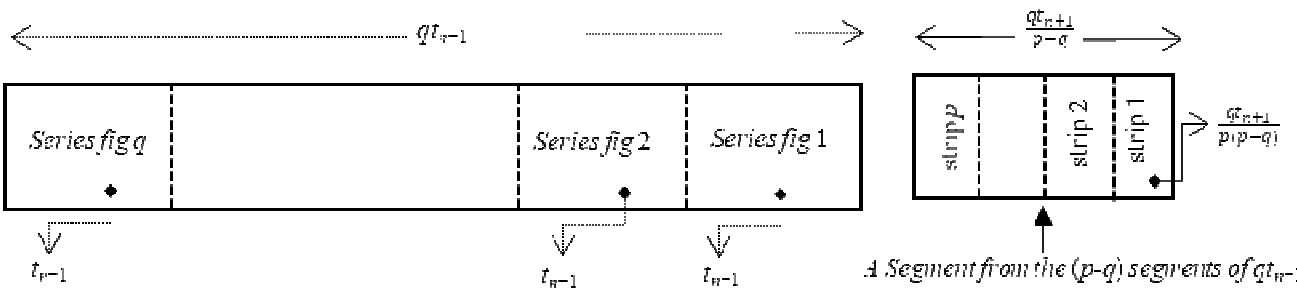
Now subdivide each of the strips representing $\left(\frac{t_n}{m+1}\right)$ into $m$ equal parts and proceeding as before to get $\frac{t_n}{m+1} = -t_{n-1} + \frac{t_{n-1}}{m+1}$ [or put all the $m$+ 1 parts together to get $t_n = -(m+1)t_{n-1} + t_{n-1}$ . Thus $t_{n+1} = -(m+1)t_n - (m+1)t_{n-1} + t_{n-1}$

Proceeding like this, $t_{n+1} = -(m+1)t_n - (m+1)t_{n-1} = (m+1)t_{n-2} - ... - (m+1)t_1 + t_1$

i.e; $t_{n+1} = -(m+1)[t_n + t_{n-1} + t_{n-2} + ... + t_1] + t_1 = -(m+1)S_n + t_1$

Thus $S_n = \frac{t_{n+1} - t_1}{-m-1} = \frac{t_{n+1} - t_1}{r-1}$

**Case 3:** $r$ is a rational number. Let $r = \frac{p}{q}$ where p and q are integers, $q \neq 0$ (We may assume $q > 0$ ).If $r > 1$, then $p > q$ , $r - 1 = \frac{(p-q)}{q}$. In this case, join rectangular strips, each representing $t_{n+1}$with breadth 1 and length $t_{n+1}$ to form a bigger rectangle of breadth 1 and length $qt_{n+1}$.This bigger rectangle is divided into $p - q$ equal segments so that each segment represents $\frac{qt_{n+1}}{p-q}$. Now subdivide each segment into $p$ equal strips so that each strip represents $\frac{1}{p}\left(\frac{qt_{n+1}}{p-q}\right)$.



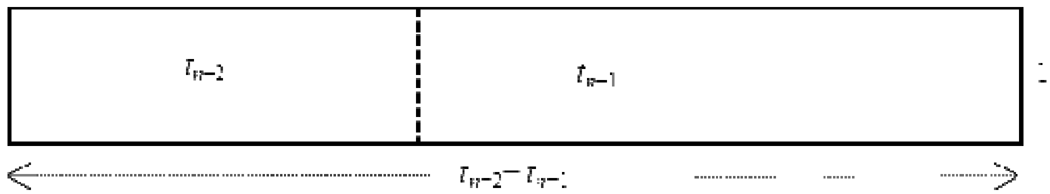

Hence $p - q$ of them taken together will represent $\frac{qt_{n+1}}{p} = \frac{t_{n+1}}{r} = t_n$. So each one of them will represent $\left(\frac{t_n}{p-q}\right)$ and hence the remaining $q$ strips will give $\frac{qt_n}{p-q}$. Since one segment is the join of $p - q$ strips and the remaining $q$ strips, $\frac{qt_{n+1}}{p-q} = t_n + \frac{qt_n}{p-q}$. Dividing the remaining strips representing $\frac{qt_n}{p-q}$ into $p$ sub strips and grouping $p - q$ of them together and remaining q separately we get $\frac{qt_{n1}}{p-q} = t_{n-1} + \frac{qt_{n-1}}{p-q}$.

Proceeding like this we get $\frac{qt_{n+1}}{p-q} = t_n + t_{n-1} + ... + t_1 + \frac{qt_1}{p-q}$.

Hence $\frac{t_{n+1}}{r-1} = S_n + \frac{t_1}{r-1}$ or $S_n = \frac{t_{n+1} - t_1}{r-1}$ .[When $r < 1$, $S_n = \frac{t_1 - t_{n+1}}{1-r}$ , $1 - r = \frac{(q-p)}{q}$]

**Case 4:** $r=\sqrt{m}$ , $m$ being positive.

Join the two rectangles representing $t_{n+2}$ and $t_{n+1}$ to form a bigger rectangle of breadth 1 and length $t_{n+2}+t_{n+1}$

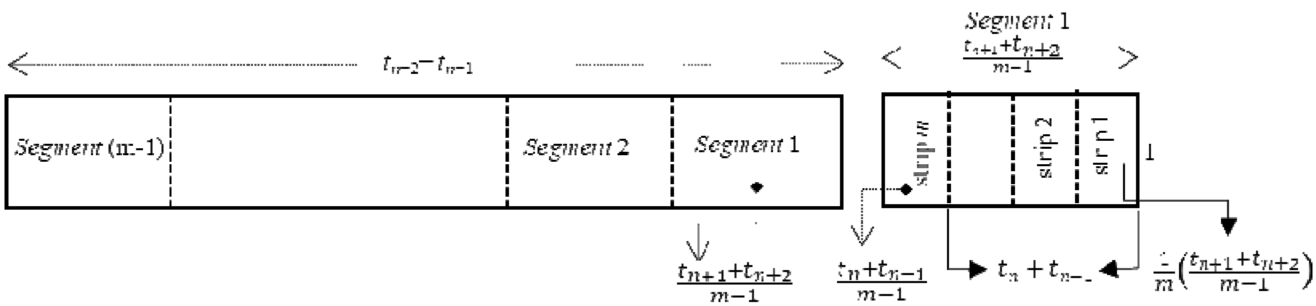


Divide this bigger rectangle into $m$ – 1 equal segments so that each segment represents $\frac{t_{n+2}+t_{n+1}}{m-1}$.

Then subdivide each of these segments into $m$ equal stripsso that each strip represents $\frac{1}{m}\left(\frac{t_{n+2}+t_{n+1}}{m-1}\right)$.

Hence $m$ – 1 such strips taken together in each segment will represent $\frac{t_{n+2}+t_{n+1}}{m}$

Now, $\frac{t_{n+2}+t_{n+1}}{m}=\frac{t_{n+2}}{r^2}+\frac{t_{n+1}}{r^2}=\frac{t_{n+1}}{r}+\frac{t_n}{r}=t_n+t_{n-1}$.The remaining one strip in each segment being same as any one of the other $m$ – 1 strips representing $t_n+t_{n-1}$ will represent $\frac{t_n+t_{n-1}}{m-1}$.




Thus all the $m$ strips in a segment taken together will represent $\frac{t_{n+2}+t_{n+1}}{m-1}=t_n+t_{n-1}+\frac{t_n+t_{n-1}}{m-1}$ and hence all the $m$ – 1 segments of $t_{n+2}+t_{n+1}$ will give $t_{n+2}+t_{n+1}=(m-1)(t_n+t_{n-1})+\{t_n+t_{n-1}\}$. Proceeding like this with the remainder strip representing $\frac{t_n+t_{n-1}}{m-1}$ in each segment, we get $t_n+t_{n-1}=(m-1)(t_{n-2}+t_{n-3})+\{t_{n-2}+t_{n-3}\}$.

Finally, as $n$ is even or odd we get,

$$t_{n+2}+t_{n+1}=\begin{cases}(m-1)[(t_n+t_{n-1})+(t_{n-2}+t_{n-3})+...+(t_2+t_1)]+t_2+t_1 \quad or\\ (m-1)[(t_n+t_{n-1})+(t_{n-2}+t_{n-3})+...+(t_1+t_0)]+t_1+t_0\end{cases}$$

i.e, $t_{n+2}+t_{n+1}=(m-1)S_n+t_2+t_1$ or $(m-1)(S_n+t_0)+t_1+t_0$

Thus $S_n=\frac{t_{n+2}+t_{n+1}}{m-1}-\frac{t_2+t_1}{m-1}$ or $\frac{t_{n+2}+t_{n+1}}{m-1}-\frac{t_1+t_0}{m-1}-t_0$

$$S_n=\frac{t_{n+2}+t_{n+1}}{(r-1)(r+1)}-\frac{t_2+t_1}{(r-1)(r+1)} \text{ or } \frac{t_{n+2}+t_{n+1}}{(r-1)(r+1)}-\frac{t_1+t_0}{(r-1)(r+1)}-t_0$$

Since $t_{n+2} = rt_{n+1}$ , $t_2 = rt_1$ , $t_1 = rt_0$ we get $S_n = \frac{t_{n+1}}{(r-1)} - \frac{t_1}{(r-1)}$ or $\frac{t_{n+1}}{(r-1)} - \frac{t_0}{(r-1)} - t_0$ . In either case, $S_n = \frac{t_{n+1} - t_1}{r-1}$.

If $r = \sqrt[k]{m}$ ,then *k* rectangular strips of unit breadth representing *k* successive *raised terms* $t_{n+k}, t_{n+k-1}, \ldots, t_{n+1}$ ,(raised from the last term $t_n$ by multiplying it successively by *r*) are needed. These *k* strips are joined to form a bigger rectangle of unit breadth and length as much as $t_{n+k} + t_{n+k-1} + \ldots + t_{n+1}$ which is then divided into $m-1$ equal segments and each segment is then divided into *m* sub parts. Each of these segments consisting of $m-1$ parts together representing $t_n + t_{n-1} + \ldots + t_{n-(k-1)}$ and are remaining part representing $\frac{t_n + t_{n-1} + \ldots + t_{n-(k-1)}}{m-1}$ will therefore give

$\frac{t_{n+k} + t_{n+k-1} + \ldots + t_{n+1}}{m-1} = t_n + t_{n-1} + \ldots + t_{n-(k-1)} + \frac{t_n + t_{n-1} + \ldots + t_{n-(k-1)}}{m-1}$ . Proceed as stated above with the remaining part $\frac{t_n + t_{n-1} + \ldots + t_{n-(k-1)}}{m-1}$ etc. to get finally the result $S_n = \frac{t_{n+1} - t_1}{r-1}$ for all *n*.

**Remark**: It may be noted that for the geometrical demonstration of series with $r = m^{1/k}$ , *k* rectangular strips representing *k* successive *raised terms* $t_{n+k}$, $t_{n+k-1}, \ldots, t_{n+1}$ are needed. Śaṅkara has given a geometric demonstration using a geometric series with $r = m^{1/k}$ , where *m* =4, *k* = 1. Since *k* = 1, only one rectangular strip representing just one raised term $t_{n+1}$ is needed.

## CONCLUSION

The practice of diagrammatic representation of arithmetic and algebraic truths is as old as geometry and the representation of series using diagrams is a special feature of Indian mathematics. The concept of diagrammatic representation of series by piling up bricks and balls is probably a heritage from vedic *citi*s and the terminology *citi*, *upaciti*, *citighana* etc for series sums used by *Āryabhaṭa* I and inclusion of the topic on series by Bhāskara I in the geometry section of his commentary on the *Āryabhaṭīya* along with a number of illustrations of series summations using piles of bricks and balls indicate their knowledge and vision of intimate connection of series with geometry. Geometric representation of series using rectangular strips is a special feature of Kerala mathematics and geometric representation of geometric series in the *Kriyākramakarī* is an important contribution made by *Śaṅkara*[15]. Such visual methods are very much effective in making the abstract concepts at once convincing for the user and thus have great importance in the pedagogical sense. The *Kriyākramakarī* is quite rich in such demonstrations.

## ACKNOWLEDGEMENT

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# SOME NEW TRILATERAL GENERATING RELATIONS INVOLVING I-FUNCTION

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**ABSTRACT**

***The I-function of one variable plays an important role in the development and study of special functions. The usefulness of this function has inspired us to find some new generating relations.***

***The aim of this paper is to establish some new trilateral generating relations involving I-function of one variable, which can be used to investigate the various useful properties of the sequences.***

***Key words : I-function , Trilateral generating relations , Horn functions , H-function.***

## 1. Introduction:

The I-function of one variable is defined by Saxena [1, p.366-375] and we will represent here in the following manner:

$$I^{m,\,n}_{p_i,\,q_i:\,r}\left[x\left|\begin{matrix}[(a_j,\alpha_j)_{1,n}],\,[(a_{ji},\alpha_{ji})_{n+1},p_i]\\ [(b_j,\beta_j)_{1,m}],\,[(b_{ji},\beta_{ji})_{m+1},q_i]\end{matrix}\right.\right] = (1/2\pi\omega)\int_L \theta(s)\,x^s\,ds \tag{1.1}$$

where $\omega = \sqrt{(-1)}$,

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\left(\sum_{i=1}^{r}\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right)},$$

integral is convergent, when (B>0, A ≤ 0), where

$$B = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{pi}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{qi}\beta_{ji} \tag{1.2}$$

$$A = \sum_{j=1}^{pi}\alpha_{ji} - \sum_{j=1}^{qi}\beta_{ji} \tag{1.3}$$

$$|\arg x| < \tfrac{1}{2}B\pi,\ \forall\, i \in (1, 2, \ldots, r). \tag{1.4}$$

Generating relations plays an important role in the investigation of various useful properties of the sequences, which they generate. They are used as z-transform in solving certain classes of difference equation which arise in a wide variety of problems in operation research (including, for example, queneing theory and related stochastic process). Generating relations can also be used with good effect for the determination of the asymptotic behavior of the generalized sequence $\{f_n\}_{n=0}^{\infty}$ as $n\to\infty$ by suitably adopting Darboux's method.

In this paper, we have discussed some trilateral generating relations involving I-function of one variable.

## 2. FORMULAE APPLIED

In the present investigation we require the following formulae:

From Shrivastava and Manocha [21, p.37 (10), 34, 44],

$$ {}_1F_1[a; a; z] = e^z . \tag{2.1}$$

$$ |z| < 1, (1-z)^{-a} = {}_1F_0[a; -; z] . \tag{2.2}$$

$$ e^z = {}_0F_0[-;-; z]. \tag{2.3}$$

$$ (\alpha)_n = (\alpha, n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} . \tag{2.4}$$

$$ (1-z)^{-a} = \sum_{n=0}^{\infty}(a)_n \frac{z^n}{n!} . \tag{2.5}$$

$$ \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} P_n^{(\alpha-n,\beta-n)}(z)t^n = F_1\left[\lambda, -\alpha, -\beta; \mu; -(z+1)\frac{t}{2}, -(z-1)\frac{t}{2}\right] . \tag{2.6}$$

$$ \sum_{n=0}^{\infty} \frac{(\lambda)_n(\delta)_n}{(\alpha+1)_n(\beta+1)_n} P_n^{(\alpha,\beta)}(z)t^n = F_4\left[\lambda, \delta; \alpha+1, \beta+1; (z-1)\frac{t}{2}, (z+1)\frac{t}{2}\right]. \tag{2.7}$$

From Rainvile [1, p.93]:

$$ {}_2F_1\left[\begin{matrix}-n, & a; \\ 1+a+n; \end{matrix} -1\right] = \frac{(1+a)_n}{(1+a/2)_n} . \tag{2.8}$$

$$ (\alpha)_{-n} = \frac{(-1)^n}{(1-\alpha,n)} . \tag{2.9}$$

$$ (\alpha', p-q) = (\alpha', -q)(\alpha'-q, p) = \frac{(-1)^q(\alpha'-q,p)}{(1-\alpha',q)}. \tag{2.10}$$

$$ (\mu, p)\,(\mu+p, r+s) = (\mu, p+r+s) . \tag{2.11}$$

$$(\lambda, p + q)\,(\lambda + p + q, r + s) = (\lambda, p + q + r + s)$$

$$= (\lambda, q)\,(\lambda + q, p + r + s)\,. \tag{2.12}$$

$$(\mu, n)\,(\mu + n, p) = (\mu, n + p) = (\mu, p)\,(\mu + p, n). \tag{2.13}$$

$$H_2(\alpha, \beta, \gamma, \delta; \varepsilon; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\varepsilon)_m m!\, n!} x^m y^n, \tag{2.14}$$

$|x| < r$, $|y| < s$, $(r + s) = 1$;

$$F_1(a, b, b'; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}\, m!\, n!} x^m y^n, \tag{2.15}$$

$\max\{|x|, |y|\} < 1$;

$$F_S[\alpha_1, \alpha_2, \alpha_2, \beta_1, \beta_2, \beta_3; \gamma_1, \gamma_1, \gamma_1; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha_1,m)(\alpha_2,n+p)(\beta_1,m)(\beta_2,n)(\beta_3,p)}{(\gamma_1,m+n+p)(1,m)(1,n)(1,p)} x^m y^n z^p, \tag{2.16}$$

## 3 TRILATERAL GENERATING RELATIONS:

In this section we establish the following trilateral generating relations:

$$\sum_{n=0}^{\infty} H_2\,[\alpha', \beta', \gamma', \delta'; \mu + n; x, y] P_n^{(\alpha-n,\beta-n)}(z)$$

$$.\, I_{p_i+1,q_i+1:r}^{m+1,k}[u|_{(\lambda+n,0),\ldots}^{\ldots,(\mu+n,0)}]t^n$$

$$= \sum_{q=0}^{\infty} \frac{(\gamma', q)(\delta', q)}{(1 - \alpha', q)(1, q)} (-y)^q I_{p_i+1,q_i+1:r}^{m+1,k}[u|_{(\lambda,0),\ldots}^{\ldots,(\mu,0)}]$$

$$.F_S[\alpha' - q, \lambda, \lambda, \beta', -\alpha, -\beta; \mu, \mu, \mu; x, -(z + 1)\tfrac{t}{2}, -(z - 1)\tfrac{t}{2}], \tag{3.1}$$

$|x| < r$, $|y| < s$, $(r + s) = 1$, $|\arg u| < ½\, B\pi$, where B is given in (1.2);

$$\sum_{n=0}^{\infty} G_1\,[\delta + n, \beta', \beta''; x, y] P_n^{(\alpha,\beta)}(z)$$

$$.I_{p_i+2,q_i+2:r}^{m+2,k}[u|_{(\gamma+n,0),(\delta+n,0),\ldots}^{\ldots,(\alpha+1+n,0)\ (\beta+1+n,0)}]t^n$$

$$= \sum_{p=0}^{\infty} \frac{(\delta, p)(\beta'', p)}{(1-\beta', p)(1, p)} (-x)^p I_{p_i+2,q_i+2:r}^{m+2,k}[u|_{(\gamma,0),(\delta,0),\ldots}^{\ldots,(\alpha+1,0)\ (\beta+1,0)}]$$

$$.\, F_E[\delta+p, \delta+p, \delta+p, \beta'-p, \gamma, \gamma; 1-\beta''-p, \alpha+1, \beta+1; -y, (z-1)\tfrac{t}{2}, (z+1)\tfrac{t}{2}], \quad (3.2)$$

$|x| < r$, $|y| < s$, $(r + s) = 1$, $|\arg u| < ½\, B\pi$;

$$\sum_{n=0}^{\infty} H_3\, [\alpha', \lambda+n; \mu+n; x, y] P_n^{(\alpha-n,\beta-n)}(z).\, I_{p_i+1,q_i+1:r}^{m+1,k}[u|_{(\lambda+n,0),\ldots}^{\ldots,(\mu+n,0)}] t^n$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha', 2p)}{(\mu, p)(1, p)} (x)^p I_{p_i+1,q_i+1:r}^{m+1,k}[u|_{(\lambda,0),\ldots}^{\ldots,(\mu,0)}]$$

$$.F_N[\alpha'+2p, -\alpha, -\beta, \lambda+r, \lambda, \lambda+r; \mu, \mu+q, \mu+q; y, -(z+1)\tfrac{t}{2}, -(z-1)\tfrac{t}{2}], \quad (3.3)$$

$|x| < 1$, $|\arg u| < ½\, B\pi$;

$$\sum_{n=0}^{\infty} H_6\, [\alpha', \lambda+n; \gamma'; x, y] P_n^{(\alpha-n,\beta-n)}(z)$$

$$.I_{p_i+1,q_i+1:r}^{m,k+1}[u|_{\ldots,(1-\mu-n,0)}^{(1-\lambda-n,0),\ldots,}] t^n$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha', 2p)}{(1-\lambda, p)(1, p)} (-x)^p I_{p_i+1,q_i+1:r}^{m,k+1}[u|_{\ldots,(1-\mu,0)}^{(1-\lambda,0),\ldots,}]$$

$$.F_G[\lambda-p, \lambda-p, \lambda-p, \gamma, -\alpha, -\beta; 1-\alpha'-2p, \mu, \mu; -y, -(z+1)\tfrac{t}{2}, -(z-1)\tfrac{t}{2}], \quad (3.4)$$

$|x| < r$, $|y| < s$, $rs^2 + s - 1$, $|\arg u| < ½\, B\pi$;

$$\sum_{n=0}^{\infty} H_7\, [\alpha', \gamma+n, \delta+n; \delta'; x, y] P_n^{(\alpha,\beta)}(z).\, I_{p_i+2,q_i+2:r}^{m+2,k}[u|_{(\gamma+n,0),(\delta+n,0),\ldots}^{\ldots,(\alpha+1+n,0)\ (\beta+1+n,0)}] t^n$$

$$= \sum_{p=0}^{\infty} \frac{(\alpha', 2p)}{(\delta', p)(1,p)} (-x)^p I^{m+2,k}_{p_i+2,q_i+2:r}[u|^{\ldots,(\alpha+1,0)\ (\beta+1,0)}_{(\gamma,0),(\delta,0),\ldots}]$$

$$. F_K[\gamma, \gamma+q, \gamma+q, \delta+r, \delta, \delta+r; 1-\alpha'-2p, \alpha+1, \beta+1; -y, (z-1)\tfrac{t}{2}, (z+1)\tfrac{t}{2}], \qquad (3.5)$$

$|x| < r$, $|y| < s$, $4r = (s^{-1} - 1)^2$, $|\arg u| < \frac{1}{2} B\pi$.

**Proof:**

To prove (3.1), consider

$$\sum_{n=0}^{\infty} H_2\,[\alpha', \beta', \gamma', \delta'; \mu+n; x, y] P_n^{(\alpha-n,\beta-n)}(z)\,.\, I^{m+1,k}_{p_i+1,q_i+1:r}[u|^{\ldots,(\mu+n,0)}_{(\lambda+n,0),\ldots}] t^n.$$

Expressing $H_2$ in series form, by[2] (2.14) and I-function (1.1) and using (2.4), we get

$$\Delta = \sum_{n=0}^{\infty} \sum_{p,q=0}^{\infty} \frac{(\alpha', p-q)(\beta', p)(\gamma', q)(\delta', q)}{(\mu+n, p)(1,p)(1,q)} x^p y^q P_n^{(\alpha-n,\beta-n)}(z)$$

$$.[\frac{1}{2\pi\omega} \int_L \theta(s) u^s \frac{(\lambda,n)\Gamma(\lambda)}{(\mu,n)\Gamma(\mu)} ds] t^n.$$

Now interchange the order of summation and integration and on using (2.13), we get

$$\Delta = \frac{1}{2\pi\omega} \int_L \theta(s) \frac{\Gamma(\lambda)}{\Gamma(\mu)} u^s$$

$$. \sum_{p,q=0}^{\infty} \frac{(\alpha', p-q)(\beta', p)(\gamma', q)(\delta', q)}{(\mu, p)(1,p)(1,q)} x^p y^q$$

$$. \left[ \sum_{n=0}^{\infty} \frac{(\lambda, n)}{(\mu+p, n)} P_n^{(\alpha-n,\beta-n)}(z) t^n \right] ds.$$

Again applying (2.6), we find that

$$\Delta = \frac{1}{2\pi\omega} \int_L \theta(s) u^s \frac{\Gamma(\lambda)}{\Gamma(\mu)} \sum_{p,q=0}^{\infty} \frac{(\alpha', p-q)(\beta', p)(\gamma', q)(\delta', q)}{(\mu, p)(1,p)(1,q)} x^p y^q$$

$$.F_1 \left[ \lambda, -\alpha, -\beta; \mu+p; -(z+1)\tfrac{t}{2}, -(z-1)\tfrac{t}{2} \right] ds.$$

Further writing $F_1$ in series form, on using (2.15), we find that

$$\Delta = \frac{1}{2\pi\omega}\int_L \theta(s)u^s \frac{\Gamma(\lambda)}{\Gamma(\mu)} \sum_{p,q=0}^{\infty} \frac{(\alpha', p-q)(\beta', p)(\gamma', q)(\delta', q)}{(\mu, p)(1, p)(1, q)} x^p y^q$$

$$.\sum_{j,k=0}^{\infty} \frac{(\lambda, j+k)(-\alpha, j)(-\beta, k)}{(\mu+p, j+k)(1, j)(1, k)} [-(z+1)\frac{t}{2}]^j [-(z-1)\frac{t}{2}]^k ds.$$

Now using relation (2.10) and (2.11), we find that

$$\Delta = \frac{1}{2\pi\omega}\int_L \theta(s)u^s \frac{\Gamma(\lambda)}{\Gamma(\mu)} \sum_{q=0}^{\infty} \frac{(\gamma', q)(\delta', q)}{(1-\alpha', q)(1, q)} (-y)^q$$

$$.\sum_{p,j,k=0}^{\infty} \frac{(\alpha'-q, p)(\lambda, j+k)(\beta', p)(-\alpha, j)(-\beta, k)}{(\mu, p+j+k)(1, p)(1, j)(1, k)} [-(z+1)\frac{t}{2}]^j [-(z-1)\frac{t}{2}]^k ds,$$

which in the light of (2.16) and (1.1) provides (3.1). Proceeding on similar lines, (3.2) to (3.5) can be derived with the help of the formulae given in section 2.

## 4. PARTICULAR CASES:

On choosing r = 1, we get following generating relations in terms of H-function of one variable:

$$\sum_{n=0}^{\infty} H_2\,[\alpha', \beta', \gamma', \delta'; \mu+n; x, y] P_n^{(\alpha-n,\beta-n)}(z)$$

$$.\,H_{p+1,q+1}^{m+1,k}[u|_{(\lambda+n,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\mu+n,0)}]t^n$$

$$= \sum_{q=0}^{\infty} \frac{(\gamma', q)(\delta', q)}{(1-\alpha', q)(1, q)} (-y)^q H_{p+1,q+1}^{m+1,k}[u|_{(\lambda,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\mu,0)}]$$

$$.F_S[\alpha'-q, \lambda, \lambda, \beta', -\alpha, -\beta; \mu, \mu, \mu; x, -(z+1)\frac{t}{2}, -(z-1)\frac{t}{2}], \quad (4.1)$$

$|x| < r$, $|y| < s$, $(r + s) = 1$, $|\arg u| < ½A\pi$, where A is given in (1.3);

$$\sum_{n=0}^{\infty} G_1\,[\delta+n,\beta',\beta'';x,y]P_n^{(\alpha,\beta)}(z)$$

$$.\,H_{p+2,q+2}^{m+2,k}[u|_{(\gamma+n,0),(\delta+n,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\alpha+1+n,0)\ (\beta+1+n,0)}]t^n$$

$$=\sum_{p=0}^{\infty}\frac{(\delta,p)(\beta'',p)}{(1-\beta',p)(1,p)}(-x)^p H_{p+2,q+2}^{m+2,k}[u|_{(\gamma,0),(\delta,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\alpha+1,0)\ (\beta+1,0)}]$$

$$.\,F_E[\delta+p,\delta+p,\delta+p,\beta'-p,\gamma,\gamma;1-\beta''-p,\alpha+1,\beta+1;-y,(z-1)\frac{t}{2},(z+1)\frac{t}{2}],\qquad (4.2)$$

$|x| < r$, $|y| < s$, $r + s = 1$, $|\arg u| < ½A\pi$;

$$\sum_{N=0}^{\infty} H_3\,[\alpha',\lambda+n;\mu+n;x,y]P_n^{(\alpha-n,\beta-n)}(z)$$

$$.H_{p+1,q+1}^{m+1,k}[u|_{(\lambda+n,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\mu+n,0)}]t^n$$

$$=\sum_{p=0}^{\infty}\frac{(\alpha',2p)}{(\mu,p)(1,p)}(x)^p H_{p+1,q+1}^{m+1,k}[u|_{(\lambda,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\mu,0)}]$$

$$.\,F_N[\alpha'+2p,-\alpha,-\beta,\lambda+r,\lambda,\lambda+r;\mu,\mu+q,\mu+q;y,-(z+1)\frac{t}{2},-(z-1)\frac{t}{2}],\qquad (4.3)$$

$|x| < r$, $|\arg u| < ½A\pi$;

$$\sum_{n=0}^{\infty} H_6\,[\alpha',\lambda+n;\gamma';x,y]P_n^{(\alpha-n,\beta-n)}(z)$$

$$.H_{p+1,q+1}^{m,k+1}[u|_{(b_j,\beta_j)_{1,q}.(1-\mu-n,0)}^{(1-\lambda-n,0),(a_j,\alpha_j)_{1,p}}]t^n$$

$$=\sum_{p=0}^{\infty}\frac{(\alpha',2p)}{(1-\lambda,p)(1,p)}(-x)^p H_{p+1,q+1}^{m,k+1}[u|_{(b_j,\beta_j)_{1,q}.(1-\mu,0)}^{(1-\lambda,0),(a_j,\alpha_j)_{1,p}}]$$

$$.\,F_G[\lambda-p,\lambda-p,\lambda-p,\gamma,-\alpha,-\beta;1-\alpha'-2p,\mu,\mu;-y,-(z+1)\tfrac{t}{2},-(z-1)\tfrac{t}{2}], \qquad (4.4)$$

$|x| < r$, $|y| < s$, $rs^2 + s - 1$, $|\arg u| < ½ A\pi$;

$$\sum_{n=0}^{\infty} H_7\,[\alpha',\gamma+n,\delta+n;\delta';x,y]P_n^{(\alpha,\beta)}(z)$$

$$.\,H_{p+2,q+2}^{m+2,l}[u|_{(\gamma+n,0),(\delta+n,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\alpha+1+n,0)\ (\beta+1+n,0)}]t^n$$

$$=\sum_{p=0}^{\infty}\frac{(\alpha',2p)}{(\delta',p)(1,p)}(-x)^p H_{p+2,q+2}^{m+2,l}[u|_{(\gamma,0),(\delta,0),(b_j,\beta_j)_{1,q}}^{(a_j,\alpha_j)_{1,p},(\alpha+1,0)\ (\beta+1,0)}]$$

$$.\,F_K[\gamma,\gamma+q,\gamma+q,\delta+r,\delta,\delta+r;1-\alpha'-2p,\alpha+1,\beta+1;-y,(z-1)\tfrac{t}{2},(z+1)\tfrac{t}{2}],$$

(4.5)

$|x| < r$, $|y| < s$, $4r = (s^{-1}-1)^2$, $|\arg u| < ½ A\pi$.

**CONCLUSION**

In this Paper we established some new Trilateral generating relations involving I-function and further we find out some special cases in which The trilateral generating relation involve H-function.

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# PERFORMANCE MEASURES OF A NUCLEAR POWER PLANT

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## ABSTRACT

***In this paper, we have considered a nuclear power plant for the performance measure in terms of evaluation of availability, reliability and MTTF. A nuclear power plant among other things is useful in the production of electricity. The whole nuclear reactor is divided into five subsystems, numbered as (1) Reactor vessel, (2) Heat exchanger, (3) Turbine, (4) Condenser, (5) Generator , which are connected in series.***

***All failures follow exponential distribution whereas repairs obey general time distribution. A set of differential difference equations has been obtained as a mathematical modeling continuous w.r.to time and discrete w.r.to space. The ergodic behavior of the system along some particular cases have been discussed. The numerical illustration together with graphical diagram have been appended to highlight the important results of study.***

***Key Words: Availability, Reliability, MTTF, Performance measure.***

## 1. INTRODUCTION

The main task of reactor vessel is to produce energy through fissioning of neutrons of nuclear fuel Uranium-235. This energy goes into heat exchanger through coolant and heat exchanger converts this energy into steam. This steam is used to rotate turbine and then it goes to condenser, which convert this again into water. This water gets back to heat exchanger. The turbine is connected to power generator, which produces the electricity. In the considered system, we take one standby redundant heat exchanger to improve system s availability. This standby heat exchanger is followed online through a perfect switching device. The whole system can fail due to failure of any of its subsystems. All the failures follow exponential time distribution while all the repairs follow general time distribution.

We have used supplementary variable technique to solve the system. System configuration diagram have shown through fig-1. A set of difference-differential equations has obtained which is continuous with respect to time and discrete with respect to space. Laplace transform has been used to solve these difference-differential equations and thus various transition-state probabilities have been obtained. Steady-state behaviour of the system and a particular case (when repairs follow exponential time distribution) has been appended at the end to illustrate practical utility of the model. Graphical illustration for a numerical example has been given at the end to highlight important results of the study.

## NOTATIONS

$P_0(t)/P_2(t)$ : Probability that the system is in operable state at time t due to working of original/ standby heat exchanger.

$P_i(j,t)\Delta$ : Probability that the system is in failed state at time t due to

| | | |
|---|---|---|
| | | failure of i[th] subsystem and elapsed repair time lies in the interval $j$ to $j+\Delta$, where i=1,3,4,5 and j= $x_1, x_3, x_4, x_5$, respectively. |
| $P_{2,i}(j,t)\Delta$ | : | Probability that the system is failed at time t due to failure of i[th] subsystem while standby heat exchanger is working and elapsed repair time lies in the interval $j$ to $j+\Delta$, where i=1,2,3,4,5 and j= $x_1, x_2, x_3, x_4, x_5$, respectively. |
| $\alpha_i$ | : | Failure rate of i[th] subsystem. |
| $\mu_i(j)$ | : | General repair rate of i[th] subsystem conditioned that it was not repaired upto the time $j$, where i=1,2,3,4,5 and j= $x_1, x_2, x_3, x_4, x_5$, respectively. |
| M.T.T.F.$_{\cdot i}$ | : | Mean time to system failure corresponding to failure rate $\alpha_i$, while other failure rates will remain constant. |

## ASSUMPTIONS

Following assumptions have been constructed to discuss the model:

(1) Initially, the whole system is in working state of full efficiency.
(2) Perfect switching device has been used to online standby heat exchanger.
(3) Repair has given to the degraded or failed state and after repair units work like new.
(4) All failures follow exponential time distribution whereas all repairs follow general time distribution.
(5) Only one change can take place in one transition.

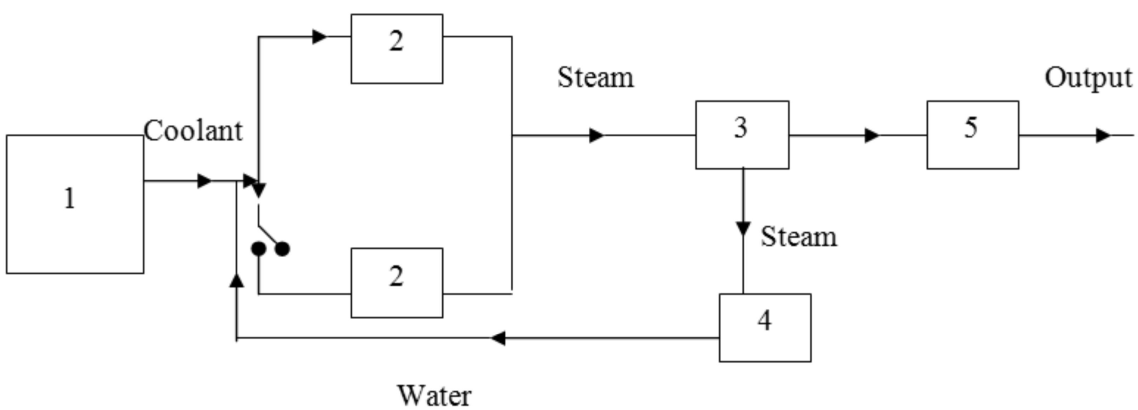


**Fig-1(Block diagram of considered system)**

## MATHEMATICAL FORMULATION

By using continuity arguments and limiting procedure, one can obtain the following set of difference-differential equations governing the behaviour of considered system:

$$\left[\frac{d}{dt}+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\right]P_0(t)=\int_0^\infty P_1(x_1,t)\mu_1(x_1)dx_1$$

$$+\int_0^\infty P_3(x_3,t)\mu_3(x_3)dx_3+\int_0^\infty P_4(x_4,t)\mu_4(x_4)dx_4$$

$$+\int_0^\infty P_5(x_5,t)\mu_5(x_5)dx_5+\int_0^\infty P_{2,2}(x_2,t)\mu_2(x_2)dx_2 \tag{1}$$

$$\left[\frac{d}{dt}+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\right]P_2(t)=\int_0^\infty P_{2,1}(x_1,t)\mu_1(x_1)dx_1$$

$$+\int_0^\infty P_{2,3}(x_3,t)\mu_3(x_3)dx_3+\int_0^\infty P_{2,4}(x_4,t)\mu_4(x_4)dx_4$$

$$+\int_0^\infty P_{2,5}(x_5,t)\mu_5(x_5)dx_5+\alpha_2 P_0(t) \tag{2}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x_1}+\mu_1(x_1)\right]P_1(x_1,t)=0 \tag{3}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x_3}+\mu_3(x_3)\right]P_3(x_3,t)=0 \tag{4}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x_4}+\mu_4(x_4)\right]P_4(x_4,t)=0 \tag{5}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x_5}+\mu_5(x_5)\right]P_5(x_5,t)=0 \tag{6}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x_1}+\mu_1(x_1)\right]P_{2,1}(x_1,t)=0 \tag{7}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x_3}+\mu_3(x_3)\right]P_{2,3}(x_3,t)=0 \tag{8}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x_4}+\mu_4(x_4)\right]P_{2,4}(x_4,t)=0 \tag{9}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x_5}+\mu_5(x_5)\right]P_{2,5}(x_5,t)=0 \tag{10}$$

$$\left[\frac{\partial}{\partial t}+\frac{\partial}{\partial x_2}+\mu_2(x_2)\right]P_{2,2}(x_2,t)=0 \tag{11}$$

Boundary conditions are:

$$P_1(0,t)=\alpha_1 P_0(t) \tag{12}$$

$$P_3(0,t)=\alpha_3 P_0(t) \tag{13}$$

$$P_4(0,t)=\alpha_4 P_0(t) \tag{14}$$

$$P_5(0,t)=\alpha_5 P_0(t) \tag{15}$$

$$P_{2,1}(0,t)=\alpha_1 P_2(t) \tag{16}$$

$$P_{2,3}(0,t)=\alpha_3 P_2(t) \tag{17}$$

$$P_{2,4}(0,t)=\alpha_4 P_2(t) \tag{18}$$

$$P_{2,5}(0,t)=\alpha_5 P_2(t) \tag{19}$$

$$P_{2,2}(0,t)=\alpha_2 P_2(t) \tag{20}$$

Initial conditions are:

$P_0(0)=1,$ otherwise 0 for all other state probabilities at t=0. (21)

## SOLUTION OF THE MODEL

Taking Laplace transforms of equations (1) through (20) and applying the initial conditions (21), we obtain:

$$[s+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5]\overline{P}_0(s) = 1+\int_0^\infty \overline{P}_1(x_1,s)\mu_1(x_1)dx_1$$

$$+\int_0^\infty \overline{P}_3(x_3,s)\mu_3(x_3)dx_3+\int_0^\infty \overline{P}_4(x_4,s)\mu_4(x_4)dx_4$$

$$+\int_0^\infty \overline{P}_5(x_5,s)\mu_5(x_5)dx_5+\int_0^\infty \overline{P}_{2,2}(x_2,s)\mu_2(x_2)dx_2 \tag{22}$$

$$[s+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5]\overline{P}_2(s) = \int_0^\infty \overline{P}_{2,1}(x_1,s)\mu_1(x_1)dx_1$$

$$+\int_0^\infty \overline{P}_{2,3}(x_3,s)\mu_3(x_3)dx_3+\int_0^\infty \overline{P}_{2,4}(x_4,s)\mu_4(x_4)dx_4$$

$$+\int_0^\infty \overline{P}_{2,5}(x_5,s)\mu_5(x_5)dx_5+\alpha_2\overline{P}_0(s) \tag{23}$$

$$\left[s+\frac{\partial}{\partial x_1}+\mu_1(x_1)\right]\overline{P}_1(x_1,s)=0 \tag{24}$$

$$\left[s+\frac{\partial}{\partial x_3}+\mu_3(x_3)\right]\overline{P}_3(x_3,s)=0 \tag{25}$$

$$\left[s+\frac{\partial}{\partial x_4}+\mu_4(x_4)\right]\overline{P}_4(x_4,s)=0 \tag{26}$$

$$\left[s+\frac{\partial}{\partial x_5}+\mu_5(x_5)\right]\overline{P}_5(x_5,s)=0 \tag{27}$$

$$\left[s+\frac{\partial}{\partial x_1}+\mu_1(x_1)\right]\overline{P}_{2,1}(x_1,s)=0 \tag{28}$$

$$\left[s+\frac{\partial}{\partial x_3}+\mu_3(x_3)\right]\overline{P}_{2,3}(x_3,s)=0 \tag{29}$$

$$\left[s+\frac{\partial}{\partial x_4}+\mu_4(x_4)\right]\overline{P}_{2,4}(x_4,s)=0 \tag{30}$$

$$\left[s+\frac{\partial}{\partial x_5}+\mu_5(x_5)\right]\overline{P}_{2,5}(x_5,s)=0 \tag{31}$$

$$\left[s+\frac{\partial}{\partial x_2}+\mu_2(x_2)\right]\overline{P}_{2,2}(x_2,s)=0 \tag{32}$$

$$\overline{P}_1(0,s)=\alpha_1\overline{P}_0(s) \tag{33}$$

$$\overline{P}_3(0,s)=\alpha_3\overline{P}_0(s) \tag{34}$$

$$\overline{P}_4(0,s)=\alpha_4\overline{P}_0(s) \tag{35}$$

$$\overline{P}_5(0,s)=\alpha_5\overline{P}_0(s) \tag{36}$$

$$\overline{P}_{2,1}(0,s)=\alpha_1\overline{P}_2(s) \tag{37}$$

$$\overline{P}_{2,3}(0,s)=\alpha_3\overline{P}_2(s) \tag{38}$$

$$\overline{P}_{2,4}(0,s)=\alpha_4\overline{P}_2(s) \tag{39}$$

$$\overline{P}_{2,5}(0,s)=\alpha_5\overline{P}_2(s) \tag{40}$$

$$\overline{P}_{2,2}(0,s)=\alpha_2\overline{P}_2(s) \tag{41}$$

Now we solve these equations one by one and final solution of above equations is given by

$$\overline{P}_0(s)=\frac{1}{B(s)} \tag{42}$$

$$\overline{P}_2(s)=\frac{A(s)}{B(s)} \tag{43}$$

$$\overline{P}_i(s)=\frac{\alpha_i D_i(s)}{B(s)};\qquad \text{i}=1,3,4,5. \tag{44}$$

$$\overline{P}_{2,i}(s)=\frac{\alpha_i D_i(s)A(s)}{B(s)};\qquad i=1,2,3,4,5. \tag{45}$$

where,

$$A(s)=\frac{\alpha_2}{s+\sum_{i=1}^{5}\alpha_i-\alpha_1\overline{S}_1(s)-\alpha_3\overline{S}_3(s)-\alpha_4\overline{S}_4(s)-\alpha_5\overline{S}_5(s)} \tag{46}$$

$$B(s)=\frac{\alpha_2}{A(s)}-\alpha_2 A(s)\overline{S}_2(s) \tag{47}$$

$$D_i(s)=\frac{1-\overline{S}_i(s)}{s} \tag{48}$$

Also, we observe that

$$\overline{P}_0(s)+\overline{P}_1(s)+\overline{P}_2(s)+\overline{P}_3(s)+\overline{P}_4(s)+\overline{P}_5(s)$$
$$+\overline{P}_{2,1}(s)+\overline{P}_{2,2}(s)+\overline{P}_{2,3}(s)+\overline{P}_{2,4}(s)+\overline{P}_{2,5}(s)=\frac{1}{s}$$

$$\Rightarrow \sum_{i=0}^{5}\overline{P}_i(s)+\sum_{i=1}^{5}\overline{P}_{2,i}(s)=\frac{1}{s} \tag{49}$$

**STEADY STATE OR ERGODIC BEHAVIOUR**

By making use of Abel s lemma, viz; $\lim_{s\to 0} s\overline{P_i}(s)=\lim_{t\to\infty}P_i(t)=P_i(say),$ provided the limit on right exists, we have the following time independent state probabilities from equations (42) through (45):

$$P_0=\frac{1}{B'(0)} \tag{50}$$

$$P_1=\frac{\alpha_1 M_1}{B'(0)} \tag{51}$$

$$P_2=\frac{1}{B'(0)} \tag{52}$$

$$P_3 = \frac{\alpha_3 M_3}{B'(0)} \tag{53}$$

$$P_4 = \frac{\alpha_4 M_4}{B'(0)} \tag{54}$$

$$P_5 = \frac{\alpha_5 M_5}{B'(0)} \tag{55}$$

$$P_{2,1} = \frac{\alpha_1 M_1}{B'(0)} \tag{56}$$

$$P_{2,2} = \frac{\alpha_2 M_2}{B'(0)} \tag{57}$$

$$P_{2,3} = \frac{\alpha_3 M_3}{B'(0)} \tag{58}$$

$$P_{2,4} = \frac{\alpha_4 M_4}{B'(0)} \tag{59}$$

$$P_{2,5} = \frac{\alpha_5 M_5}{B'(0)} \tag{60}$$

where, $M_i = -\overline{S}'_i(0) =$ mean time to repair i$^{\text{th}}$ unit. (61)

**PARTICULAR CASE**

***When repairs follow exponential time distribution***

Setting $\overline{S}_i(j) = \dfrac{\mu_i}{j+\mu_i}$ or $D_i(j) = \dfrac{1}{j+\mu_i}$ in equations (42) through (45), we get the following Laplace transforms of various transition-state probabilities, in this case:

$$\overline{P}_0(s) = \frac{1}{B_1(s)} \tag{62}$$

$$\overline{P}_1(s) = \frac{\alpha_1}{B_1(s)}\left(\frac{1}{s+\mu_1}\right) \tag{63}$$

$$\overline{P}_2(s) = \frac{A_1(s)}{B_1(s)} \tag{64}$$

$$\overline{P}_3(s) = \frac{\alpha_3}{B_1(s)}\left(\frac{1}{s+\mu_3}\right) \tag{65}$$

$$\overline{P}_4(s) = \frac{\alpha_4}{B_1(s)}\left(\frac{1}{s+\mu_4}\right) \tag{66}$$

$$\overline{P}_5(s) = \frac{\alpha_5}{B_1(s)}\left(\frac{1}{s+\mu_5}\right) \tag{67}$$

$$\overline{P}_{2,1}(s) = \frac{\alpha_1 A_1(s)}{B_1(s)}\left(\frac{1}{s+\mu_1}\right) \tag{68}$$

$$\overline{P}_{2,2}(s) = \frac{\alpha_2 A_1(s)}{B_1(s)}\left(\frac{1}{s+\mu_2}\right) \tag{69}$$

$$\overline{P}_{2,3}(s) = \frac{\alpha_3 A_1(s)}{B_1(s)}\left(\frac{1}{s+\mu_3}\right) \tag{70}$$

$$\overline{P}_{2,4}(s) = \frac{\alpha_4 A_1(s)}{B_1(s)}\left(\frac{1}{s+\mu_4}\right) \tag{71}$$

$$\overline{P}_{2,5}(s) = \frac{\alpha_5 A_1(s)}{B_1(s)}\left(\frac{1}{s+\mu_5}\right) \tag{72}$$

where,

$$A_1(s) = \frac{\alpha_2}{s + \sum_{\mathrm{i}=1}^{5}\alpha_{\mathrm{i}} - \alpha_1\left(\frac{\mu_1}{s+\mu_1}\right) - \alpha_3\left(\frac{\mu_3}{s+\mu_3}\right) - \alpha_4\left(\frac{\mu_4}{s+\mu_4}\right) - \alpha_5\left(\frac{\mu_5}{s+\mu_5}\right)} \tag{73}$$

$$B_1(s) = \frac{\alpha_2}{A_1(s)} - \alpha_2 A_1(s)\left(\frac{\mu_2}{s+\mu_2}\right) \tag{74}$$

## EVALUATION OF THE WORK

We have $\overline{P}_{up}(s) = \overline{P}_0(s) + \overline{P}_2(s)$

Availability of the system $P_{up}(t) = e^{-\sum_{i-1}^{5}\alpha_i t}\left[1+\alpha_2 t\right]$ (75)

and $\overline{R}(s) = \dfrac{1}{s+\sum_{\text{i=1}}^{5}\alpha_{\text{i}}}$

Reliability of the system as: $R(t) = e^{-\sum_{i=1}^{5}\alpha_i t}$ (76)

Also, mean time to failure or mean time to system failure is given by:

$$\text{M.T.T.F.} = \int_0^\infty R(t)dt \quad = \frac{1}{\sum_{\text{i=1}}^{5}\alpha_{\text{i}}} \qquad \ldots(77)$$

## NUMERICAL COMPUTATION

For a numerical computation, we consider some particular values of failure rates, repair rates and cost as follows: $\alpha_1$=0.001, $\alpha_2$=0.003, $\alpha_3$=0.005, $\alpha_4$=0.007, $\alpha_5$=0.009, $\mu_1$=0.01, $\mu_2$=0.03, $\mu_3$=0.05, $\mu_4$=0.07, $\mu_5$=0.09 on the basis of these values and for t = 0,2,4,6,8…………, we can compute the table-1 and 2 . Their corresponding graphs have shown through fig-2 and 3, respectively.

## RESULT AND DISCUSSION

In this paper, we have measured performance of the considered nuclear power plantin terms of availability, reliability and mean time to failure by equations (85), (86) and (87), respectively. Figure (2) shows the comparative changes in R(t) and $P_{up}$(t) with respect to time and result shows that initially Reliability and Availability of the system is 1 and as time increases Reliability decreases more rapidly than Availability. Figure (3) shows the changes in MTTF with respect to the failure rate $\alpha_i$ (failure rate of $i^{th}$ subsystem).

| t | R(t) | $P_{up}$(t) |
|---|---|---|
| 0 | 1 | 1 |
| 6 | .86 | .88 |
| 12 | .74 | .77 |
| 18 | .63 | .67 |
| 24 | .55 | .59 |
| 30 | .47 | .51 |
| 36 | .41 | .45 |
| 42 | .35 | .39 |
| 48 | .30 | .34 |
| 54 | .26 | .30 |
| 60 | .22 | .26 |
| 66 | .19 | .23 |

**Table-1**

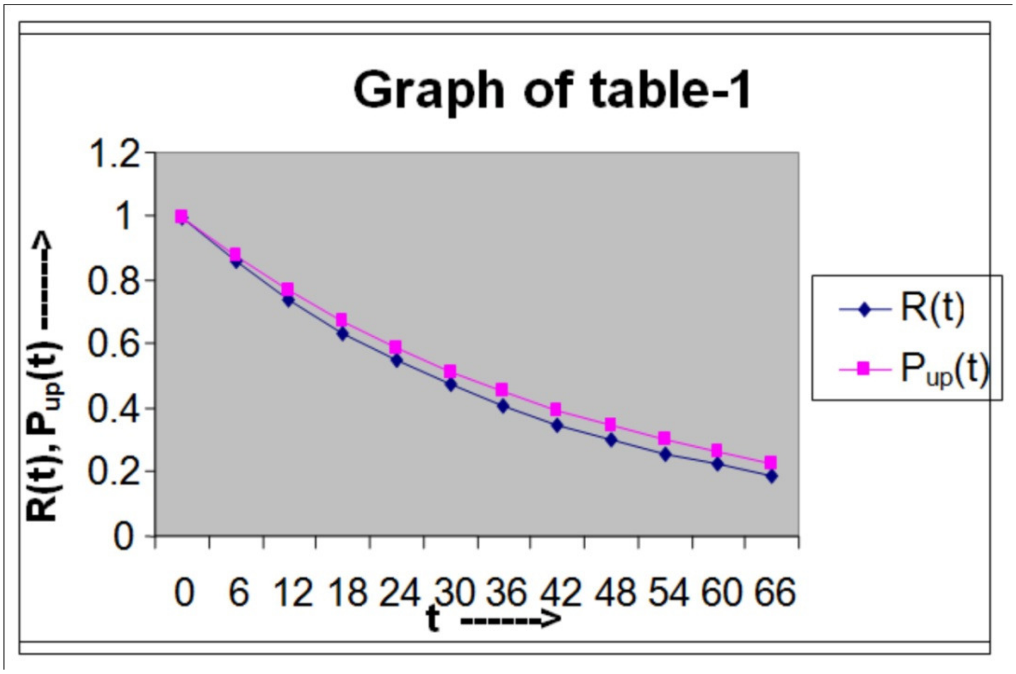


**Fig-2**

| $\alpha_i$ | MTTF$_1$ | MTTF$_2$ | MTTF$_3$ | MTTF$_4$ | MTTF$_5$ |
|---|---|---|---|---|---|
| 0 | 41.667 | 45.455 | 50 | 55.556 | 62.5 |
| .005 | 34.482 | 37.037 | 40 | 43.478 | 47.619 |
| .013 | 27.027 | 28.571 | 30.303 | 32.258 | 34.483 |
| .021 | 22.222 | 23.256 | 24.390 | 25.641 | 27.027 |
| .029 | 18.868 | 19.608 | 20.408 | 21.277 | 22.222 |
| .035 | 16.949 | 17.544 | 18.182 | 18.868 | 19.608 |

**Table-2**

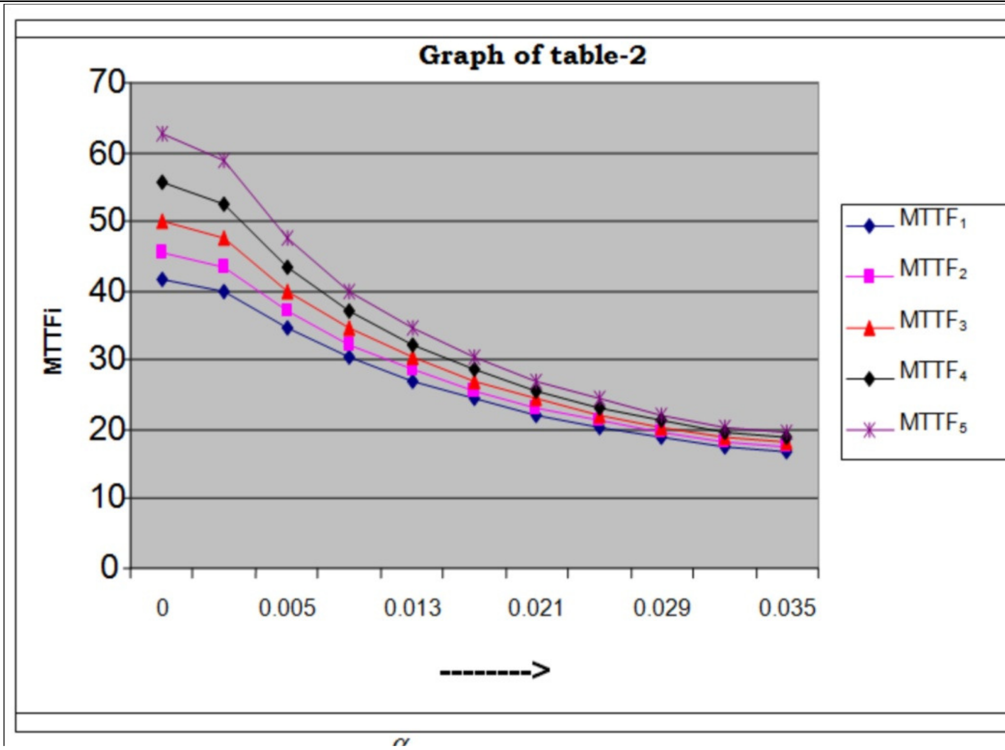


**Fig-3**

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# ALLOCATION OF SUBJECTS IN AN EDUCATIONAL INSTITUTION BY ROBUST RANKING METHOD

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**ABSTRACT**
***The classical assignment problems cannot be successfully used for real world situation it is because the work efficiency vary to some extent from person to person hence, the use of fuzzy assignment problem is more appropriate. Based on the performance of previous record in this study, on taking the scores obtained by the Chairman, by Course Director, by faculty members, and the student's feedback, we applied Robust ranking technique in fuzzy assignment problem for allocating the subjects in a department of coming semester. A study of the technique for 4 jobs and 4 persons have been made.***
***Keywords: Centroid ranking, Robust ranking, Magnitude ranking, Fuzzy assignment, Membership function.***

## 1. INTRODUCTION

Effective teaching and learning are critically important to all students and especially for those with special educational needs. In this, allocation of subjects plays a vital role.

Andrew and Collins [1] developed a procedure for assigning the subjects to the teachers. Hawood and Lawless [8] used a goal programming to solve the teacher assignment problem. Aldy Gunawan, K. M. Ng and H. L. Ong [2] used genetic algorithm for the teacher assignment problem.

Assignment problem with fuzzy parameters have been studied by several authors such as Balinski [3] and Chi-Jen-Lin [4] and Chen [5], Kuhn Liu and Gao [6], Sathi, Mukherjee and Kajla Basu [7].

Suppose there are ‘n’ people and ‘n’ jobs. Each job must be done by exactly one person; also, each person can do, at most, one job. The problem is to assign jobs to the people so as to minimize the total cost of completing all the jobs. The general assignment problem can be mathematically stated as follows : -

Minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad for \;\; i = 1, 2, \ldots \ldots . n \quad ( one\ job\ is\ done\ by\ the\ i^{th}\ person,$$

$$i = 1, 2, \ldots \ldots n) \sum_{i=1}^{n} x_{ij} = 1 \quad for \;\; j$$

$$= 1, 2, \ldots \ldots n \quad (only\ one\ person\ should\ be\ assigned\ the\ \ j^{th}\ job, \qquad j = 1, 2, \ldots n)$$

$$x_{ij} = \begin{cases} 1, & if\ i^{th}\ person\ assigned\ j^{th}\ job \\ 0, & if\ \ not \end{cases}$$

## FUZZY ASSIGNMENT PROBLEM

Minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} \widetilde{c_{ij}}\, x_{ij}$

Subject to

$$\sum_{j=1}^{n} x_{ij} = 1 \quad for \;\; i = 1,2, \ldots \ldots . n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad for \;\; j = 1,2, \ldots \ldots n$$

$$x_{ij} = 0 \; or \; 1.$$

## TRIANGULAR FUZZY NUMBER

A fuzzy number A is a triangular fuzzy number denoted by $(a_1, a_2, a_3)$ and its membership function $\mu_A(x)$ is given below:

$$\mu_A(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & a_1 \le x \le a_2 \\ 1 & x = a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & a_2 \le x \le a_3 \\ 0 & otherwise \end{cases}$$

**CENTROID RANKING METHOD**

The centroid of a triangle fuzzy number $\tilde{a} = (a, b, c; w)$ as $G_{\tilde{a}} = (\frac{a+b+c}{3}, \frac{w}{3})$. The ranking function of the generalized fuzzy number $\tilde{a} = (a, b, c; w)$ which maps the set of all fuzzy numbers to a set of real numbers is defined as $R(\tilde{a}) = (\frac{a+b+c}{3})(\frac{w}{3})$

**ROBUST RANKING TECHNIQUE**

Robust ranking technique which satisfies compensation, linearity, additive properties and provides results which are consists human intuition. If $\tilde{a}$ is a fuzzy number then the Robust ranking is defined by

$$R(\tilde{a}) = 0.5 \int_0^1 (\, a_\alpha^L, a_\alpha^U)\, d\alpha$$

where $(a_\alpha^L, a_\alpha^U)$ is the $\alpha$- level cut of the fuzzy number $\tilde{a}$ and

$$(a_\alpha^L, a_\alpha^U) = \{\big((b-a)+a\big), \big(d-(d-c)\big)\}$$

**MAGNITUDE RANKING METHOD**

For an arbitrary triangular fuzzy number $\tilde{a} = (a_1, a_2, a_3)$ with parametric form$\tilde{a} = (a(r), \bar{a}(r))$, the magnitude of the triangular fuzzy number $\tilde{a}$ by

$$Mag\ (\tilde{a}) = \frac{1}{2} \int_0^1 (a_3 + 3a_1 - a_2)\, f(r)\, dr$$

Where the function $f(r)$ is a non- negative and increasing function on [0,1]. In the real-life applications $f(r)$ can be chosen by the decision maker according to the situation.

**METHODOLOGY**

Let there are four faculty members A, B, C, D and we have to assign four subjects I, II, III, IVto each of them. Each faculty member has obtained the scores by the Chairman, Course Director and the students of the department based on the performance in previous semester.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline \end{array}$$

$$\begin{array}{c|cccc} & \text{I} & \text{II} & \text{III} & \text{IV} \\ \hline A & (1, 4, 7) & (3, 6, 9) & (7, 10, 13) & (2, 5, 8) \\ B & (8, 11, 14) & (5, 8, 11) & (4, 7, 10) & (6, 9, 12) \\ C & (9, 12, 15) & (0, 3, 6) & (1, 4, 7) & (4, 7, 10) \\ D & (7, 10, 13) & (8, 11, 14) & (2, 5, 8) & (3, 6, 9) \end{array}$$

**By Robust ranking method**

$\boldsymbol{R(1,4,7)}$The membership function of the triangular fuzzy number$(1,4,7)$ is

$$\mu(x) = \begin{cases} \frac{x-1}{3}, & 1 \le x \le 7 \\ 1, & x = 4 \\ \frac{7-x}{3}, & 4 \le x \le 7 \\ 0, & otherwise. \end{cases}$$

The $\alpha -$ cut of the fuzzy number $(1,4,7)$ is $(a_\alpha^L, a_\alpha^U) = (3\alpha + 1, 7 - 3\alpha)$.

$$R\,(1,4,7) = 0.5\int_0^1 (\,a_\alpha^L, a_\alpha^U)\,d\alpha = \;0.5\int_0^1 (3\alpha + 1 + \;7 - 3\alpha)\;d\alpha = 0.5\int_0^1 8\,d\alpha = 4$$

$\boldsymbol{R(3,6,9)}$The membership function of the triangular fuzzy number $(3,6,9)$ is

$$\mu(x) = \begin{cases} \frac{x-3}{3}, & 3 \le x \le 6 \\ 1, & x = 6 \\ \frac{9-x}{3}, & 6 \le x \le 9 \\ 0, & otherwise. \end{cases}$$

The $\alpha -$ cut of the fuzzy number $(3,6,9)$ is $(a_\alpha^L, a_\alpha^U) = (3\alpha + 3, 9 - 3\alpha)$.

$$R\,(3,6,9) = 0.5\int_0^1 (\,a_\alpha^L, a_\alpha^U)\,d\alpha = \;0.5\int_0^1 (3\alpha + 3, 9 - 3\alpha)\;d\alpha = 0.5\int_0^1 12\,d\alpha = 6$$

$\boldsymbol{R(7,10,13)}$The membership function of the triangular fuzzy number $(7,10,13)$ is

$$\mu(x) = \begin{cases} \frac{x-7}{3}, & 7 \le x \le 10 \\ 1, & x = 10 \\ \frac{13-x}{3}, & 10 \le x \le 13 \\ 0, & otherwise. \end{cases}$$

The $\alpha-$ cut of the fuzzy number $(7,10,13)$ is $(a_\alpha^L, a_\alpha^U) = (3\alpha + 7, 13 - 3\alpha)$.

$$R\,(7,10,13) = 0.5\int_0^1 (a_\alpha^L, a_\alpha^U)\, d\alpha = 0.5\int_0^1 (3\alpha + 7, 13 - 3\alpha)\; d\alpha = 0.5\int_0^1 20\, d\alpha = 10$$

$\boldsymbol{R(2,5,8)}$ The membership function of the triangular fuzzy number $(2,5,8)$ is

$$\mu(x) = \begin{cases} \frac{x-2}{3}, & 2 \le x \le 5 \\ 1, & x = 5 \\ \frac{8-x}{3}, & 5 \le x \le 8 \\ 0, & otherwise. \end{cases}$$

The $\alpha-$ cut of the fuzzy number $(2,5,8)$ is $(a_\alpha^L, a_\alpha^U) = (3\alpha + 2, 8 - 3\alpha)$.

$$R\,(2,5,8) = 0.5\int_0^1 (a_\alpha^L, a_\alpha^U)\, d\alpha = 0.5\int_0^1 (3\alpha + 2, 8 - 3\alpha)\; d\alpha = 0.5\int_0^1 10\, d\alpha = 5$$

$\boldsymbol{R(8,11,14)}$ The membership function of the triangular fuzzy number $(8,11,14)$ is

$$\mu(x) = \begin{cases} \frac{x-8}{3}, & 8 \le x \le 11 \\ 1, & x = 11 \\ \frac{14-x}{3}, & 11 \le x \le 14 \\ 0, & otherwise. \end{cases}$$

The $\alpha-$ cut of the fuzzy number $(8,11,14)$ is $(a_\alpha^L, a_\alpha^U) = (3\alpha + 8, 14 - 3\alpha)$.

$$R\,(8,11,14) = 0.5\int_0^1 (\,a_\alpha^L, a_\alpha^U)\;d\alpha = \;0.5\int_0^1 (3\alpha + 8, 14 - 3\alpha)\;\;d\alpha = 0.5\int_0^1 22\;d\alpha = 11$$

$\boldsymbol{R(5,8,11)}$ The membership function of the triangular fuzzy number $(5,8,11)$ is

$$\mu\,(x) = \begin{cases} \frac{x-5}{3}, & 5 \le x \le 8 \\ 1, & x = 8 \\ \frac{11-x}{3}, & 8 \le x \le 11 \\ 0, & otherwise. \end{cases}$$

The $\alpha -$ cut of the fuzzy number $(5,8,11)$ is $(a_\alpha^L, a_\alpha^U) = (3\alpha + 5, 11 - 3\alpha)$.

$$R\,(5,8,11) = 0.5\int_0^1 (\,a_\alpha^L, a_\alpha^U)\;d\alpha = \;0.5\int_0^1 (3\alpha + 5, 11 - 3\alpha)\;\;d\alpha = 0.5\int_0^1 16\;d\alpha = 8$$

$\boldsymbol{R(4,7,10)}$ The membership function of the triangular fuzzy number $(4,7,10)$ is

$$\mu\,(x) = \begin{cases} \dfrac{x-4}{3}, & 4 \le x \le 7 \\ 1, & x = 7 \\ \dfrac{10-x}{3}, & 7 \le x \le 10 \\ 0, & otherwise. \end{cases}$$

The $\alpha -$ cut of the fuzzy number $(4,7,10)$ is $(a_\alpha^L, a_\alpha^U) = (3\alpha + 4, 10 - 3\alpha)$.

$$R\,(4,7,10) = 0.5\int_0^1 (\,a_\alpha^L, a_\alpha^U)\;d\alpha = \;0.5\int_0^1 (3\alpha + 4, 10 - 3\alpha)\;\;d\alpha = 0.5\int_0^1 14\;d\alpha = 7$$

$\boldsymbol{R(6,9,12)}$The membership function of the triangular fuzzy number $(6,9,12)$ is

$$\mu\,(x) = \begin{cases} \dfrac{x-6}{3}, & 6 \le x \le 9 \\ 1, & x = 9 \\ \dfrac{12-x}{3}, & 9 \le x \le 12 \\ 0, & otherwise. \end{cases}$$

The $\alpha -$ cut of the fuzzy number $(6,9,12)$ is $(a_\alpha^L, a_\alpha^U) = (3\alpha + 6, 12 - 3\alpha)$.

$$R\ (6,9,12) = 0.5 \int_0^1 (a_\alpha^L, a_\alpha^U)\ d\alpha = 0.5 \int_0^1 (3\alpha + 6, 12 - 3\alpha)\ d\alpha = 0.5 \int_0^1 18\ d\alpha = 9$$

$\boldsymbol{R(9,12,15)}$ The membership function of the triangular fuzzy number $(9,12,15)$ is

$$\mu\ (x) = \begin{cases} \frac{x-9}{3}, & 9 \le x \le 12 \\ 1, & x = 12 \\ \frac{15-x}{3}, & 12 \le x \le 15 \\ 0, & otherwise. \end{cases}$$

The $\alpha -$ cut of the fuzzy number $(9,12,15)$ is $(a_\alpha^L, a_\alpha^U) = (3\alpha + 9, 15 - 3\alpha)$.

$$R\ (9,12,15) = 0.5 \int_0^1 (a_\alpha^L, a_\alpha^U)\ d\alpha = 0.5 \int_0^1 (3\alpha + 9, 15 - 3\alpha)\ d\alpha = 0.5 \int_0^1 24\ d\alpha = 12$$

$\boldsymbol{R(0,3,6)}$The membership function of the triangular fuzzy number $(0,3,6)$ is

$$\mu\ (x) = \begin{cases} \frac{x-0}{3}, & 0 \le x \le 3 \\ 1, & x = 3 \\ \frac{6-x}{3}, & 3 \le x \le 6 \\ 0, & otherwise. \end{cases}$$

The $\alpha -$ cut of the fuzzy number $(0,3,6)$ is $(a_\alpha^L, a_\alpha^U) = (3\alpha + 0, 6 - 3\alpha)$.

$$R\ (0,3,6) = 0.5 \int_0^1 (a_\alpha^L, a_\alpha^U)\ d\alpha = 0.5 \int_0^1 (3\alpha + 0, 6 - 3\alpha)\ d\alpha = 0.5 \int_0^1 6\ d\alpha = 3$$

After putting these values, we get the assignment problem

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{bmatrix} I & II & III & IV \\ 4 & 6 & 10 & 5 \\ 11 & 8 & 7 & 9 \\ 12 & 3 & 4 & 7 \\ 10 & 11 & 5 & 3 \end{bmatrix}$$

By Hungarian method, the optimal assignment is as follows:

$A \rightarrow III$

$B \rightarrow IV$

$C \rightarrow I$

$D \rightarrow II$

The optimum value is 42.

## CONCLUSION

In an educational institution, allocation of subjects to the faculty members plays an important role. In this paper, we apply Robust ranking method, to allocate the subjects to the faculty members. Here, we have used the scores given to the faculty members by the Chairman, Course Director and the students of the department in the previous semester. We can apply this method to different sectors like health, human resource etc.

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# DIOPHANTINE EQUATION RELATED TO ARITHMETIC PROGRESSION

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## ABSTRACT

***In this paper, it has been shown that for an arithmetic progression of n terms it is possible that their product is of the form $x^m$, where m is of the form m=np+1, p is a positive integer.***

## INTRODUCTION

**Brian Micheli** (2021) presented a problem 'Some five distinct integers form an arithmetic progression. Is it possible that their product equals $x^{2021}$ for some positive integer $x$?' It has been shown that it is possible that product equals $x^{2021}$ for some positive integer $x$.

In this article, the above problem has been generalized. It has been shown that for an arithmetic progression of $n$ terms it is possible that their product is of the form $x^m$ where $m$ is of the form $m = np + 1$, $p$ is a positive integer.

**Analysis:**

We consider an arithmetic progression of $n$ terms whose first term is $a$ and common difference is $d$. Then it can be written as

$$a, a + d, a + 2d, \dots, a + (n - 1)d. \qquad \dots(1)$$

Let their product is equal to $x^m$ where $m$ is of the form $m = np + 1$. Thus we have

$$a(a + d)(a + 2d) \dots (a + (n - 1)d) = x^m. \qquad \dots(2)$$

We have to find the possible values of $a, d, x$ and $m$ given in expression (2). We consider $a = d$ in (2). Then expression (2) can be written as

$a(a+a)(a+2a)\ldots(a+(n-1)a)=x^m$,

Or $a.2a.3a\ldots.na=x^m$,

Or $n!\,a^n=x^m$. …(3)

Now if we consider $x=n!$ and $m=np+1$, $p$ is a positive integer, then from (3), we have

$n!\,a^n=(n!)^{np+1}$.

This gives $a^n=(n!)^{np}$. This implies $a=(n!)^p$. Thus we have

$a=d=(n!)^p$, $x=n!$, and $m=np+1$.

This shows that it is possible to find the solution of the Diophantine equation related to arithmetic progression.

**Illustrations:**

(1) For $n=4$, we have

$a(a+d)(a+2d)(a+3d)=x^m$

Taking $a=d$ and $x=4!$, we have

$4!\,a^4=(4!)^{4p+1}$.

This gives $d=a=(4!)^p$, $x=4!$.

(2) For $n=10$, we have

$a(a+d)(a+2d)\ldots(a+9d)=x^m$

Taking $a=d$ and $x=10!$, we have

$10!\,a^4=(10!)^{10p+1}$.

This gives $d=a=(10!)^p$, $x=10!$.

(3) For $n=15$, we have

$a(a+d)(a+2d)\ldots(a+14d)=x^m$

Taking $a = d$ and $x = 15!$, we have

$$15!\,a^{15} = (15!)^{15p+1}.$$

This gives $d = a = (15!)^p, x = 15!.$

(4) For $n = 20$, we have

$$a(a+d)(a+2d)\dots(a+19d) = x^m$$

Taking $a = d$ and $x = 20!$, we have

$$20!\,a^{20} = (20!)^{20p+1}.$$

This gives $d = a = (20!)^p, x = 20!.$

(5) For $n = 50$, we have

$$a(a+d)(a+2d)\dots(a+49d) = x^m$$

Taking $a = d$ and $x = 50!$, we have

$$50!\,a^{50} = (50!)^{50p+1}.$$

This gives $d = a = (50!)^p, x = 50!.$

(6) For $n = 100$, we have

$$a(a+d)(a+2d)\dots(a+99d) = x^m$$

Taking $a = d$ and $x = 100!$, we have

$$100!\,a^{100} = (100!)^{100p+1}.$$

This gives $d = a = (100!)^p, x = 100!.$

## CONCLUSION

Here it has been proved that the product of terms of an arithmetic progression can be expressed as $x^m$, where x is a positive integer.

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# EVALUATION OF BREAST SELF-EXAMINATION AND CLINICAL BREAST EXAMINATION AMONG RURAL FEMALE POPULATION IN TAMILNADU : A PILOT STUDY

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## ABSTRACT

***This research paper attempts a cross-sectional descriptive study conducted on Female rural population at KarpakaVinayagar Institute of Medical Science and Research Center, Chengalpet, Tamilnadu, India, among female population regarding their awareness of Breast Self-Examination (BSE) and Clinical Breast Examination (CBE). In the recent days, breast cancer is play vital disease for women and men also. In this connection, the primary sources of samples were collected from voluntary basis. Secrecy and privacy of the responses was assured. The questionnaire consists of threesections: socio-economic parameters, BSE and CBE. All the questions are closed ended and the total samplesize is 200. Initially, the descriptive statistics were described in the form of frequency tables and percentages. Principal Component Analysis, k-mean cluster and Multiple Discriminant analysis used to identify the structure, pattern and Cross Validation of BSE and CBE.Principal Component Analysis (PCA) is used for data reduction or variable reduction and this method extracted 9 factors with 70.95%, the nine factor regression scores are statistically significant except 3 factor regression score. The k-mean cluster identified three meaningful clusters and cross validate with the help of Multiple Discriminant Analysis. The extracted factors are named as Knowledge of BSE 1, Initial Stage of BSE, use of BSE, tool for BSE, practice of BSE, Benefits of BSE, Mammography and CBE, Knowledge of BSE 2, without awareness of CBE and BSE. The k-means cluster analysis achieved three clusters with 88, 71 and 43 respondents based on the centroids of their cluster. The clusters are assessed by high awareness in second cluster, moderate awareness in third cluster and low awareness in first cluster respectively. Finally, three cluster cross validation using Multiple Discriminant Analysis (MDA) accounts to 96.5% of original grouped cases correctly classified in first iteration itself***

***Keywords: Descriptive Statistics, CBE, BSE, PCA, K-mean Clustering and Multiple Discriminant Analysis.***

## 1. INTRODUCTION

World Health Organization (WHO) recommends developing nations those suffering the dual saddle of cervical and breast cancer to implement economical and inexpensive interventions to confront these greatly preventable diseases [1] Although there is no proof of the worth of screening through BSE, the BSE practice has been appreciated to enable women, taking concern for their health [2] BSE is a suitable and cheap means that can be implemented regularly. The American Cancer Society and other leading cancer agencies recommended the monthly practice of BSE [3][4]. A country with inadequate resource facilities and poor health systems ought to promote early diagnosis programs based on breast self-exam, awareness of early signs and symptoms, and prompt referral to diagnosis and treatment.[3].

Machine learning is all about learning rules from the data set. In this paper, I use classification and analysis processes on the breast cancer dataset. Naïve Bayes classifier, SMO (Support Vector Machine), Decision Tree, KStar (Instance basedClassifier), Artificial Neural Networks (ANNs) have been used in order to analyze the results. Artificial neural networks (ANNs) supply a general, practical method for learning real valued examples. Algorithms such as back propagation use gradient descent to adjust network parameters to be best fitted. ANN learning minimizes the errors in the training data and has been successfully applied to the problems such as interpreting visual demonstrations [5]. In this paper, machine learning algorithms determined above and ANNs used on retrieving medical data from breast cancer patients. The main objectives of this study are to assess the knowledge and practice of BSE and CBE among rural women of Chengalpet, Tamilnadu, India.

## 2. DATABASE

The cross-sectional database collected from rural Female population at Karpaka Vinayaga Institute of Medical Science and Research Center, Chengalpet, Tamilnadu, India, among female population regarding their awareness of Breast Self-Examination (BSE) and Clinical Breast Examination (CBE). The database consists of 199 samples with 24 parameters. In this connection, the primary sources of samples were collected from voluntary basis. Secrecy and privacy of the responses was assured.

## 3. METHODOLOGY

### 3.0.1 Factor Analysis

In the present study, factor analysis is initiated to uncover the patterns underlying Breast Self-Examination (BSE) and Clinical Breast Examination (CBE) variables. Factor analysis reduces the variable space to a smaller number of patterns that retain most of the information contained in the original data matrix. In factor extraction method the number of factors is decided based on the proportion of sample variance explained. Orthogonal rotations such as Varimax and Quartimax rotations are used to measure the similarity of a variable with a factor by its factor loading. In factor analysis, the interest is centered on the parameter in the factor model that estimated values of the common factor, called *factor scores.*These scores are subjected to further analysis to mine the data.

### 3.0.2k-Means Clustering Algorithm

A nonhierarchical clustering algorithm suggested by MacQueen (1967) also known as *unsupervised classification* is the next technique in data mining. This process divides the data set into mutually exclusive group such that the members of each group are as close as possible to one another and different groups are as far as possible from another. Generally this technique uses Euclidean distances measures computed by variables. Since the group labels are unknown for the data set, **k**-means clustering is one such technique in applied statistics that discovers acceptable classes. Thus forming the nuclei of clusters or groups as seed points exhibited in factor analysis. The number of cluster **k** is determined as part of the clustering procedure [6].

### 3.0.3. Discriminant Analysis

Many researchers have used aprior group information for classification and model buildings using discriminant Analysis (DA) to achieve their objectives. In the present study, discriminant analysis is used to exhibit groups graphically and judge the nature of overall performance of the companies.

### 3.0.4 Algorithm

A brief algorithm to classify the BSE and CBE during the study period based on their overall performance is described below:

**Step 1**: Factor analysis is initiated to find the structural pattern underlying the database and scores were extracted.

**Step 2**: **k** –means analysis partitioned the data set into **k**-clusters using factor scores as input matrix.

**Step 3**: Repeat Steps 1 and 2 until meaningful groups are obtained, by removing outliers in each cycle.

**Step 4**: Discriminant analysis is then performed with the original BSE and CBE by considering the groups formed by the **k**-means algorithm.

## 4. RESULT AND DISCUSSION

As mentioned in Section 3.0.1 Varimax and Quartimax criterion for orthogonal rotation have been used for the pruned data. Even though the results obtained by both the criterions were very similar, the varimax rotation provided relatively better clustering of BSE and CBE. Consequently, only the results of varimax rotation are reported here. We have decided to retain 71 percent of total variation in the data, and thus accounted consistently nine factors for this study with eigen values little less than or equal to unity. *Table 1* shows variance accounted for each factors. The extracted factors are named as Knowledge of

BSE 1, Initial Stage of BSE, use of BSE, tool for BSE, practice of BSE, Benefits of BSE, Mammography and CBE,Knowledge of BSE 2, without awareness of CBE and BSE.

**Table 1.Percentage of Variance explained by factors (BSE and CBE)**

| Component | Extraction Sums of Squared Loadings | | | Rotation Sums of Squared Loadings | | |
|---|---|---|---|---|---|---|
| | Total | % of Variance | Cumulative % | Total | % of Variance | Cumulative % |
| 1 | 6.271 | 25.086 | 25.086 | 3.608 | 14.434 | 14.434 |
| 2 | 2.290 | 9.160 | 34.246 | 2.748 | 10.993 | 25.426 |
| 3 | 1.685 | 6.738 | 40.984 | 2.000 | 8.001 | 33.428 |
| 4 | 1.590 | 6.361 | 47.345 | 1.875 | 7.498 | 40.926 |
| 5 | 1.353 | 5.412 | 52.757 | 1.670 | 6.681 | 47.608 |
| 6 | 1.262 | 5.049 | 57.806 | 1.525 | 6.099 | 53.707 |
| 7 | 1.144 | 4.575 | 62.381 | 1.514 | 6.057 | 59.764 |
| 8 | 1.104 | 4.417 | 66.798 | 1.505 | 6.022 | 65.786 |
| 9 | 1.039 | 4.156 | 70.954 | 1.292 | 5.168 | 70.954 |

After performing factor analysis, the next stage is to assign initial group labels to each BSE and CBE parameters. Step 2 of the algorithm is explored with factor score extracted by Step 1, by conventional **k**-means clustering analysis. Formations of clusters are explored by considering 2-clusters, 3-clusters, 4-cluster and so on. Isolated groups with few BSE and CBE are discarded from the analysis as outliers. A few BSE and CBE for these outlier companies are comparatively high or low awareness to those excelled in the analysis. Out of all the possible trials, 3-cluster exhibited meaningful interpretation than two, four and higher clusters. Having decided to consider only 3 clusters, it is possible to assess BSE and CBE of k-means cluster analysis achieved three clusters with 88, 71 and 43 respondents based on the centroids of their cluster. The clusters are assessed by high awareness in second cluster, moderate awareness in third cluster and low awareness in first cluster respectively.

*Table 2*. The first column in *Table 2* provides the groupings done by cluster analysis. The second column gives the groupings after the application of discriminant analysis until 97 percent classification is achieved.

**Table 4. Number of BSE and CBE in the Clusters**

| **BSE and CBE** | K.-means INITIAL CLUSTER | | | **Discriminant Classification** | | |
|---|---|---|---|---|---|---|
| | **1** | **2** | **3** | **1** | **2** | **3** |
| | 86 | 71 | 43 | 82 | 70 | 40 |

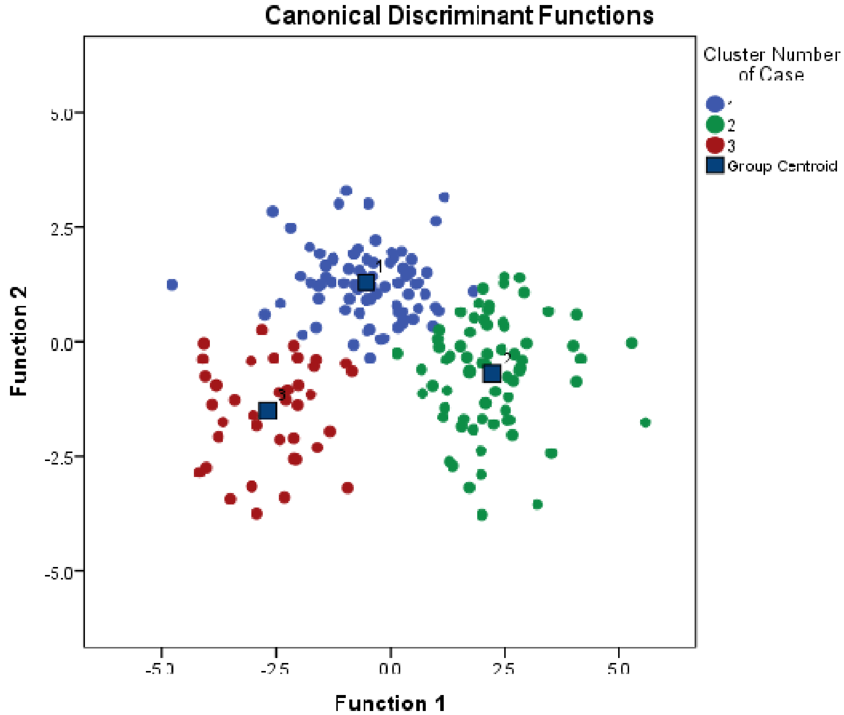


**Figure 1. Classification of BSE and CBE Clinical Parameters**

*Figure1* shows that the groupings of BSE and CBE parameters into 3 clusters of the study span.The clusters are assessed by high awareness in second cluster, moderate awareness in third cluster and low awareness in first cluster respectively. Three percent cases are misclassified, its considered as a outlier of the database. In the both methods of clustering and classification methods achieved good models of BSE and CBE parameter with few outliers.

## 5. CONCLUSION

The results of BSE and CBE clinical parameters were extracted nine factors and are named as Knowledge of BSE 1, Initial Stage of BSE, use of BSE, tool for BSE, practice of BSE, Benefits of BSE, Mammography and CBE,Knowledge of BSE 2, without awareness of CBE and BSE. The k-means cluster analysis achieved three clusters with 88, 71 and 43 respondents based on the centroids of their cluster. The clusters are assessed by high awareness in second cluster, moderate awareness in third cluster and low awareness in first cluster respectively.Finally, three cluster cross validation using Multiple Discriminant Analysis (MDA) accounts to 96.5% of original grouped cases correctly classified in first iteration itself. It is achieved that the PCA, k-means clustering and MDA gives best efficiency results of CBE and BSE attributes of the pilot study

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# A RHOMBIC SHAPED UWB ANTENNA WITH DUAL B AND-RECONFIGURABLE CHARACTERISTICS

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## ABSTRACT

***This article presents a planar ultra wideband (UWB) Antenna with dual band frequency-reconfigurable capability. A rhombic metal patch and 50 Ω coplanar waveguide (CPW) are built on FR4 substrate of 1 mm height. Two slots one is C-shape and another is L-shape have been etched out from the metal patch and ground plane respectively, which are responsible to achieve the reconfiguration characteristics with the help of RF-MEMS switches. Three RF-MEMS switches have been used to achieve the reconfigurable characteristics on two bands. The designed Antenna is satisfying all broadband conditions such as matched impedance, steady radiation patterns and constant gain along with reconfigurable characteristics.***

***Index Terms- Coplanar waveguide (CPW) fed, reconfiguration characteristics, ultrawideb and Antennas.***

## 1. INTRODUCTION

Nowadays, the multiple operating frequency device in wireless communication is in high demand. So to fulfil this demand the multiple Antennas are required in the device, which makes the device costly and species or in other words the complexity will increase [1]. As a solution a reconfigurable Antenna can be helpful, which can switch or tune the operating frequency by changing the effective area or effective length of Antenna. A reconfigurable Antenna can reduce the size, cost and complexity of the front-end circuitry of wireless system. Reconfiguration parameters of an Antenna can be segregated in several types such as frequency, [2–14] radiation pattern, [15-16] and polarization [17–19]. However, this article only focused on frequency reconfiguration parameter of Antenna.

Previously the maximum work on reconfigurable Antenna focused on frequency reconfiguration such as, wideband to narrowband [2–4], narrowband to another narrowband [5-9], however, a less work have been done on wideband to multiband reconfiguration [10, 11]. A planar ultra-wideband (UWB) Antenna has been designed which is having operating frequency from 2.6 to 11 GHz and also capable to switched to narrowband mode at 5 GHz [3]. The reconfigurability is achieved by switching within the parts of the Antenna. A slot Antenna, which is able to switch from one narrowband to other seven narrowband has been proposed [5]. The effective length of Antenna is varies with the help of switching mechanism. The coveringrange of frequency of the Antenna is 6 to 10.6 GHz. A conventional meander-type PIFA is presented [8], which is able to modify the effective Antenna length by using a switch. When the switch is

ON state then the electrical length of Antenna is high so the design is capable to operate multiband mode while in off state, the Antenna is switched to operate in narrowband mode.Wideband to multiband mode is proposed in,10 this isaccomplished through a combination of UWB fractal and Shashank EV at. al in 2015 have been proposed a wideband to narrowband reconfigurable Antenna [10]. The proposed UWB fractal Antenna covering 3.1 to 11 GHz and the reconfigurability at dual band has been achieved by changing the Antenna feed using two port. The reconfigurable dual are 3.5 GHz and 6 GHz respectively. A wide to multiband mode Antenna from 1 to 3 GHz is proposed in 2017 [11], in which the reconfigurability has been achieved by loading adjustment of the structure. The proposed structure is suitable for the band-out filtering technique in order to run in single and dual-band mode. When the all diodes are switched OFF then the Antenna operates on triple-band mode at center frequencies 1, 1.6 and 3.2 GHz respectively. Unlike in, [10] the proposed work here demonstrates wide-band to multiband mode by using a single port.

An Antenna with multiband (2.4, 3.5, and 5.2 GHz) and one wideband (2-6GHz) has been presented [11]. The proposed Antenna is a combination of two Antenna namely multiband slot dipole and wideband monopole. The overall size of this Antenna is similar with Antenna proposed in Ref. [11] with more flexibility in terms of frequency control. To achieve the switching between the bands a pair of PIN diode has been used in the Antenna structure.

Based on the background research this paper presents a simple and compact CPW-fed planar UWB Antenna with dual band-reconfigurable characteristics in 3.4 GHz (3.3–3.8 GHz) and 5.5 GHz (5–6 GHz). This reconfigurable operations are achieved by embedded one RF-MEMS switch in C-shaped slots [21] and two RF-MEMS switch in L-shaped slots in ground plane, which are etched out in the rectangular metal radiating patch and ground plane respectively. The reconfigurability between the frequencies bands are depends on the total length of the C-shaped slots, which is approximately equal to half-wavelength of the desired operating frequency bands. If the switch is OFF state a destructive interference can take place, causing the Antenna nonresponsive at that frequency and in ON state, the path of current will not change so the Antenna will work in the desired bands (WiMAX and WLAN). The proposed Antenna yields an impedance bandwidth of 3–11 GHz.. The stable radiation patterns and constant gain are also obtained

## 2. ANTENNA DESIGN CONFIGURATIONS AND RESULT DISCUSSION

Chu, Q. X., & Yang, Y. Y. in 2008 proposed a compact ultrawideband Antenna of size 26 by 30 mm [21], but it was having fixed dual band-notched characteristics. In this paper, we have proposed the same size wideband Antenna with dual band reconfigurable characteristics. Simply by embedding the MEMS switch in C-shaped slotsthe reconfigurable characteristic from 3.3 to 3.8 GHz and 5 to 6 GHz have been obtained. Antenna design details, simulation performance and the measured results are given of the designedAntennas.

Antenna 1

The geometry and configuration of proposed UWB Antenna shown in Fig. 1 [21]. The Antenna 1 is designed and fabricated on FR4 epoxy substrate with h=1 mm height. A 50 Ohm CPW has been used to feed the truncated radiator as shown in the figure 1. A single sided metallized PCB has been used to fabricate the Antenna which makes it very low cost [3]. The good impedance matching for wideband and switching between one resonant modes to another prove the radiating patch and the ground plane are beveled [8, 9].

The optimize dimensional of the Antenna 1 are given in table 1

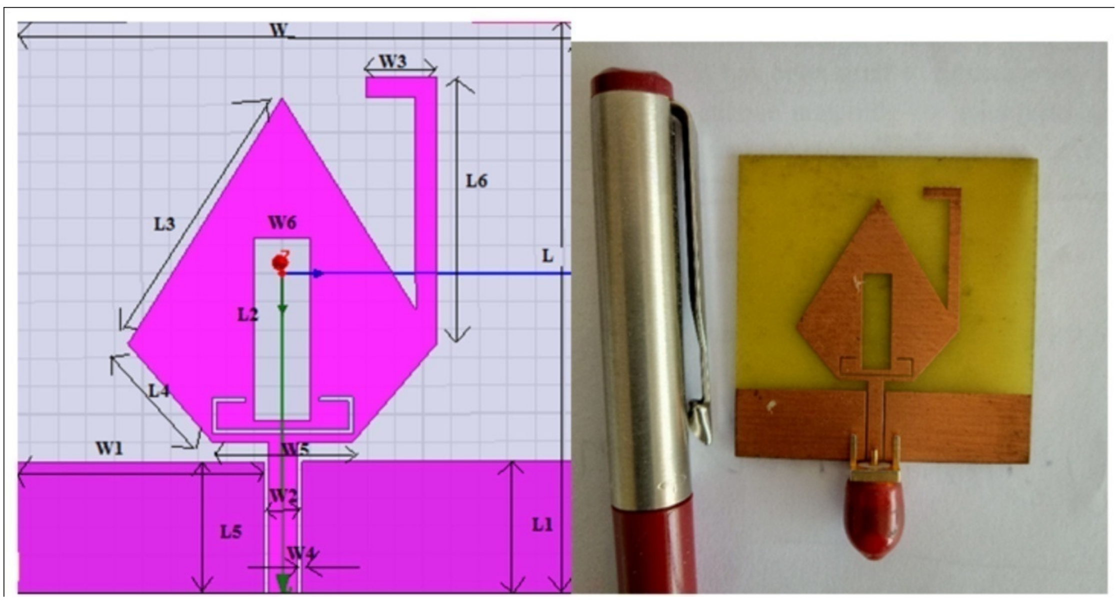


**Fig. 1. Geometry of Antenna 1.**

***Table 1 optimize dimensional of the Antenna 1***

| Parameters | Dimensions (mm) | Parameters | Dimensions (mm) |
|---|---|---|---|
| L | 41 | W | 40 |
| L1 | 9.6 | W1 | 18.65 |
| L2 | 13 | W2 | 2 |
| L3 | 17.5 | W3 | 5 |
| L4 | 7 | W4 | 0.35 |
| L5 | 9.6 | W5 | 8 |
| L6 | 19 | W6 | 4 |

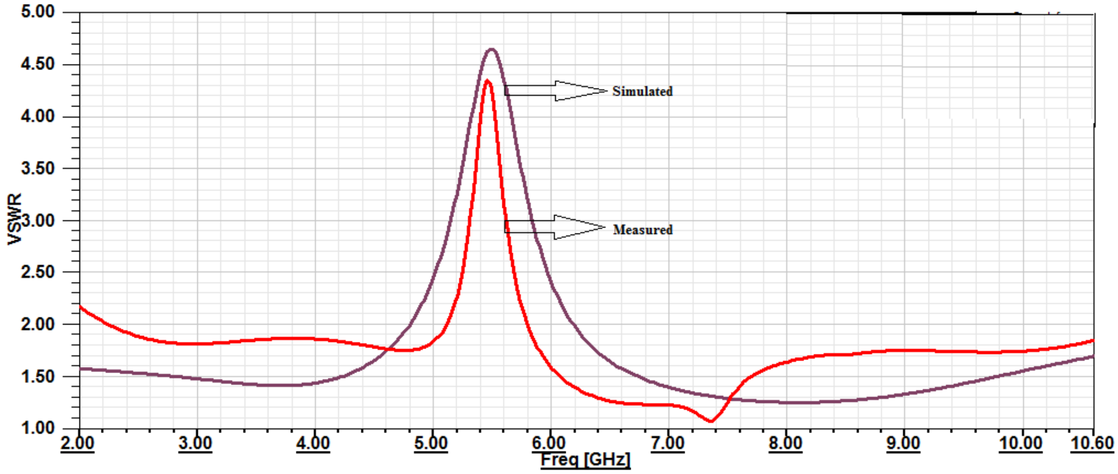


**Fig. 2. Simulated and Measured VSWR characteristics of Antenna 1**

As described above a potential interference dominate between UWB and WLAN systems, so to minimize this interference the reconfiguration has been achieved by embedded a MEMS switch in C-shaped slot of radiating patch. Fig. 3 shows the geometry Antenna 2 and MEMS switch possession in the slot. The ON and OFF condition of the switch is responsible to filter out or not of WLAN (5-6 GHz) band form UWB band. At the 5-6 GHz frequency band the maximum current follow the path around the C-shape slot so the destructive interference is happened and Antenna isnot responsible for radiation.

Antenna 2

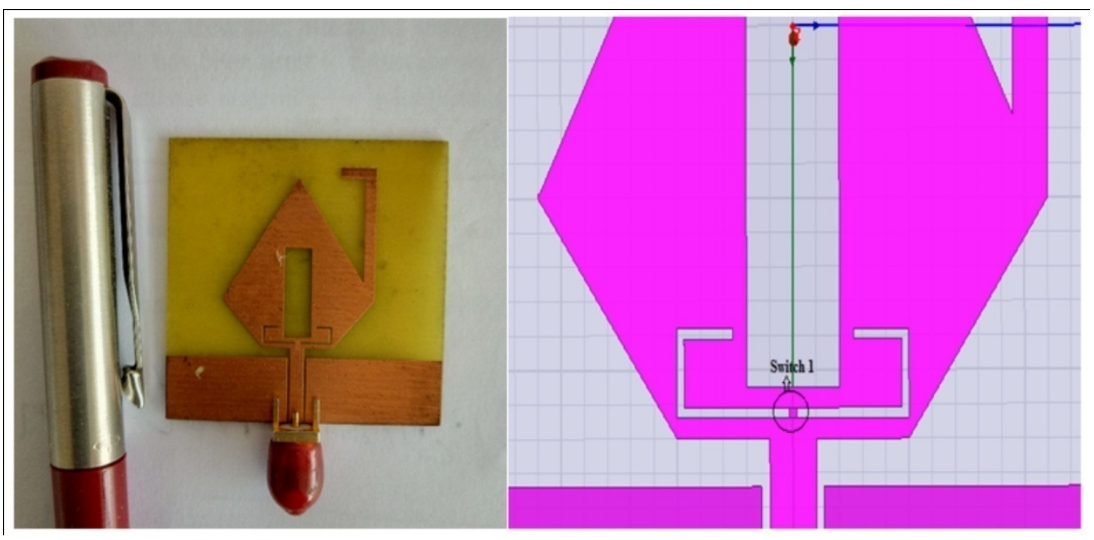


**Fig. 3. Geometry of Antenna 2.**

The measured and simulated VSWR characteristics of Antenna 2 shown in Fig. 4 and it is found a better relative agreement between measurement and simulation over the entire bandwidth of wideband with VSWR < 2.

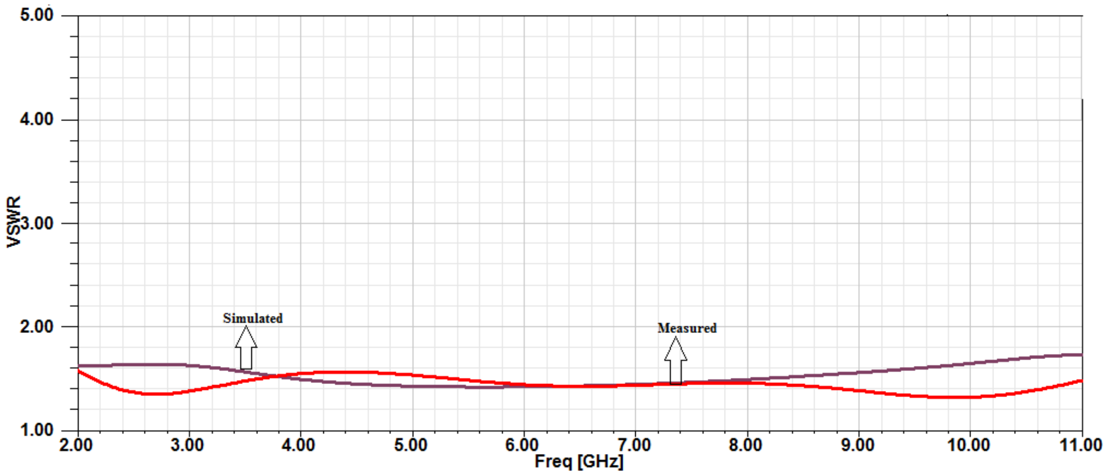


**Fig. 4. Simulated and Measured VSWR characteristics of Antenna 2**

The reconfigurableb and function is depends on the total length of the slots, which is approximately equal to the half-wavelength at the desired reconfigurable band, which makes the input impedance singular. The slot length for the desired reconfigurable frequency band is given by the equation 1.

$$f_{notch} = \frac{C}{2L\sqrt{\varepsilon_{eff}}} \qquad (1)$$

Where

L =Total length of the C-shaped slot

Ɛr= The effective dielectric constant

C =Speed of the light

The reconfigurable frequency can be tuned by the total length.

Antenna 3: The geometry of Antenna 3 is shown in Fig. 5. The Antenna structure is having additional two L-shape slots in ground plane, which are responsible to get the reconfigurability in WiMAX band.

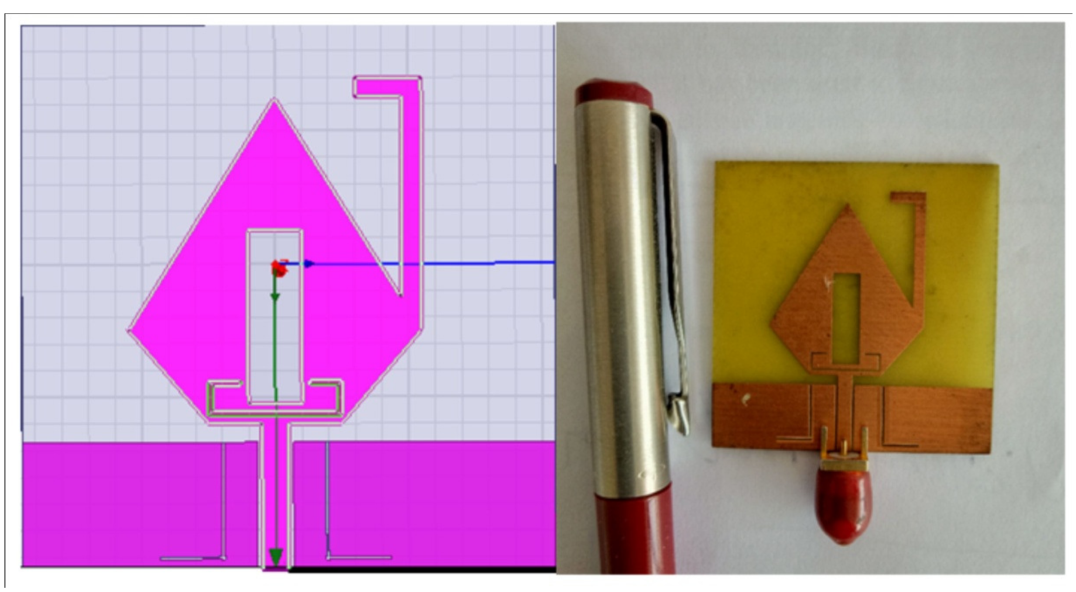

**Fig. 5. Geometry of Antenna 3**

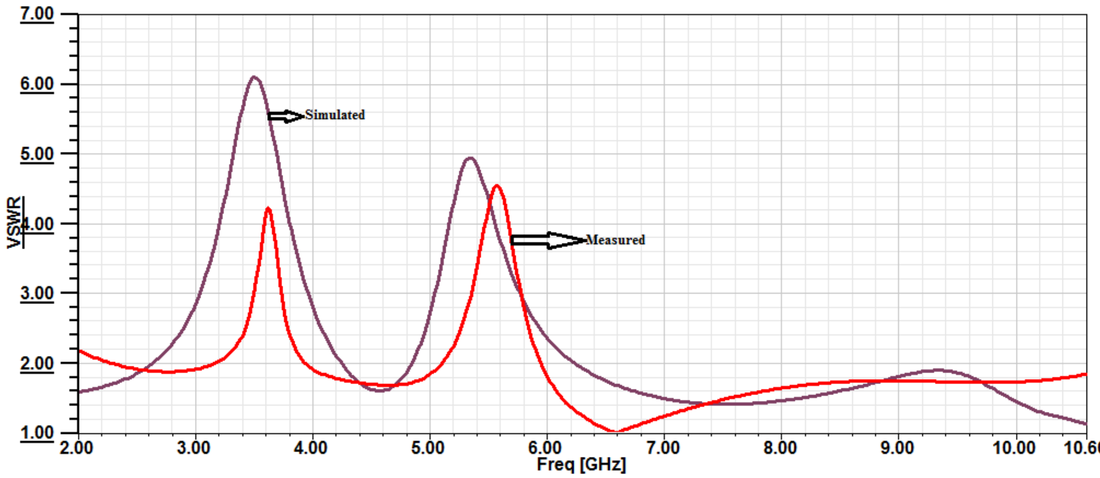


**Fig. 6. Simulated and Measured VSWR characteristics of Antenna 3**

The measured VSWR characteristics of Antenna3for different states of MEMS switch (ON and OFF) shown in Fig. 6. From the figure, as expected the reconfigurable frequency band has been achieved by embedded a MEMS switches in the slots.

Antenna 4: As we know WiMAX from 3.3–3.6 GHz also operates in the UWB band. A potential interference again dominate between UWB and WiMAX systems, so to minimize this interference the reconfiguration has been achieved by embedding the two MEMS switches in L-shaped slot of ground plane. Fig. 7 shows the geometry and MEMS switch possession in the slot. The ON and OFF condition of the switch is responsible to filter out or not of WLAN (3.3-3.8 GHz) band form UWB band and again there is no retuning required for the previously determined dimensions Antenna 1 and Antenna 2 and Antenna 3.

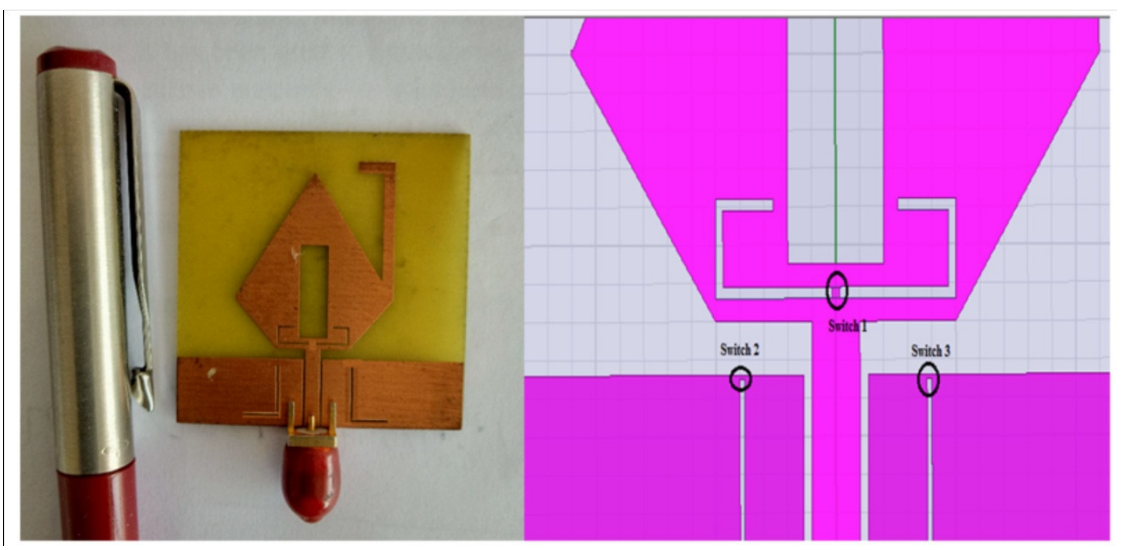


**Fig. 7. Geometry of Antenna 4**

When all three switchesare in OFF sate the maximum current flows around the slots. In this case, destructive interference for the excited surface currents in the Antenna will occur, which causes the Antenna to be nonresponsive at that frequency. Similarly if all switches are ON sate the current path change and Antenna will work on the same frequency bands (3.3-3.8 GHz and 5-6 GHz).

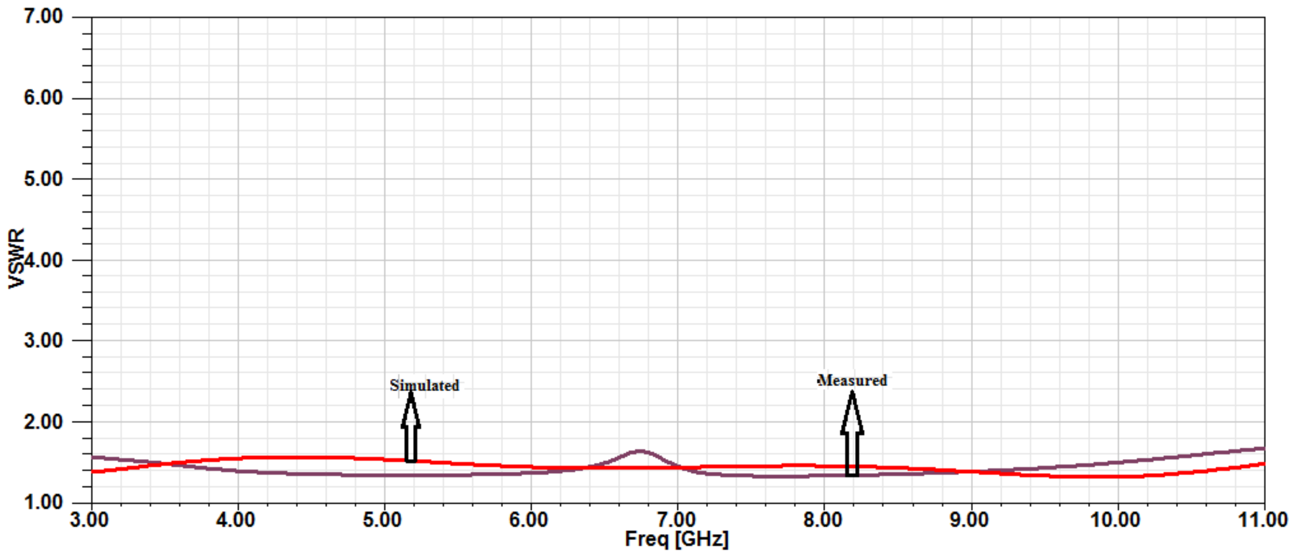


**Fig. 8. Measured VSWR characteristics of Antenna 4**

The measured VSWR characteristics of Antenna4 for ON states of all three MEMS switches shown in Fig. 8. From the figure, as expected the reconfigurable frequency band has been achieved by embedded a MEMS switch in the slots.Thedual band-reconfiguration property successfully achieved.

Fig. 9 is described the measured and simulated radiation patterns of Antenna4 in the E-plane (xz-plane) at frequencies 3, 8 and 10 GHz.As expected the patterns in the E-plane are quite omnidirectional. To the best of authors' knowledge, the proposed reconfigurable structurewith single and double band-reconfiguration characteristics have little influence on the radiation patterns of the UWB Antenna.

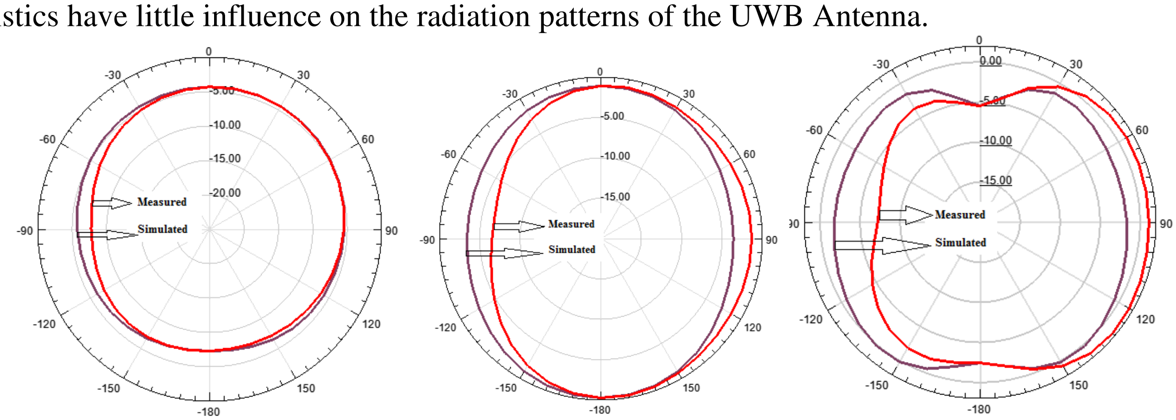


**Fig. 9. Measured E-Plane characteristics of Antenna 3**

Fig. 10 and Fig. 11 shows the group delay and gain characteristics with the different position of switches. A linear phase response and constant gain are showing in UWB range except reconfigurable band. The variation of the group delay of Antenna when the both switch are OFF is less than 1 ns in entire covered band. Similarly the gain variation is less than 1.5 dBi.

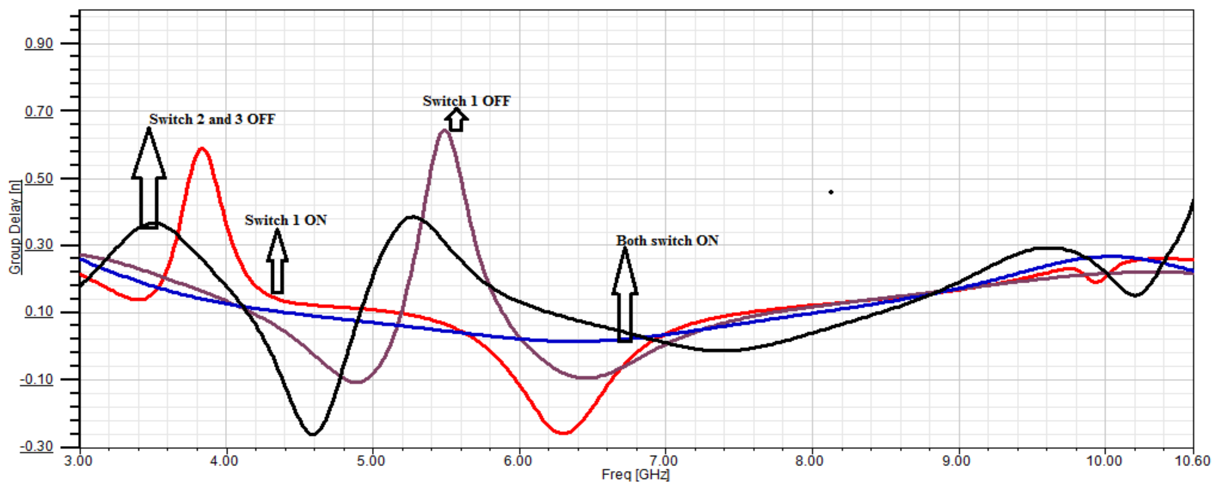


**Fig. 10. Group delay characteristics with different position of switches**

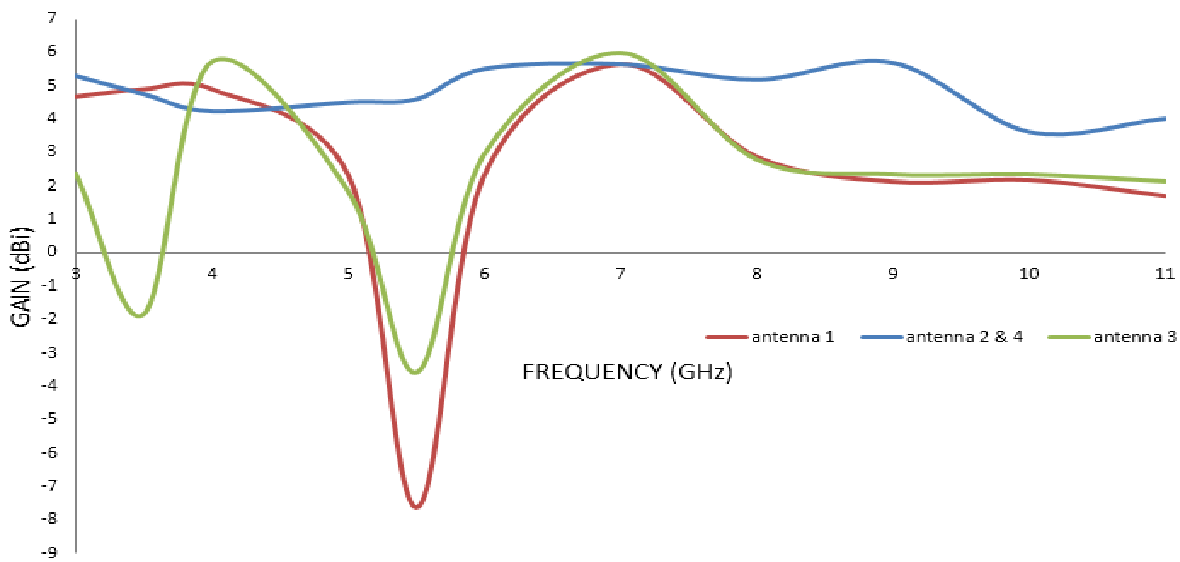


**Fig. 11. Gain characteristics with different position of switches**

## CONCLUSION

The proposed Antenna structure has been discussed with the help of suitable and sufficient results. The proposed structure is very helpful to minimize the potential interferences between the UWB system and the narrowband systems with the reconfigurable features. By embedding the three RF-MEMS switch the single and dual band reconfiguration characteristics has been achieved. The frequency band reconfiguration features of Antenna is depends on the length of slots, which has been described in this paper. Fixed radiation patterns and almost constant gain in the entire UWB band has been obtained. The simulation and measuredresults of the proposed Antenna show a good agreement in term of the VSWR, Antenna gain and radiation patterns. The proposed Antenna is best suited for wideband applications with reconfigurable features.

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# $(Super)(a, d) - H-$ ANTIMAGIC TOTAL LABELING OF SUPER SUB DIVISION OF CYCLE

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## ABSTRACT

***A labeling of a graph is a mapping that carries some set of graph elements i.e. vertices or edges or both into numbers (usually positive integers). An edge-covering of G is a family of subgraphs H1, H2, ..., Ht such that each edge of E(G) belongs to at least one of the sub graphs Hi, i = 1, 2, ..., t.***
***If every subgraph Hi is isomorphic to a given graph H then G is said to admit an H-covering.***
***In 2009, G. Sethuraman [8] has introduced concept of arbitrary super- subdivision of graph and posed a problem of labeling of it. As in su- per subdivision of cycles obvious sub graphs are seen, it generated a curiosity to study if super sub divisions of cycles admit H covering. This inspiration lead me to check even further if they can be labeled using (Super)(a, d) – H- anti magic total labeling. We prove that Super subdivision of cycle is (Super)(a, d) – H-anti magic total graph.***
***AMS subject classification: 05C78***
***Keywords: Cycle, subdivision of graphs, super sub division of graphs, weight of a subgraph, edge-covering ,total labeling f of G ,(a, d)-H – anti magic labeling, Super(a, d) – H - antimagic total labeling.***

## 1 INTRODUCTION

We cosider finite undirected graphs without loops and multiple edges. In 2005, Guti,errez and Llad,o [1] gave definition of H-supermagic graph; more- over they proved the star $K_{1,n}$ and the complete bipartite graphs Kn,m are K1,h-supermagic for some h; the path Pn and the cycle Cn are Ph supermagic for some h. Selvagopal and Jeyanthi proved for any positive integer n, the k-polygonal snake of length n is Ck-supermagic [7] They also proved that for $m \geq 2$, $n = 3$, or $n > 4$, Cn × Pm is $C_4$-supermagic, $P_2 \times Pn$ and $P_3 \times Pn$ are $C_4$ supermagic for all $n \geq 2$ [6]In 2009 Inayah, Salman and Simanjuntak introduced the (a, d) H antimagic total graphs. In 2010 Ngurah, Salman and Susilowati proved that chains, wheels, triangles, ladders and grids are cycle- supermagic. [4] has proved that subdivision of ladder are graceful. G Sethu-raman and P. Selvaraju [2] have introduced supersubdivision of graphs. Later in 2018, Dr. Ujwala Deshmukh and Smita Bhatavadekar [8] proved that there exists a graceful arbitrary super subdivision of Cn, $n \geq 3$ with certain conditions. Later, in 2004, K.M. Kathiresan [5] also proved that arbitrary super subdivision of stars are graceful. V. Ramchandran and C. Sekar [9] have given graceful labeling of super subdivision of ladder.

In this paper we have made an effort to study that the super-subdivision of cycle admit H-covering and proved that it has ( super) (a,d) -H- anti-magic total graph.

## 2. DEFINITIONS

**Definition 2.1. Cycle:** Cycle is a graph with an equal number of vertices and edges where vertices can be placed around circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The cycle is denoted by Cn.

**Definition 2.2. Subdivision of a Graph:** Let G be a graph with p vertices and q edges. A graph H is said to be a subdivision of G if H is obtained by subdividing every edge of G exactly once. H is denoted by S(G). Thus,

$|V| = p + q$ and $|E| = 2q$.

**Definition 2.3. Super subdivision of a graph:**

Let G be a graph with p vertices and q edges. A graph H is said to be a supersubdivision of G if it is obtained from G by replacing every edge e of G by a complete bipartite graph $K_2$,m. H is denoted by SS(G). Thus, $|V| = p + mq$ and $|E| = 2mq$.

**Definition 2.4. Edge-covering of G :** An edge-covering of G ... is a family of subgraphs $H_1, H_2, ..., H_t$ such that each edge of E(G) belongs to at least one of the subgraphs Hi, i = 1, 2, ..., t.

If every subgraph Hi is isomorphic to a given graph H then G is said to admit an H-covering.

**Definition 2.5. Total Labeling :** G is a simple graph admitting H-covering. A total labeling f of G is a bijection $f : V(G) \rightarrow E(G) \{1, 2, ..., |V(G)| + |E(G)|\}$.

**Definition 2.6. Weight of a graph:** The value or weight of a graph is defined by the sum of the numbers associated by a labeling, to the edges of the graph.

**Definition 2.7. (a, d) - H - antimagic Total labeling:** A graph is called (a, d) - H - antimagic total if it admits (a, d) - H - antimagic total labeling.

i.e. if its vertices and edges are labeled with distinct positive integers from the set $\{1, 2, ..., |V(G)| + |E(G)|\}$ such that the set of all the H weights forms an arithmetic sequence starting from a and having a common difference d , i.e., a, a + d, . . . , a + (t-1)d , where t is the number of all sub graphs of G isomorphic to H.

Moreover, if the smallest possible labels appear on the vertices, the graph is called (super)(a, d) – H-antimagic total. Let us see an example.

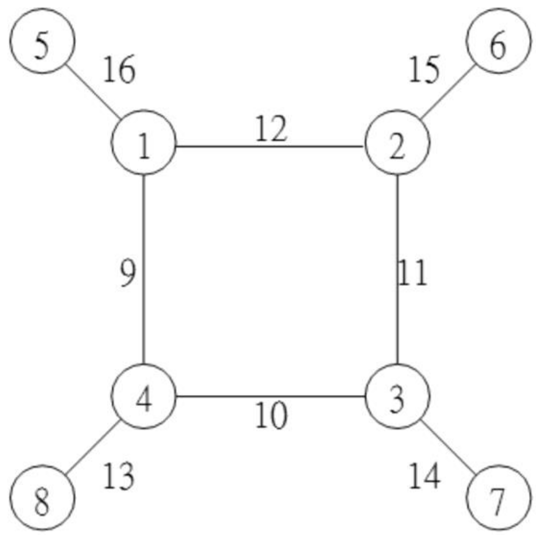
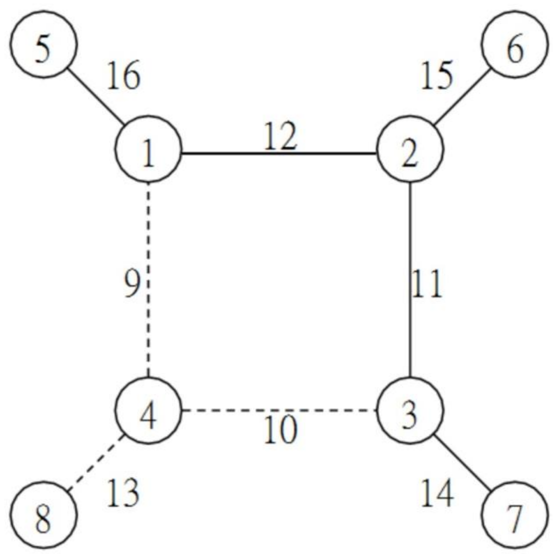
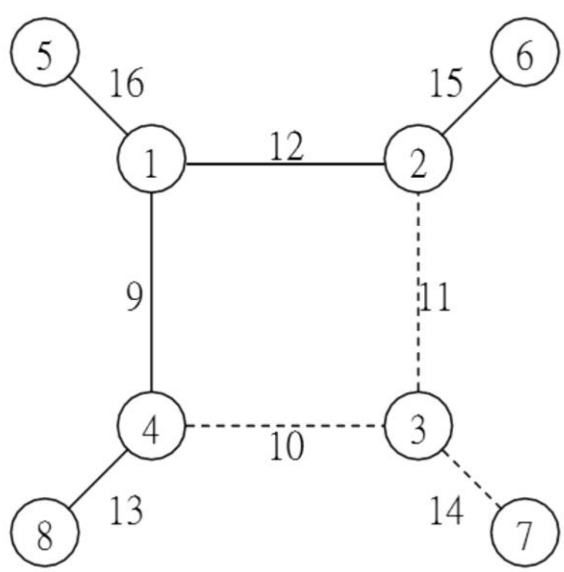
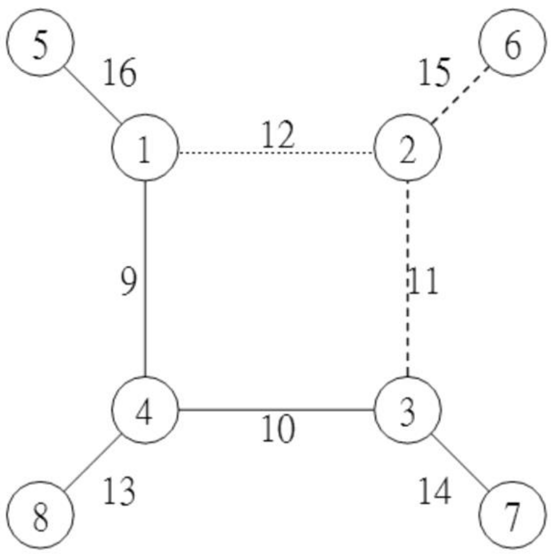
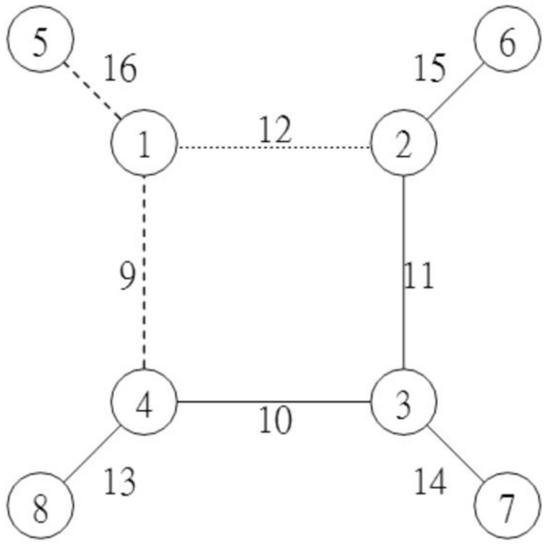



**Figure 1: super (48, 1)-K1,3-antimagic total labeling**






## 3. NOTATIONS

In the complete bipartite graph $K_2$,m, we call the part consisting of two vertices, the 2-vertices part of $K_2$,m and the part consisting of m vertices, the m-vertices part of $K_2$,m

Let $C_n$ be a cycle of length n. Let $c_1, c_2, \ldots, c_n$ be the vertices of cycle. Let $c^k_{i,i+1}$, ., k = 1, 2, . . . , m be the vertices of the m-vertices part of $K_2$,m merged with the edge $c_i c_{i+1}$ for $i = 1, 2, \ldots, n - 1$ and k = 1, 2, . . . , m. Let $c^k_n$,1 k = 1, 2, . . . , m be the vertices of the m-vertices part of $K_{2,m}$ merged with the edge $c_n c_1$ and k = 1, 2, . . . , m.

$$|V| = n + nm = (m + 1)n$$

$$|E| = 2mn$$

For example if $n = 7$ and $m = 3$ then vertex labeling is as follows.

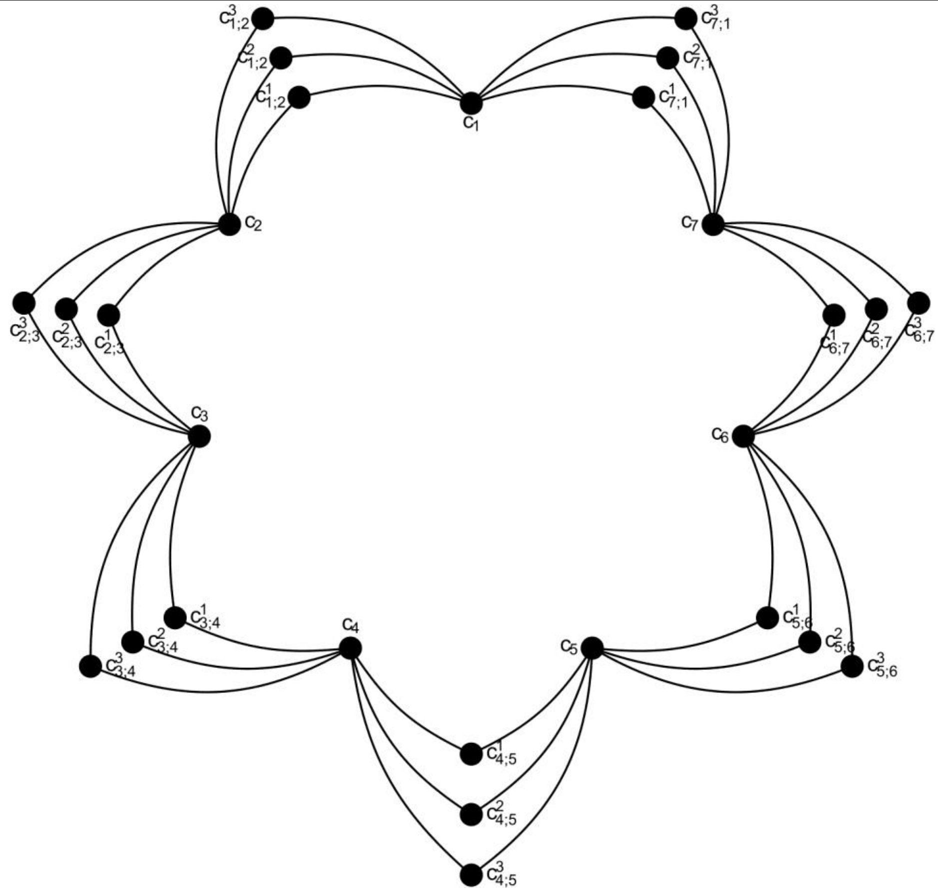


**Figure 2: Graph with *n* = 7 with general vertex labels**

## 4 MAIN RESULT

**Theorem 4.1.** Super sub division of $C_n$, $n \geq 3$ i.e. SS ($C_n$) is Super (a, d) - antimagic total for any *m*.

**Proof.**

$$|V| = n + nm = (m + 1)n, \qquad |E| = 2mn$$

$$f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, |V(G)| + |E(G)| = 3mn + n\}$$

be vertex and edge labeling as follows. We first label vertices.

$f(c_i) = i$ for $i = 1, 2, 3, \ldots, n.$

For $m = 1$

$f\left(c^1_{n,1}\right) = n + 1$

$f\left(C^1_{i,i+1}\right) = 2n - (i - 1)$ for $i = 1, 2, 3, \ldots, n - 1.$

For $m \geq 2$ for $j = 2, 3, \ldots, m.$

$f\left(c^j_{n,1}\right) = jn + 1$

$f\left(c^j_{1,2}\right) = jn + 2$

$f\left(c^j_{i,i+1}\right) = f\left(c^j_{i-1,i}\right) + 1$ for $i = 2, 3, \ldots, n - 1.$

We now label edges.

For $j = 1, 2, \ldots, m$.

$$f\left(C_1, c^j_{n,1}\right) = (m + 2j - 1)n + 1$$

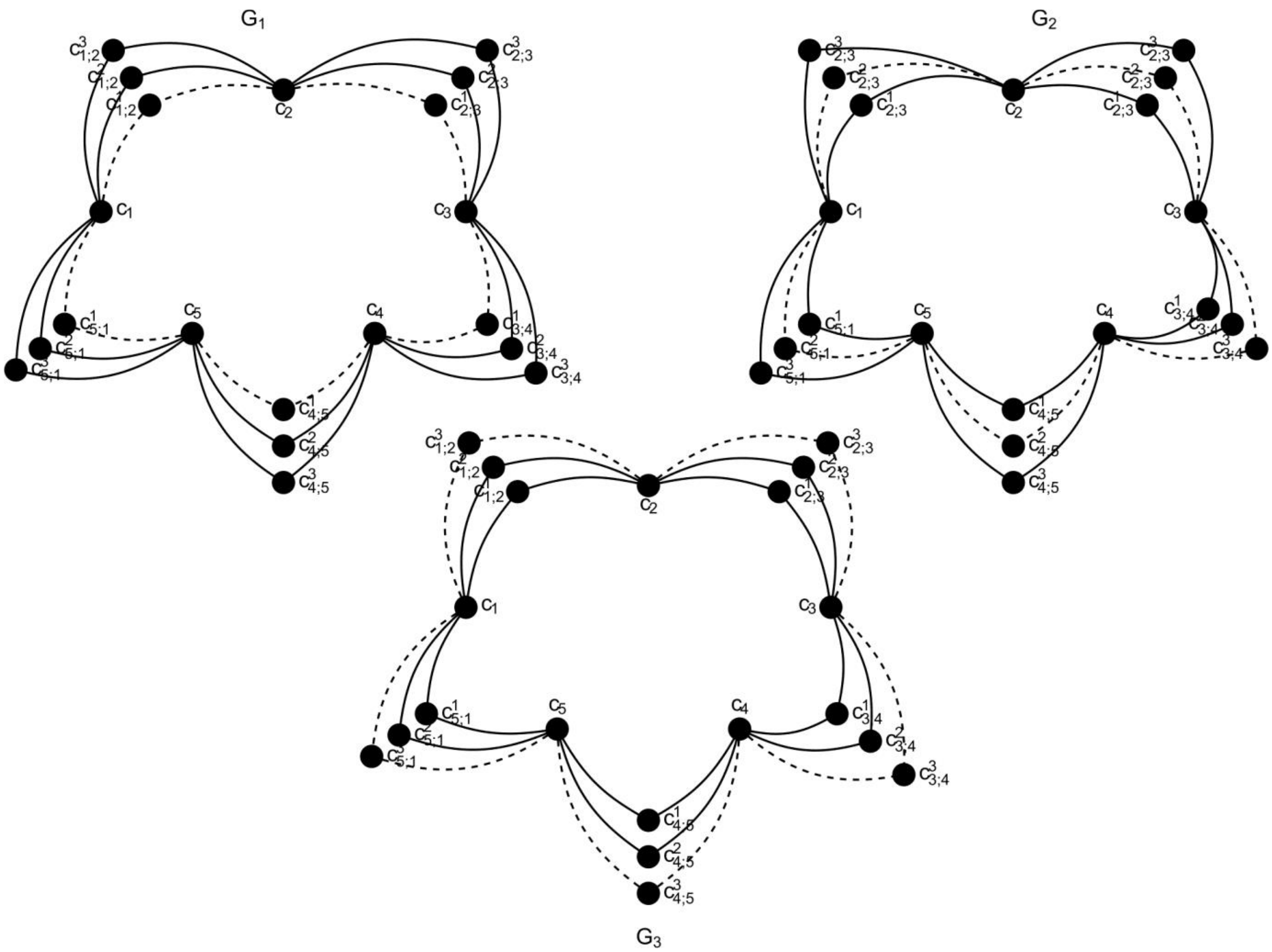
$$f\left(C_{i+1}, c^j_{i,i+1}\right) = (m + 2j - 1)n + i + 1 \qquad \text{for } \ i = 1, 2, \ldots, n - 1$$

$$f\left(C_n, c^j_{n,1}\right) = (m + 2j)n + 1$$

$$f\left(C_i, c^j_{i,i+1}\right) = (m + 2j)n + i + 1 \qquad \text{for } \ i = 1, 2, \ldots, n - 1.$$

At every stage of subdivision i.e. for $j = 1$ to $m$, we get an increment in earlier graph. Let this incremented graph at each step be denoted by $G_j$. In following figures $G^s_j$ are indicated by dotted lines.

At every stage of subdivision i.e. for $j = 1$ to m, we get an increment in earlier graph. Let this incremented graph at each step be denoted by $G_j$. In following figures $G_j^s$ are indicated by dotted lines.




With this notation we can see vertex labels clearly as follows.

**Table 1: Range of vertex labels for $C_n$.**

| Graphs | Labels | | |
|---|---|---|---|
| $G_i$ | $in+1$ | to | $(i+1)n$ |
| $G_1$ | $n+1$ | to | $2n$ |
| $G_2$ | $2n+1$ | to | $3n$ |
| $G_3$ | $3n+1$ | to | $4n$ |
| : | : | : | : |
| $G_m$ | $mn+1$ | to | $(m+1)n$ |

**Table 2: Range of edge labels for $C_n$.**

| Graphs | Labels | | |
|---|---|---|---|
| $G_i$ | $(m+2i-1)n+1$ | to | $(m+2i)n+n$ |
| $G_1$ | $(m+1)n+1$ | to | $(m+2)n+n$ |
| $G_2$ | $(m+3)n+1$ | to | $(m+4)n+n$ |
| $G_3$ | $(m+5)n+1$ | to | $(m+6)n+n$ |
| : | : | : | : |
| $G_m$ | $(3m-1)n+1$ | to | $(3m)n+n$ |

The labels of vertices of original cycle before subdividing are first 1 to $n$ numbers. From Table 1, labels of vertices, of each $G_i$ where $i$ = 1 *to* $m$, are following arithmetic progressions starting with $n$ + 1. Thus vertex labels of $G$ are distinct. Also there are $mn + n = (m + 1)n$ vertices in G and first $(m + 1)n$ numbers are used as labels for vertices of $G$.

The labels of edges starts with $(m + 1)n + 1$ and goes till $3mn + n$ as needed. Also from table 2 it is clear that edge labels for all $G_i$ and subsequently of $G$ are distinct.

Now we consider sub graphs $H_i$ covering $SS(C_n)$. Let the Subgraph $H = K_{2,m}$ having 2− vertices part as $C_i$ and $C_{i+1}$ be denoted by $H_i$ and let the Subgraph $H = K_{2,m}$ having 2− vertices part as $C_n$ and $C_1$ be denoted by $H_n$. Here, family of subgraphs $H_1, H_2, ..., H_n$ is each copy of subgraph $H = K_{2,m}$ such that each edge of $E(G)$ belongs to at least one of the subgraphs $H_i$, $i = 1, 2, . . . , n$. Therefore as every subgraph $H_i$ is isomorphic to a graph $H = K_{2,m}$, thus covering above mentioned here is an $H-$ covering of $G$. Following figure indicates a sample $H_i$ with $m = 3$.

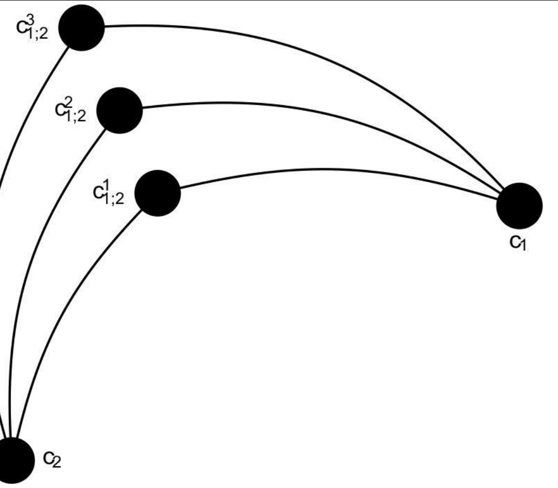



**Figure 3: Subgraph $H = K_{2,m}$ with $m = 3$, .**

We see sum of vertex labels in each $H_i$ in the following table.

**Table 3: Sums of vertex labels for $H_i$.**

| Graphs | Sum of Vertex Labels |
|---|---|
| $H_1$ | $1\ (2)\ +(2n)\ +(2n+2)\ +(3n+2)\ +(4n+2)\ \ldots+(mn+2)$ |
| $H_2$ | $2\ +(3)\ +(2n+1)\ +(2n+3)\ +(3n+3)\ +(4n+3)\ \ldots+(mn+3)$ |
| $H_3$ | $3\ +(4)\ +(2n+2)\ +(2n+4)\ +(3n+4)\ +(4n+4)\ \ldots+(mn+4)$ |
| $\vdots$ | $\vdots$ |
| $H_i$ | $i\ +(i+1)\ +(2n-(i-1))\ +(2n+i+1)\ +(3n+i+1)\ +(4n+i+1)\ \ldots+(mn+i+1)$ |
| $\vdots$ | $\vdots$ |
| $H_{n-1}$ | $n-1\ +(n)\ +(n+1))\ +(3n)\ +(4n)\ +(5n)\ \ldots+(m+1)n$ |
| $H_n$ | $n\ +1\ +(2n+1)\ +(3n+1)\ +(4n+1)\ \ldots(\frac{m(m+1)}{2}+1)n+m+1$ |

Thus, for $i = 1$ to $n-1$,
Sum of vertex labels of $H_i$ is

$$=i+(i+1)+(2n-(i-1)+(2n+i+1)+(3n+i+1)+(4n+i+1)+\ldots+(mn+i+1)$$

$$=(\frac{m(m+1)}{2}+1)n+m+1+mi\text{———-}(A)$$

Sum of vertex labels of $H_n$ is

$$=n+1+(2n+1)+(3n+1)+(4n+1)\ldots+(\frac{m(m+1)}{2}+1)n+m+1$$

$$=(\frac{m(m+1)}{2}+1)n+m+1.\text{————}(B)$$

We see sum of edge labels in each H*i* in the following table.

**Table 4 : Sums of edge labels for H*i*.**

| Graphs | Sum of Edge Labels |
|---|---|
| $H_1$ | $((m+1)n+2)+((m+2)n+2)+((m+3)n+2)\ldots+(\frac{m(m+1)}{2})n+2)$ |
| $H_2$ | $((m+1)n+3)+((m+2)n+3)+((m+3)n+3)\ldots+(\frac{m(m+1)}{2})n+3)$ |
| $\vdots$ | $\vdots$ |
| $H_i$ | $((m+1)n+i+1)+((m+2)n+i+1)+((m+3)n+i+1)\ldots+(\frac{m(m+1)}{2})n+i+1)$ |
| $\vdots$ | $\vdots$ |
| $H_{n-1}$ | $((m+1)n+n((+)m+2)n+n((+)m+3)n+n)\ldots+(\frac{m(m+1)}{2})n+n)$ |
| $H_n$ | $n+1+(2n+1)+(3n+1)+(4n+1)\ldots+(\frac{m(m+1)}{2}+1)n+n$ |

Thus, for $i=1$ to$n-1$,
Sum of edge labels of $H_i$ is

$$((=m+1)n+i+1)+((m+2)n+i+1)+((m+3)n+i+1)\ldots+(\frac{m(m+1)}{2})n+i+1)$$

$$=(4m^2+m)n+2m+2mi$$

Sum of edge labels of $H_n$ is

$$=n+1+(2n+1)+(3n+1)+(4n+1)\ldots(\frac{m(m+1)}{2}+1)n+n$$

$$=(4m^2+m)n+2m$$

Thus Sum of labels of $H_i$is

$$=(\frac{m(m+1)}{2}+1)n+m+1+mi+(4m^2+m)n+2m+2mi$$

$$=(\frac{9m^2+3m+2}{2}n+3m+1+3mi$$

Thus Sum of labels of $H_n$is

$$=(\frac{m(m+1)}{2}+1)n+m+1+(4m^2+m)n+2m$$

$$=(\frac{9m^2+3m+2}{2})n+3m+1$$

Hence it can be seen that sums of labels of $H_n, H_1, H_2, \ldots, H_{n-1}$ respectively are $(\frac{9m^2+3m+2}{2})n+3m+1$, $(\frac{9m^2+3m+2}{2})n+3m+1+3m(\frac{9m^2+3m+2}{2})n+3m+1+6m(\frac{9m^2+3m+2}{2})n+3m+1+9m$ and so on till $(\frac{9m^2+3m+2}{2})n+3m+1+3m(n-1)$.

Thus set of $H-$weights form an arithmetic sequence starting from $a=(\frac{9m^2+3m+2}{2})n+3m+1$ and having common difference $d=3m$. Moreover, the smallest possible labels appear on the vertices. Thus, $SS(C_n)$ is $(Super)(a,d)-H-$antimagic total.

## 5. ILLUSTRATIONS

**5.1** We illustrate $Super(a,d)-$antimagic total labeling for $SS(C_{10})$.
$n=10, m=3$

$$|V|=n+nm=40, \qquad |E|=2mn=60$$

$$f:V(G)\cup E(G)\rightarrow \{1,2,3,\ldots,100\}$$

Vertex labels:

$$f(C_1)=1,\quad f(C_2)=2,\quad f(C_3)=3,\quad f(C_4)=4,\quad f(C_5)=5,$$
$$f(C_6)=6,\quad f(C_7)=7,\quad f(C_8)=8,\quad f(C-9)=9,\quad f(C\ {}_{10}=10.$$

For $j=1$.

$$f(C^1_{10,1})=11,\quad f(C^1_{1,2})=20,\quad f(C^1_{2,3})=19,\quad f(C^1_{3,4})=18,\quad f(C^1_{4,5})=17,$$
$$f(C^1_{5,6})=16,\quad f(C^1_{6,7})=15,\quad f(C^1_{7,8})=14,\quad f(C^1_{8,9})=13,\quad f(C^1_{9,10})=12,$$

For $j=2$.

$$f(C^2_{10,1})=21,\quad f(C^2_{1,2})=22,\quad f(C^2_{2,3})=23,\quad f(C^2_{3,4})=24,\quad f(C^2_{4,5})=25,$$
$$f(C^2_{5,6})=26,\quad f(C^2_{6,7})=27,\quad f(C^2_{7,8})=28,\quad f(C^2_{8,9})=29,\quad f(C^2_{9,10})=30$$

For $j=3$.

$$f(C^3_{10,1})=31,\quad f(C^3_{1,2})=32,\quad f(C^3_{2,3})=33,\quad f(C^3_{3,4})=34,\quad f(C^3_{4,5})=35,$$
$$f(C^3_{5,6})=36,\quad f(C^3_{6,7})=37,\quad f(C^3_{7,8})=38,\quad f(C^3_{8,9})=39,\quad f(C^3_{9,10})=40$$

Edge Labels:

For $j=1$.

$$f(C_1,C^1_{10,1})=41,\ f(C_2,C^1_{1,2})=42,\ f(C_3,C^1_{2,3})=43,\ f(C_4,C^1_{3,4})=44,\ f(C_5,C^1_{6,7})=45$$
$$f(C_6,C^1_{5,6})=46,\ f(C_7,C^1_{6,7})=47,\ f(C_8,C^1_{7,8})=48,\ f(C_9,C^1_{8,9})=49,\ f(C_{10},C^1_{9,10})=50$$
$$f(C_{10},C^1_{10,1})=51,\ f(C_1,C^1_{1,2})=52,\ f(C_2,C^1_{2,3})=53,\ f(C_3,C^1_{3,4})=54,\ f(C_4,C^1_{6,7})=55$$
$$f(C_5,C^1_{5,6})=56,\ f(C_6,C^1_{6,7})=57,\ f(C_7,C^1_{7,8})=58,\ f(C_8,C^1_{8,9})=59,\ f(C_9,C^1_{9,10})=60$$

For $j$= 2.

$f(C_1, C^2_{10,1}) = 61,\ f(C_2, C^2_{1,2}) = 62,\ f(C_3, C^2_{2,3}) = 63,\ f(C_4, C^2_{3,4}) = 64,\ f(C_5, C^2_{6,7}) = 65$

$f(C_6, C^2_{5,6}) = 66,\ f(C_7, C^2_{6,7}) = 67,\ f(C_8, C^2_{7,8}) = 68,\ f(C_9, C^2_{8,9}) = 69,\ f(C_{10}, C^2_{9,10}) = 70$

$f(C_{10}, C^2_{10,1}) = 71,\ f(C_1, C^2_{1,2}) = 72,\ f(C_2, C^2_{2,3}) = 73,\ f(C_3, C^2_{3,4}) = 74,\ f(C_4, C^2_{6,7}) = 75$

$f(C_5, C^2_{5,6}) = 76,\ f(C_6, C^2_{6,7}) = 77,\ f(C_7, C^2_{7,8}) = 78,\ f(C_8, C^2_{8,9}) = 79,\ f(C_9, C^2_{9,10}) = 80$

For $j$= 3.

$f(C_1, C^3_{10,1}) = 81,\ f(C_2, C^3_{1,2}) = 82,\ f(C_3, C^3_{2,3}) = 83,\ f(C_4, C^3_{3,4}) = 84,\ f(C_5, C^3_{6,7}) = 85$

$f(C_6, C^3_{5,6}) = 86,\ f(C_7, C^3_{6,7}) = 87,\ f(C_8, C^3_{7,8}) = 88,\ f(C_9, C^3_{8,9}) = 89,\ f(C_{10}, C^3_{9,10}) = 90$

$f(C_{10}, C^3_{10,1}) = 91,\ f(C_1, C^3_{1,2}) = 92,\ f(C_2, C^3_{2,3}) = 93,\ f(C_3, C^3_{3,4}) = 94,\ f(C_4, C^3_{6,7}) = 95$

$f(C_5, C^3_{5,6}) = 96,\ f(C_6, C^3_{6,7}) = 97,\ f(C_7, C^3_{7,8}) = 98,\ f(C_8, C^3_{8,9}) = 99,\ f(C_9, C^3_{9,10}) = 100$

Thus weights of all subgraphs $H_i^s$ can be easily obseved to be as follows.
Weight of $H_1$ is 479.
Weight of $H_2$ is 488.
Weight of $H_3$ is 497.
Weight of $H_4$ is 506.
Weight of $H_5$ is 515.
Weight of $H_6$ is 524.
Weight of $H_7$ is 533.
Weight of $H_8$ is 542.
Weight of $H_9$ is 551.
Weight of $H_{10}$ is 470.

Arithmetic progression has first term as 470 and common difference 9. It starts with $H{-}10$ and last term is $H_9$.

Given above is the $Super(470{,}9)-$antimagic total labeling for $SS(C_{10})$ with $m$= 3.

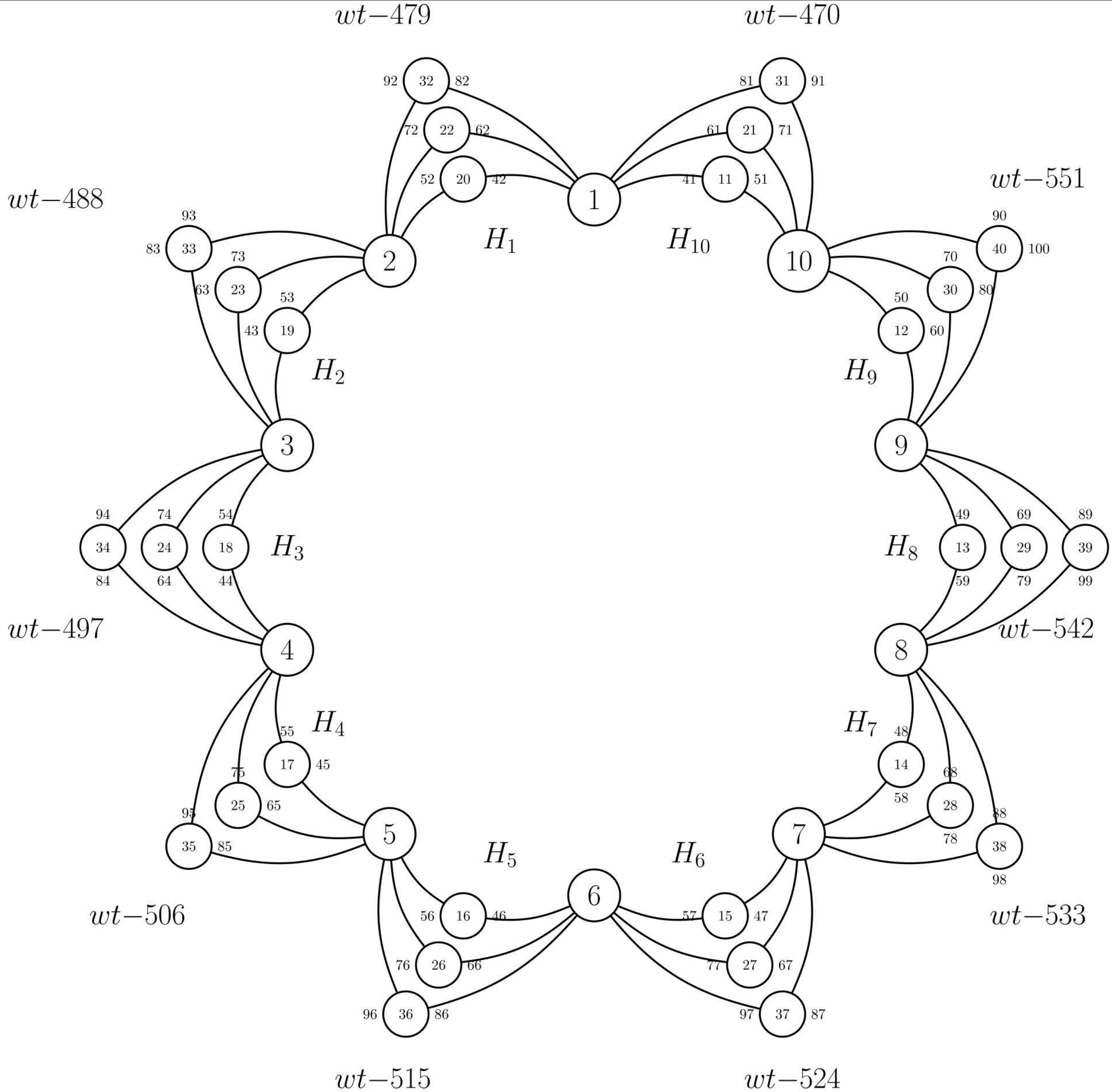



**Figure 4: $(Super)(470{,}9)$−antimagic total $SS(C_{10})$ with $m$= 3.**

**5.2** We illustrate $Super(a, d)$−antimagic total labeling for $SS(C_7)$ with $m$=5.

$n$= 7, $m$= 5

$$|V|=n+nm= 42, \qquad |E|= 2mn= 70$$

$$f:V(G)\cup E(G)\rightarrow \{1,2,3,\ldots,112\}$$

Vertex labels:

$f(C_1) = 1,\ f(C_2) = 2,\ f(C_3) = 3,\ f(C_4) = 4,\ f(C_5) = 5\ f(C_6) = 6,\ f(C_7) = 7.$

For$j$= 1.

$f(C^1_{7,1}) = 8,\ f(C^1_{1,2}) = 14,\ f(C^1_{2,3}) = 13,\ f(C^1_{3,4}) = 12,\ f(C^1_{4,5}) = 11,\ f(C^1_{5,6}) = 10,\ f(C^1_{6,7}) = 9.$

For$j$= 2.

$f(C^2_{7,1}) = 15,\ f(C^2_{1,2}) = 16,\ f(C^2_{2,3}) = 17,\ f(C^2_{3,4}) = 18,\ f(C^2_{4,5}) = 19,\ f(C^2_{5,6}) = 20,\ f(C^2_{6,7}) = 21.$

For$j$= 3.

$f(C^3_{7,1}) = 22,\ f(C^3_{1,2}) = 23,\ f(C^3_{2,3}) = 24,\ f(C^3_{3,4}) = 25,\ f(C^3_{4,5}) = 26,\ f(C^3_{5,6}) = 27,\ f(C^3_{6,7}) = 28,$

For$j$= 4.

$f(C^4_{7,1}) = 29,\ f(C^4_{1,2}) = 30,\ f(C^4_{2,3}) = 31,\ f(C^4_{3,4}) = 32,\ f(C^4_{4,5}) = 33,\ f(C^4_{5,6}) = 34,\ f(C^4_{6,7}) = 35,$

For$j$= 5.

$f(C^5_{7,1}) = 36,\ f(C^5_{1,2}) = 37,\ f(C^5_{2,3}) = 38,\ f(C^5_{3,4}) = 39,\ f(C^5_{4,5}) = 40,\ f(C^5_{5,6}) = 41,\ f(C^5_{6,7}) = 42,$

Edge Labels:

For$j$= 1.

$f(C_1, C^1_{7,1}) = 43,\quad f(C_2, C^1_{1,2}) = 44,\quad f(C_3, C^1_{2,3}) = 45,\quad f(C_4, C^1_{3,4}) = 46,$

$f(C_5, C^1_{6,7}) = 47, f(C_6, C^1_{5,6}) = 48,\quad f(C_7, C^1_{6,7}) = 49$

$f(C_7, C^1_{7,1}) = 50,\quad f(C_1, C^1_{1,2}) = 51,\quad f(C_2, C^1_{2,3}) = 52,\quad f(C_3, C^1_{3,4}) = 53$

, $f(C_4, C^1_{6,7}) = 54\quad f(C_5, C^1_{5,6}) = 55,\quad f(C_6, C^1_{6,7}) = 56.$

For$j$= 2.

$f(C_1, C^2_{7,1}) = 57,\quad f(C_2, C^2_{1,2}) = 58,\quad f(C_3, C^2_{2,3}) = 59,\quad f(C_4, C^2_{3,4}) = 60,$

$f(C_5, C^2_{6,7}) = 61, f(C_6, C^2_{5,6}) = 62,\quad f(C_7, C^2_{6,7}) = 63.$

$f(C_7, C^2_{7,1}) = 64,\quad f(C_1, C^2_{1,2}) = 65,\quad f(C_2, C^2_{2,3}) = 66,\quad f(C_3, C^2_{3,4}) = 67,$

$f(C_4, C^2_{6,7}) = 68, f(C_5, C^2_{5,6}) = 69,\quad f(C_6, C^2_{6,7}) = 70,$

For$j$= 3.

$f(C_1, C^3_{7,1}) = 71,\quad f(C_2, C^3_{1,2}) = 72,\quad f(C_3, C^3_{2,3}) = 73,\quad f(C_4, C^3_{3,4}) = 74,$

$f(C_5, C^3_{6,7}) = 75,\quad f(C_6, C^3_{5,6}) = 76,\quad f(C_7, C^3_{6,7}) = 77.$

$f(C_7, C^3_{7,1}) = 78,\quad f(C_1, C^3_{1,2}) = 79,\quad f(C_2, C^3_{2,3}) = 80,\quad f(C_3, C^3_{3,4}) = 81,$

$f(C_4, C^3_{6,7}) = 82,\quad f(C_5, C^3_{5,6}) = 83,\quad f(C_6, C^3_{6,7}) = 84.$

For $j$= 4.

$f(C_1, C^4_{7,1}) = 85$, $f(C_2, C^4_{1,2}) = 86$, $f(C_3, C^4_{2,3}) = 87$, $f(C_4, C^4_{3,4}) = 88$,

$f(C_5, C^4_{6,7}) = 89$, $f(C_6, C^4_{5,6}) = 90$, $f(C_7, C^4_{6,7}) = 91$.

$f(C_7, C^4_{7,1}) = 92$, $f(C_1, C^4_{1,2}) = 93$, $f(C_2, C^4_{2,3}) = 94$, $f(C_3, C^4_{3,4}) = 95$,

$f(C_4, C^4_{6,7}) = 96$, $f(C_5, C^4_{5,6}) = 97$, $f(C_6, C^4_{6,7}) = 98$.

For $j$= 5.

$f(C_1, C^5_{7,1}) = 99$, $f(C_2, C^5_{1,2}) = 100$, $f(C_3, C^5_{2,3}) = 101$, $f(C_4, C^5_{3,4}) = 102$,

$f(C_5, C^5_{6,7}) = 103$, $f(C_6, C^5_{5,6}) = 104$, $f(C_7, C^5_{6,7}) = 105$.

$f(C_7, C^5_{7,1}) = 106$, $f(C_1, C^5_{1,2}) = 107$, $f(C_2, C^5_{2,3}) = 108$, $f(C_3, C^5_{3,4}) = 109$,

$f(C_4, C^5_{6,7}) = 110$, $f(C_5, C^5_{5,6}) = 111$, $f(C_6, C^5_{6,7}) = 112$.

Thus weights of all subgraphs $H_i^s$ can be easily obseved to be as follows:

Weight of $H_1$ is 878.

Weight of $H_2$ is 893.

Weight of $H_3$ is 908.

Weight of $H_4$ is 923.

Weight of $H_5$ is 938.

Weight of $H_6$ is 953.

Weight of $H_7$ is 863.

Arithmetic progression has first term as 863 and common difference 15. It starts with $H-7$ and last term is $H_6$.

Given above is the $Super(863{,}15)-$antimagic total labeling for $SS(C_7)$ with $m$= 5.

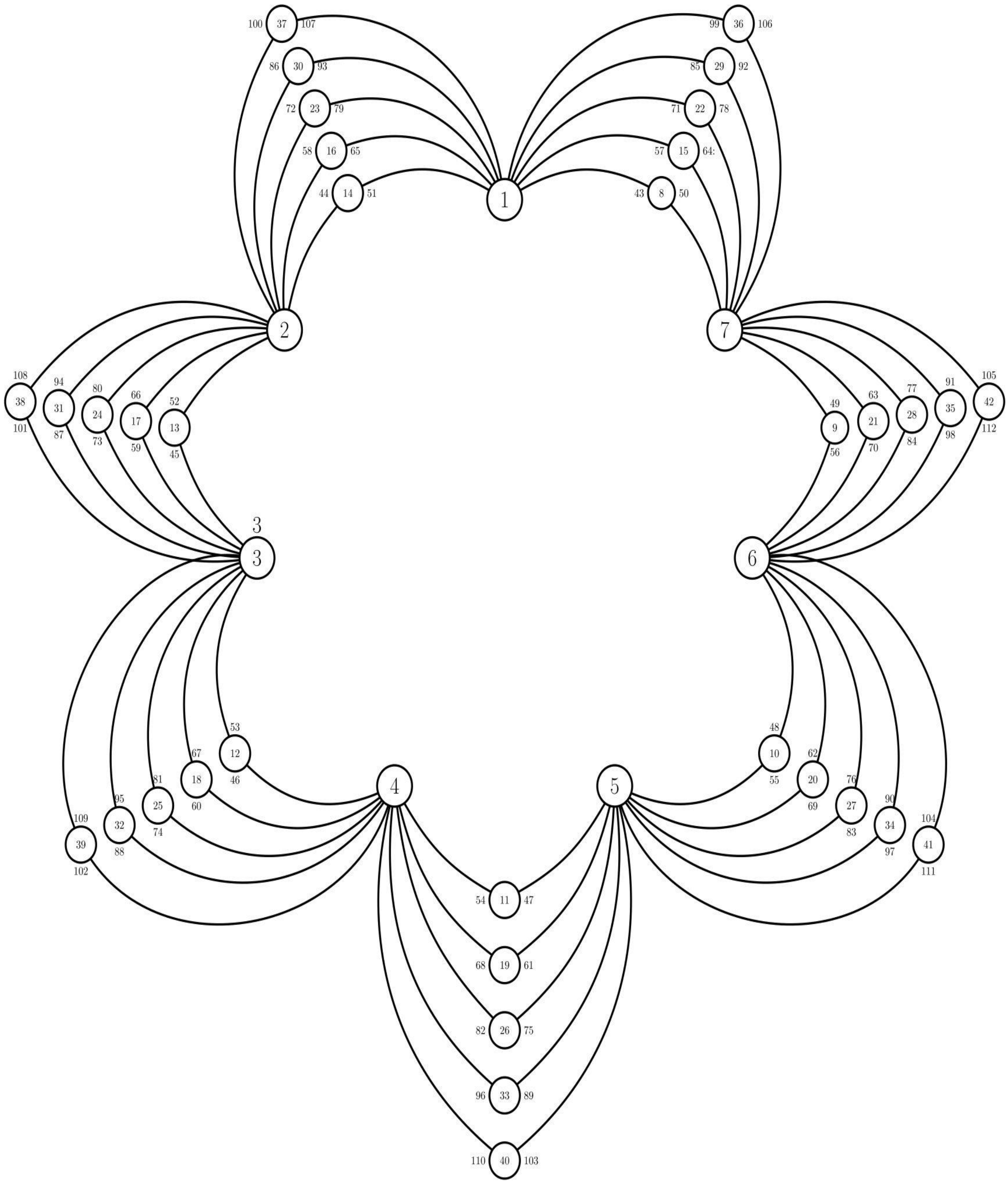



Figure 5: $(Super)(863{,}15)-$antimagic total $SS(C_7)$ with $m = 5$.

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