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PHYSICAL SIGNIFICANCE OF € AND δ IN THE €-δ DEFINITION OF CONTINUITY OF A FUNCTION

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ABSTRACT :
The €-δ definition of continuity of a function is theoretically taught in classroom but generally the teachers fail to clarify the concept & subject matter in real world situation. In this paper we have made an attempt to explore the physical significance of € & δ occurring in the definition to make it more clear and understandable. On the basis of graphical study and explaining the meaning of definition through practical situation the concept is made self explanatory. Even the layman can understand the meaning and concept easily.
Keywords: Function, Continuity, Uniform Continuity, Deviation, €-nhd etc.

INTRODUCTION

The €-δ definition of continuity of a function f(x) at any point 'a' defined in an interval (α,β) is given as "The function f(x) denoting $f : X \rightarrow Y$ is said to be continuous us at point $a \in (\alpha,\beta)$ if for a very small positive value € > 0 there exists a real number $\delta > 0$ (depends upon €) such that

$$| f(x) - f(a) | < € \text{ whenever, } | x- a | < \delta \qquad (1)$$

i.e. $f(x) \in (f(a) - €, f(a) + €)$ whenever, $x \in (a- \delta, a+ \delta) \cap X$"

In case δ depends upon € only then f(x) is said to be uniformly continuous and if δ depends upon both € and selection of the point 'a' at which the continuity of f(x) is being considered then the function is said to be continuous. In other words if for a fix value of €, the size of all the values of δ remain constant irrespective of selection of point in the domain of function then it is uniformly continuous and if for that particular €, the value of δ changes for at least one point of domain then it is said to be continuous or we may say uniform continuity of a function is a global property while continuity is a local one. We talk of uniform continuity on a set and not at a point.

Uniform continuity ⇒ continuity but converse need not be true

ADDRESS OF QUERIES :

For a beginner, the physical concepts of € & δ are very hard to visualize, for example a particular person say till the age of 25 years has been alive since his birth. The process of his being alive without any break for this much period of 25 years may be called a continuous function by a real analyst but he may fail in visualizing the € & δ occurring in this process. Usually a real analysis student may secure full marks in the examination after only reproducing the crammed definition, yet chances are very much there that he may not at all be able to understand what actually is the physical meaning of € and δ in any continued process. To make the things practically crystal clear, we shall concentrate on the following queries :

i) Why € & δ are physically greater than zero only. What happens to continuity if either one or both are less than or equal to zero ?

ii) Why δ depends upon € and not the other way round ?

iii) Why $| f(x) – f(a) | \nleq €$ and $| x– a | \nleq \delta$ i e why $| f(x) – f(a) | < €$ and $| (x-a) | < \delta$ only ?

iv) What is the practical meaning of δ depending on € only and δ depending upon € and selection of any point 'a' in (α,β)

Let us find an answer, to the above said queries by elaborating a particular situation pertaining to the continuous process of survival mentioned above. We start with a realistic situation arising to such a medical complication which leads to the breakdown of continued survival process of any surviving fellow. We all know that whenever our body develops some health problems the same are experienced by us because of arising of relevant symptoms like fever, body pain, extra growth in body parts etc.

We now, undertake the problem of body fever symptoms which is depicted in fig. 1 as :

Graphical Representation

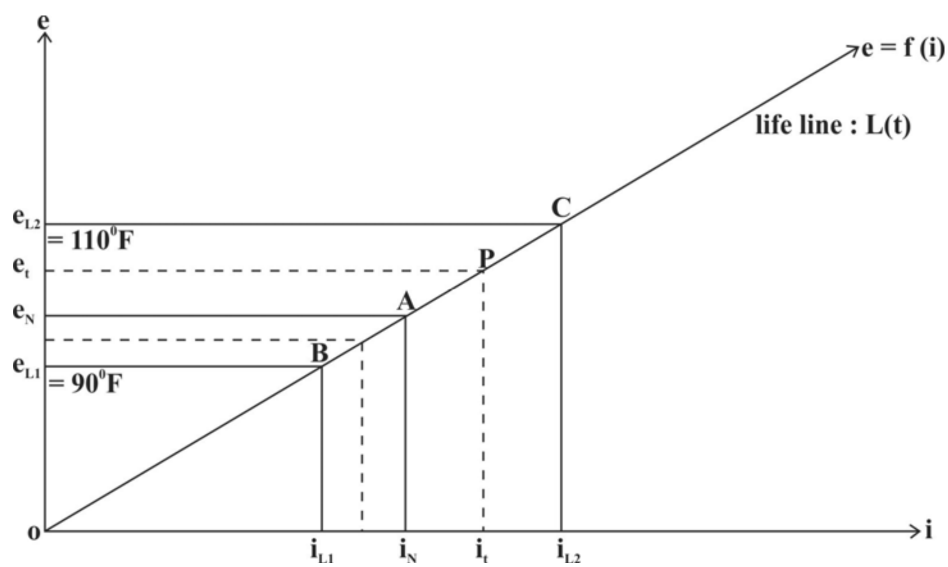

Fig. 1

SYMBOLS & NOTATIONS :

i-axis stands for system or body intake (input axis)

e-axis stands for effect axis (output axis)

Assumed lower and upper temperature limits which body can bear are in the range of $(90^0F, 110^0F)$ where as 98.4^0F is the normal temperature denoted by $\mathbf{e_N}$ which corresponds to daily normal intake $\mathbf{i_N}$.

Geometrically the problem is divided into two sections one for the above normal situation and another for below normal situation. Let $\mathbf{e_t}$ denotes the body temperature and $\mathbf{i_t}$ corresponding intake at a particular instant. The body will breakdown if $\mathbf{e_t} \geq \mathbf{e_{L2}}$ or $\mathbf{i_t} \geq \mathbf{i_{L2}}$ for upper part.

L (t) $\longrightarrow$ time based lifeline

$\mathbf{i_N}$ $\longrightarrow$ normal intake (Input)

$\mathbf{i_{L1}}, \mathbf{i_{L2}}$ $\longrightarrow$ lower & upper intake limits respectively

$\mathbf{e_N}$ $\longrightarrow$ normal effect (98.4^0F)

$\mathbf{e_{L1}}, \mathbf{e_{L2}}$ $\longrightarrow$ permissible lower (90^0F) & upper (110^0F) breaking point limits of discontinuity of the system

$\mathbf{e_t}$ $\longrightarrow$ the body temp. at any time t

$\mathbf{i_t}$ $\longrightarrow$ the intake at any time t

$\mathbf{e_t} - \mathbf{e_N} = d_e$, the deviation from normal temperature.

& $\mathbf{i_t} - \mathbf{i_N} = d_i$, the deviation from normal intake.

The system breaks down if $\mathbf{e_t} \geq \mathbf{e_{L2}}$ **or** $\mathbf{i_t} \geq \mathbf{i_{L2}}$. We have assumed that the life graph will be continued if the body temperature lies in (90^0F, 110^0F) otherwise the system will break down.

In fig (i) we depict the geometry representing the body fever symptoms ranging between 90^0F to 110^0F i e $\mathbf{e_{L1}}$ **to** $\mathbf{e_{L2}}$ meaning there by that the life line is discontinued if $\mathbf{e_t \leq 90^0F}$ or $\mathbf{e_t \geq 110^0F}$.

DISCUSSION AND ANALYSIS :

For the sake of convenience, let at a particular time t, $e_t = 104^0$F, the temperature deviation $d_e = 104 - 98.4 = 5.6^0$F is an alarming situation which indicates the daily life routine will be disrupted. The concerned doctor after examination will declare that the patient might have deviated sizably from his normal intake of which that level of temperature is the indication of side effect and a wise doctor would advise such a patient to be vigilant so that the temperature limits are well within the range (90^0F,110^0F) and starts the medical treatment with a suitable level of medicine dose prescribed for that matter which changes the intake to $\mathbf{i_t}$. After 3 or 4 days time period the doctor calls the patient again and notes down the treatment effect. If there is no considerable effect of that level of dose and the doctor is of the confirmed opinion about his selection of medicine then he may increase the dose size for few more days and on the other hand, if the said treatment has already considerable positive effect then the size of dose may be decreased accordingly. That way he goes on repeating the iterations till the temperature is almost normal. Here we find that the dose size level d_i depends upon the size of temperature symptoms i.e. d_e and not the other way round.

There may be a drastic situation when in spite of giving all possible treatments with fullest dose level of concerned available medicines the temperature is not under control and it reaches to the life line breaking point of 110^0F. Similar may be situation when the temperature level crosses the breaking point of 90^0F for the lower part. In both the cases doctor tries his level best to save the patient by giving higher & higher dose level but keeping in view that the dose level should not in either case cross the prescribed limits (i.e. should not be increased unreasonably) otherwise in the process of controlling the temperature the patient will loose his life line due to the side effect of excess medicine dose. In either case the life line is discontinued if either the temperature limits or the dose size is not within the prescribed limits.

Thus, we have seen that continuity of life line in no case allows the usage of very very large i.e. infinite size of medicines dose (i.e. d_i can't be infinitely large) also nor it gives the freedom to the patient to allow the temperature limits to very large level (meaning there by $\mathbf{d_e}$ can't be infinite)

In case of constant function one can experience either $\mathbf{d_i}$ or $\mathbf{d_e}$ or both can be zero. The function may appear constant in macro world but at the micro world a continuous change goes on because of eternal cosmic dance of existence. Since the definition of continuity concept comes under the branch of mathematics called calculus which is the study of continuous change, hence the definition of continuity under consideration is basically meant for changes at infinitesimal level or quantum level and not at the finite one. Thus the events may appear happening constantly at the finite level for practical purposes but in fact at infinitesimal levels the changes for which the definition is being developed nothing is ever constant. It is because at that level the change is ever happening. This implies that in real sense neither $\mathbf{d_e} = 0$ nor $\mathbf{d_i} = 0$, it is because approaching a doctor and taking medicine without any symptoms makes no sense as the survival process in this case already continuous.

Thus through the above said practical example we conclude as :

i) None of d_i and d_e can approach infinite.

ii) None of d_i and d_e is equal to zero.

iii) d_i depends upon d_e and not the other way round.

Symbolically,

$$\mathbf{e_t - e_N = d_e} \quad \text{where} \quad \mathbf{e_N < e_t < e_{L2}} \tag{2}$$

and $$\mathbf{i_t - i_N = d_i} \quad \text{where} \quad \mathbf{i_N < i_t < i_{L2}} \tag{3}$$

The above real world situation has been discussed for the temperature range 98.4^0F to 110^0F. A similar explanation can be given for the temperature between 90^0F to 98.4^0F which symbolically can be expressed as :

$$\mathbf{e_t - e_N = - d_e} \quad \text{where} \quad \mathbf{e_{L1} < e_t < e_N} \tag{4}$$

and $$\mathbf{i_t - i_N = - d_i} \quad \text{where} \quad \mathbf{i_{L1} < i_t < i_N} \tag{5}$$

Replacing the symbols used in the above explanation with those being used conventionally denoting **i-axis by x-axis, e-axis by y-axis, $\mathbf{i_N}$ by 'a', $\mathbf{e_N}$ by f (a), $\mathbf{i_t}$ by x, $\mathbf{e_t}$ by f (x) the life line graph e = f (i) by y = f (x) in the interval ($\mathbf{i_{L1}, i_{L2}}$) by (α,β),** in equations (2), (3), (4) & (5) we get

$$\mathbf{f(x) - f(a) = d_e} \quad \text{whenever} \quad \mathbf{x - a = d_i} \tag{6}$$

and $$\mathbf{f(x) - f(a) = -d_e} \quad \text{whenever} \quad \mathbf{x - a = -d_i} \tag{7}$$

Combining (6) & (7) we get

$| f(x) - f(a) | = d_e$ whenever $| x - a | = d_i$ (8)

As per the continuity condition laid in the above system both d_e and d_i are finite, non zero, positive number. Since the problem under study pertains to the changing at infinitesimal level where practically the evaluation of exact value of d_e and d_i is not possible hence, we select non zero +ve numbers € and δ representing those micro changes at that level with the condition that $d_e <$ € & $d_i < \delta$ hence equation (8) in final form can be written as.

$| f(x) - f(a) | <$ € whenever $| x - a | < \delta$

PRACTICAL DIFFERENCE BETWEEN CONTINUITY & UNIFORM CONTINUITY

To explain the practical difference between continuity and uniform continuity of a function we discuss another daily life problem i.e. **Car Journey Problem** in relation with fuel consumption and distance covered per unit fuel. Draw the car Journey graph as depicted in fig. 2 over a unit amount of fuel.

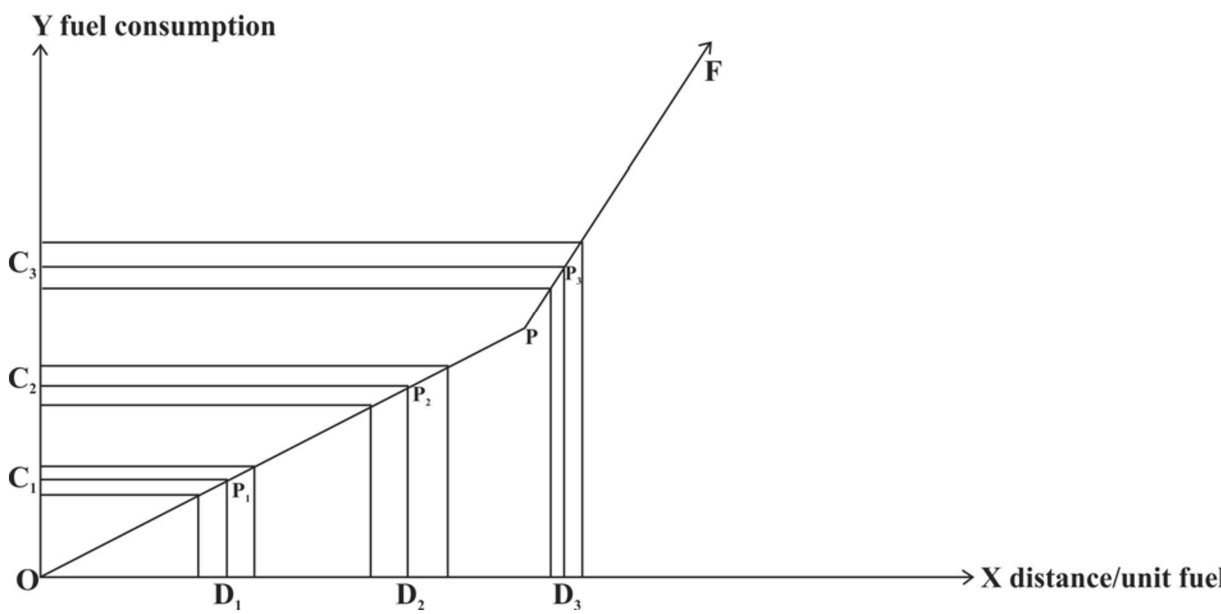

Fig. 2 Car Journey Graph

Let O & F be starting and finishing points between which the journey is assumed to be without break i.e. it is a continuous flow. We assume that from O to P the driver comes across the normal traffic whereas from P to F he faces comparatively a traffic jam like situation.

To study the problem, we consider three arbitrary locations **P_1, P_2** and **P_3** on journey path with a constraint of utilizing 1 liter fuel in each case and noting down the correspond to the distances covered. Let the points **C_1, C_2, C_3** corresponding to optimal fuel consumption & **D_1, D_2, D_3** are the corresponding points on the distance axis. Since practically, it is not possible to keep the rate of consumption of fuel constant therefore, the driver as per requirement may make minor changes in the €-nhds of three locations

C_i - € < C_i < C_i + € (i = 1, 2, 3) while on the distance axis the corresponding distances may be covered in the respective δ - nhds of $D_i - \delta < D_i < D_i + \delta$ (i = 1, 2) and $D_3 - \delta' < D_3 < D_3 + \delta'$ where $\delta_1 = \delta_2 \neq \delta'$.

Now, we observe from starting journey point O to P that the same size of € yields the equal sizes δ whereas for the journey path P to F the value of δ for any arbitrary point $\mathbf{P_3}$ is not equal to δ_i even though the € - nhd is of same magnitude for this case also. So we find that for a fixed value of €, the value of δ for the every point part of journey from O to P has got constants value whereas for the other part of journey from P to P_3 it has got different value meaning thereby that from O to P, δ depends upon € only and not on the choice of point that way defining the uniform continuity but on the other hand when the journey is considered jointly from O to F it is said to be continuous because the size of δ depends upon the selection of point as well, thereby differentiating practically the uniform continuity from continuity.

For the sake of convenience, we have considered only two daily life practical situations to explain the ideas of continuity and uniform continuity. Actually in existence around, we experience every natural phenomenon like the formation/deformation of galaxies, growth/decay of creatures and vegetation, social human behavior and relation etc is occurring in continuous process By providing the €-δ margins in the occurrence process of all universal cause effect phenomenon the existence has permitted to function every entity as per its need and convenience in a controlled democratic way to maintain the ongoing cosmos (smoothness) without the creation of any chaos. Had there been no provision for €-δ margins the existence could then have been forced to undergo a dictatorial set up instead of democratic one, that way leading the system to a hell like situation instead of the ongoing heaven like one. On the basis of above discussions the author is of the confirmed opinion that the €-δ margins can very well be used as shock obsorbers in maintaining the daily life harmonious relations between nears & dears both at the social as well as professional levels.

CONCLUSION :

In the present paper, we have highlighted the €-δ definition of continuity of a function practically in real world situation. We are sure that this type of explanation of complex definition of continuity will enable to clarify the subject matter in an effective, interesting and understandable manner to the learners.

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RAMANUJAN-TYPE DIOPHANTINE EQUATION INVOLVING JARASANDHA NUMBERS

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ABSTRACT :
In this paper, we accomplish integer solutions to the Ramanujan-type diophantine equation involving jarasandha numbers.
Keywords : Jarasandha numbers, Diophantine equation, integer solutions, Ramanujan-type Diophantine equation.

INTRODUCTION:

Number theory is fascinating on the grounds that it has such a large number of open problems that seem accessible from the outside. Of course, open problems in number theory are open for a reason. Numbers, despite being simple, have an incredibly rich structure which we only understand to a limited degree. In the mid twentieth century, Carmichael [1] made an important breakthrough in the study of diophantine equations. His proof is one of the first examples of the polynomial method. His proof impacted a great deal of later work in number theory, including diophantine equations. So number theory and its different subfields will keep on enthralling the brains of mathematicians for a very long time [1-3].

Apart from the polygonal numbers we have some more fascinating patterns of numbers namely Jarasandha numbers, Nasty numbers and Dhuruva numbers. These numbers have been presented in [4-12]. These results motivated us to examine for integer solutions to Ramanujan-type Diophantine equations involving Jarasandha numbers. In this communication, we search for integer solutions to the Ramanujan-type diophantine equation involving jarasandha numbers.

JARASANDHA NUMBERS:

In our Indian epic Mahabharatha, we come across a Person named 'JARASANDHA'. He had a boon that if he was split into two parts and thrown apart, the parts would rejoin and return to life. In fact, he was given life by the two halves of his body. In the field of Mathematics, we have numbers exhibiting the same property as Jarasandha.

Consider a number of the form XC. This may split as two numbers X and C and if these numbers are added and squared we get the same number XC.

(i.e) $XC = (X+C)^2 = XC$

Note: If C is an n-digit number, then $(X+C)^2 = (10^n)(X)+C$

Method of Analysis:

In this paper, we will show that the Ramanujan-type diophantine equation, $x^2+11J = J(12)^n$ has integer solution for suitable choice of J, where J is a jarasandha number. For all the Jarasandha numbers, the solution (x,n) is represented by $(\pm\sqrt{J},1)$.

Theorem:

$(\pm\sqrt{J},1)$ is the only solution to the Ramanujan-type diophantine equation, $x^2+11J = J(12)^n$ for suitable choice of J, where J is a jarasandha number.

Proof:

The Ramanujan-type diophantine equation under consideration is,

$$x^2+11J = J(12)^n \tag{1}$$

Rewriting (1) as, $(x+\sqrt{J}\sqrt{-11})(x-\sqrt{J}\sqrt{-11}) = J(12)^n$ (2)

It is treated as an equation in $Q(\sqrt{-11})$ whose ring of integers Q_{-11} has unique factorization domain whose units are ± 1.

$$\text{Let } \alpha = 1+\sqrt{-11} \;\&\; \beta = 1-\sqrt{-11} \tag{3}$$

The above choices of α,β leads to

$$\alpha^2 - 2\alpha + 12 = 0 \tag{4}$$

$$\text{b}\,\beta^2 - 2\beta + 12 = 0 \tag{5}$$

The power of α is given by

$$\alpha^n = r_n\alpha + s_n \quad \text{for } n>1 \tag{6}$$

In a similar way, we obtain

$$\beta^n = r_n\beta + s_n \quad \text{for } n>1 \tag{7}$$

where $\begin{pmatrix} r_{n+1} \\ s_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -12 & 0 \end{pmatrix}\begin{pmatrix} r_n \\ s_n \end{pmatrix}$ for $n>1$

$$\Rightarrow \begin{pmatrix} r_n \\ s_n \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -12 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A^{n-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } n>1$$

Here $A \quad \begin{matrix} 2 & 1 \\ 12 & 0 \end{matrix} \quad \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \pmod 2$

$$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod 2$$

……

& $A^{n-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod 2$

Therefore, $r_n \equiv 0 \pmod 2$ (8)

In another way, using (3) we get $\alpha^n - \beta^n = \pm 2\sqrt{-11} = \pm(\alpha - \beta)$ (9)

From (6) & (7), $\alpha^n - \beta^n = r_n(\alpha - \beta)$ (10)

Comparing $r_n = \pm 1$

which is a contradiction to (8).

Hence, $(\pm\sqrt{J}, 1)$ is the only solution for the Ramanujan-type diophantine equation, $x^2 + 11J = J(12)^n$

The Ramanujan-type diophantine equation (1) for the jarasandha numbers and their corresponding integer solutions are given in the following table:

S.No	J	Ramanujan-type diophantine equation	Integer solutions
1.	81	$x^2 + 891 = 81(12)^n$	$(\pm 9, 1)$
2.	2025	$x^2 + 22275 = 2025(12)^n$	$(\pm 45, 1)$
3.	3025	$x^2 + 33275 = 3025(12)^n$	$(\pm 55, 1)$
4.	9801	$x^2 + 107811 = 9801(12)^n$	$(\pm 99, 1)$
5.	88209	$x^2 + 970299 = 88209(12)^n$	$(\pm 297, 1)$

Remarkable Observation:

1. Also, Ramanujan-type Diophantine equations of the form

$x^2 + 11(J/81) = J(12)^n$ whenever, $J/81$ is an integer

& $x^2 + 11(J/25) = J(12)^n$ whenever, $J/81$ is not an integer.

for the jarasandha numbers with their integer solutions are given in the table below.

S.No	J	Ramanujan-type diophantine equation	Integer solutions
1.	81	$x^2 + 11 = 81(12)^n$	$(\pm 31, 1)$
2.	2025	$x^2 + 275 = 2025(12)^n$	$(\pm 155, 1)$
3.	3025	$x^2 + 1331 = 3025(12)^n$	$(\pm 187, 1)$
4.	9801	$x^2 + 1331 = 9801(12)^n$	$(\pm 341, 1)$
5.	88209	$x^2 + 11979 = 88209(12)^n$	$(\pm 1023, 1)$

CONCLUSION:

In this paper, we have presented integer solutions to the Ramanujan-type diophantine equation involving jarasandha numbers. To conclude that, one may search for other equations with some other numbers.

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TWO STAGE SPECIALLY STRUCTURED FLOWSHOP SCHEDULING INCLUDING TRANSPORTATION TIME, PROBABILITIES ASSOCIATED WITH PROCESSING TIME AND JOB BLOCK CRITERIA : A REVIEW PAPER

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ABSTRACT :
The present paper is aimed to provide algorithm minimizing the total waiting time of jobs for specially structured two stage flowshop scheduling. The model includes the transportation time , job block criteria , probabilities are associated with processing time of machines. The algorithm is made clear by numerical illustration. The paper is a review study made by various researchers of the related field. The lemma has been provided on which study is based.
Keywords : Waiting time of jobs, Transportation time, Flow shop scheduling, Processing time, Job block.

INTRODUCTION

The problems in which the facilities are fixed and the sequence of servicing the waiting customers or items/ jobs is subject to control for a set criteria comes under the heading of Scheduling theory. The scheduling problems involve placing items in a certain order for service before one or more machines in order to optimize the measure of effectiveness. We consider a two-stage flow-shop scheduling problem with the objective to minimize the total waiting time of jobs. When the jobs arrive for processing, the waiting time for their turn on the first machine is considered to be zero. But in order to process a job on second machine they may have to wait for their turn for many reasons such as the previous job can take some time for operation on second machine, machine take set up time, machine break down etc. The time which is consumed on waiting for their turn is called waiting time of job. A lot of work has been done in minimizing total waiting time for obtaining an optimal schedule of jobs. The waiting time is important for scheduling jobs on the machines. The idea of minimizing the waiting time or cost may be an economical expression for industrial /factory manager's point of view. Today’s large-scale markets and immediate interactions mean that clients expect high-class goods and services at time they require, anywhere they require them. Organizations, whether public or private, have to make available these products and services as effectively and efficiently as possible. Moreover, due to technological constraint or some externally imposed restriction it becomes significant to process some set of specified jobs together as a block called group technology in sequencing problems hence equivalent job block concept is included in the study. The group technology has application to a variety of production concerns for the purpose of improving productivity. The present study is a review work based on study made by various researchers of the field. The related lemma and a theorem based on the heuristic approach has also been given.

LITERATURE REVIEW

In order to find optimal sequence of jobs the fundamental study was made by Johnson[1] using heuristic approach for n jobs 2 and restrictive case 3 machines flow-shop scheduling . Ignall and Schrage[2] developed branch and bound algorithms for the permutation flow-shop problem minimizing make-span . Lockett et.al.[3] studied sequencing problems which involves sequence dependent changeover times. Maggu and Das [4] introduced the equivalent job concept for job block in scheduling problems. Singh T.P.[5] extended the study by introducing various parameters like transportation time, break down interval, weightage of jobs etc. The work was further extended by Gupta J.N.D.[6], Rajendran C. et. [7], Singh T.P. etal.[8,9] considering criteria other than make-span. Further Singh T.P., Gupta D. etal.[10] made an attempt to minimize the rental cost of machines including job block through simple heuristic approach. Gupta D. and Bharat Goyal etal.[11] studied specially structured two stage Flow Shop scheduling models with the objective to optimize the total waiting time of jobs. This paper is an extension of study done by Gupta D. and Bharat Goyal[12] in the sense job block concept is taken into consideration. In fact this paper is a review paper based on the study made by various researchers.

PRACTICAL SITUATION

Manufacturing units/industries play a momentous role in the economic progress of a country. Flow shop scheduling occurs in various offices, service stations, banks, airports etc. In our routine working in industrial and manufacturing units diverse jobs are done on a variety of machines. In textile industry different types of fabric is produced using different types of yarn. Here, the maximum equivalent time taken in dying of yarn on first machine is always less than or equal to the minimum equivalent time taken in weaving of yarn on the second machine.

Lemma1: Let p jobs 1,2,3,...,p are to be processed through two machines M and N in the order MN with no passing allowed. Let job i (i = 1, 2, 3, ….., p) has processing times M_i and N_i on each machine respectively assuming their respective probabilities s_i and t_i such that $0 \leq s_i \leq 1$; $0 \leq t_i \leq 1$ and $\sum s_i = \sum t_i = 1$.Expected processing times are defined as $M_i' = M_i * s_i$ and $N_i' = Ni * t_i$ satisfying expected processing times structural relationship :Max M_i' $\leq$Min N_i' then for the p job sequence S:α_1, α_2, ………,α_p where $T_{\alpha N}$ is the completion time of job α on machine N.

Proof: Applying mathematical induction hypothesis on p:

Let the statement S(p):$T_{\alpha_p N} = M'_{\alpha_1} + N'_{\alpha_2} + N'_{\alpha_2} + \ldots\ldots\ldots + N'_{\alpha_p}$

$T_{\alpha_1 M}$ M'_{α_1}

$T_{\alpha_1 N} = M'_{\alpha_1} + N'_{\alpha_1}$Hence for p = 1 the statement S(1) is true.

Let for p = k, the statement S(k) be true, i.e.$T_{\alpha_k N} = M'_{\alpha_1} + N'_{\alpha_2} + N'_{\alpha_2} + \ldots\ldots\ldots + N'_{\alpha_k}$Now,$T_{\alpha_{k+1} N}$ $= \mathrm{Max}(T_{\alpha_{k+1} M}, T_{\alpha_k N}) + N'_{\alpha_{k+1}} = \mathrm{Max}(M'_{\alpha_1} + M'_{\alpha_2} + \ldots\ldots + M'_{\alpha_{k+1}}, M'_{\alpha_1} + N'_{\alpha_1} + N'_{\alpha_2} + \ldots + N'_{\alpha_k}) + N'_{\alpha_{k+1}}$As Max$M'_i \leq$Min$N'_i$ Hence$T_{\alpha_{k+1} N} = M'_{\alpha_1} + N'_{\alpha_2} + N'_{\alpha_2} + \ldots\ldots\ldots + N'_{\alpha_k} + N'_{\alpha_{k+1}}$

Hence, for p = k + 1 the statement S(k+1) holds true. Since S(p) is true for p = 1; p = k; p = k + 1 and k being arbitrary. Hence, S(p):$T_{\alpha_p N} = M'_{\alpha_1} + N'_{\alpha_2} + N'_{\alpha_2} + \ldots\ldots + N'_{\alpha_p}$ is true.

Lemma2: With the same notations as that of Lemma 1 , for p job sequence S:α_1 , α_2, ……. ,α_p

$W_{\alpha_1}= 0$

$W_{\alpha_k} = M'_{\alpha_1} + \sum_{r=1}^{k-1} (x_{\alpha_r}) - M'_{\alpha_k}$

where W_{α_k} is the waiting time of job α_k for the sequence(α_1 , α_2 , ..., α_p) and $x_{\alpha_r} = N'_{\alpha_r} - M'_{\alpha_r}$, $\alpha_r \epsilon (1, 2, 3, \ldots , p)$

Proof: $W_{\alpha_1} = 0$ $W_{\alpha_k} = Max(T_{\alpha_k M} , T_{\alpha_{k-1} N}) - T_{\alpha_k M} = Max(M'_{\alpha_1}+M'_{\alpha_2}+ \ldots + M'_{\alpha_k} , M'_{\alpha_1}+N'_{\alpha_1}+N'_{\alpha_2}+\ldots+N'_{\alpha_{k-1}}) - (M'_{\alpha_1}+M'_{\alpha_2}+\ldots+M'_{\alpha_k}) = M'_{\alpha_1} + N'_{\alpha_1} + N'_{\alpha_2}+\ldots + N'_{\alpha_{k-1}}-M'_{\alpha_1}-M'_{\alpha_2}\ldots-M'_{\alpha_k} = M'_{\alpha_1} + (N'_{\alpha_1}- M'_{\alpha_1}) + (N'_{\alpha_2}- M'_{\alpha_2}) + \ldots + (N'_{\alpha_{k-1}}- M'_{\alpha_{k-1}}) - M'_{\alpha_k} = M'_{\alpha_1} +\sum_{r=1}^{k-1} (N'_{\alpha_r}- M'_{\alpha_r}) - M'_{\alpha_k} = M'_{\alpha_1} + \sum_{r=1}^{k-1} (x_{\alpha_r})- M'_{\alpha_k}$

Theorem 1: Let p jobs 1,2,3,...,p are to be processed through two machines M and N in the order MN with no passing allowed. Let job i(i = 1,2,3,…, p) has processing times M_i and N_i on each machine respectively assuming their respective probabilities si and ti such that $0 \le s_i \le 1$, $0 \le t_i \le 1$ and $\sum s_i = \sum t_i = 1$.Expected processing times are defined as $M'_i = M_i * s_i$ and $N'_i = N_i * t_i$ satisfying expected processing times structural relationships :Max $M'_I \le$ Min N'_i then for the p job sequence S:α_1,α_2, …..,α_p the total waiting time T_w(say)$T_w = pM'_{\alpha_1}+ \sum_{r=1}^{p-1} (z_{\alpha_r}) - \sum_{i=1}^{p} (M'_i) z_{\alpha_r} = (p - r)x_{\alpha_r}$;$\alpha_r \epsilon$ (1, 2, 3, …., p)

Proof: From Lemma 2 we have,$W_{\alpha_1} = 0$

k = 2, k - 1 = 1 $W_{\alpha_2 N} = M'_{\alpha_1} +\sum_{r=1}^{1} (x_{\alpha_r})- M'_{\alpha_2}$

$= M'_{\alpha_1}+ x_{\alpha_1}- M'_{\alpha_2}$

k = 3, k - 1 = 2 $W_{\alpha_3} = M'_{\alpha_1}+ \sum_{r=1}^{2} (x_{\alpha_r}) - M'_{\alpha_3}$

$= M'_{\alpha_1} + x_{\alpha_1}+ x_{\alpha_2}- M'_{\alpha_3}$ continuing in this way

k = p,k -1 = p - 1 $W_{\alpha_p}= M'_{\alpha_1} +\sum_{r=1}^{p-1} (x_{\alpha_r})- M'_{\alpha_p}= M'_{\alpha_1} + x_{\alpha_1} + x_{\alpha_2} +\ldots + x_{\alpha_{p-1}}- M'_{\alpha_p}$

Hence, total waiting time

$T_w = W_{\alpha_1} +W_{\alpha_2}+W_{\alpha_3} + \ldots +W_{\alpha_p}$ $T_w = 0 + (M'_{\alpha_1} + x_{\alpha_1}- M'_{\alpha_2}) + (M'_{\alpha_1} + x_{\alpha_1} + x_{\alpha_2}- M'_{\alpha_3}) + \ldots + (M'_{\alpha_1} + x_{\alpha_1} +x_{\alpha_2} + \ldots + x_{\alpha_{p-1}}- M'_{\alpha_p})$ $T_w = (M'_{\alpha_1} +M'_{\alpha_1} +\ldots+(p-1)\text{times})+(x_{\alpha_1} +x_{\alpha_1} +\ldots+(p-1)\text{times})+(x_{\alpha_2} +x_{\alpha_2} + \ldots + (p-2)\text{times}) + \ldots(x_{\alpha_{p-1}}- (M'_{\alpha_2} +M'_{\alpha_3} + \ldots +M'_{\alpha_p}))$ $T_w = (p-1)M'_{\alpha_1} + (p- 1)(x_{\alpha_1} + (p - 2)x_{\alpha_2} + \ldots + x_{\alpha_{p-1}}- (\sum_{i=1}^{p} M'_{\alpha i}- M'_{\alpha_1}) = pM'_{\alpha_1} + \sum_{r=1}^{k-1} (p-r)(x_{\alpha_r})- \sum_{i=1}^{p}(M'_{\alpha_i})$

Equivalent Job Block Theorem: In processing a schedule s=(1,2,3,...,p)of p jobs on two machines M and N in the order MN with no passing allowed. A job i (i=1,2,3,...,p) has processing time M_i and N_i on each machine respectively. The job block (k,m) is equivalent to the single job . Now the processing times of job on the machine M and N are denoted respectively by M, N are given by

$M_\alpha = M_k- M_m- \min(M_m,N_k)$

$N_\alpha= N_k- N_m- \min(M_m,N_k)$

The proof of the theorem is given by Maggu P.L. and Das G.

PROBLEM FORMULATION

Let the machines A and B be dealing out n jobs in order AB. Let a_k and b_k be the relevant processing time together with the probabilities p_k and q_k of the kth job correspondingly, let t_k be the transportation time of k^{th} job from machine A to machine B. Our goal is to come across a best possible sequence A_i of jobs with minimum total waiting time of jobs. Expected processing time of k^{th} job on machines A and B are defined as $a'_k = a_k \times p_k$, $b'_k = b_k \times q_k$. Define the two fictitious machine X and Y with processing time of kth job X'_k and Y'_k defined as $X'_k = a'_k + t_k$ and $Y'_k = b'_k + t_k$ satisfying processing time structural relationship Max $X_k' \leq$ Min Y_k' .Let an equivalent job α be defined as (i,j) where i,j are any jobs among the given n jobs such that the job i occurs before job j in the order of job block (i,j).

Tableau 1: Matrix form of the problem

Jobs	Machine A		Transportation Time (A→B)	Machine B	
J	a_k	p_k	t_k	b_k	q_k
1	a_1	p_1	t_1	b_1	q_1
2	a_2	p_2	t_2	b_2	q_2
3	a_3	p_3	t_3	b_3	q_3
. . .	. . .	. . .	. . .	. . .	. . .
N	a_n	p_n	t_n	b_n	q_n

ASSUMPTIONS

For the given flowshop problem following assumptions are made

1) There are n number of jobs (J) and two machines(A and B)

2) $\sum_{k=1}^{n} (p_k) = \sum_{k=1}^{n} (q_k) = 1$

3) Pre-emption is not allowed i.e. the process can't be interrupted until a job which is started on a machine can't be fully completed.

ALGORITHM

Step 1: Calculate expected processing times, a'_k and b'_k on machines A and Bdefined as follows:
$a'_k = a_k * p_k, b'_k = b_k * q_k$.

Step 2: Define the fictitious machines X and Y with processing times X'_k and Y'_k as follows:

$X'_k = a'_k + t_k$ and $Y'_k = b'_k + t_k$ and verify the structural relationship

Step 3: Take equivalent job α= (i , j) and define processing time using equivalent job block theorem and replace the pair of jobs (i,j) in this order by a single job α. Present the data in tabular form as below:

Tableau 2

Jobs (J)	Machine X (X'_k)	Machine Y (Y'_k)	$x_k = Y'_k - X'_k$
1	X'_1	Y'_1	x_1
2	X'_2	Y'_2	x_2
3	X'_3	Y'_3	x_3
. . α . .	. . X'_α . .	. . Y'_α . .	. . x_α . .
n	X'_n	Y'_n	x_n

Step 4: Arrange the jobs in increasing order of x_k. Let the sequence be($\mu_1, \mu_2, \dots, \mu_n$).

Step 5: Find Min X'_k two cases arise;

a) If $X'_{\mu 1}$ = Min X'_k, then schedule according to step 3 is required optimal sequence.

b) If $X'_{\mu 1} \neq$ Min X'_k, then go to next step.

Step 6: Consider the different sequences of jobs $S_1, S_2, \dots, S_r$ where S_1 is the sequence obtained in step 3 , sequence S_k($k = 2,3, \dots, r$) can be obtained by placing k^{th} job in the sequence S_1 to the first position and rest of the sequence remaining same.

Step 7: Fill the values in the following table

Tableau 3

Jobs (J)	Machine X (X'_k)	MachineY (Y'_k)	$X_k=Y'_k - X'_k$	$Z_{kr}=(n-r)x_k$			
				r=1	r=2	. . . .	r=(n-1)
1	X'_1	Y'_1	x_1	z_{11}	z_{12}	. . . .	$z_{1(n-1)}$
2	X'_2	Y'_2	x_2	z_{21}	z_{22}	. . . .	$z_{2(n-1)}$
3	X'_3	Y'_3	x_3	z_{31}	z_{32}	. . . .	$z_{3(n-1)}$
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
N	X'_n	Y'_n	x_n	z_{n1}	z_{3n}	. . . .	$z_{n(n-1)}$

Step 8: Calculate the waiting time T_w for all the sequences $S_1, S_2, \dots, S_r$ using the formula:
$T_w = nx'_{\mu_1}+\sum_{r=1}^{n-1} (Z_{ar})- \sum_{k=1}^{n} (X'_k)$ where $X'_{\mu 1}$ =Equivalent processing time of first job on machine X in sequence S_k.

$Z_{ar} = (n - r)x_{ar}$; $a = \mu_1, \mu_2, \dots, \mu_n$

The sequence with minimum waiting time is required optimal sequence.

NUMERICAL ILLUSTRATION

Assume 5 jobs 1,2,3,4,5 are to be processed on two machines A and B withprocessing times a_k, b_k and p_k, q_k are their respective probabilities and t_k is the transportation time of k^{th} job from machine A to machine B.

Tableau 4

Jobs	Machine A		Transportation Time (A→B)	Machine B	
J	a_k	p_k	t_k	b_k	q_k
1	6	0.2	4	12	0.2
2	7	0.2	3	21	0.2
3	12	0.2	2	34	0.2
4	11	0.3	3	22	0.2
5	13	0.1	2	24	0.2

Our propose is to achieve a most favorable schedule, minimizing the total waiting time for the jobs.

As per step 1- Expected processing times a'_k and b'_k on machines A and B are calculated in the following table

Tableau 5

Jobs (J)	Machine A (a'_k)	Transportation Time (A→B)	Machine B (b'_k)
1	1.2	4	2.4
2	1.4	3	4.2
3	2.4	2	6.8
4	3.3	3	4.4
5	1.3	2	4.8

As per step 2- Defining the fictitious machines X and Y with processing times $X'_k = a'_k + t_k$ and $Y'_k = b'_k + t_k$ respectively.

Tableau 6

Jobs (J)	Machine X (X'_k)	MachineY (Y'_k)
1	5.2	6.4
2	4.4	7.2
3	4.4	8.8
4	6.3	7.4
5	3.3	6.8

$\text{Max}X'_k = 6.3 < \text{Min}Y'_k = 6.4$

As per step3: Take equivalent job = (2, 5). Then processing times are defined as follows

$X'_\alpha = X'_2 + X'_5 - \text{Min}(X'_5 - Y'_2) = 4.4$ and $Y'_\alpha = Y'_2 + Y'_5 - \text{Min}(X'_5 - Y'_2) = 10.7$

Tableau 7

Jobs (J)	Machine X (X'_k)	MachineY (Y'_k)	$x_k = Y'_k - X'_k$
1	5.2	6.4	1.2
α	4.4	10.7	6.3
3	4.4	8.8	4.4
4	6.3	7.4	1.1

As per step 4- Arrange the jobs in increasing order of x_k i.e. the sequence found to be 4,1,3,α

Tableau 8

Jobs (J)	Machine X (X'_k)	MachineY (Y'_k)	$x_k = Y'_k - X'_k$
4	6.3	7.4	1.1
1	5.2	6.4	1.2
3	4.4	8.8	4.4
α	4.4	8.8	4.4

As per step 5- Min $X'_k = 4.4 \neq 6.3$

As per step 6- The sequences obtained are

$S_1 = (4, 1, 3, \alpha)$

$S_2 = (1, 4, 3, \alpha)$ $S_3 = (3, 4, 1, \alpha)$

$S_4 = (\alpha, 4, 1, 3)$

As per step 7- Fill the values in the following table

Tableau 9

Jobs (J)	Machine X (X'_k)	MachineY (Y'_k)	$x_k=Y'_k - X'_k$	$Z_{kr}=(n-r)x_k$			
				r=1	r=2	r=3	r=4
1	5.2	6.4	1.2	4.8	3.6	2.4	1.2
2	4.4	7.2	2.8	11.2	8.4	5.6	2.8
3	4.4	8.8	4.4	17.6	13.2	8.8	4.4
4	6.3	7.4	1.1	4.4	3.3	2.2	1.1
5	3.3	6.8	3.5	14.0	10.5	7.0	3.5

As per step 8- Calculate the total waiting time for the sequences S_1, S_2, S_3, S_4.

$\sum_{k=1}^{n} (X'_k) = 23.6$

For the sequence $S_1 = (4, 1,3,\alpha)$ or $(4, 1,3, 2, 5)$

Total waiting time

$T_w = 5\times 6.3 + 4.4 + 3.6 + 8.8 + 2.8 - 23.6 = 18.5$

For the sequence $S_2 = (1, 4, 3,\alpha)$ or $(1, 4, 3,2, 5)$

Total waiting time $T_w = 5 \times 5.2 + 4.8 + 3.3 + 8.8 + 2.8 - 23.6 = 22.1$

For the sequence $S_3 = (3, 4, 1, \alpha)$ or $(3, 4, 1, 2, 5)$

Total waiting time $T_w = 5 \times 4.4 + 17.6 + 3.3 + 2.4 + 2.8 - 23.6 = 24.5$

For the sequence $S_4 = (\alpha, 4, 1, 3)$ or $(2, 5, 4, 1, 3)$

Total waiting time

$T_w = 5 \times 4.4 + 11.2 + 10.5 + 2.2 + 1.2 - 23.6 = 23.5$

Hence the sequence $S_1 = (4, 1, 3, 2,5)$ is the required sequence which minimizes total waiting time for the said problem.

CONCLUSION

The present study deals with the flow-shop scheduling model with the main idea to optimize the total waiting time of jobs. However, it may increase the other costs like machine idle cost or penalty cost of the jobs, yet the idea of minimizing the waiting time is a matter of importance that cannot be avoided in the cases when there is a minimum time contract with the customers. The study can be extended by adding various parameters like weights of jobs, setup time of machines, breakdown interval of machines etc. Many more concepts fuzzy scheduling, Scheduling vs Rescheduling can be a subject of review & study as mentioned in [13].

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SOME NON-UNIQUE FIXED POINT THEOREMS OF ĆIRIĆ TYPE ON PARAMETRIC B-METRIC SPACES

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ABSTRACT :

In this paper, we establish some fixed point theorem for some non-unique fixed point theorems of ?iri? type on parametric b-metric space, inspired by the interesting results of Ćirić [2]. We investigate some existing non unique fixed points of certain operators in the context of parametric b-metric spaces. Our main result generalizes and unifies several existing results in the literature.

Keywords: Fixed point, parametric metric spaces, parametric b-metric spaces, self map.

INTRODUCTION

With celebrated Banach's fixed point theorem, since 1922 fixed point theory has fascinated many researchers. Being very active field of research, at present a vast literature is present on fixed point. Fixed point theorems are important tool for proving existence and uniqueness of various mathematical model solutions.

The importance of notion of non unique fixed point was given by Ćirić in 1974. For certain operators possessing non unique fixed point he proposed a criteria and inspired by the work of Ćirić many authors and researchers have worked on non unique fixed point of the operators fulfilling different conditions, for more see [1,5,8, 10-12, 14].

The concept of metric space is generalized by many authors and Czerwik [4] gave notion of a b-metric. Hussain et al. [6,7] by generalizing metric space gave notion of parametric metric space and introduced parametric b-metric space by generalizing both metric and b-metric space.

In this paper, inspired by the interesting results of Ćirić [2], we presented some fixed point theorems for non unique fixed point theorems of Ćirić type on parametric b-metric space by investigating some existing non unique fixed points of certain operators in the context of parametric b-metric spaces.

DEFINITION AND PRELIMINARIES

Definition 2.1[3] Let X be a nonempty set and $\mathcal{P}: X \times X \times (0, +\infty) \to [0, +\infty)$ be a function. We say $\mathcal{P}$ is a parametric metric on Xif,

(1) $\mathcal{P}(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$;

(2) $\mathcal{P}(x, y, t) = \mathcal{P}(y, x, t)$ for all $t > 0$;

(3) $\mathcal{P}(x,y,t) \leq \mathcal{P}(x,z,t) + \mathcal{P}(z,y,t)$ for all $x,y,z \in X$ and all $t > 0$:
and one says the pair $(X,\mathcal{P})$is a parametric metric space.

Definition 2.2[3] Let X be a nonempty set, $s \geq 1$ be a real number and $\mathcal{P}: X \times X \times (0,+\infty) \to [0,+\infty)$be a function. We say $\mathcal{P}$ is a parametric b-metric on X if,
(1) $\mathcal{P}(x,y,t) = 0$ for all $t > 0$ if and only if $x = y$;
(2) $\mathcal{P}(x,y,t) = \mathcal{P}(y,x,t)$ for all $t > 0$;
(3) $\mathcal{P}(x,y,t) \leq s[\mathcal{P}(x,z,t) + \mathcal{P}(z,y,t)]$ for all $x,y,z \in X$ and all $t > 0$, where $s \geq 1$,and one says the pair$(X,\mathcal{P})$is a parametric metric space with parameter $s \geq 1$.
Obviously, for $s = 1$, parametric b-metric reduces to parametric metric.

Definition 2.3[7] Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric b-metric space $(X,\mathcal{P},s)$.
(1) Let $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $lim_{n\to\infty}x_n = x$, for all $t > 0$, if $lim_{n\to\infty}\mathcal{P}(x_n,x,t) = 0$.
(2) $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in X if for all $t > 0$, for all if $lim_{n,m\to\infty}\mathcal{P}(x_n,x_m,t) = 0$.
(3) $(X,\mathcal{P},s)$ is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 2.4[13, 16] A function$\phi : [0,\infty) \to [0,\infty)$is called a comparison function if it is increasing and $\phi^n(t) \to 0$ as $n \to 0$for every$t \in [0,\infty)$where $\phi^n$is the n-th iterate of ϕ.
One can find properties and examples of comparison functions in [13,16].The following Lemma gives an important property of comparison functions.

Lemma 2.5[13, 16] If $\phi : [0,\infty) \to [0,\infty)$is a comparison function, then
(1) each iterate ϕ^k of $\phi, k \geq 1$ is also a comparison function;
(2) ϕ is continuous at 0;
(3) $\phi(t) < t$for all $t > 0$.

Lemma 2.6[15] Let$\phi : [0,\infty) \to [0,\infty)$ be a (b)-comparison function. Then,
(1) the series $\sum_{k=0}^{\infty} s^k\phi^k(t)$converges for any $t \in [0,\infty)$;
(2) the function $b_s: [0,\infty) \to [0,\infty)$defined as $b_s = \sum_{k=0}^{\infty} s^k\phi^k(t)$ is increasing and continuous at $t = 0$.

Remark 2.7 By the Lemma 2.6, we conclude that every (b)-comparison function is a comparison function and hence, by the Lemma 2.5, any (b)-comparison function ϕsatisfies$\phi(t) < t$.

Definition 2.8 [10] A mapping Ton bMS $(X,d)$is said to be orbitally continuous if $lim_{i\to\infty}T^{n_i}(x) = z$implies $lim_{i\to\infty}T(T^{n_i}(x)) = Tz$.A$bMS$ $(X,d)$is called T-orbitally complete if every Cauchy sequence of the form $\{T^{n_i}(x)\}_{i=1}^{\infty}, x \in X$converges in (X,d).

MAIN RESULTS

Theorem 3.1: Let T be an orbitally continuous self-map on the T-orbitally complete Parametric b-metric space$(X,\mathcal{P},s)$. If there is $\phi \in \Phi$ such that

$$\min\{d(Tp,Tq),d(p,Tp),d(q,Tq)\} - \min\left\{d(p,Tq),d(Tp,q),\left(1+\frac{d(Tp,Tq)}{d(q,Tq)}\right)\right\}$$
$$\leq \phi\,(d(p,q)) \qquad (1)$$

for all $p,q \in X$, then for each $p_0 \in X$ the sequence $\{T^n p_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Proof: Let for any $p \in X$, we construct a sequence $\{p_n\}$ such that

$p_0 = p, p_n = Tp_{n-1}, \forall\, n \in \mathbb{N}$(2)

and let us suppose $p_n \neq p_{n-1}\ \forall\, n \in \mathbb{N}$.(3)

Now if we have $p_n = Tp_{n-1} = p_{n-1}$, then the proof is obvious. Now substituting $p = p_{n-1}$ and $q = p_n$ in inequality (1), we have

$$\min\{d(Tp_{n-1}, Tp_n), d(p_{n-1}, Tp_{n-1}), d(p_n, Tp_n)\} - \min\left\{d(p_{n-1}, Tp_n), d(Tp_{n-1}, p_n), \left(1 + \frac{d(Tp_{n-1}, Tp_n)}{d(p_n, Tp_n)}\right)\right\} \leq \phi\big(d(p_{n-1}, p_n)\big). \quad (4)$$

It implies

$$\min\{d(p_n, p_{n+1}), d(p_n, p_{n-1})\} \leq \phi\big(d(p_{n-1}, p_n)\big). \quad (5)$$

Since $\phi(t) < t, \forall\, t > 0$, the case $d(p_n, p_{n-1}) \leq \phi\,(d(p_{n-1}, p_n))$ is not possible. Hence taking

$$d(p_n, p_{n+1}) \leq \phi\big(d(p_{n-1}, p_n)\big). \quad (6)$$

By Remark 2.7, we have

$$d(p_n, p_{n+1}) \leq \phi\big(d(p_{n-1}, p_n)\big) \leq \phi^2\big(d(p_{n-2}, p_{n-1})\big) \leq \cdots \leq \phi^n\big(d(p_0, p_1)\big). \quad (7)$$

By lemma 2.6, we have

$$\lim_{n\to\infty} d(p_{n+1}, p_n) = 0 \quad (8)$$

Following this we can show that $\{p_n\}$ is a Cauchy Sequence.

Let us consider $d(p_n, p_{n+k})$ for $k \geq 1$. On repeatedly using triangle inequality, we get

$$\begin{aligned} d(p_n, p_{n+k}) &\leq s[d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+k})] \\ &\leq sd(p_n, p_{n+1}) + s\{s[d(p_{n+1}, p_{n+2}) + d(p_{n+2}, p_{n+k})]\} \\ &= sd(p_n, p_{n+1}) + s^2 d(p_{n+1}, p_{n+2}) + s^2 d(p_{n+2}, p_{n+k}) \\ &\ \ \vdots \\ &\leq sd(p_n, p_{n+1}) + s^2 d(p_{n+1}, p_{n+2}) + \cdots \\ &\quad + s^{k-1} d(p_{n+k-2}, p_{n+k-1}) + s^{k-1} d(p_{n+k-1}, p_{n+k}) \end{aligned} \quad (9)$$

As $s \geq 1$, from (7)and (9) we have

$$\begin{aligned} d(p_n, p_{n+k}) &\leq s\phi^n\big(d(p_0, p_1)\big) + s^2\phi^{n+1} d(p_0, p_1) + \cdots + s^{k-1}\phi^{n+k-2} d(p_0, p_1) + s^k \phi^{n+k-1} d(p_0, p_1) \\ &= \frac{1}{s^{n-1}}[s^n\phi^n\big(d(p_0, p_1)\big) + s^{n+1}\phi^{n+1}\big(d(p_0, p_1)\big) + \cdots \\ &\quad + s^{n+k-2}\phi^{n+k-2}\big(d(p_0, p_1)\big) + s^{n+k-1}\phi^{n+k-1}\big(d(p_0, p_1)\big) \end{aligned} \quad (10)$$

It implies

$$d(p_n, p_{n+k}) \leq \frac{1}{s^{n-1}}[R_{n+k-1} - R_{n-1}], \qquad n \geq 1, k \geq 1, \quad (11)$$

here $R_n = \sum_{j=0}^{n} s^j \phi^j\big(d(p_0, p_1)\big), n \geq 1$. From Lemma 2.6, series $\sum_{j=0}^{n} s^j \phi^j\big(d(p_0, p_1)\big)$ is convergent and since $s \geq 1$, in (12)on taking limit $n \to \infty$ we have

$$\lim_{n\to\infty} d(p_n, p_{n+k}) \leq \frac{1}{s^{n-1}}[R_{n+k-1} - R_{n-1}] = 0. \quad (12)$$

Hence sequence $\{p_n\}$ is Cauchy in $(X, \mathcal{P})$.

As we have constructed $x_n = T^n x_0$ and as $(X, \mathcal{P})$ is$T$- orbitally complete, there is $w \in X$ such that $p_n \to w$. As T is orbital continuous, then $p_n \to Tz$. Hence $w = Tw$, this completes proof.

Corollary 3.2 [Non-unique fixed point theorem of Ćirić [8]] Let T be an orbitally continuous self-map on the T-orbitally complete standard metric space (X, d). If there is $k \in [0,1)$ such that

$\min\{d(Tx,Ty),d(x,Tx),d(y,Ty)\}-\min\{d(x,Ty),d(Tx,y)\}\le kd(x,y)$, for all $x,y\in X$, then for each $x_0\in X$ the sequence $\{T^nx_0\}_{n\in\mathbb{N}}$ converges to a fixed point of T.

Theorem 3.3: On the T- orbitally complete Parametric b-metric space $(X,\mathcal{P},s)$, let T be an orbitally continuous self-map. Suppose there exist a self mapping $T:X\to X$ and m_1,m_2,m_3,m_4,m_5 be real numbers satisfying conditions

$$m_1+m_2+m_3\ge 1, m_3-m_5\ge 0, m_1+m_2\ne 0 \text{ and } 0\le\frac{m_4-m_2}{m_1+m_2}<1 \quad (13)$$

$$m_1d(Tp,Tq)+m_2[d(p,Tp)+d(q,Tq)]+m_3[d(q,Tp)+d(p,Tq)] \le m_4d(p,q)+m_5d(p,T^2p) \quad (14)$$

for all $p,q\in X$, then there is atleast one fixed point in T.

Proof: Let for any $p_0\in X$, we construct a sequence $\{p_n\}$ such that

$$p_{n+1}=Tp_n \qquad n=0,1,2 \quad (15)$$

On taking $p=p_n$ and $q=p_{n+1}$, from equation (14) we have

$$m_1d(Tp_n,Tp_{n+1})+m_2[d(p_n,Tp_n)+d(p_{n+1},Tp_{n+1})]+m_3[d(p_{n+1},Tp_n)+d(p_n,Tp_{n+1})] \le m_4d(p_n,p_{n+1})+m_5d(p_n,T^2p_n) \quad (16)$$

From (15) we have

$$m_1d(p_{n+1},p_{n+2})+m_2[d(p_n,p_{n+1})+d(p_{n+1},p_{n+2})]+m_3[d(p_{n+1},p_{n+1})+d(p_n,p_{n+2})] \le m_4d(p_n,p_{n+1})+m_5d(p_n,p_{n+2}) \quad (17)$$

$$m_1d(p_{n+1},p_{n+2})+m_2d(p_n,p_{n+1})+m_2d(p_{n+1},p_{n+2})+m_3d(p_{n+1},p_{n+1})+m_3d(p_n,p_{n+2}) \le m_4d(p_n,p_{n+1})+m_5d(p_n,p_{n+2})$$

$$(m_1+m_2)d(p_{n+1},p_{n+2})+(m_3-m_5)d(p_n,p_{n+2})\le(m_4-m_2)d(p_n,p_{n+1}) \quad (18)$$

It implies

$$d(p_{n+1},p_{n+2})\le hd(p_n,p_{n+1}) \quad (19)$$

Here $h=\frac{m_4-m_2}{m_1+m_2}$. From (13) we have $0\le h<1$, hence from above equation

$$d(p_n,p_{n+1})\le hd(p_{n-1},p_n)\le h^2d(p_{n-2},p_{n-1})\le\cdots h^nd(p_0,p_1) \quad (20)$$

Now we prove that $\{p_n\}_{n\in\mathbb{N}}$ is Cauchy sequence

$$\begin{aligned} d(p_n,p_{n+m}) &\le sd(p_n,p_{n+1})+s^2d(p_{n+1},p_{n+2})+\cdots+s^{m-1}d(p_{n+m-2},p_{n+m-1}) \\ &\quad +s^md(p_{n+m-1},p_{n+m}) \\ &\le sh^nd(p_0,p_1)+s^2h^{n+1}d(p_0,p_1)+\cdots+s^{m-1}h^{n+m-2}d(p_0,p_1) \\ &\quad +s^mh^{n+m-1}d(p_0,p_1) \\ &\le \frac{1}{s^nh}[s^{n+1}h^{n+1}d(p_0,p_1)+\cdots+s^{n+m-1}h^{n+m-1}d(p_0,p_1) \\ &\quad +s^{n+m}h^{n+m}d(p_0,p_1)] \\ &= \frac{1}{s^nh}\sum_{j=n+1}^{n+m}s^jh^j\,d(p_0,p_1) \end{aligned}$$

$$< \frac{1}{s^n h} \sum_{j=n+1}^{\infty} s^j h^j \, d(p_0, p_1).$$

Resulting inequality is

$$d(p_n, p_{n+m}) \le \frac{1}{s^n h} \sum_{j=n+1}^{\infty} s^j h^j \, d(p_0, p_1) \to 0 \; as \; n \to \infty.$$

Thus proved that$\{p_n\}_{n\in\mathbb{N}}$ is Cauchy sequence.

As from Theorem 3.1, $p_n = T^n p_0$ and $(X, \mathcal{P}, s)$ is T- orbitally complete, for $w \in X, p_n \to w$. Again T being orbital continuous we find that $p_n \to Tw$. Hence $w = Tw$.

Corollary 3.4(See [9]) Let T beanorbitallycontinuousself-maponthe T -orbitallycompletestandardmetric space (X, d). Suppose there exist real numbersa_1, a_2, a_3, a_4, a_5 and a self mapping $T: X \to X$ satisfies the conditions

$$0 \le \frac{a_4 - a_2}{a_1 + a_2} < 1, a_1 + a_2 \ne 0, a_1 + a_2 + a_3 > 0, and \; 0 \le a_3 - a_5 \tag{21}$$

$$\begin{aligned} &a_1 d(Tx, Ty) + a_2[d(x, Tx) + d(y, Ty)] + a_3[d(y, Tx) + d(x, Ty)] \\ &\le a_4 d(x, y) + a_5 d(x, T^2 y) \end{aligned} \tag{22}$$

hold for all $x, y \in X$. Then, Thas at least one fixed point.

CONCLUSION

Thus we have proved some fixed point theorems fornonunique fixed point theorems of Ćirić type on parametric b-metric space, which is generalization of the existing results.

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CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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SIMULATION AND ANALYSIS OF QUADRATIC ASSIGNMENT PROBLEMS (QAP) USING ANT COLONY OPTIMIZATION

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ABSTRACT :
It is the human nature to look for proficient and effectual optimization of any problem. This paper presents optimization of quadratic assignment problem. This article describes ant colony optimization for better results in QAP. The algorithm is based on the development of a mathematical model to tackle the problems of allocation of distributors and wholesalers/retailers. The Simulation is done on 15 distributors and 15 wholesalers/retailers and a matrix of 15 X 15 has been prepared for the distances between them. MATLAB is used for the design of GUI which is helpful for the Assignment Problem. Both situations of Assignment Problem i.e. Maximization or Minimization have been covered in this paper. The results have been tested for Laxmi Plywood data.

INTRODUCTION

From the primordial times, each organization whether big or small has been burning midnight oil to unearth the best achievable results. The well known quantitative model for assigning workers to jobs is the assignment model, which has been utilized for a long time in numerous scheduling applications.

Initially, the Quadratic Assignment Problem was used to model a location problem in 1950. What attributes it to be perfect choice for research work is that back then too it was computationally very difficult to solve. The Quadratic Assignment Problem (QAP) is a standard problem in location theory. It was introduced by Koopmans and Beckmann in 1957 and is a model for many practical problems.

The QAP is a NP-hard optimization problem. While some NP-hard combinatorial optimization problems can be solved exactly even for relatively large instances, as exemplified by the traveling salesman problem (TSP) The QAP is considered as one of the hardest optimization problems, because exact algorithms show a very poor performance on it. Therefore, several heuristics have been proposed for finding near-optimum solutions for large QAP instances. The outline of this paper is to describe Swarm Intelligence as an adaptive learning tool for the QAP. To test the optimization procedure, the proposed algorithm is implemented in MATLAB and GUI (Graphic User Interface) is designed. The results of the quadratic problem are found to be very encouraging.

Section II of this research paper, gives a glimpse of the Swarm intelligence, Section III deals with brief introduction to Ant colony optimization. Section IV & V throws light on the problem formulation and methodology used to solve the problem and obtain the best possible results. Laxmi Plywood, Yamuna Nagar data has been used to verify the proposed technique and simulation results of experiments are illustrated in Section VI. The conclusion and future scope is discussed in end of this paper.

SWARM INTELLIGENCE

An Artificial Intelligence technique which lays emphasis on studying the collective behavior of a decentralized system made up by a population of simple agents interacting locally with each other and with the environment is known as SWARM INTELLIGENCE. Although there is typically no centralized control dictating the behavior of the agents, local interactions among the agents often cause a global pattern to emerge.

Most successful two types of swarm intelligence techniques currently in existence are Ant Colony Optimization (ACO) and Particle Swarm Optimization (PSO). ACO is a metaheuristic technique that can be used to calculate approximate solutions to various types of combinatorial optimization problems.

The QAP (Quadratic Assignment Problem) has been put forth as one of the mainstream, eye-catching, fascinating and the most demanding combinatorial optimization problems by Commander in 2005. Grosan (2009) presented a novel meta-heuristic technique for job scheduling on computational grids. Leandro et al. (2012) concluded that the term “swarm intelligence” is used to describe algorithms and distributed problem solvers inspired by the collective behavior of insect colonies and other animal societies. The basic drawback of preceding design methods is that the large amount of computation time is required.

ANT COLONY OPTIMIZATION

An exhaustive portrayal of the foraging behavior of ants is illustrated in figure 1. In first step, three ants leave their nest in random directions to search for food. During wander around, they deposit certain amount of pheromone trails, which will evaporate slowly but are detectable by other ants.

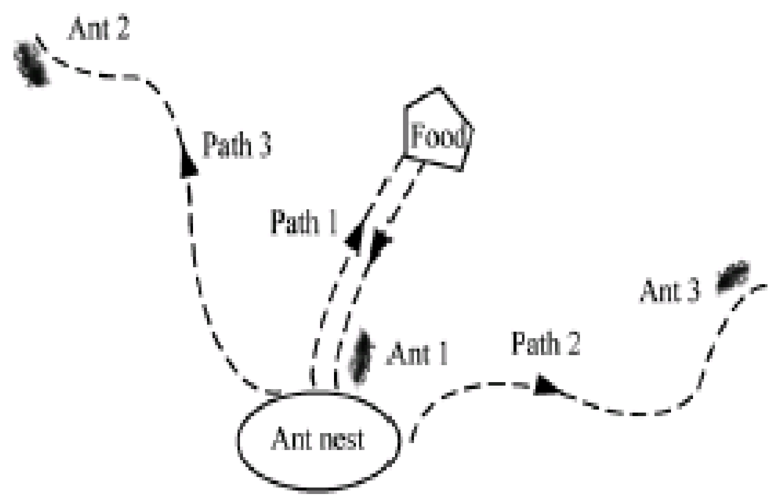

Fig. 1: Rummaging behavior of ants.

It is presumed that *Ant* 1 locates a food source. *Ant 1,* then goes, picks up some food and proceeds back to the nest, following its own pheromone trail, along with laying additional pheromone on the same path while *Ant*2 and *Ant* 3 are till now moving arbitrarily. When the next group of ants leaves their nest to search for food, they detect twice as much pheromone on *Path* 1 than on *Path* 2 and *Path* 3, assuming the evaporation of pheromone is negligible. Since the probability for a path to be followed is proportional to its pheromones value, more ants will follow *Path* 1 in this second round of search for food. In this way, the ants can establish the optimized path from their colony to the feeding sources.

An apt example of meta-heuristic algorithms used for combinatorial optimization problems is Ant Colony Optimization (ACO). Ant Colony Optimization (ACO) is a meta-heuristic in which a colony of artificial ants cooperates in finding good solution to difficult discrete optimization problems. ACO algorithms have the few desirable characteristics such as versatile, robustness and population-based heuristic. The artificial ant colonies will have some major differences with a real (natural) like artificial ants have some memory. They are not completely blind and can live in an environment where time is discrete.

Ant Colony Optimization (ACO) can be utilized to solve both static and dynamic combinatorial optimization problems.. Static problems are those in which the characteristics of the problem are given once and for all when the problem is defined and don't changes while the problem are solved. A paradigmatic example for such problem is Traveling salesman problem (TSP) (Dorigo and Gambardella, 1997). On the contrary, dynamic problems are defined as a function of some quantities whose value is set by the dynamics of an underlying system. The problem instance changes therefore at run time and algorithm must be capable of adapting on line to change environment. An example of this situation is network routing problems.

By and large, an ACO algorithm is likely considered to be interplay of three procedures: Construct Ants Solutions, Update Pheromones, and Daemon Actions. Construct Ants Solutions manages a colony of ants that concurrently and asynchronously visit adjacent states of the considered problem by moving through neighbor nodes of the problem's construction graph G. Update Pheromones is the second stage in which the pheromone trails are modified. The trails value can either increase, as ants deposit pheromone on the components of connections they use, or decrease, due to pheromone evaporation. Finally, the Daemon Actions procedure is used to implement centralized actions which cannot be performed by single ants.

PROBLEM FORMULATION

This paper deals with devising of a mathematical model to embark upon the problem of allocation of distributors and wholesalers/retailers in an optimized way so as to make the distributors more streamlined to steer clear of any chaos created due to wrong allocations and to maximize salesmen profits. Laxmi Plywood, Yamuna Nagar, is constantly facing this problem which in turn consumes lot of valuable manpower and management's precious time due to improper allocation because of non-availability of any such system in the Laxmi Plywood. Laxmi Plywood has number of distributors in New Delhi who are supplying different categories of papers to different wholesalers/retailers.

Simulation is done on 15 distributors and 15 wholesalers/retailers and a matrix of 15 X 15 has been prepared for the distances between them.

METHODOLOGY

Assignment model can be expressed as $x_{ij} = \begin{cases} 0, \textit{if the } i^{th} \textit{ facility is not assigned to } j^{th} \textit{ job}, \\ 1, \textit{if the } i^{th} \textit{ facility is assigned to } j^{th} \textit{ job.} \end{cases}$

Then, the model is given by minimize Z

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

subject to constraints $\sum_{j=1}^{n} x_{ij} = 1,\ i = 1,2,3,.......,n,$ (available or supply constraints)

$\sum_{i=1}^{n} x_{ij} = 1,\ j = 1,2,3,.......,n,$ (requirements constraints)

and $x_{ij} = 0$ or 1.

If the last condition is replaced by $x_{ij} \geq 0$, the transportation model with all requirements and available resources will be equal to 1. The methodology is illustrated as given below:-

STEP 1: Prepare matrix & Set Parameters as per data.
STEP 2: Subtract minimum row element from all the elements of row.
STEP 3: Subtract minimum column element from all the elements of column
STEP 4: Evaluate *J(w)*as per fitness function:

$$J(w) = \frac{1}{N} \sum_{k=1}^{N} \left(d(k) - y(k) \right)^2$$

where *d(k)* and *y(k)* are the desired and actual responses.
STEP 5: Applied the ant search behavior
STEP 6: Calculate Pheromones Trail Evaporation & Daemon actions
STEP 7: Update the pheromone factor

SIMULATION RESULTS

The Screen shot of GUI designed in MATLAB is shown below in figure 2.

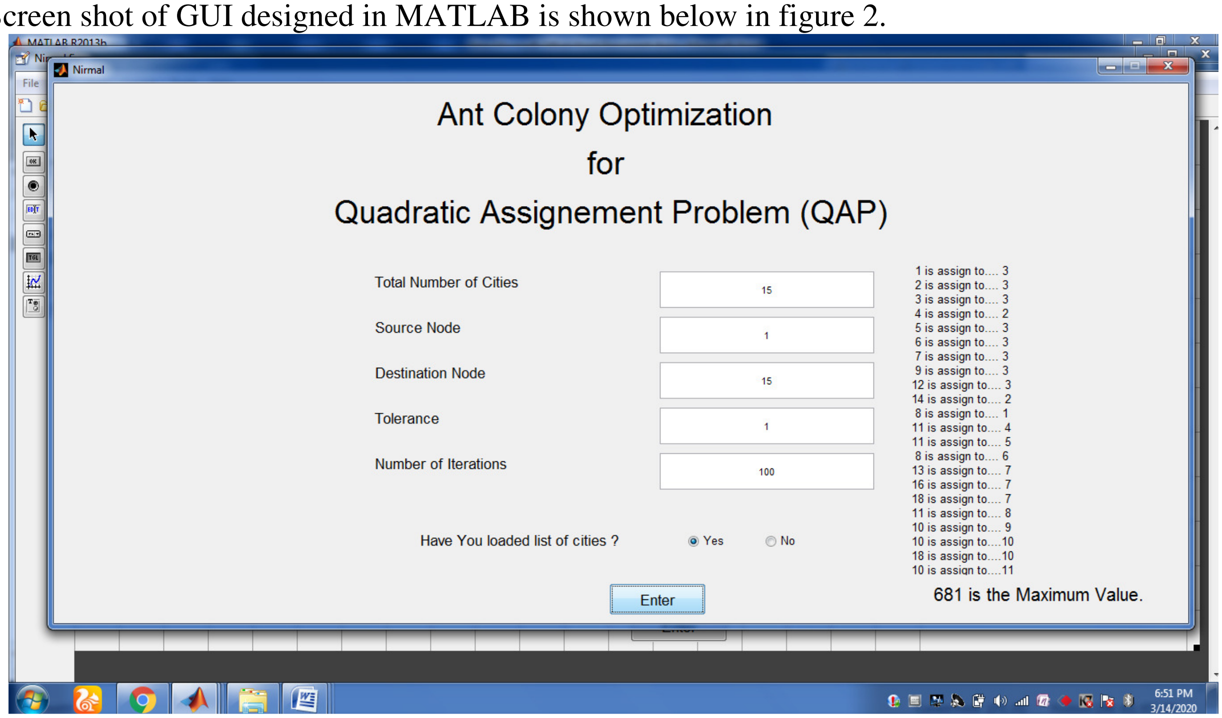

Fig. 2: Simulation results through GUI designed

CONCLUSION

In Assignment Problem there are two situations either Maximization or Minimization. The GUI is designed for the Assignment Problem. In this paper, the simulation has been done for Laxmi Plywood data. In the minimization problem, the minimum distance from fifteen different distributors to fifteen different wholesalers/retailers comes out to be 681 Km.. The algorithm assigned the best suitable retailer to each distributor because this will minimize the distance by approximately 20% as compared to the distance taken from random movement.

In the maximization problem, the maximum profit of eight salesmen from eight different areas turns out to be about Rs. 7500. The algorithm assigned the best suitable location to each salesman to achieve the maximum possible profit giving 10% hike as compared to the average profit.

FUTURE SCOPE

The work can be further carried out in the few areas such as Bacterial Forging Algorithm (BFA), Cat Algorithm Techniques (CAT) etc.. These types of evolutionary methods can be applied over this work for better results. The optimization can be done to test convergence.

Further, the hybridization of two algorithms can be done for better results but computation time has to be kept in mind, while designing these types of algorithms.

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SUMMATION FORMULAE INVOLVING GENERALIZED I–FUNCTION OF TWO VARIABLES

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ABSTRACT :
The aim of this paper is to derive some summation formulae involving generalized I–function of two variables.

INTRODUCTION:

The I–function of two variables introduced by Sharma & Mishra [3], will be defined and represented as follows:

$$I\begin{bmatrix}x\\y\end{bmatrix} = I^{0,n:m_1,n_1:m_2,n_2}_{p_i,q_i:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}[(a_j:\alpha_j,A_j)_{1,n}],[(a_{ji}:\alpha_{ji},A_{ji})_{n+1,p_i}]\\ [(b_{ji}:\beta_{ji},B_{ji})_{1,q_i}]\end{matrix}\right.$$

$$\left.\begin{matrix}:[(c_j;\gamma_j)_{1,n_1}],[(c_{ji'};\gamma_{ji'})_{n_1+1,p_{i'}}];[(e_j;E_j)_{1,n_2}],[(e_{ji''};E_{ji''})_{n_2+1,p_{i''}}]\\ :[(d_j;\delta_j)_{1,m_1}],[(d_{ji'};\delta\gamma_{ji'})_{m_1+1,q_{i'}}];[(f_j;F_j)_{1,m_2}],[(f_{ji''};F_{ji''})_{m_2+1,q_{i''}}]\end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\,x^{\xi}y^{\eta}d\xi d\eta, \qquad (1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\sum_{i=1}^{r}[\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi-A_{ji}\eta)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi+B_{ji}\eta)]},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_1}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_1}\Gamma(1-c_j+\gamma_j\xi)}{\sum_{i'=1}^{r'}[\prod_{j=m_1+1}^{q_{i'}}\Gamma\left(1-d_{ji'}+\delta_{ji'}\xi\right)\prod_{j=n_1+1}^{p_{i'}}\Gamma\left(c_{ji'}-\gamma_{ji'}\xi\right)]},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_2}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_2}\Gamma(1-e_j+E_j\eta)}{\sum_{i''=1}^{r''}[\prod_{j=m_2+1}^{q_{i''}}\Gamma(1-f_{ji''}+F_{ji''}\eta)\prod_{j=n_2+1}^{p_{i''}}\Gamma(e_{ji''}-E_{ji''}\eta)]},$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, $p_{i'}$, $p_{i''}$, q_i, $q_{i'}$, $q_{i''}$, n, n_1, n_2, n_j and m_k are non negative integers such that $p_i \geq n \geq 0$, $p_{i'} \geq n_1 \geq 0$, $p_{i''} \geq n_2 \geq 0$, $q_i > 0$, $q_{i'} \geq 0$, $q_{i''} \geq 0$, (i = 1, …, r; i′ = 1, …, r′; i′′ = 1, …, r′′; k = 1, 2) also all the A's, α's, B's, β's, γ's, δ's, E's and F's are assumed to be positive quantities for standardization purpose; the definition of I-function of two variables given above will however, have a meaning even if some of these quantities are zero. The contour L_1 is in the ξ–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j\xi)$ (j = 1, ..., m_1) lie to the right, and the poles of $\Gamma(1-c_j+\gamma_j\xi)$ (j = 1, ..., n_1), $\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$ (j = 1, ..., n) to the left of the contour.

The contour L_2 is in the η–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j-F_j\eta)$ (j=1,....., n_2) lie to the right, and the poles of $\Gamma(1-e_j+E_j\eta)$ (j = 1, ..., m_2), $\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$ (j = 1, ..., n) to the left of the contour. Also

$$R'=\sum_{j=1}^{p_i}\alpha_{ji}+\sum_{j=1}^{p_i'}\gamma_{ji'}-\sum_{j=1}^{q_i}\beta_{ji}-\sum_{j=1}^{q_i'}\delta_{ji'}<0,$$

$$S'=\sum_{j=1}^{p_i}A_{ji}+\sum_{j=1}^{p_i''}E_{ji''}-\sum_{j=1}^{q_i}B_{ji}-\sum_{j=1}^{q_i''}F\delta_{ji'}<0,$$

$$U=\sum_{j=n+1}^{p_i}\alpha_{ji}-\sum_{j=1}^{q_i}\beta_{ji}+\sum_{j=1}^{m_1}\delta_j-\sum_{j=m_1+1}^{q_i'}\delta_{ji'}+\sum_{j=1}^{n_1}\gamma_j-\sum_{j=n_1+1}^{p_i'}\gamma_{ji'}>0, \tag{2}$$

$$V=-\sum_{j=n+1}^{p_i}A_{ji}-\sum_{j=1}^{q_i}B_{ji}-\sum_{j=1}^{m_2}F_j-\sum_{j=m_{21}+1}^{q_i''}F_{ji''}+\sum_{j=1}^{n_2}E_j-\sum_{j=n_2+1}^{p_i''}E_{ji''}>0, \tag{3}$$

and $|\arg x| < \frac{1}{2}U\pi$, $|\arg y| < \frac{1}{2}V\pi$.

RESULT REQUIRED:

The following results are required in our present investigation:

From Rainvile [1]:

$$\Gamma(a+k)=(a)_k\,\Gamma(a), \tag{4}$$

$$\sum_{k=0}^{\infty}\frac{(a)_k(b)_k(1)^k}{(c)_k k!}={}_2F_1\left[{a,b \atop c}\ ;1\right]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}. \tag{5}$$

MAIN RESULT:

In this paper we will establish the following summation formulae:

$$\sum_{k=0}^{\infty}\frac{(a_1')_k(a_2')_k}{k!}\times$$

$$\times\, I^{0,\ n\ :m_1,n_1:m_2,n_2}_{p_i,q_i+1:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[{x \atop y}\middle|{\ldots,\ldots:\ldots\ldots,\ldots:\ldots\ldots\ldots,\ldots\ldots\ldots \atop \ldots,\ldots\ldots,(w_1-k;\omega_1,W_1):\ldots\ldots,\ldots:\ldots\ldots\ldots,\ldots\ldots\ldots}\right]$$

$$=I^{0,\ n+1\ :m_1,n_1:m_2,n_2}_{p_i+1,q_i+2:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[{x \atop y}\middle|{(w_1+a_1'+a_2';\omega_1,W_1),\ldots:\ldots\ldots,\ldots:\ldots\ldots\ldots,\ldots\ldots\ldots \atop \ldots\ldots,(w_1+a_1';\omega_1,W_1),(w_1+a_2';\omega_1,W_1):\ldots\ldots,\ldots:\ldots\ldots\ldots,\ldots\ldots\ldots}\right] \tag{6}$$

where $|\arg x| < \frac{1}{2}U\pi$, $|\arg y| < \frac{1}{2}V\pi$, where U and V are given in (2) and (3) respectively.

$$\sum_{k=0}^{\infty}\frac{(a_1')_k}{k!}\times$$

$$\times\, I^{0,\ n+1\ :m_1,n_1:m_2,n_2}_{p_i+1,q_i+1:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[{x \atop y}\middle|{(u_1-k;\rho_1,U_1),\ldots:\ldots\ldots,\ldots:\ldots\ldots\ldots,\ldots\ldots\ldots \atop \ldots,\ldots\ldots,(w_1-k;\omega_1,W_1):\ldots\ldots,\ldots:\ldots\ldots\ldots,\ldots\ldots\ldots}\right]$$

$$=I^{0,\ n+2\ :m_1,n_1:m_2,n_2}_{p_i+2,q_i+2:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[{x \atop y}\middle|\right.$$

$$\left.\begin{matrix}(u_1;\rho_1,U_1),(1+w_1+a_1'-u_1;\omega_1-\rho_1,W_1-U_1),\ldots:\ldots\ldots,\ldots:\ldots\ldots\ldots,\ldots\ldots\\ \ldots\ldots,(w_1+a_1';\omega_1,W_1),(1+w_1-u_1;\omega_1-\rho_{1_1},W_1-U_1):\ldots\ldots,\ldots:\ldots\ldots\ldots,\ldots\ldots\end{matrix}\right] \tag{7}$$

Where $|\arg x| < \frac{1}{2}U\pi$, $|\arg y| < \frac{1}{2}V\pi$, where U and V are given in (2) and (3) respectively.

Proof:

To establish (6), we use for the I-function of two variables Mellin-Barnes types of contour integral as given in (1), on the left-hand side of (6), change the order of integration and summation (which is justified under the conditions given with (6)), we then obtain

Left-hand side of (6)

$$\sum_{k=0}^{\infty}\frac{(a_1')_k(a_2')_k}{k!}\times$$

$$\times I^{0,\ n\ :m_1,n_1:m_2,n_2}_{p_i,q_i+1:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}\ldots,\ldots:\ldots\ldots,\ldots:\ldots\ldots\ldots,\ldots\ldots\\ \ldots,\ldots\ldots,(w_1-k;\omega_1,W_1):\ldots\ldots,\ldots:\ldots\ldots\ldots,\ldots\ldots\end{matrix}\right]$$

$$=\frac{(-1)}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\left[\sum_{k=0}^{\infty}\frac{(a_1')_k(a_2')_k}{k!\Gamma[1-(w_1-k)+\omega_1\xi+W_1\eta]}\right]x^{\xi}y^{\eta}d\xi d\eta$$

Now using the results (4) and (5), we obtain

$$=\frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\frac{\Gamma[1-(w_1+a_1'+a_2')+\omega_1\xi+W_1\eta]}{\Gamma[1-(w_1+a_1')+\omega_1\xi+W_1\eta]\Gamma[1-(w_1+a_2')+\omega_1\xi+W_1\eta]}x^{\xi}y^{\eta}d\xi d\eta$$

Now interpreting the above result as the I-function of two variables, we once get the right-hand side of (6). Proceeding on the similar way, the result (7) can be obtained.

4. Special Cases:

On choosing r = 0, r' = 0, r'' = 0 in main results, we get following partial derivatives in terms of H-function of two variables given in [2, (7.6.2) to (7.6.3), pp. 114]:

$$\sum_{k=0}^{\infty}\frac{(a_1')_k(a_2')_k}{k!}\times$$

$$\times H^{0,n_1;m_2,n_2;m_3,n_3}_{p_1,q_1+1;p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(w_1-k;\omega_1,W_1):(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right]$$

$$=H^{0,n_1+1;m_2,n_2;m_3,n_3}_{p_1+1,q_1+2;p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(w_1+a_1'+a_2';\omega_1,W_1),(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3}\\(b_j,\beta_j;B_j)_{1,q_1},(w_1+a_1';\omega_1,W_1),(w_1+a_2';\omega_1,W_1):(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3}\end{matrix}\right] \tag{8}$$

$$\sum_{k=0}^{\infty} \frac{(a_1')_k}{k!} \times$$

$$\times\, H^{0,n_1+1;m_2,n_2;m_3,n_3}_{p_1+1,q_1+1;p_2,q_2;p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (u_1-k;\rho_1,U_1),(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(w_1-k;\omega_1,W_1):(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3} \end{matrix}\right]$$

$$= H^{0,n_1+2;m_2,n_2;m_3,n_3}_{p_1+2,q_1+2;p_2,q_2;p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\right|$$

$$\left.\begin{matrix} (u_1;\rho_1,U_1),(1+w_1+a_1'-u_1;\omega_1-\rho_1,W_1-U_1),(a_j,\alpha_j;A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2}:(e_j,E_j)_{1,p_3} \\ (b_j,\beta_j;B_j)_{1,q_1},(w_1+a_1';\omega_1,W_1),(1+w_1-u_1;\omega_1-\rho_{1_1},W_1-U_1):(d_j,\delta_j)_{1,q_2}:(f_j,F_j)_{1,q_3} \end{matrix}\right] \tag{9}$$

where $|\arg x| < \frac{1}{2} U'\pi$, $|\arg y| < \frac{1}{2} V'\pi$, where U' and V' are given as follows respectively:

$$U' = -\sum_{j=n_1+1}^{p_1} \alpha_j - \sum_{j=1}^{q_1} \beta_j + \sum_{j=1}^{m_2} \delta_j - \sum_{j=m_2+1}^{q_2} \delta_j + \sum_{j=1}^{n_2} \gamma_j - \sum_{j=n_2+1}^{p_2} \gamma_j > 0,$$

$$V' = -\sum_{j=n_1+1}^{p_1} A_j - \sum_{j=1}^{q_1} B_j + \sum_{j=1}^{m_3} F_j - \sum_{j=m_3+1}^{q_3} F_j + \sum_{j=1}^{n_3} E_j - \sum_{j=n_3+1}^{p_3} E_j > 0,$$

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IMPACT OF GLOBAL RECESSION ON MODERN BUSINESS - A CRITICAL STUDY

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ABSTRACT :
Entrepreneurship in India is rightly defined as a rich man's monopoly & a poor man's golden dream. With the end of the license raj, a good amount of opportunity was given to upcoming entrepreneurs in India but without a proper policy & sufficient financial support from the government, the result is a haphazard growth of industries & business. With oil prices reaching a high, inflation touching Himalayan peaks it is a make or break for the interested entrepreneurs. During the recession time, many banks including international ones having gone bust, there is no financial support in the form of loans, etc to be given to the prospective entrepreneurs. The outbreak of pandemic Corona Virus (Covid-19) all over the world has disturbed the political, social, economic, religious and financial structures of the whole world. The situation of the World has become worst. As a result, the normal process of establishment & nurturing, the young entrepreneurs are in danger and on the other hand, institutions which depend upon industries & business people for fulfilling their placement commitments to their students are increasingly at disadvantageous position. It appears that both the modern entrepreneurs & the technical institutions are at the cross roads, whether to meet for a healthier growth & development or depart on different tunes in different directions leaving the scenario unchanged for the worse to come. In this situation , what should be the role of budding entrepreneurs coming out of deemed universities & other higher institutions and that of industry leaders whether they are MNC'S or SSI'S is the topic of discussion in this paper.

1. INTRODUCTION:

The new Global Entrepreneurship continues to be a great source of interest among business school students. With current crisis in the market, they have to be better prepared to convince investors and improve their probability of success not just in local market but in global business cultures *[Ref 1]*.

The global recession has thrown several challenges to world economy. With the collapse of **laymen Brothers** an investment banker, there's filtery across the world to overcome the challenges of slow down. Every economy is like a cycle where there will be peaks & troughs. The present situation is a reflection where we need for trough. It is a matter of time to recover the scenario to set for peak. Whenever any economy is overheated it needs cooling off. Like a human body needs a breathing space after a long marathon. The economy also needs fresh oxygen to set into motion further. *[Ref 2*] It all depends on the mindset of the people how they view the present scenario. The situation is not as bleak as depicted by media. No doubt, there are challenges for changes in the present economy trends & policies. When we recall the "Great

The outbreak of pandemic Corona Virus (Covid-19) all over the world has disturbed the political, social, economic, religious and financial structures of the whole world. The situation of the World has become worst. Depression" started in USA, it comes to our notice that many companies evolved during such tough times. Sam Walton grew in USA during the great depression. He sold news papers to support his family. He developed the right attitude to handle customers. He learns the psyche and pulse of the customers. Subsequently, he started his retail venture "Wal-Mart". Financial hardships toughened and taught him many lessons. It paid him off in his later life evolving as a retailing.

Similarly the Second World War brought massive destruction to mankind but led to several inventions and innovations resulting into comforts and luxuries to mankind. The current global meltdown has not adversely affected India as of now but pinch of it will be felt shortly. But it provides several opportunities to set up enterprises at this time and a few of them evolving as entrepreneurial giants like Sam Walton. In the past also, the dot combust of 2000 led to setting up several internet companies resulting into huge success as on today companies like Hyatt, FedEx, GE, hp, in overseas and Wipro-BPO, & Mind tree in India were born in a show down. Besides, spectra mind was born during the peak of dotcom bust in 2000 opportunities. Majority of Indians live in villages and there are 600, 000 villages where a huge potential is lying untapped. Indian rural market is like a sleeping giant and if woken up and tapped properly it opens the flood gates for consumer demand. Hitherto, companies focused on urban market buoyed by the swing of IT, IIT's and knowledge sectors. It is time to shift the focus and tap rural Indian market by providing tailor made products & services. Besides, there are several other sectors which were neglected and untapped till now due to the quick gains from knowledge sectors.

2. THE EXPERT'S OPINION:

According to PC Reddy, Chairman of Apollo Hospital – 3P's purity of purpose, patience & persistence – are needed to succeed in India. Entrepreneur must aim for ideas that impact the lives of other, and are accessible to all. Recession is an excellent opportunity for the laid of people who hitherto had been in their comfort zones. They need to reinvent themselves, explore and experiment entrepreneurship and evolve as entrepreneurs. Talent is cheaper and the opportunity costs are relatively lower during recession. One has to look at existing gaps in products & services through extensive research and then precede setting up the business plan for extensions. Recession will change consumer preferences. The startups should smell that changing preferences and evolved business plans accordingly for viability & feasibility. In this process, they can approach the experts in the domain for their comments and feedback on the business plan.

2.1 The Effect of Recession on Youngsters : Economists say that when the economy takes a dive, it is common for people to turn to their inner entrepreneur to try to make their own work. The task of finding a job is daunting and in turn, encouraging many entrepreneurs to make work for themselves even in prosperous times; Entrepreneurs have a daunting failure rate. Entrepreneurship is recession is designed to capture best practices for launching a business during a recession. *[Ref 3]*

Periods of high unemployment tend to be particularly hard as teenagers, who wind up completing for jobs with more experienced laid-off adults. In USA, unemployment for 16-19 yrs. olds is at its highest rate since 1992-at 22.7 % in May 09 according to bureau of labor statistics. This is causing some teenagers to rethink their notion of work and to embrace entrepreneurship. Many teenagers have also seen the turmoil in the auto industry and layoffs of parents or other adults. They no longer associate financial security with big corporations. In a survey conducted by the Kauffman foundation for entrepreneurship in Dec'07out of 10 people from the ages of 8-21years, said that they would like to start their own business in the future.

Interest in entrepreneurship education among teenagers is rising. The distributive education clubs of America (or DECA) which provides high school and college students with training in marketing, management and entrepreneurship says. It has found a 20 % event increase this year in interest in its entrepreneurship. Amy Roren, CEO of Network for teaching entrepreneurship says that the USA kids are concerned that the world their parents grew up in no longer exists and the notion of taking control and owning one's future is really appealing. [*Ref 4]*

In **"THE ABC'S of making money for Teens"** by Alan lysaght, the internet may be the most significant catalyst for teenager's entrepreneurship. The ability to start a business online has lowered many barriers to self-employment faced by young people.

Ex: of laure durst (18 year), a recent high school, graduate in Woodstock count's, said that there were so few jobs for teenagers there that two years ago. She began setting up a web based business working My Room.com. it provides teenagers with info and online resources to find jobs that can done from home. Teenagers start a wide range of business from selling art jewelry or collectible on line to web site creation and design. They also do non-web-based things like. Yard work, house cleaning, dog walking pool care, in tutoring and party planning.

2.2 The Effect on Markets: With the global recession, businesses are tightening their belts and several of them are cutting their marketing budget. It is clear that times are tough and a lot of companies are even cutting down manpower to save on operating costs. But, businessmen should think twice about cutting their marketing budget. Studies show that companies. Who continue to market their wares even during recession often see growth despite the tough economic times. Laura Lake, author of About.com says tough economic times call for a revolution of marketing strategies Asian analysis of what is effective and what is not. [*Ref 5]*

It is a good time to discover marketing alternatives that would cost less but still provide the results one wants. It is necessary to understand the significance of continuing to market a business even during tough time Exploiting social media marketing to one's advantage and setting a pricing strategy would work in this economy,[ref.5]

2.3 The Role of the Governments: Governments around the world have spent trillions of dollars to same banks automakers and insurance co's .Big corporations that benefit the most from bailouts won't save the world from economic crisis. The bulk of new job's, products, services, economic stimulus and philanthropy well come from entrepreneurs are ready to pounce on new opportunities in the changed economy. The entrepreneur's community is the one that is going to bring the economy back. They will make the greatest contribution in net employment in terms of new products & services. It is not going to be the large corporation which is where the governments are spending money.

The best things the governments can do are to provide capital for start ups and stay out the way. The economic crisis has created hardships for entrepreneurs around the world, but the downturn also has worked as a motivator to spur some gazelle into action. some people sitting on good ideas will get the push they need and when job security is no longer there, they are more prone to go for it the downturn also will expose and eliminate weak co's that do not provide real value to society and consumer always wins when government allows this process of natural selection to run its course *[Ref 6]*. The biggest hurdle for entrepreneurs in the prolonged downturn has been access to debt financing .Bank are different to deal with, they cut back line of credit even for good companies . The positive and negative effects of the down turn will balance themselves out in the end.

3.0 CASE STUDIES

3.1. Example of Walker Center: It has expended its programmes & services following an 11 million pledge in 2008 from Texas Entrepreneurs Scott walker'81. Thunder birds expose every student to at least one course in entrepreneurship which is enough for many students with corporate aspirations. 3 levels of people: (*1) **Some are born entrepreneurs .they do not require any formal degree etc. (2) Some are born to only work under others they do not require any clauses on entrepreneurship. (3) Majority of the young people are caught in the middle who posses varying capacities for entrepreneurs that can be developed though formal education and education.***

It is a really a fascinating thing to watch student get an idea and take it and do the whole thing and get turned on. The global mind set that may student bring to the institute often makes then ideal candidates for entrepreneurship people who think like entrepreneurs think always big they have a clear vision.

Program: (1) Advanced entrepreneurs for full time students. (2) Access to thunder birds global entrepreneurs incubators. (3) Support programs for women entrepreneurs in emerging markets (Afghanistan, Jordan). (4) A seeped venture fund for budding entrepreneurs. (5) Support program for family business. Small entrepreneurs can move much faster than large corporations. Bureaucracy stamps out creativity, that's a big hindrance in today's environment where product life cycle is short, hyper-completion and new markets opening all over the world.

3.2 The Example of 3 B-Schools: Entrepreneurs need to understand how to do business in global, rather than just domestic markets if they are to fulfill their true potential. In fact, this type of entrepreneurship may be the only viable way of escaping the threat of global recession. Three business schools have come together to educate next generation of international global entrepreneurs. The program, they are offering. The new entrepreneurship program, they teach student across campus in (TNGEP) Europe, USA and China .it is a twelve month program. It offer the opportunity to work in a semester, long working project with an operating company is in each of the three regions. Entrepreneurship continues to be a great source of interest among business school students with the current crisis in the markets. They have to be prepared to convince investors and improve their probability of success, not just in a local market but in a global business cultures. This program offers a rare opportunity to work in culturally diverse team with the best international teachers. While living and breathing are the three of the most important business environments in the world. All three contributing school's have ranked as No.1 for the teaching of entrepreneurship in their respective regions.*[Ref 7]* When a brands share of voice *(sov)* is greater than its share of market *(som),* it is likely to grow its market share in the coming year. Therefore, companies that increase their marketing investment when others are cutting back have an opportunity to substantially improve the standing of their brands. *[Ref 8]* Manufacturing workforce reduction and outsourcing of manufacturing operations oversees have reportedly cost 2.7 million American workers their jobs in the last four years. Yet, many manufacturing jobs lie unfilled for months as companies seek worker with the skills they need for these jobs. *[Ref 9]* There is a continual need for engineers in our society to build and renew our infrastructure and to serve our commercial, research, industrial and manufacturing sectors.*[Ref 10]*

3.3 The Example of Entrepreneurship Development Cell: EDC established in 1988 under the aegis of the DST, GO1,it organizes, a short term technical modular trade courses such as, EDP, EAC (Entrepreneur Awareness Campus) and SIMAP (Small Industry Management Assistant Program). *[Ref 11]* Today India has hundreds of engineering college that have churned out thousands of graduates in the past few years. But, obviously, not all colleges are made equal. India is known to have one of the best education systems to world over, be it in the fields of computer education, arts, commerce, or medicine. The IIT'S are the most prestigious institutions in the country. They demand excellence and impart nothing short of it. Today brand I IT is global phenomenon with graduates from this elite institution sought after by the biggest and best engineering firms and institution around the world. *[Ref 12]* The words "you're fired" will probably forever be linked to Donald Trump and his NBC reality shown the apprentice, in which candidates compete on business tasks to get hired by trump. Shortly after its 2004 debut, a phenomenon began to blossom on college campus nation wide –academic programs based on the sense everyone was talking about. The program are helping students land jobs after graduation-both directly and indirectly through networking opportunities.*[Ref-13]*

3.4 The Example E-Summit: The E-summit is a unique initiative by E-cell, IIT Bombay which serves as community a social forum for the entrepreneurial. Work shop at community E-summit'09 emphasized on critical issues faced by entrepreneurs and start-ups at early stage, of development and will be taken by the best in biz.*[Ref 14]* Apart from serving as a big platform for ASBVC's (Academicians, students, Business people and Venture- capitalists). It showcased the launch of globle venture labs. It is a collaborative effort of IIT-KGP, University of California Berkely (VCB), University of Jyvaskyle (Finland) to promote entrepreneurship and innovation*. [Ref 15]* The India chapter of Asia-Pacific Student Entrepreneurship Society (ASES-India) operating out of IIT- madras is set conduct workshops running up to competition in which the best social biz-plan will be rewarded. *[REF16]* An E-cell has been established to guide students and promote entrepreneurship. SINE," society for innovation and entrepreneurship" hosted by IIT-Bombay is an umbrella for promotion of entrepreneurship at IIT-Bombay. SINE-administers is a biz-incubator, which provides support for technology-based entrepreneurship. It extends the role of IIT-Bombay by facilitating the conversion of research activity into entrepreneurial ventures. *[Ref17 & 18]*

3.5 The Example of Nexus: Nexus India capital is looking for new business to fund. They are looking for companies in their early stages, driven by innovation in technology or business model. With over 320 million U S $ capital under management, Nexus has invested in 14 students, in area ranging from internet to solar energy. The parameters on which a VC judges the potential of a business; (a) Market size, (b) Technology used, (c) Scalability, (d) Core-team competency and (e) Absence of regulatory hurdles.
Opportunities**,** (a) Mobile –communication, (b) Internet-commerce and (c) Clean technology*[Ref 19].*

4.0 ROLE MODELS:

4.1 Pallav Nadhani: an MS in computer science who graduated from the University of Edinburgh, Scotland founded Infosoft global Ltd. In 2002 when he was still a teenager a leading provider of a visual web application and solutions, specializes in making data visualization [Ref 20].

4.2 Hufrish Gandhi: Founder EATT (Experience at the Table) Women are expected to be great cooks and women in particular have a lot expected of then in the kitchen hufrish Gandhi was one such women who took her talent and he passion for inducing very seriously. She found her entrepreneurial recipe in home cooked food made at EATT in 2006 *[Ref 20].*

Eatt's USP: **(1) Home Cooked, (2) Customized, (3) Fresh food supplied and (4) Numerous Varieties.**

Background: **Professional Training at Hyatt, Intercontinental, The grand and Oberoi.**

Motto for growth**: (1) Slow &steady, (2) Try & hire more cooks and (3) Train & nurture.**

The inner call: **(1) Don't waste your life, (2) Be your own boss, (3) Start your own stuff and (4) Take the risk in order to grow.**

Practical difficulties: **(a) Business logistics, (b) Finances and (c) Image management**

Entrepreneurial life: **(1Tthe thirst to innovate in the MENU'S, (2) The hunger to keep clients asking for more and (3) The drive to keep the biz going.**

Advice: **When you stumble upon**
that thing called "DREAM"
don't just roll over the pillow
if it is really what you want to do,
then wake up and smell the coffee
and go ahead &pursue it
make your dream, your job,
It will become your "DREAM JOB".

The report" education - entrepreneurship "by researches from Duke University, UCB, & the Kauffman foundation found that only 15% of Indian American entrepreneurship surveyed received their under graduate degrees from IIT'S. Vivek Wadhwa lead author of the report says "The report tells more about Indians than their education. The fact is that no matter what school they graduate from, given the right opportunities, they flourish .The IIT's are great school but are not the 'only ones that produces Entrepreneurs" By contrast, China's five leading universities accounted for half of all Chinese- American entrepreneur; **(1) Peking university, (2) Nanjing University, (3) Shanghai University,(4) Jiao University and (5) Tianjin University.** However almost 90% of Indian founders have received their graduate degree in India, whereas only a third of Chinese founders went to undergraduate school in China.

4.3 Warren Buffet: the legendary investor hailed as "the oracle of Omaha" has singled out Ajit Jain who manages his reinsurance division for special praise in his annual letter to share holders of his holding company, Berkshire Hathaway. Ajit Jain a former Mckinsey executive, and graduate of IIT Kharagpur and has Harvard University has been tipped as Buffett's possible successor at Berkshire. Buffet praised the reinsurance division "one of the most remarkable business in the world "generating billions with just 31 people in office. [*Ref 22]*

5.0 GLOBAL RECESSION AND ITS IMPACT ON EXPATRIATES (REVERSE EXODUS):

The shrinking Gulf economy has seen 500,000 non resident Keralites coming back home, The annual remittance was $ 8 billion sent by NRK'S (Non Residents Keralites).The reason for the reverse migration have been well- documented. The gulf economy has been suffering after the crash in oil prices. Dubai has been hit particularly hard because it was the region key financial centre. It is mainly the real estate and property segments along with bankers who have been hard hit. India gets the heighest remittances of any country in the world ($27 billion in 2007- $32 billion in 2008). In kerala, these are self –created social un-employment. *[Ref 23]*

6.0 HOW BUSINESS, INDUSTRY AND INSTITUTIONS CAN WORK TOGETHER?

6.1 Kern Family Foundation: has awarded a grant of $ 13, 22,500 to the entrepreneurship program at IIT (Illinois, USA). Under the leadership of Dr. David Pistrui, Coleman Foundation Chair and entrepreneurship, this grant is used to create and support a student-centered flagship curriculum. The entrepreneurship program at IIT, strive to foster an entrepreneurial mindset among the students, faculty, alumni, business and civic organizations that make up the IIT community. The KEEN industrial entrepreneurship initiative has developed into an important demonstration model on how business and engineering colleges can work together to build new and innovation curriculum offerings over the past four years; nearly 1,200 students have taken an entrepreneurship course at IIT. IIT educational history has always included a focus on fostering innovation and harnessing the creativity of its students and faculty. The University understands the importance of practical preparation for entrepreneurs. The entrepreneurship program at IIT teaches, student how to create companies and grow existing enterprises; provides students with the knowledge of key factors central to successful technology commercialization and company formation*[Ref 24]*. On 26 January' 09 Republic day, over 200 IIT'ans launched a unique 'entrepreneurship movement" across the country to coach 1,000 budding entrepreneurs. The program was held under aegis of "***Pan IIT Alumni, India***", an umbrella organization covering alumni of all 7 IIT's. The program was organized by TIE (The Indus entrepreneur) and NEN (National entrepreneurship network)*[Ref 25]*.

6.2 "***Stay Hungry, Stay Foolish***" is a story of 25 IIM'Ahmedabad graduates who chose the rough road of entrepreneurship. Three varieties (a) The believers (b) The opportunists (c) The alternate vision holders (social entrepreneurs who cared about things beyond money). The sheer breadth of advice from entrepreneurs proves one big point; there is no consistent formula for success *[Ref 26]*.

6.3 Pro- quest entrepreneurship bridges theory with practice by providing unprecedented access to wide array of resources and tools for starting one's own business or supporting the study of entrepreneurship. Pro-quest entrepreneurship offer's easy access to a range of multimedia and traditional resources useful in creating and growing successful new ventures and is a valuable resource for educators, students and entrepreneurs worldwide *[Ref 27]*.

6.4 "***Enterprenuer-Sarath Babu*** is not the typical IIM, Ahamdabad student one comes across in the daily times. He did not take the easy path to earn mega bucks in spite of being offered plum job from top companies'. Instead, he chose like a true entrepreneur to carve out his own path to success. So, he started *Food king*. He had earlier studied in BITS Pilani. In the beginning, he took a loan of twenty lakhs and started "Food king" in August 2006. Initially, the losses were poised at Rs.2000 a day. The Cafeterias he set up initially did not work according to plan and he soon came to the conviction that only by selling in large volumes that any profit could be made. In March 2007, he got an offer to start a unit at BITS, Pilani. This contract proved profitable. He move to BITS-GOA campus and his earning stood at Rs. 65,000 per day. Then he got an opportunity to serve at SRM Deemed University, Chennai which boasts of a student strength of 17,000. Right now, his earnings stood at Rs. 3.5 crores and expectations are around twenty crores in the coming years *[Ref 28]*.

6.5 Suhas Gopinath: A child prodigy-Entrepreneur at the tender age of 14, Suhas Gopinath was born in Bangalore and had a dream of becoming a veterinary doctor. He was enthralled by the power of internet, web, e-mail etc. Within a year, he had taught himself how to create and design a web site, HTML, ASP, and all other related software. He was a "good "student at academics till then but became an *average student* when he started exploring his new found world in *cyber cafes*. He started' "COOL HINDUSTAN.Com" along with his friends. For funding he wrote to network solutions headquartered in California. They readily agreed and offered him a job, and would also have paid for his education in USA. He became the youngest Indian to start a company, when he founded "**Globals Inc**."At California in USA at the age of 14 in 2000. It is an MNC with offices in more than 11 countries that offers quantity solutions to web, mobile, multimedia, e-commerce etc. The company expanded from 4 people to 400 in a very short time. In 2005, a Houston investment firm approached him and offered him one billion US$ for a majority stake. The answer was 'NO'and Suhas said "Why should I sell my baby" *[Ref 29]*.

6.6 NSTEDB; established by GOI in 1982 is an institutional mechanism with a broad objective of promoting gainful "*self employment amongst the science and technology man power in the country and to set up knowledge based and innovation driven enterprises*." NSTEDB has initiated programs jointly with the international organizations such as UNDP on vocational training. This has broadened the scope and content of entrepreneurship development activities and has brought in international expertise for socio-economic upliftment. A special emphasis is now being given to cluster development where in a few selected clusters have been adopted for development using S and T methods and intervention under TBI (Technology Business Incubator), 85 units have been set up and under STST(Skill development through science and technology) nearly 1,13,000 people have been trained *[Ref30]*.

6.7 Endeavor is transforming the economies of emerging market by identifying and supporting high-impact enterpreneurers. Endeavor differentiates itself from others in the entrepreneurship-support sphere in two-critical ways. (1) High-impact entrepreneurs have ground- breaking ideas, dogged determination and infinite ambition. (2) Their innovations make people's lives easier, safer and richer. Their business creates hundreds of jobs that stimulates economic growth and spur development. These incomes enable families to afford health services, purchase of food and clean water, develop domestic savings, receive primary and secondary educations and reach new standards of living. Their stories inspire others to become empowered entrepreneurs. High impact entrepreneurs from emerging markets are often over looked and face significant barriers to success. They encounter; *(a) Few role models, (b) Lack of trust, (c) A limited pool of management talent, (d) An inability to access smart capital and (e) Insufficient contact.* Endeavor provides *(1) MBA interns, (2) Mentorship from biz-leaders, (3) Pro-bono consultant services, (4) Access to capital via road-shows and (5) An entrepreneurial peer group.* Results of Endeavors; (1) 91,000+high value jobs created, (2) 2.1 billion USD revenue generated in year 2007, (3) 908 million USD in financing raised and (4) 409 End-Entrepreneurs selected *[Ref 31]*.

6.8 Innovating technical and business solutions on the way to college: If one is interested in solving climate and energy issues and if one is interested in converting one's research into a potential business idea and be an entrepreneur, then participation in the "*Global challenge award*" competition by developing a technology innovation plan for climate issues is recommended. Achievements: (1) Flex fuel car, (2) Nanotech. Catalytic converter, (3) Capturing CO2 in Zeolitic Imidezolate framework, (4) Multipurpose Electric generator and (5) Solar and Piezoelectric sidewalk Tile system [Ref 32].

6.9 5 Tips for using Colleges and Universities to help Business:

(1) Volunteering the company to be a business school case study. One will learn much about his company in the process and get good ideas for the future.
(2) Obtaining management and technical assistance from small biz.-development centers.
(3) Participating in special programs ex. Venture capital forums, family business programs.
(4) Working with the business school to offer internships to graduate students.
(5) Exploring the expertise in a business school *[Ref 33]*.

7.0 INSTITUTIONAL-TBI CENTERS IN INDIA:

(1) ICICI Knowledge park-life sciences Incubator (Hyderabad).
(2) ICRISAT (Hyderabad).
(3) University of Hyderabad (Hyderabad).
(4) Indian Institute of chemical technology (Hyderabad).
(5) Department of Microbiology university of Delhi.
(6) National design business incubator, NID, (Ahemdabad).
(7) Nirma LABS, Nirma University (Ahemdabad).
(8) Centre for innovation incubation, & Entrepreneurship (CIIE), Vastrapur, (Ahmedabad).
(9) EDC, Madera institute of Communications, Shele (Ahemdabad).
(10) Composite technology park, Kengeri.Sat. Township (Bangalore).
(11) NDRI, Karnal, (Haryana).
(12) E –health TBI, Pesit Technology Park, Bangalore.
(13) Amrita Viswa Vidhyapeetham, Kollam(Kerla).

8.0 SOFT-LANDING FACILITY FOR INDIAN STARTUP IN NEW ZEALAND:

VIT-TBI has signed a letter of intent for co-operation with a NZ-based business incubator "ICE HOUSE". The ice house is a NZ. - Business incubator operating in Auckland which strives to facilitate the growth and international success of emerging Knowledge-based Companies. It offers its resident companies office space, access to networks and business support services.

Project under incubator 'ICE-HOUSE'

Name of the Project	Brief description	Remarks
(1) Mobile veda.	**A simplified web platform for Mobile value added services.**	
(2) Acharya green peace silk saree.	**Manufactured as a special handloom fabric called KUTNI fabric made out of natural vegetables dyed, veg.silk and cotton yarns.**	
(3) MSI Biotech.	**Production of enzymes to treat industrial effluents**	
(4) Kaapi TV	**Interactive. HDTV content and advertisement.**	
(5) Ocown solutions	**Design, develop and provide online solutions for learning, brains forming, messaging, mailing, meeting/presentations.**	
(6) Max value On line education.	**E-learning platform and tool and programs.**	
(7)Think core Technologies.	**An enterprise productized software development organization targeting SME'S.**	

Success stones at SINE (Society for innovation and entrepreneurship, IIT Bombay)

Name of The Company	Brief Description
1. Geosyndicate Power Pvt. Ltd.	**Aims at promoting of non-conventional energy mechanisms like geo-thermal to deliver high efficiency and low cost electricity to the Indian rural and power sector.**
2. Voyager 2 InfoTech.	**Raised US $250K, built a creative ideas portal and was bought out by purple yogi in all stock deal.**
3 My2US Technology.	**Raised US $600 k, develops products and services in the area of wireless gateways and connectivity bridges.**
4. E-Infinitus.	
5. Herald Logic Pvt. Ltd.	**Raised Rs.1.2 crores, develops products in network operations, real time band width provisioning and specialized router software.**
	Develops products in enterprise info, rule-based engine

8.1 Accelerating Entrepreneurship in emerging Economies: "The economic greatness of a country is fueled by the strength and vitality of its entrepreneurs." Ramesh Wadhwani/ President- Wadhwani foundation. The wadhwani foundation funds not for profit efforts that inspire educate and support new entrepreneurs and create environments where they can succeed.

8.2 Reducing Poverty through agriculture innovations: partnership and product for poors. Bio-products research consortium (BRC, ICRISAT) World wide about 3 billion kgs. of pesticides are applied each year at a cost of over US $ 40 billion. Despite this, problems continue, with resurgence of secondary pests, environmental pollution and pesticides residues in food drinking water and milk. Bio pesticides have shown great potential to replace or supplement chemical pesticides. It is in this backdrop that the BRC, an initiative of ICRISAT was formalized in 2005 to deliver the research outputs and technologies leading to mass scale production of quantity microbial bio pesticides at low cost through effective public-private partnership. BRC currently has 11 private companies as members supporting research.

9.0 IMPACT OF CORONA VIRUS (COVID-19) ON GLOBAL ECONOMY:

The outbreak of pandemic Corona Virus (Covid-19) *[Ref 34]* all over the world has disturbed the political, social, economic, religious and financial structures of the whole world. The situation of the World has become worst. Global cases of Corona Virus affected figure have touched more than 4,148,529 including 282,388 deaths globally as on May 11, 2020..World's topmost economies such as the US, China, UK, Germany, France, Italy, Japan and many others are at the verge of collapse. Besides, Stock Markets around the world have been crushed and oil prices have fallen off a cliff. Experts on economic and financial matters have warned about the worsening condition of global economic and financial structure.

Moreover, Covid-19 is harming the global economy because the world has been experiencing the most difficult economic situation since World War-II. When it comes to the human cost of the Corona Virus pandemic it is immeasurable therefore all countries need to work together with cooperation and coordination to protect the human beings as well as limit the economic damages. For instance, the lockdown has restricted various businesses and most of the nations are going through recession and collapse of their economic structure that points out the staggering conditions for them in this regard almost 80 countries have already requested International Monetary Fund (IMF) for financial help. The Organization for Economic Cooperation and Development (OECD) stated that global growth could be cut in half to 1.5% in 2020 if the virus continues to spread. Most of the economists have already predicted about the recession to happen because there is no surety and still no one knows that how for this pandemic fall and how long the impact would be is still difficult to predict.

Corona Virus has already become a reason for closing the multiple businesses and closure of supermarkets which seems empty nowadays. Therefore, many economists have fear and predicted that the pandemic could lead to inflation. There are various sectors and economies that seem most vulnerable because of this pandemic, such as, both the demand and supply have been affected by the virus, as a result of depressed activity Foreign Direct Investment flows could fall between 5 to 15 percent. Besides, the most affected sectors have become vulnerable such as tourism and travel-related industries, hotels, restaurants, sports events, consumer electronics, financial markets, transportation, and overload of health systems.

10. CONCLUSIONS:

1. Recession all over the world has resulted in massive unemployment in almost all the industrial sectors. This has affected largely middle class employees and others. The future of the economy depends upon the number of entrepreneurs a country can nurture and develop.
2. It is the morale responsibility of all knowledge- technology driven people to encourage and promote entrepreneurship not only in colleges and universities but also in various SME's & vocational centres.
3. The establishment of The New Global Entrepreneurship programme (TNGEP), EDP, SINE, E-Summit, ASES-India, TIE, Kern Foundation, Proquest Entrepreneurship, NSTEDB and Endeavor are all positive signs of economic recovery and entrepreneurial enhancement.
4. India can take the help of leading institutions such as IIT's, NIT's and Industrial Research Organizations for developing entrepreneurship culture all over the country.

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BEHAVIORIAL ANALYSIS OF A HETEROGENEOUS FEEDBACK QUEUE SYSTEM

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ABSTRACT :
The paper studies the behavior analysis of a network of queue system having two heterogeneous service channels wherefeed-backing are allowed for defective items. The arrivals and departures at each channel follows poissonlaw. Connectingthe various probabilities, the differential difference equations have been formed. The marginal mean queue size for each channel and for the entire system has been explored. On the basis of graphical study the behavior of the system has been studied. An effort has been made to analyze how the system behaves on changing the parameter.
Keywords: Feedback, cyclic queues, poisson process. Heterogeneous service etc.

1. INTRODUCTION

In literature, most of the queue models with multi-servers have been studied under the assumption that the servers are identical and work with the constant rate, i.e a homogeneous service process. However, in real life situation it is not so. The homogeneous service type situation practically can only prevail when the service stations are highly mechanized.In situations where services are provided by human beings such as at check- out counters, in grocery stores, in barber shop,in saloon,in banks etc. one can't expect to get the services with same rate. The services provided with different rates in a queue networkis termed as heterogeneous service. Satty (1961) applied the heterogeneous server concept in order to find the transient solution of multi-server Markovian queue. Krishnamoorthy (1963)studied a poisson queue with two heterogeneous service channels. Singh V.P(1970) extended the heterogeneous servers study with balking. Sharma & Dass (1989) analyzed M/M/2/N queue system with heterogeneous servers to derive the probability density function of busy period.

Finch P.D (1959) made the study of cyclic queues with feed-backing. Singh T.P (1986,2005) extended the idea of queues in series and discussed various queue models in which each of two non- serial service channel was separately linked in series with a common channel. Singh T.P. & Kusum (2010) studied transient behavior of feedback queue system considering the service rate proportional to queue numbers. The feedback is taken from second and third channel to initial one, where the channels are arranged in a series. Further Singh T.P. & Kusum (2011) developed a steady state heterogeneous feed- back queue model.

The present paper analyses the behavior of the model developed by Singh T.P. & Kusum(2011).The first channel provides service to two heterogeneous channels not linked in successive order, there is a feedback from these two service channels to initial one. We have made an effort to find out how the system behaves on changing one parameter. The differential difference equations formed in the model has been solved by using generating function technique, laws of calculus and other statistical formulae. The objective of the study is to investigate the mean queue size of each channel and for entire system. On the basis of graphical study the system analysis has been presented.

The paper is organized as follows. In Section 2, we provide a formal description of the model and some notations. Section 3, form mathematical modeling and state the steady state solution for the marginal queue length for each channel and for entire system on the basis of g.f.t, laws of calculus and statistical formulae as made by Singh T.P. & Kusum (2011). Finally behavioral aspect of the system has been traced out on the basis of graphical study.

2. THE MODEL:

The queue system consists of three service channels S_1, S_2 and S_3; where S_2 and S_3 are heterogeneous channels and S_1 is commonly linked in serial order both with S_2 and S_3. The feed backing is allowed at S_2 and S_3. The items arrive for service before service channel S_1according to poissonlaw with mean rate λ. Arriving customers form a single waiting line based on the order of their arrivals. First cum first served (FCFS) is the queue discipline.

The items/customers after getting service at S_1 go through either service channel S_2 or S_3 with probabilities p_{12} or p_{13}such that $p_{12}+ p_{13}=1$. After getting service at S_2either an item/ customer departs from the system with probability p_2 or feedbacks to S_1 for completion of service with probability p_{21} such that $p_2+ p_{21}=1$.

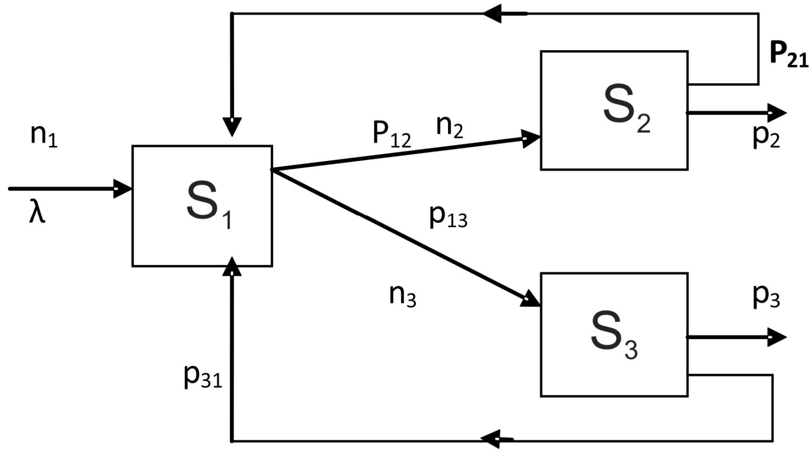
In similar fashion, after service at S_3, an item/ customer either departs from the system with probability p_3 or feeds back to initial channel S_1 with probability $p_{31}$for completion of service where $p_3+ p_{31}=1$.

We assume that the service times at each S_1, S_2 and S_3 are distributed exponentially with service parameters μ_1,μ_2 and μ_3 at channels S_1, S_2, S_3 respectively. It has also been assumed that that there will be no any change in probability when the items are served again by the same server.

3. FORMULATION OFMATHEMATICAL MODEL:

Let $P_{n_{1,}n_{2,}n_3}(t)$: the probability that at time't' there are n_1, n_2,n_3 customers/items waiting in service at the queues in front of service channels S_1, S_2 and S_3respectively where $n_1, n_2, n_3 \geq 0$.

Considering different events at time t and $t+\Delta t$ & $\Delta t \to 0$,



The steady state equations governing the queue model can be depicted as follows:

$(\lambda + \mu_1 + \mu_2 + \mu_3)P_{n_1,n_2,n_3}P_{n_1-1,n_2,n_3} + \mu_1 p_{12}P_{n_1+1,n_2-1,n_3} + \mu_2 p_2 P_{n_1,n_2+1,n_3} + \mu_2 p_{21}P_{n_1-1,n_2+1,n_3} + \mu_1 p_{13}P_{n_1+1,n_2,n_3-1} + \mu_2 p_{31}P_{n_1-1,n_2,n_3+1} + \mu_3 p_3 P_{n_1,n_2,n_3+1}$,

$n_1,n_2,n_3 \geq 0$ (1)

Six more equations depending upon the conditions of n_1, n_2, n_3

$\lambda P_{0,0,0} = \mu_2 p_2 P_{0,1,0} + \mu_3 p_3 P_{0,0,1}$ $n_1, n_2, n_3 = 0$ (2)

Singh T.P. & Kusum (2011) derived the ultimate solution of the above set of differential difference equations using generating function technique , laws of calculus and standard statistical formulae.

3.1. Expected queue length:

The marginal expected queue length for Q_1 in front of S_1 denoted by L_{Q_1} is given by statistical formulae,

$L_{Q_1} = \frac{\lambda}{(1-p_{12}p_{21}-p_{13}p_{31})(\mu_1-\lambda-\mu_2 p_{21}-\mu_3 p_{31})}$ (3)

$L_{Q_2} = \frac{\lambda p_{12}}{(1-p_{12}p_{21}-p_{13}p_{31})(\mu_2-\mu_1 p_{12})}$ (4)

$L_{Q_3} = \frac{\lambda p_{13}}{(1-p_{12}p_{21}-p_{13}p_{31})(\mu_3-\mu_1 p_{13})}$ (5)

3.2 The Expected queue length of the entire system:

Mean queue size

$$L_q = \sum_{n_3=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_1=0}^{\infty} (n_1 + n_2 + n_3) P_{n_1,n_2,n_3}$$

$$L_q = L_{Q_1} + L_{Q_2} + L_{Q_3}$$

$$L_q = \frac{\lambda}{(1-p_{12}p_{21}-p_{13}p_{31})}\left\{\frac{1}{(\mu_1-\lambda-\mu_2 p_{21}-\mu_3 p_{31})} + \frac{p_{12}}{(\mu_2-\mu_1 p_{12})} + \frac{p_{13}}{(\mu_3-\mu_1 p_{13})}\right\}$$

Here, traffic intensities $\rho_1 = \frac{\lambda}{\mu_1}<1$, $\rho_2 = \frac{\lambda}{\mu_2}<1$, $\rho_3 = \frac{\lambda}{\mu_3}<1$

4. BEHAVIORAL ANALYSIS OF THE MODEL:

Now, we discuss the behavior analysis of performance measure as mean queue size of the stated system on changing the values of arrival parameter and service parameter in two different ways:

4.1 Behavior of marginal mean queue lengths and the mean queue length of entire system with respect to arrival rate λ is depicted in table: 1

TABLE:1

Service rate	$\mu_1 = 12$	$\mu_2 = 13$, $\mu_3 = 15$		
Probability	$p_{12} = .4$, $p_{21} = .2$	$p_{13} = .6$, $p_{31} = .1$		
Arrival rate (λ)	L_{Q_1}	L_{Q_2}	L_{Q_3}	L_Q=L_{Q_1}+L_{Q_2}+L_{Q_3}
2	.3941	.1134	.1788	.6863
2.5	.5383	.1418	.2236	.9037
3	.7119	.1701	.2683	1.1503
3.5	.9249	.1985	.3130	1.4364
4	1.1926	.2268	.3577	1.7771
4.5	1.5389	.2552	.4025	2.1966
5	2.004	.2836	.4472	2.7348
5.5	2.6647	.3119	.4919	3.4685
6	3.6719	.3403	.5366	4.5488
6.5	5.3986	.3686	.5813	6.3485
7	9.0439	.3970	.6261	10.067
7.5	21.802	.4254	.6708	22.8982

Table 1: Marginal & Mean queue length of entire system with respect to variable arrival parameter λ keeping other parametric values constant.

Graphical Representation of the Model:

Graphical presentation has been made for marginal mean queue lengths & mean queue length of the entire system in fig.1 to 4.

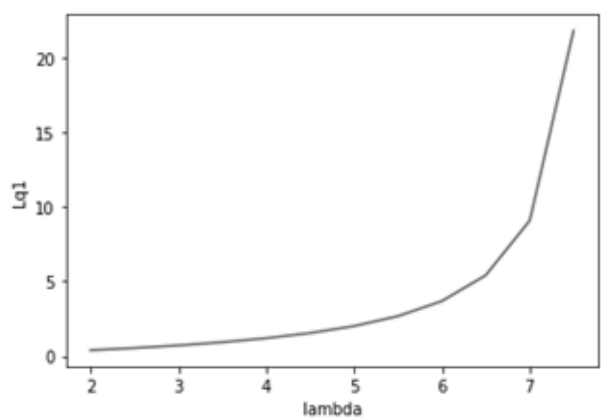


Fig. 1 : L_{q1} vs λ

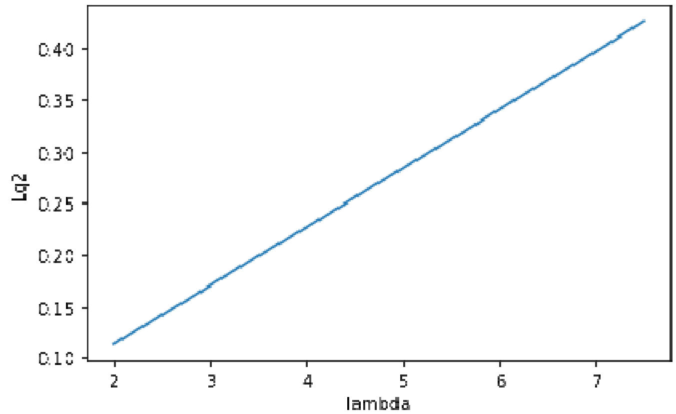


Fig. 2 : Lq2 vsλ

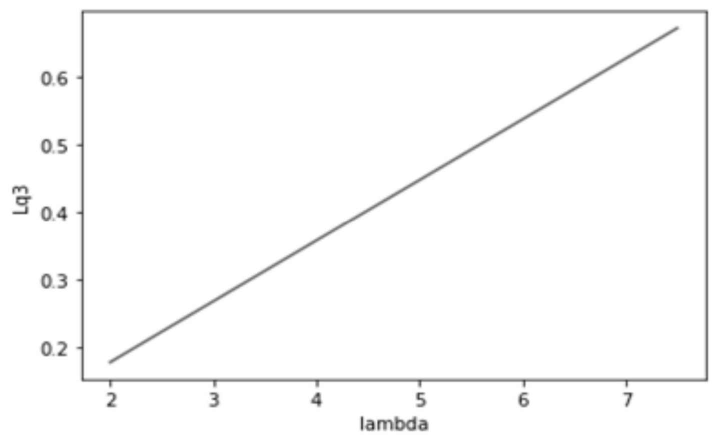


Fig. 3 : Lq3vsλ

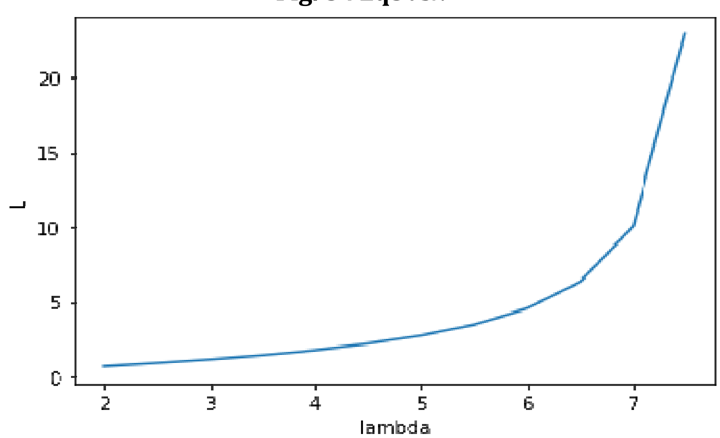


Fig 4: L vsλ (where L=Lq)

4.1 Result & Discussion :

From table 1 & fig.1 to 4, following observations have been noticed:

1. Marginal queue lengths & mean queue length of entire system increases with increase in the mean arrival rate.
2. Gradual increase in the mean queue length of entire system has been observed for λ upto7.A sudden increase can be noticed for the values of mean queue length beyond 7, A quick increase is being noticed for the values of lambda beyond 7.

4.2 Behavior of mean queue length for the system with respect to service rate; μ_1 is depicted in table2 and through graph in fig.5 to 8.

TABLE:2

Service rate	$\mu_2 = 6$	$\mu_3 = 5$	Arrival rate (λ)=3	
Probability	$p_{12} = .6$, $p_{21} = .2$	$p_{13} = .4$, $p_{31} = .1$		
Service rate (μ_1)	L_{Q_1}	L_{Q_2}	L_{Q_3}	L_Q=L_{Q_1}+L_{Q_2}+L_{Q_3}
5	11.904	.7142	.4761	13.0943
5.5	4.4642	.7936	.5102	5.768
6	2.7472	.8928	.5494	4.1894
6.5	1.9841	1.0204	.5952	3.5997
7	1.552	1.1904	.6493	3.3917
7.5	1.2755	1.4285	.7142	3.4182
8	1.0822	1.7857	.7936	3.6615
8.5	.9398	2.3809	1.344	4.6647

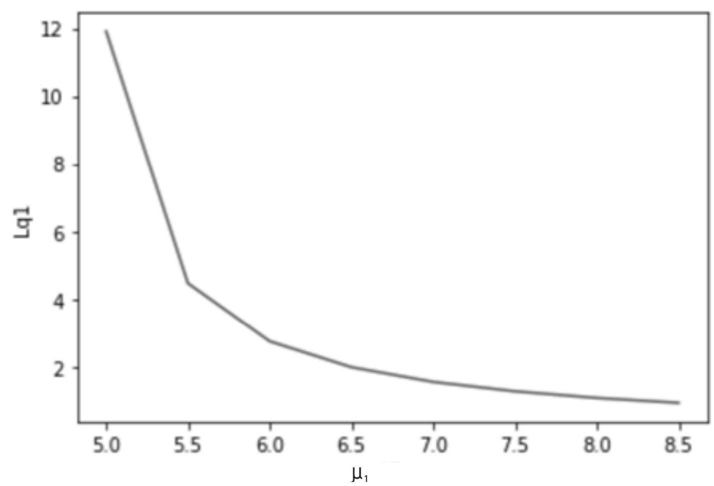


Fig.5 L_{Q_1} vsμ_1

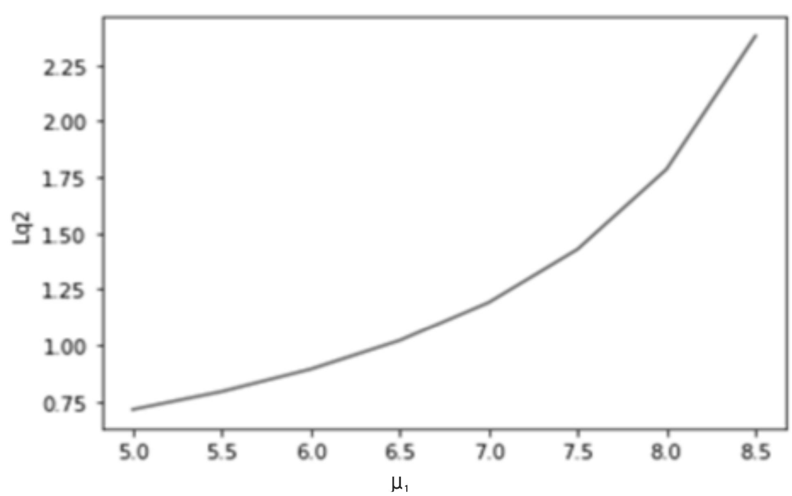


Fig.6 L_{Q_2} vsμ_1

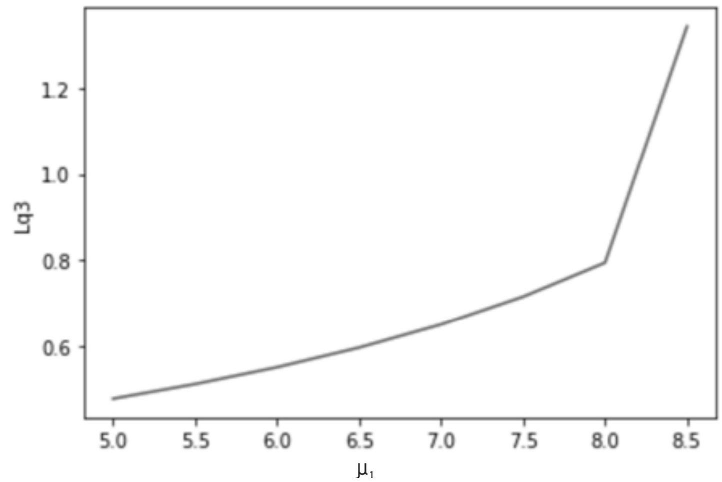


Fig.7 L_{Q_2} **vs**μ_1

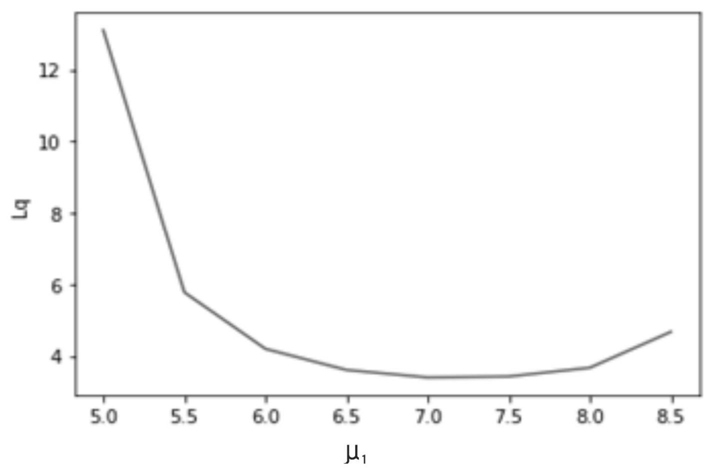


Fig.8 L_Q **vs**μ_1

Mean queue length of entire system vs μ_1

From fig.5 to 8 we observe that initially the mean queue length for the entire system decreases gradually with increase in the value of service rate of first server i.e. μ_1 and attains its minimum value at μ_1=8, after that it starts increasing. Hence, at μ_1 =8 we obtain the point of minima. The minimum queue length is obtained for different values of μ_1. we can also obtain other parameters such as variance, idle time of server etc.

CONCLUSION:

The mean queue length of entire system helps the decision maker how to redesign the queue system so that congestion can be minimized and the service system may no longer be idle. The model finds the practical applications in administrative setup, super market, office management, hospital management, urban transportation, communication networks, steel making process, air traffic control etc. In administrative set up the files are sent from initial collective center to two different clearance service center each with different rate of working and those found defective or incomplete are sent back to initial one to remove shortcomings.

LIMITATION OF MODEL:

The model fails to clarify the following aspects which are arise while dealing the practical situation:

1. On serving the items/ customers again by the same server, there is a chance to change the probability. This concept has not been taken in account.
2. The number of repetition by a server for an item/ customer has not been specified.

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PERFORMANCE ANALYSIS OF A PAPER MACHINE CONSIDERING ITS MINOR/MAJOR FAULTS AND POWER FAILURE

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ABSTRACT :
The purpose of present paper is to analyze the performance of a paper machine using stochastic modeling of a single unit considering various types of minor/major faults and removal thereof by repair/replacement of different parts as well as power failure during operation of the machine. On occurrence of minor faults, operation of the machine is partially stopped while due to major faults, machine stopped working completely. A real data related to different failures is collected from a famous paper mill in Northern Haryana namely Sri Jagdumbe Paper Mills Ltd. located at Sirsa (Haryana). Various measures of system- effectiveness such as MTSF, Reliability, Availability and Busy period etc. are derived by using Semi-Markov process and Regenerative Point technique. Using collected data, various numerical results and graphs are drawn and performance of the system along with profit is analyzed, which may be helpful for the maintenance team of the paper mill.
Keywords: availability, busy period, performance analysis, profit, reliability, repair, replacement.

INTRODUCTION:

Paper industry plays an important role in the industrial as well as social growth of a country as paper can be put to many uses. It contributes a lot of not only to the growth of economic development but also to cultural development. Paper is an important mean of communication and is necessary for the growth of education, industry and other sectors of the economy. Since the paper industry is playing an important role in the society and is also vital for overall industrial growth, it is necessary that the paper industry performs well. Therefore, the main concern of the industry is to maintain system performance measures such as reliability, availability, busy period of repairmen etc. to achieve high profit and productivity of the system. Real data of paper machine is collected by visiting the mill premises from time to time. The paper machine is a single unit system with various sub systems wherein different types of minor/ major faults occur besides power failure. Using Stochastic Models with Regenerative Points Technique, the system is analysed for various measures of system effectiveness such as MTSF, Reliability and Availability etc. and finally, performance measure is calculated with the help of numerical methods and graphs using real data.

A number of Researchers and Scientists are trying to improve the performance of industries in India. The semi-Markov process was adopted by **Branson and Shah (1971)** to discuss a system with exponential failure and arbitrary repair distributions. **Nakagawa (1976)** considered the replacement of the unit at a certain level of damage while **Mine and Kaiwal (1979)** enhanced the system reliability by assigning priority repair discipline. **Goel et al. (1986)** obtained the reliability analysis of a system with preventive maintenance. **Kumar et al. (2010)** carried out profit analysis of a three stage operational single unit system under warranty and two types of service facility. **Kumar and Bhatia (2013)** studied MTSF and cost analysis of a stochastic model on centrifuge system with inspection on halt, rest and failure. **Bhatia and Kumar (2013)** studied Performance and Profit Evaluations of a Stochastic Model on Centrifuge System Working in Thermal Power Plant Considering Neglected Faults**. Renu and Bhatia (2017)** dealt with reliability analysis for removing shortcomings using stochastic processes and applied for maintenance in industries. **Rajaprasad (2018)** investigated the reliability, availability and maintainability (RAM) characteristics of a paper converting machine from a paper mill. A few of the Researchers have worked for real data of paper machine. In view of importance of paper industry and work on this industry, an effort has been made in present paper to obtain performance measures of the paper machine using real data of its various failures. For this purpose, a Stochastic model is developed by using Regenerative Point Technique.

Following measures of system effectiveness are obtained:

- Transition Probabilities
- Mean Sojourn Time
- Mean Time to System Failure (MTSF)
- Reliability
- Availability
- Busy Period of Service man (Inspection, Repair, Replacement time)
- Performance Analysis (Profit)

Assumptions:

- The system consists of a single unit,
- The system is as good as new after each repair and replacement.
- The Service man reaches the system in negligible time.
- A single Service man facility is provided to the system for inspection, repair and replacement of the components.
- Time distributions of various faults i.e. minor/major/ power failure are Exponential distribution and other time distributions are general.
- A minor fault leads to degradation/ failure whereas a major fault leads to complete failure.
- Due to power failure/degradation the machine temporary stops for few minutes.

Notations:

O	:	Operative Unit.
$\lambda_1/\lambda_2/\lambda_3$	:	Rate of minor faults/ major faults/ power failure.
a/b	:	Probability that a minor fault to be repairable/ non- repairable.
$i_1(t)/I_1(t)$	:	pdf/cdf of rate of inspection of a minor fault w.r.t. time.
$g_1(t)/G_1(t)$	:	pdf/cdf of repair rate of minor faults w.r.t. time.
$h_1(t)/H_1(t)$	:	pdf/cdf of replacement rate of minor faults w.r.t. time.
$h_2(t)/H_2(t)$	:	pdf/cdf of replacement rate of major faults w.r.t. time.
$k_1(t)/K_1(t)$	:	pdf/cdf of rate of power degradation/ failure w.r.t. time.
©/Ⓢ	:	Laplace convolution/ Laplace stieltjes convolution.
*/**	:	Laplace transformation/ Laplace stieltjes transformation.
$q_{ij}(t)/Q_{ij}(t)$	:	pdf/cdf for the transition of the system from one regenerative state S_i to another regenerative state S_j or to a failed state S_j.

Model Description: Different states of the system model according to Semi Markov process and Regenerative Point Technique are as follows:

State 0 : Initial operative state.

State 1 : Operative unit partially failed due to some minor faults.

State 2 : Unit completely failed due to some major faults.

State 3 : Unit temporarily failed due to power degradation/ failure.

State 4 : After inspection unit undergoes for repair and system is operative.

State 5 : After inspection unit undergoes for removal of minor faults by replacement of components/ parts.

Here, states 0, 1 and 4 are operative states.

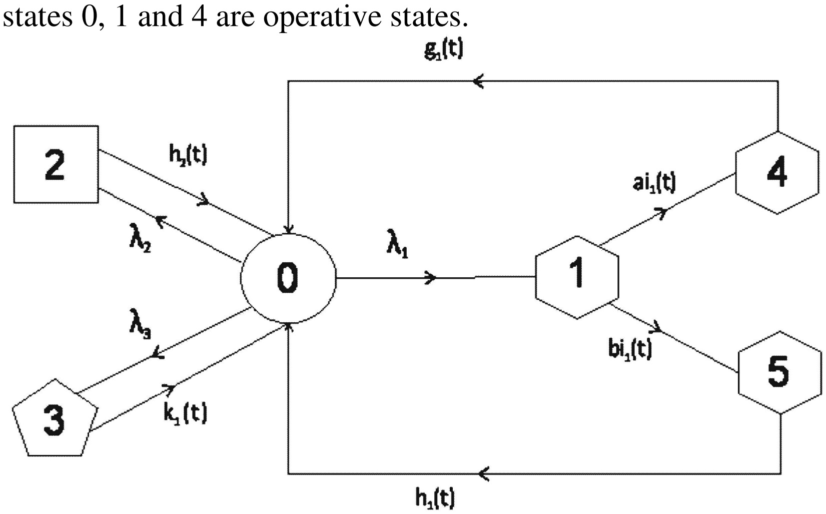

Fig.1 Model

◯ Operative State ☐ Failed State ⬡ Degraded State ⬠ Temporary/short Failed State

Transition Probabilities:

By simple probabilistic arguments, we can find transition probabilities given by:

$p_{ij} = \lim_{s\to 0} Q_{ij}^{**}(s)$ where, $Q_{ij}^{**}(s) = \int_0^\infty e^{-st} dQ_{ij}(t)dt$

$p_{01} = \dfrac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}$ $p_{02} = \dfrac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}$ $p_{03} = \dfrac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}$

$p_{14} = \mathbf{a} i_1^*(0) = \mathbf{a}$ $p_{15} = \mathbf{b} i_1^*(0) = \mathbf{b}$

$p_{20} = h_2^*(0) = 1$ $p_{30} = k_1^*(0) = 1$ $p_{40} = g_1^*(0) = 1$ $p_{50} = h_1^*(0) = 1$

It is simple to verify that

$p_{0i} + p_{02} + p_{03} = 1,$ $p_{14} + p_{15} = \mathbf{a}+\mathbf{b} = 1$ $p_{20} = p_{30} = p_{40} = p_{50} = 1$

Means Sojourn Time:

The unconditional mean time taken by the system to transit from any regenerative state S_i into state S_j when time is counted from epoch of entrance is given by: $m_{ij} = \int_0^\infty t dQ_{ij}(t) = -Q_{ij}^{*\prime}(0)$

Thus, $m_{01} = \dfrac{\lambda_1}{(\lambda_1+\lambda_2+\lambda_3)^2}$ $m_{02} = \dfrac{\lambda_2}{(\lambda_1+\lambda_2+\lambda_3)^2}$ $m_{03} = \dfrac{\lambda_3}{(\lambda_1+\lambda_2+\lambda_3)^2}$

$m_{14} = -\mathbf{a} i_1^{*\prime}(0)$ $m_{15} = -\mathbf{b} i_1^{*\prime}(0)$ $m_{20} = -h_2^{*\prime}(0)$

$m_{30} = -k_1^{*\prime}(0)$ $m_{40} = -g_1^{*\prime}(0)$ $m_{50} = -h_1^{*\prime}(0)$

Also, Mean Sojourn Time in state S_i is given by: $\mu_i = \int_0^\infty P(T > t)dt$, so that

$\mu_0 = \dfrac{1}{\lambda_1+\lambda_2+\lambda_3}$ $\mu_1 = -i_1^{*\prime}(0)$ $\mu_2 = -h_2^{*\prime}(0)$

$\mu_3 = -k_1^{*\prime}(0)$ $\mu_4 = -g_1^{*\prime}(0)$ $\mu_5 = -h_1^{*\prime}(0)$

Thus, we see that

$m_{01} + m_{02} + m_{03} = \mu_0,$ $m_{14} + m_{15} = \mu_1,$ $m_{20} = \mu_2,$ $m_{30} = \mu_3,$ $m_{40} = \mu_4,$ $m_{50} = \mu_5$

Measures of System Effectiveness:

Using probabilistic arguments for regenerative processes, various recursive relations are obtained and are solved to find different measures of system effectiveness, which are as follows:

Mean time to system failure (MTSF) = $\dfrac{N_1}{D_1}$,

Where, $N_1 = \mu_0 + \mu_1 p_{01} + \mu_4 p_{14} p_{01}$ and $D_1 = 1 - p_{40} p_{01} p_{14}.$

Availability per unit time (A_0) = $\dfrac{N_2}{D_2}$, Where, $N_2 = \mu_0 + \mu_1 p_{01} + \mu_4 p_{14} p_{01}$

and $D_2 = \mu_0 + \mu_1 p_{01} + \mu_2 p_{02} + \mu_3 p_{03} + \mu_4 p_{14} p_{01} + \mu_5 p_{15} p_{01}.$

Busy period of service man (inspection time) $B_0^I = \dfrac{N_3}{D_2}$, where $N_3 = \mu_1 p_{01}$ and D_2 is as above.

Busy period of service man (repair time) (B_0^R) = $\frac{N_4}{D_2}$,where $N_4 = \mu_4 p_{01} p_{14}$ and D_2 is as above.

Busy period of service man (replacement time) (B_0^{Rp})= $\frac{N_5}{D_2}$, where $N_5 = \mu_5 p_{01} p_{15}$ and D_2 is as defined above.

Performance (Profit) Analysis: The performance of the system can be figured as follows:

$P = C_0 A_0 - C_1 B_0^I - C_2 B_0^R - C_3 B_0^{Rp} - C_4$

where, C_0 = Revenue per unit availability of the system;

C_1= Cost of inspection per unit time;

C_2 = Cost of repair per unit time;

C_3 = Cost of replacement per unit time;

C_4 = Miscellaneous cost of machine.

Numerical Study and Graphical Analysis:

Giving some specific values to the parameters and considering

$k_1(t) = \alpha_1 e^{-\alpha_1(t)}$, $h_1(t) = \gamma_1 e^{-\gamma_1(t)}$, $h_2(t) = \gamma_2 e^{-\gamma_2(t)}$, $g_1(t) = \beta_1 e^{-\beta_1(t)}$, $i_1(t) = \eta_1 e^{-\eta_1(t)}$,

we get $p_{01} = \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}$, $p_{02} = \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}$, $p_{03} = \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}$, $p_{14} =$ **a**, $p_{15} =$ **b**

$p_{20} = 1$, $p_{3o} = 1$, $p_{4o} = 1$, $p_{5o} = 1$

and $\mu_0 = \frac{1}{\lambda_1+\lambda_2+\lambda_3}$, $\mu_1 = \frac{1}{\eta_1}$, $\mu_2 = \frac{1}{\gamma_2}$ $\mu_3 = \frac{1}{\alpha_1}$, $\mu_4 = \frac{1}{\beta_1}$, $\mu_5 = \frac{1}{\gamma_1}$.

For the above specified cases, taking values from the collected data and assuming the values

$\lambda_1 = 0.012$, $\lambda_2 = 0.007$, $\lambda_3 = 0.003$, $\alpha_1 = 4.5$, $\beta_1 = 4.71$, $\gamma_1 = 3.25$, $\gamma_2 = 0.53$, $\eta_1 = 0.9$,

a = 0.8, **b** = 0.2, we obtained the following values for the measures of system effectiveness:

Mean Time to System Failure (MTSF) = 81.88

Availability per unit time = 0.9858

Busy period of Service man (Inspection time) (B_0^I) = 0.01294

Busy period of Service man (Repair time) (B_0^R) = 0.001979

Busy period of Service man (Replacement time) (B_0^{Rp}) = 0.000717

Using above numerical values, various graphs are drawn for MTSF, Availability and profit of the system for different values of rates of minor as well as major faults.

Fig.2 below shows the graph between MTSF and various rates of minor faults for different values of rates of major faults. It is concluded that MTSF decreases as the rate of minor fault increases and it is lower for higher values of rate of major faults.

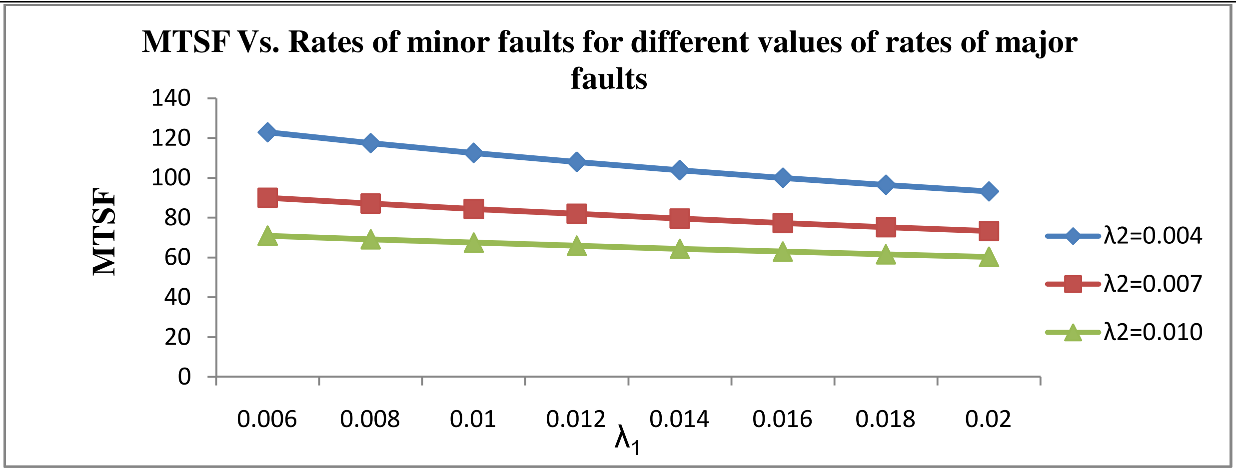

Fig.2: Mean time to system failure Vs. Rates of minor faults.

Fig.3 below shows the graph between availability per unit time and values of rates of minor faults for different values of rates of major faults. It shows that availability decreases with the increase in the rate of minor faults and its value is lower for higher rate of major faults. It can also be observed that rates of major faults put more effect upon availability than the rates of minor faults which have a very low effect and there are very minor differences in availability due to changes in rates of minor faults.

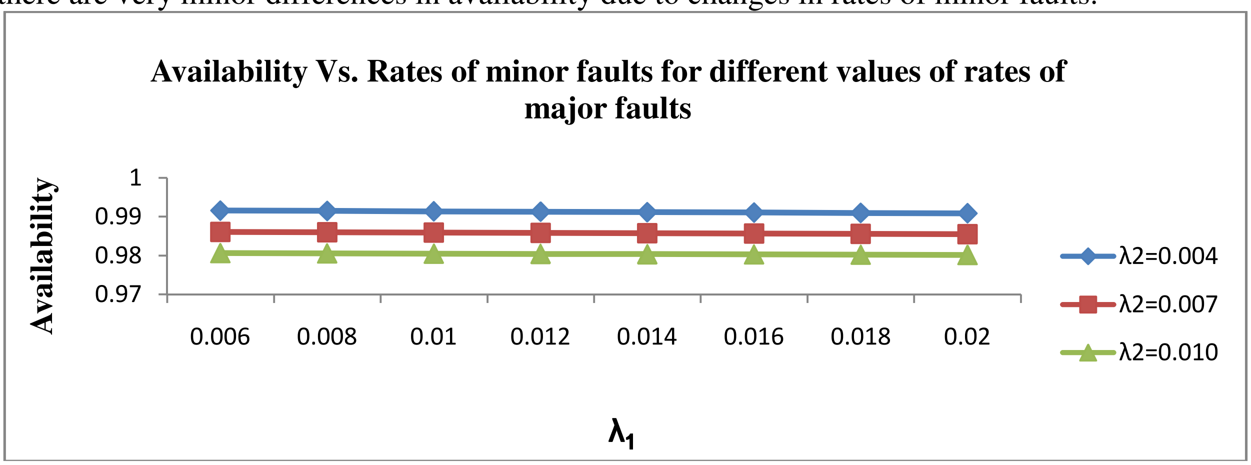

Fig.3: Availability Vs. Rates of minor faults.

Fig.4 below is the graph between profit and rates of minor faults for different values of rates of major faults. It reveals that profit also decreases with the increase in the rate of minor faults and is lower for higher values of rates of major faults. It is further observed that there are more changes in profit due to changes in the rates of major faults but changes in profit are very less due to changes in rates of minor faults.

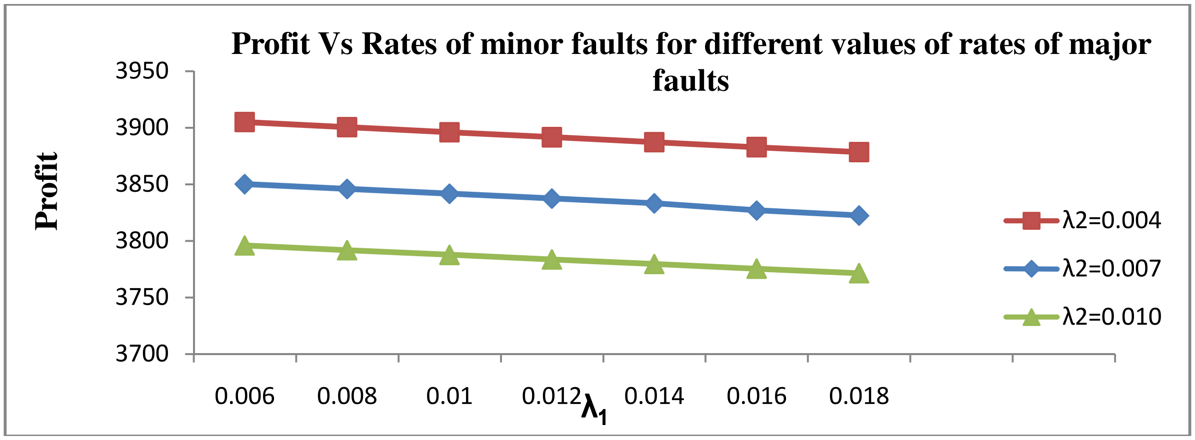

Fig.4: Profit Vs Rates of minor faults.

CONCLUSION:

From analysis of the graphs above, we conclude that mean time to system failure, availability and the profit per unit time of the paper machine decreases with the increase in the values of the rate of minor as well as major faults. Further, it can also be concluded that Major faults put more effect on profit and availability than minor faults and, therefore, for more profit and maximum availability, more attention should be given for removal of major faults.

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MINIMIZING TOTAL WAITING TIME OF J0BS IN SPECIALLY STRUCTURED TWO STAGE FLOWSHOP WITH DISJOINT JOB BLOCK CONCEPT

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ABSTRACT :
The present paper is aimed to provide algorithm for minimizing the total waiting time of jobs for specially structured two stage flow-shop scheduling. The model includes the processing times associated with probabilities and transportation time with disjoint job block criteria. The algorithm is made clear by numerical illustration. The lemma has been provided on which study is based.
Keywords : Waiting time of jobs, Transportation time, Flow shop scheduling, Processing time, string of disjoint Job block etc.

INTRODUCTION

Scheduling means to obtain a sequence of jobs for a set of machines such that certain performance measures are optimized. Scheduling is generally considered to be one of the most important issues in the planning and operation of a manufacturing system. Better scheduling system has vital influence in decreasing the cost, increasing the output, client contentment and on the whole provides competitive assistance to the organization. Manufacturing units and service centers play an important part in the economic growth of nation. Productivity can be increased if existing assets are used in an optimal manner. In the routine working of production houses and service providers numerous applied and experimental situations exist relating to flow shop scheduling. In today’s manufacturing and distribution systems, scheduling have significant role to meet customer requirements as quickly as possible while maximizing the profits. In a flow shop scheduling problem n-jobs are processed on m-machines in a fixed order i.e. the order in which various machines function for completing the job is given. The common objectives in flow shop scheduling problems are to minimize some performance measures such as make-span, mean flow time, mean tardiness, mean setup time, number of tardy jobs and mean number of setups.

LITERATURE REVIEW

To find optimal sequence of jobs the fundamental study was made by Johnson[1] using heuristic approach for n jobs 2 and in restrictive case 3 machines flow-shop scheduling . Ignall and Schrage[2] developed branch and bound algorithms for the permutation flow-shop problem minimizing make-span . Lockett et.al.[3] studied sequencing problems which involves sequence dependent change over times. Maggu and Das and et. al. [4] introduced the equivalent job concept for job block in scheduling problems.

Singh T.P.[5] extended the study by introducing various parameters like transportation time, break down interval, weightage of jobs etc. The work was further extended by Gupta J.N.D.[6], Rajendran C. et. al.[7], Singh T.P. et.al.[8] considering criteria other than make-span. Further Singh T.P., Gupta D. et.al.[9-10] made an attempt to minimize the rental cost of machines including job block through simple heuristic approach. Gupta D. and Bharat Goyal [11] studied specially structured two stage Flow Shop scheduling models with the objective to optimize the total waiting time of jobs. Heydari [13] studied flow shop scheduling problem with processing of jobs in a string of disjoint job blocks. Singh T.P., Kumar, R. and Gupta, D. [14] studied n×2 flow-shop scheduling problem in which processing time, set up time each associated with probabilities along with jobs in a string of disjoint job blocks.

This paper is an extension of study done by Gupta D. and Bharat Goyal [12] in the sense that a string of disjoint job block criteria is taken in account. The concept of job block is significant in scheduling systems where certain orderings of jobs are prescribed either by technological constraints or by externally imposed policy. The string of disjoint job blocks consist of two disjoint job blocks such that in one block the order of jobs is fixed while in other block the order of jobs is arbitrary. Infact the paper is a combination of the study made by [12], [13] and [14]. The objective of the study is to obtain an optimal sequence of jobs to minimize the total waiting time of the machines. An algorithm is proposed to solve the problem and is validated with the help of a numerical illustration.

PRACTICAL SITUATION

Manufacturing units/industries play a momentous role in the economic progress of a country. Flow shop scheduling occurs in various offices, service stations, banks, airports etc. In our routine working in industrial and manufacturing units diverse jobs are practiced on a variety of machines. In textile industry different types of fabric is produced using different types of yarn. Here, the maximum equivalent time taken in dying of yarn on first machine is always less than or equal to the minimum equivalent time taken in weaving of yarn on the second machine. Flow-shop scheduling occurs in various offices, service stations, airports etc. Routine working in industries and factories have diverse jobs which are to be processed on various machines. Sometimes the manufacturer has a minimum time contract with the customers to complete their job. This condition leads to enquire about the best way to schedule the task so that waiting times for the jobs are reduced and greater satisfaction is achieved. The idea of minimizing the waiting time may be a reasonable aspect from manager's point of view in factories/ industries perspective when he has to make decision on minimum time bond with a profit making party to complete the jobs.

Lemma1: Let p jobs 1,2,3,...,p are to be processed through two machines M and N in the order MN with no passing allowed. Let job i (i = 1, 2, 3, ….., p) has processing times M_i and N_i on each machine respectively assuming their respective probabilities s_i and t_i such that $0 \leq s_i \leq 1$; $0 \leq t_i \leq 1$ and $\sum s_i = \sum t_i = 1$.Expected processing times are defined as $M_i' = M_i * s_i$ and $N_i' = Ni * t_i$ satisfying expected processing times structural relationship :Max $M_i' \leq$ Min N_i' then for the p job sequence S:$\alpha_1, \alpha_2, \ldots\ldots, \alpha_p$ where $T_{\alpha N}$ is the completion time of job α on machine N.

Proof: Applying mathematical induction hypothesis on p:

Let the statement S(p): $T_{\alpha_p N} = M'_{\alpha_1} + N'_{\alpha_2} + N'_{\alpha_2} + \ldots\ldots\ldots + N'_{\alpha_p}$

$T_{\alpha_1 M} = M'_{\alpha_1}$ $\quad T_{\alpha_1 N} = M'_{\alpha_1} + N'_{\alpha_1}$

Hence for p = 1 the statement S(1) is true.

Let for p = k, the statement S(k) be true, i.e. $T_{\alpha_k N} = M'_{\alpha_1} + N'_{\alpha_2} + N'_{\alpha_2} + \ldots\ldots\ldots + N'_{\alpha_k}$

Now, $T_{\alpha_{k+1} N} = \mathrm{Max}(T_{\alpha_{k+1} M}, T_{\alpha_k N}) + N'_{\alpha_{k+1}} = \mathrm{Max}(M'_{\alpha_1} + M'_{\alpha_2} + \ldots\ldots + M'_{\alpha_{k+1}}, M'_{\alpha_1} + N'_{\alpha_1} + N'_{\alpha_2} + \ldots.. + N'_{\alpha_k}) + N'_{\alpha_{k+1}}$

As $\mathrm{Max}M'_i \leq \mathrm{Min}N'_i$

Hence $T_{\alpha_{k+1} N} = M'_{\alpha_1} + N'_{\alpha_2} + N'_{\alpha_2} + \ldots\ldots\ldots + N'_{\alpha_k} + N'_{\alpha_{k+1}}$

Hence for p = k + 1 the statement S(k+1) holds true. Since S(p) is true for p = 1; p = k; p = k + 1 and k being arbitrary.

Hence S(p): $T_{\alpha_p N} = M'_{\alpha_1} + N'_{\alpha_2} + N'_{\alpha_2} + \ldots\ldots + N'_{\alpha_p}$ is true.

Lemma2: With the same notations as that of Lemma 1 , for p job sequence

S: $\alpha_1, \alpha_2, \ldots\ldots, \alpha_p$

$W_{\alpha_1} = 0$

$W_{\alpha_k} = M'_{\alpha_1} + \sum_{r=1}^{k-1} (x_{\alpha_r}) - M'_{\alpha_k}$

where W_{α_k} is the waiting time of job α_k for the sequence($\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_p$) and $x_{\alpha_r} = N'_{\alpha_r} - M'_{\alpha_r}$,

$\alpha_r \epsilon$ (1, 2, 3, ….. , p)

Proof: $W_{\alpha_1} = 0$

$W_{\alpha_k} = \mathrm{Max}(T_{\alpha_k M}, T_{\alpha_{k-1} N}) - T_{\alpha_k M}$

$= \mathrm{Max}(M'_{\alpha_1} + M'_{\alpha_2} + \ldots.. + M'_{\alpha_k}, M'_{\alpha_1} + N'_{\alpha_1} + N'_{\alpha_2} + \ldots.. + N'_{\alpha_{k-1}}) - (M'_{\alpha_1} + M'_{\alpha_2} + \ldots\ldots. + M'_{\alpha_k})$

$= M'_{\alpha_1} + N'_{\alpha_1} + N'_{\alpha_2} + \ldots.. + N'_{\alpha_{k-1}} - M'_{\alpha_1} - M'_{\alpha_2} \ldots\ldots - M'_{\alpha_k} = M'_{\alpha_1} + (N'_{\alpha_1} - M'_{\alpha_1}) + (N'_{\alpha_2} - M'_{\alpha_2}) + \ldots\ldots + (N'_{\alpha_{k-1}} - M'_{\alpha_{k-1}}) - M'_{\alpha_k}$

$= M'_{\alpha_1} + \sum_{r=1}^{k-1} (N'_{\alpha_r} - M'_{\alpha_r}) - M'_{\alpha_k} = M'_{\alpha_1} + \sum_{r=1}^{k-1} (x_{\alpha_r}) - M'_{\alpha_k}$

THEOREM : Let p jobs 1,2,3,...,p are to be processed through two machines M and N in the order MN with no passing allowed. Let job i(i = 1,2,3,…, p) has processing times M_i and N_i on each machine respectively assuming their respective probabilities si and ti such that $0 \leq s_i \leq 1$, $0 \leq t_i \leq 1$ and $\sum s_i = \sum t_i = 1$. Expected processing times are defined as $M'_i = M_i * s_i$ and $N'_i = N_i * t_i$ satisfying expected processing times structural relationships : Max $M'_I \leq$ Min N'_i then for the p job sequence S: $\alpha_1, \alpha_2, \ldots.., \alpha_p$ the total waiting time T_w(say)

$T_w = pM'_{\alpha_1} + \sum_{r=1}^{p-1} (z_{\alpha_r}) - \sum_{i=1}^{p} (M'_i) z_{\alpha_r} = (p - r)x_{\alpha_r}$; $\alpha_r \epsilon$ (1, 2, 3, …., p)

Proof: From Lemma 2 we have, $W_{\alpha_1} = 0$

k = 2, k - 1 = 1 $W_{\alpha_2 N} = M'_{\alpha_1} + \sum_{r=1}^{1} (x_{\alpha_r}) - M'_{\alpha_2}$

$= M'_{\alpha_1} + x_{\alpha_1} - M'_{\alpha_2}$

k = 3, k - 1 = 2 $W_{\alpha_3} = M'_{\alpha_1} + \sum_{r=1}^{2} (x_{\alpha_r}) - M'_{\alpha_3}$

$= M'_{\alpha_1} + x_{\alpha_1} + x_{\alpha_2} - M'_{\alpha_3}$ Continuing in this way

k = p, k -1 = p - 1

$W_{\alpha_p} = M'_{\alpha_1} + \sum_{r=1}^{p-1} (x_{\alpha_r}) - M'_{\alpha_p} = M'_{\alpha_1} + x_{\alpha_1} + x_{\alpha_2} + \ldots.. + x_{\alpha_{p-1}} - M'_{\alpha_p}$ Hence total waiting time

$T_w = W_{\alpha_1} + W_{\alpha_2} + W_{\alpha_3} + \ldots\ldots + W_{\alpha_p}$

$T_w = 0 + (M'_{\alpha_1} + x_{\alpha_1} - M'_{\alpha_2}) + (M'_{\alpha_1} + x_{\alpha_1} + x_{\alpha_2} - M'_{\alpha_3}) + \ldots. + (M'_{\alpha_1} + x_{\alpha_1} + x_{\alpha_2} + \ldots.. + x_{\alpha_{p-1}} - M'_{\alpha_p})$

$T_w = (M'_{\alpha_1} + M'_{\alpha_1} + \ldots\ldots + (p\text{-}1)\text{times}) + (x_{\alpha_1} + x_{\alpha_1} + \ldots\ldots. + (p\text{-}1)\text{times}) + (x_{\alpha_2} + x_{\alpha_2} + \ldots.. + (p\text{-}2)\text{times}) + \ldots..(x_{\alpha_{p-1}} - (M'_{\alpha_2} + M'_{\alpha_3} + \ldots. + M'_{\alpha_p}))$

$T_w = (p\text{-}1)M'_{\alpha_1} + (p\text{-}1)(x_{\alpha_1} + (p-2)x_{\alpha_2} + \ldots.. + x_{\alpha_{p-1}} - (\sum_{i=1}^{p} M'_{\alpha i} - M'_{\alpha_1})$

$= pM'_{\alpha_1} + \sum_{r=1}^{k-1} (p-r)(x_{\alpha_r}) - \sum_{i=1}^{p} (M'_{\alpha_i})$

Equivalent Job Block Theorem: In processing a schedule s=(1,2,3,...,p) of p jobs on two machines M and N in the order MN with no passing allowed. A job i (i=1,2, 3...,p) has processing time M_i and N_i on each machine respectively. The job block (k,m) is equivalent to the single job . Now the processing times of job on the machine M and N are denoted respectively by M, N are given by $M_\alpha = M_k - M_m - \min(M_m,N_k)$

$N_\alpha = N_k - N_m - \min(M_m,N_k)$

The proof of the theorem is given by Maggu P.L. and Das G.

PROBLEM FORMULATION

Let n jobs are to be processed through two machines A and B in order AB. Let a_k and b_k denotes the processing time associated with probabilities for k^{th} job on these machines. Let t_k be the transportation time of k^{th} job from machine A to machine B. Let two job blocks be α and β such that block α consists of i jobs out of n jobs in which the order of jobs is fixed and β consists of r jobs out of n in which order of jobs is arbitrary such that i+r=n. Let $\alpha \cap \beta = \phi$ i.e. the two job blocks α and β form a disjoint set in the sense that the two job blocks have no job in common. Also we consider the structural relationship i.e. Min $a_k \geq$ Max b_k holds good.

Tableau 1: Matrix form of the problem

Jobs	Machine A		Transportation Time (A→B)	Machine B	
J	a_k	p_k	t_k	b_k	q_k
1	a_1	p_1	t_1	b_1	q_1
2	a_2	p_2	t_2	b_2	q_2
3	a_3	p_3	t_3	b_3	q_3
. . .	. . .	. . .	. . .	. . .	. . .
N	a_n	p_n	t_n	b_n	q_n

Our aim is to find the best possible sequence S_i of jobs with minimum total waiting time of jobs.

ASSUMPTIONS

In the given flow shop scheduling the following assumptions are made

1) There are n number of jobs (J) and two machines (A and B)
2) Time intervals for processing time are independent of the order in which operations are performed.
3) $\sum_{k=1}^{n} p_k = \sum_{k=1}^{n} q_k = 1$
4) A job is an entity i.e. even though the job represents a lot of individual part, no job may be processed by more than one machine at a time.
5) Each operation once started must performed till completion.

ALGORITHM

Step 1: Calculate expected processing times, a'k and b'k on machines A and B defined as follows: $a'_k = a_k \times p_k$, $b'_k = b_k \times q_k$.

Step 2: Define the fictitious machines X and Y with processing times $X_k$and Y_k as follows: $X_k = a'_k + t_k$ and $Y_k = b'_k + t_k$ and verify the structural relationship Min $X_k \geq$ MaxY_k.

Step 3: Take equivalent job α for the job block (r, m) and calculate the processing time$A_{\alpha 1}$and $A_{\alpha 2}$ on the guidelines of Maggu and Das [6] as follows:

$X_{\alpha 1} = X_{r1} + X_{m1} - \min(X_{m1}, X_{r2})$.$Y_{\alpha 2} = Y_{r2} + Y_{m2} - \min(X_{m1}, X_{r2})$.

If a job block has three or more than three jobs then to find the expected flow times we use the property that the equivalent job for job-block is associative i.e. $((i_1, i_2), i_3) = (i_1, (i_2, i_3))$.

Step 4: Obtain the new job block from the job block (disjoint from job block) by the proposed algorithm. Obtain the processing times and as defined in step 2.

Step 5: Now, reduce the given problem to a new problem by replacing s-jobs by job block α with the processing times $X_{\alpha 1}$ and $X_{\alpha 2}$ and remaining r= (n–s) jobs by a disjoint job block β_k with processing times and as defined in step 2.Now form the table in the following format-

Tableau 2

Jobs	Machine X_k	Machine Y_k	$x_k = Y_k - X_k$
A	$X_{\alpha 1}$	$Y_{\alpha 2}$	x_1
β_k	$X_{\beta k 1}$	$Y_{\beta k 2}$	x_2

Step 6: Arrange the jobs in increasing order of x_k. Let the sequence be$(\mu_1, \mu_2, \ldots, \mu_n)$.

Step 7: Find Min X_kNow two cases arise

a) If X_{μ_1} = Min X_k, then schedule according to step 5 is required optimal sequence.

b) If $X_{\mu_1} \neq$ Min X_k, then go to next step.

Step 8: Consider the different sequences of jobs $S_1, S_2, \ldots, S_r$ where S_1 is the sequence obtained in step 3 , sequence S_k(k = 2,3, …., r) can be obtained by placing k^{th} job in the sequence S_1 to the first position and rest of the sequence remaining same.

Step 9: Form the table in the following format:

Tableau 3

Jobs (J)	Machine X (X_k)	MachineY (Y_k)	$x_k = Y_k - X_k$	$Z_{kr} = (n-r)x_k$			
				r=1	r=2	. . . .	r=(n-1)
1	X_1	Y_1	x_1	z_{11}	z_{12}	. . . .	$z_{1(n-1)}$
2	X_2	Y_2	x_2	z_{21}	z_{22}	. . . .	$z_{2(n-1)}$
3	X_3	Y_3	x_3	z_{31}	z_{32}	. . . .	$z_{3(n-1)}$
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
N	X_n	Y_n	x_n	z_{n1}	z_{3n}	. . . .	$z_{n(n-1)}$

Step 10: Calculate the waiting time T_w for all the sequences $S_1, S_2, \ldots, S_r$ using the formula:
$T_w = nX_{\mu_1}+\sum_{r=1}^{n-1} (Z_{ar})- \sum_{k=1}^{n} (X_k)$ where $X_{\mu 1}$ =Equivalent processing time of first job on machine X in sequence S_k. $Z_{ar} = (n - r)x_{ar}$; $a = \mu_1, \mu_2, \ldots, \mu_n$

The sequence with minimum waiting time is required optimal sequence.

NUMERICAL ILLUSTRATION

Assume 5 jobs 1,2,3,4,5 are to be processed on two machines A and B with processing times a_k, b_k and t_k is the transportation time of k^{th} job from machine A to machine B.

Tableau 4

Jobs	Machine A		Transportation Time (A→B)	Machine B	
J	a_k	p_k	t_k	b_k	q_k
1	6	0.2	4	12	0.2
2	7	0.2	3	21	0.2
3	12	0.2	2	34	0.2
4	11	0.3	3	22	0.2
5	13	0.2	2	24	0.2

Our objective is to find an optimal sequence to minimize the total waiting time.

As per step 1- Expected processing times a'_k and b'_k on machines A and B are calculated in the following table

Tableau 5

Jobs (J)	Machine A (a'_k)	Transportation Time (A→B)	Machine B (b'_k)
1	1.2	4	2.4
2	1.4	3	4.2
3	2.4	2	6.8
4	3.3	3	4.4
5	2.6	2	4.8

As per step 2- Defining the fictitious machines X and Y with processing times $X_k = a'_k + t_k$ and $Y_k = b'_k + t_k$ respectively.

Tableau 6

Jobs (J)	Machine X (X_k)	MachineY (Y_k)
1	5.2	6.4
2	4.4	7.2
3	4.4	8.8
4	6.3	7.4
5	4.6	6.8

$\text{Max}X_k = 4.6 < \text{Min}Y_k = 6.4$

As per step3: Take equivalent job $\alpha = (2, 5)$. Then processing times are defined as follows $X_\alpha = X_2+X_5 - \text{Min}(X_5, Y_2) = 4.6$ and $Y_\alpha = Y_2 + Y_5 - \text{Min}(X_5, Y_2) = 9.6$

As per step 4: Taking new job block $\beta = (1,3,4)$ or $(\gamma,4)$ where $\gamma = (1,3)$.

As per step 5:Then processing times are defined as described in step 2. And forming a table in following format-

Tableau 7

Jobs	Machine X_k	Machine Y_k	$x_k = Y_k - X_k$
Β	6.3	13.0	6.7
Α	4.6	9.6	5.0

As per step 6: Arranging the jobs in increasing order of x_k

Tableau 8

Jobs	Machine X_k	Machine Y_k	$x_k=Y_k - X_k$
A	4.6	9.6	5.0
B	6.3	13.0	6.7

As per step 7:Min X_k= 4.6=4.6

As per step 8- The sequence obtained is the optimal sequence. S= (α,β) =(2,5,1,3,4)

As per step 9: Form the table in the following format:

Tableau 9

Jobs (J)	Machine X (X_k)	MachineY (Y_k)	$x_k=Y_k - X_k$	$Z_{kr}=(n-r)x_k$			
				r=1	r=2	r=3	r=4
1	5.2	6.4	1.2	4.8	3.6	2.4	1.2
2	4.4	7.2	2.8	11.2	8.4	5.6	2.8
3	4.4	8.8	4.4	17.6	13.2	8.8	4.4
4	6.3	7.4	1.1	4.4	3.3	2.2	1.1
5	4.6	6.8	2.2	8.8	6.6	4.4	2.2

As per step 10- Calculate the total waiting time for the sequence S $\sum_{k=1}^{n}(X_k) = 24.9$

For the sequence S = (α,β) or (2, 5,1,3,4) Total waiting time $T_w = 5 \times 11.2 + 6.6 + 2.4 + 4.4 - 24.9$

$= 44.5$

Hence the sequence, S = (2,5,1,3,4) is the required sequence which minimizes total waiting time for the said problem.

CONCLUSION

The present study deals simple flowshop scheduling model with the main idea to optimize the total waiting time of jobs. However, it may increase the other costs like machine idle cost or penality cost of the jobs, yet the idea of minimizing the waiting time is a matter that cannot be avoided in cases when there is a minimum time contract with the customers. The study can be extended by introducing various parameters like weightage of jobs , setup time of machines, breakdown interval of machines, fuzzy processing time etc.

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CERTAIN SUBCLASSES OF UNIFORMLY SPIRALLIKE FUNCTIONS ASSOCIATED WITH THE GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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ABSTRACT :

For the hypergeometric function $F(a,b,c;z)=\sum_{n=0}^{\infty}\left\{\frac{(a)_n(b)_n}{(c)_n(1)_n}\right\}z^n$, ***if conditions are applied on a, b, c then zF(a, b, c; z) will be in certain subclasses of uniformly spirallike functions. An integral operator related to this hypergeometric function is also considered.***

Keywords and Phrases: Gaussian hypergeometric functions, univalent functions, uniformly spirallike functions, integral operator.
2010 Mathematics Subject Classification: 30C45.

1. INTRODUCTION AND DEFINITIONS

$\mathcal{S}$ denotes the class of functions of the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ that are analytic and univalent in the open unit disk U = {z ∈ C : |z| < 1} normalized by f(0) = 0, f'(0)= 1. $\mathcal{S}^*$ and C denote the subclasses of $\mathcal{S}$ that are star like and convex respectively. In [1, 2] certain subclass of star like functions called uniformly star like functions UST and subclass of convex functions namely uniformly convex functions UCV were introduced.

A new class Sp of starlike functions was introduced by Ronning in [3]. UCV and Sp got extended further by Kanas and Wisniowska in [4] as k-UCV(α) and k-ST(α). Ravichandran et al. in [5] introduced the classes of uniformly spirallike and uniformly convex spiral like which got further generalized in [6] as UCSP(α, β).

In [14], Herb Silverman introduced the subclass T of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

Motivated by [7] new subclasses with negative coefficients UCSPT(α, β) and $SP_pT(\alpha, \beta)$ were introduced and studied in [8] and got extended as $UCSPT_c(\alpha, \beta)$ and $SP_pT_c(\alpha, \beta)$ in [9, 10].

$UCSPT_c(\alpha, \beta)$ is the class of functions in $UCSPT(\alpha,\beta)$ of the form

$$f(z) = z - \frac{c(\cos\alpha - \beta)z^2}{2(4 - \cos\alpha - \beta)} - \sum_{n=3}^{\infty} a_n z^n, \tag{1.2}$$

($a_n \geq 0$), where $0 \leq c \leq 1$. When c = 1$UCSPT_1(\alpha, \beta) = UCSPT(\alpha, \beta)$.

$SP_pT_c(\alpha, \beta)$ is the subclass of functions in $SP_pT(\alpha, \beta)$ of the form

$$f(z) = z - \frac{c(\cos\alpha - \beta)z^2}{4 - \cos\alpha - \beta} - \sum_{n=3}^{\infty} a_n z^n, \tag{1.3}$$

($a_n \geq 0$), where $0 \leq c \leq 1$. With c = 1,$SP_pT_1(\alpha, \beta) = SP_pT(\alpha, \beta)$.

The Hadamard product of 2 analytic functions f and g in U is $f * g = \sum_{n=0}^{\infty} a_n b_n z^n$ where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

The (Gaussian) hypergeometric function F(a, b, c; z) is defined by $F(a,b,c;z) = \sum_{n=0}^{\infty} \left\{ \frac{(a)_n (b)_n}{(c)_n (1)_n} \right\} z^n$, where c $\neq 0, -1, -2$ and$(\beta)_n$ is the Pochhammer symbol defined by

$$(\beta)_n = \begin{cases} 1, & \text{if } n = 0 \\ \beta(\beta+1)\ldots(\beta+n-1), & \text{if } n \in N = \{1,2,\ldots\} \end{cases}.$$

Also, $F(a,b,c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}$, $Re(c-a-b) > 0$. The sufficient condition for zF(a,b, c; z) to be in $SP_p(\alpha, \beta)$, $UCSP(\alpha, \beta)$ and $UCSPT(\alpha, \beta)$ was established in [11].

In this paper we extend the results to $UCSPT_c(\alpha, \beta)$ and $SP_pT_c(\alpha, \beta)$. The operator $I_{a,b,c}(f)$ of Hohlov [13]; which maps $\mathcal{S}$ into itself is defined by

$[I_{a,b,c}(f)](z) = zF(a, b, c; z) * f(z)$, where * denotes the Hadamard product, is considered.

2. PRELIMINARY RESULTS

The following lemmas are needed to establish our main results.

Lemma 2.1. [12]

Let a, b $\in$ C \{0}, c > 0. Then we have the following:

(i) For a, b > 0, c > a+b+1,

$$\sum_{n=0}^{\infty} (n+1) \frac{(a)_n (b)_n}{(c)_n (1)_n} = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{ab}{c-1-a-b} + 1 \right].$$

(ii) For a, b > 0, c > a+b+2,

$$\sum_{n=0}^{\infty}(n+1)^2\frac{(a)_n(b)_n}{(c)_n(1)_n}=\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{(a)_2(b)_2}{(c-2-a-b)_2}+\frac{3ab}{c-1-a-b}\right].$$

(iii) For $a \neq 1$, $b \neq 1$, $c \neq 1$ with $c > \max\{0, a+b-1\}$,

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_{n+1}}=\frac{1}{(a-1)(b-1)}\left[\frac{\Gamma(c+1-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}-(c-1)\right].$$

Lemma 2.2. [9]

The function f(z) defined by (1.2) belongs to $UCSPT_c(\alpha, \beta)$ if and only if

$$\sum_{n=3}^{\infty}(2n-\cos\alpha-\beta)na_n \leq (1-c)(\cos\alpha-\beta).$$

Lemma 2.3. [10]

The function f(z) defined by (1.3) belongs to $SP_pT_c(\alpha, \beta)$ if and only if

$$\sum_{n=3}^{\infty}(2n-\cos\alpha-\beta)a_n \leq (1-c)(\cos\alpha-\beta).$$

3. MAIN RESULTS

We prove in this section certain theorems concerning the sufficient condition for the hypergeometric function to be in various subclasses of uniformly spiral like functions such as $UCSPT_c(\alpha,\beta)$ and $SP_pT_c(\alpha, \beta)$. To avoid confusion of the alphabet c in F(a, b, c; z) and $UCSPT_c(\alpha, \beta)$ and $SP_pT_c(\alpha, \beta)$ we use c_1 for the suffix.

Theorem 3.1.

If $a, b \in C \setminus\{0\}$, $c > a + b + 1$ and

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{2ab}{c-1-a-b}+2-\cos\alpha-\beta\right]$$
$$\leq -\left(\frac{ab}{c}+c_1\right)\cos\alpha+\left(-2+c_1-\frac{ab}{c}\right)\beta$$
$$+2\left(1+\frac{2ab}{c}\right),$$

then, $zF(a, b, c; z) \in SP_pT_{c_1}(\alpha,\beta)$.

Proof.

Using the condition for a function to be in $SP_pT_{c_1}(\alpha,\beta)$ (Lemma 2.3), we have to prove that

$$\sigma=\sum_{n=3}^{\infty}(2n-\cos\alpha-\beta)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq (1-c_1)(\cos\alpha-\beta).$$

But

$$\sigma = \sum_{n=2}^{\infty}(2(n+1)-\cos\alpha-\beta)\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$\leq 2\sum_{n=0}^{\infty}(n+1)\frac{(a)_n(b)_n}{(c)_n(1)_n}-(\cos\alpha+\beta)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$
$$-2+\cos\alpha+\beta-(4-\cos\alpha-\beta)\frac{ab}{c}$$

$$=\frac{2\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{ab}{c-1-a-b}+1\right]-(\cos\alpha+\beta)\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$
$$-2+\cos\alpha+\beta-(4-\cos\alpha-\beta)\frac{ab}{c}$$
$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{2ab}{c-1-a-b}+2-\cos\alpha-\beta\right]$$
$$-2+\cos\alpha+\beta-\frac{4ab}{c}+(\cos\alpha+\beta)\frac{ab}{c}$$
$$\leq(1-c_1)(\cos\alpha-\beta)$$

if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{2ab}{c-1-a-b}+2-\cos\alpha-\beta\right]$$
$$\leq-\left(\frac{ab}{c}+c_1\right)\cos\alpha+\left(-2+c_1-\frac{ab}{c}\right)\beta+2\left(1+\frac{2ab}{c}\right),$$

with $c > a + b + 1$. □

Theorem 3.2.

If $a, b \in C\setminus\{0\}$, $c > a + b + 2$ and

$$\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}\left\{2\left[1+\frac{(a)_2(b)_2}{(c-1-a-b)_2}+\frac{3ab}{c-1-a-b}\right]\right.$$
$$\left.-(\cos\alpha+\beta)\left[\frac{ab}{c-1-a-b}+1\right]\right\}$$
$$\leq-\left(\frac{2ab}{c}+c_1\right)\cos\alpha+\left(-2+c_1-\frac{2ab}{c}\right)\beta+2\left(1+\frac{4ab}{c}\right)$$

then, $zF(a, b, c; z) \in UCSPT_{c_1}(\alpha,\beta)$.

Proof.

From Lemma 2.2 we have to prove that

$$\sigma = \sum_{n=3}^{\infty} (2n - \cos\alpha - \beta) n \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$\leq (1 - c_1)(\cos\alpha - \beta).$$

But

$$\begin{aligned}\sigma &= \sum_{n=2}^{\infty} (2(n+1) - \cos\alpha - \beta)(n+1) \frac{(a)_n (b)_n}{(c)_n (1)_n} \\ &\leq 2\sum_{n=0}^{\infty} (n+1)^2 \frac{(a)_n (b)_n}{(c)_n (1)_n} - (\cos\alpha + \beta) \sum_{n=0}^{\infty} (n+1) \frac{(a)_n (b)_n}{(c)_n (1)_n} \\ &\quad - (2 - \cos\alpha - \beta) - (4 - \cos\alpha - \beta) 2 \frac{(a)_1 (b)_1}{(c)_1 (1)_1} \\ &= \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \left\{ 2\left[1 + \frac{(a)_2 (b)_2}{(c-2-a-b)_2} + \frac{3ab}{c-1-a-b} \right] \right. \\ &\quad \left. - (\cos\alpha + \beta)\left[\frac{ab}{c-1-a-b} + 1 \right] \right\} - 2 + \cos\alpha + \beta \\ &\quad - \frac{8ab}{c} + (\cos\alpha - \beta) 2 \frac{ab}{c} \\ &\leq (1 - c_1)(\cos\alpha - \beta)\end{aligned}$$

if and only if

$$\begin{aligned}&\frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \left\{ 2\left[1 + \frac{(a)_2 (b)_2}{(c-2-a-b)_2} + \frac{3ab}{c-1-a-b} \right] \right. \\ &\quad \left. -(\cos\alpha + \beta)\left[\frac{ab}{c-1-a-b} + 1 \right] \right\} \\ &\leq -\left(\frac{2ab}{c} + c_1 \right)\cos\alpha + \left(-2 + c_1 - \frac{2ab}{c} \right)\beta + 2\left(1 + \frac{4ab}{c} \right),\end{aligned}$$

with $c > a + b + 2$. □

Corollary 3.3.

With $\beta = 0$ in theorem 3.1 and theorem 3.2 we get the sufficient conditions for $zF(a, b, c; z)$ to be in $SP_pT_{c_1}(\alpha)$ and $UCSPT_{c_1}(\alpha)$.

Theorem 3.4.

Let $a \in C \setminus \{0\}$, $c > \max\{2 + 2\,\mathrm{Re}\, a, 0\}$,

(i) If

$$\frac{\Gamma(c)\Gamma(c-2\mathrm{Re}\, a)}{\Gamma(c-a)\Gamma(c-\bar{a})} \left[\frac{2|a|^2}{c-1-2\mathrm{Re}\, a} + 2 - \cos\alpha - \beta \right]$$
$$\leq -\left(\frac{|a|^2}{c} + c_1 \right)\cos\alpha + \left(-2 + c_1 - \frac{|a|^2}{c} \right)\beta + 2\left(1 + \frac{2|a|^2}{c} \right),$$

then $zF(a,\bar{a},c;z)\in SP_pT_{c_1}(\alpha,\beta)$.

(ii) If

$$\frac{\Gamma(c)\,\Gamma(c-2\operatorname{Re} a)}{\Gamma(c-a)\,\Gamma(c-\bar{a})}\left\{2\left[1+\frac{|(a)_2|^2}{(c-1-2\operatorname{Re} a)_2}+\frac{3|a|^2}{c-1-2\operatorname{Re} a}\right]\right.$$
$$\left.-(\cos\alpha+\beta)\left[\frac{|a|^2}{c-1-2\operatorname{Re} a}+1\right]\right\}$$
$$\le -\left(\frac{2|a|^2}{c}+c_1\right)\cos\alpha+\left(-2+c_1-\frac{2|a|^2}{c}\right)\beta+2\left(1+\frac{4|a|^2}{c}\right),$$

then $zF(a,\bar{a},c;z)\in UCSPT_{c_1}(\alpha,\beta)$.

Proof.

By taking $b=\bar{a}$ in theorem 3.1 and theorem3.2 we get the proof. □

Theorem3.5.

If a, b ∈ C \{0}, a, b > 0 and c > a+b+1,

$$2(1-\beta)\cos\eta\left\{\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}\left[\frac{2ab}{c-1-a-b}+2-\cos\alpha-\beta_1\right]\right.$$
$$\left.-2+\cos\alpha+\beta_1-\frac{4ab}{c}+(\cos\alpha+\beta_1)\frac{ab}{c}\right\}$$
$$\le(1-c_1)(\cos\alpha-\beta_1),$$

then the operator $I_{a,b,c}(f)$ maps $R_\eta(\beta)$ into $UCSPT_{c_1}(\alpha,\beta_1)$.

Proof.

Let f(z) which is defined in (1.1) be in $R_\eta(\beta)$.From the coefficient estimate we have $a_n\le\frac{2}{n}(1-\beta)\cos\eta$.

By Lemma 2.2, it is enough to prove that

$$\sum_{n=3}^{\infty}(2n-\cos\alpha-\beta_1)2(1-\beta)\cos\eta\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\le(1-c_1)(\cos\alpha-\beta_1).$$

Considering the LHS of the above inequality, we get

$$2(1-\beta)\cos\eta\sum_{n=3}^{\infty}(2n-\cos\alpha-\beta_1)\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$=2(1-\beta)\cos\eta\left\{2\sum_{n=0}^{\infty}(n+1)\frac{(a)_n(b)_n}{(c)_n(1)_n}\right.$$

$$\left.-(\cos\alpha+\beta_1)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-(2-\cos\alpha-\beta_1)-(4-\cos\alpha-\beta_1)\frac{ab}{c}\right\}$$

$$=2(1-\beta)\cos\eta\left\{\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{2ab}{c-1-a-b}+2-\cos\alpha-\beta_1\right]\right.$$

$$\left.-2+\cos\alpha+\beta_1-\frac{4ab}{c}+(\cos\alpha+\beta_1)\frac{ab}{c}\right\}$$

$$\leq(1-c_1)(\cos\alpha-\beta_1),$$

then $I_{a,b,c}(f)$ maps $R_\eta(\beta)$ into $UCSPT_{c_1}(\alpha,\beta_1)$. □

Corollary 3.6.

If $a \in C\setminus\{0\}$, $c > \max\{0, 1 + 2\,\mathrm{Re}\, a\}$ and satisfies the condition

$$2(1-\beta)\cos\eta\left\{\frac{\Gamma(c)\Gamma(c-2\mathrm{Re}\,a)}{\Gamma(c-a)\Gamma(c-\overline{a})}\left[\frac{2|a|^2}{c-2\mathrm{Re}\,a-1}+2-\cos\alpha-\beta_1\right]\right.$$

$$\left.-2+\cos\alpha+\beta_1-\frac{4|a|^2}{c}+(\cos\alpha+\beta_1)\frac{|a|^2}{c}\right\}$$

$$\leq(1-c_1)(\cos\alpha-\beta_1),$$

then the operator $I_{a,\overline{a},c}(f)$ maps $R_\eta(\beta)$ into $UCSPT_{c_1}(\alpha,\beta_1)$.

CONCLUSION

More such properties can be proved for $UCSPT_c(\alpha,\beta)$ and $SP_pT_c(\alpha,\beta)$. Also the results can be extended to similar subclasses of uniformly spirallike functions.

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ON SOME NON-EXTENDABLE GAUSSIAN TRIPLES INVOLVING MERSENNE AND GNOMONIC NUMBER

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ABSTRACT :
In this paper we made an attempt to generate some non-extendable Gaussian triples involving Mersenne number, Gnomonic number under three sections with the properties D (1), D (9) and D (64).

INTRODUCTION

The Gaussian integer z[i] is the handiest generalization of ordinary integer Z and that they behave in plenty the identical way. Carl Friedrich Gaussian first fathomed this in his second treatise on quartic reciprocity in the year 1832. z[i] adores distinct prime factorization and this permits z[i] to manage as the same way we do about Z. Gaussian integers are identified as $z[i]=\{a+ib; a, b \,\varepsilon\, Z\}$.

[1-18] has been studied for revamping the basic ideologies on Pythagorean and Dio phantine problems. [19-22] gives diverse methodologies to generate the Dio triples. [23-27] provides various triples generated from several numbers. In [28,29] Gaussian triples and quadruples were generated. In this paper we try to outline some non-extendable Gaussian triples(k, l, m) involving Mersenne number and Gnomonic number under three sections with the properties D (1), D (9) and D (64).

Notations:

M_n = Mersenne number of rank n.

Gno_n = Gnomonic number of rank n.

METHOD OF ANALYSIS

Here we outline a few non-extendable triples beneath 3 sections. In section A, the non-extendables are generated from Mersenne numbers with the property D(1), in Section B it is generated from Gnomonic number with the property D(9) and we present a widespread case for the non-extendables in section C with the property D (64).

Here, D (1) refers to the property that for any a, b we have a * b+1 = square number.

Similarly the property D(9) and D(64) replicates that a* b + 9 = square number and a* b+ 64 = square number.

Section-A:

Let $k=M_n-2+i; l=M_n-4+i$ be two Gaussian integers.

We have $k\cdot l+1=\alpha^2$ (1)

where $\alpha=M_n-3+i$

Hence (k, l) is a Gaussian double with the property D(1).

Introduce m such that m is any non-zero Gaussian integer such that,

$k \cdot m+1=\beta^2$ (2)

$l \cdot m+1=\gamma^2$ (3)

Taking $\beta = k+\alpha$ and $\gamma = l+\alpha$, we have from (2) and (3)

$m = 4(M_n - 3 + i)$ (4)

Thus we have the Gaussian triples $\{M_n - 2 + i, M_n - 4 + i, 4(M_n - 3 + i)\}$ with the property D (1).

Non-Extendibility of the triple:

Now the fourth tuple for the property D(1) can be obtained from the formula $n = k + l + m + 2[klm + \alpha\beta\gamma]$, and is given by

$n = 838(2^n) + 24(2^{3n}) - 252(2^{2n}) + i[626 + 52(2^{2n}) - 364(2^n)]$ (5)

The 'n' obtained from the formula (5) when commuted with k, l or m and increased by one does not give a perfect square and for this reason this Gaussian triple cannot be extended to a Gaussian quadruple.

Section-B:

Let $k = 4Gno_p + 9 + i8q$; $l = 4Gno_p + 15 + i8q$ be two Gaussian integers.

We have $k \cdot l + 9 = \alpha^2$ (6)

where, $\alpha = 8p + 8 + i8q$

Hence (k, l) is a Gaussian double with the property D(9).

Introduce m such that m is any non-zero Gaussian integer such that,

$k \cdot m + 9 = \beta^2$ (7)

$l \cdot m + 9 = \gamma^2$ (8)

Taking $\beta = k+\alpha$ and $\gamma = l+\alpha$, we have from (7) and (8),

$m = 16(Gno_p + 3 + i2q)$ (9)

Thus we have the Gaussian triples $\{4Gno_p + 9 + i8q, 4Gno_p + 15 + i8q, 16(Gno_p + 3 + i2q)\}$ with the property D (9).

NON-EXTENDIBILITY OF THE TRIPLE:

Now the fourth tuple for the property D (9) can be obtained from the formula

$n = k + l + m + \frac{2}{9}[klm + \alpha\beta\gamma]$ which is given by,

$$n = 1/9[(24288p + 7904 + 8192p^3 + 24576p^2 - 24576pq^2 - 24576q^2) + i(49152pq + 24288q + 24576p^2q - 8192q^3)] \quad (10)$$

The value of 'n' obtained from (10) when commuted with k, l or m and increased by nine does not give a perfect square and for this reason this Gaussian triple cannot be extended to a Gaussian quadruple.

Section-C:

General case:

For any a > 0, let $k = ap - 4 + iaq$, $l = ap + 12 + iaq$ are two Gaussian integers.

Proceeding as in section-A, we get the double (k, l) as a Gaussian double with the property D (64).

Again this double can be extended into a Gaussian triple (k, l, m) where $m = 4(ap + 4 + iaq)$.

The fourth tuple can be generated using the formula

$n = k + l + m + \frac{2}{64}[klm + \alpha\beta\gamma]$, which is given by

$$\begin{aligned}n = \frac{1}{8}[&(-36pa^2q^2 + 8ap + 16a^2p^2 - 16a^2q^2 - 3a^3pq^2 + a^3p^3 - 6a^3p^2q^2 + 32ap^2 - 32q^2a \\ &+ 12a^2p^3 + a^3p^4 + q^4a^3) + i(3a^3p^2q + 64aq + 64paq + 4a^3p^3q - 4q^3a^3p + 32a^2pq \\ &- 12q^3a^2 + 36p^2a^2q - a^3q^3)]\end{aligned} \tag{11}$$

Now for the same reason as discussed in previous sections, this triple cannot be extended in to a Gaussian quadruple.

CONCLUSION

In this paper we have outlined a few non-extendable triples. One may also search for similar type of non-extendable Gaussian triples with suitable property.

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TIME HORIZON INVENTORY COST MODEL HAVING WEIBULL DISTRIBUTED DETERIORATION WITH SHORTAGES AND PARTIAL BACKLOGGING

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ABSTRACT :
In the present study, an inventory cost models for weibull rate of deteriorating items with constant holding cost and quadratic demand rate, partial backlogging over a time horizon is proposed. The demand rate is considered as quadratic function of time. Shortages are allowed and partially backlogging case is discussed. The main objective of this paper is to minimize the total cost per unit time in an inventory system. Numerical problems are presented to demonstrate the result. Also, the effect of changes in different parameters on optimal total cost is graphically presented.
Keywords : weibull deterioration, quadratic demand, holding cost, shortages, Partial backlogging.

INTRODUCTION

Deterioration plays a vital role in Inventory model. Deterioration can be considered as decay or damage, spoilage evaporation and loss of utility of the product. In inventory problem deterioration is a realistic feature and can't be ignored. Often we encounter products such as fruits, medicines, fashion goods, alcohol, milk, vegetables, electronic components, photographic films and many more that have a defined period of life time and start loosing their values with passage of time .Such items are refereed as deteriorating items. Due to deterioration, inventory system faces the problem of shortages and loss of goodwill or loss of profit. Shortage can be supposed a fraction of those customers whose demand is not satisfied in the current period reacts to this by not returning the next period. Moreover, the sales for the product may decline on adding more competitive product or change in consumer's behavior or priority or high-tech products .Longer the waiting time, lower will be backlogging rate which leads to a larger fraction of lost sales and yield less profit. Hence, the factor of **partial backlogging** becomes significant to be considered.

Ghare and Schrader (1963), Shah and Jaiswal (1977) proposed an inventory model for deteriorating items with a constant rate of deterioration. Later Dave and Patel (1981) extended Ghare and Schrader's work by considering an inventory model for deteriorating items with time-proportional demand when shortages were not allowed. Park K.S 1982, Chang.H.J and Dye.C.Y. (1999), Zeng (2001) and Wu et al. (2005) developed inventory models for deteriorating items using partial backlogging approach and minimize total cost function. The models identify the conditions for partial backordering policy to be feasible.

Dye et al. (2007) finds an optimal inventory model with exponential partial backlogging. R.Begum, S.K.Sahu and R.R.Sahoo (2012) introduced an optimal inventory model with qsuadratic demand and partial backlogging. Sarkar, Ghosh & Chaudhry (2012) explored an inventory replenishment policy for a deteriorating item with quadratic demand and time dependent partial backlogging with shortages in all cycles. Nanda gopal Rajeswari and Thirumalaisamy Vanjikodi (2012) considered an inventory model with Weibull distribution deterioration and backlogging. Hari kishan etal (2012), Trailokyanth Singh and Hadibandhu Pattanayak (2013) presented an Inventory model with variable deterioration and time dependent demand rates. Rangarajan etal (2016) and Aggarwal etal (2016) has also worked on partial backlogging.

Recently, Yadav H.K. & Singh T.P.(2019) explored an inventory model with variable demand , Weibull distributed deterioration rate, time holding cost without shortage constraints. this paper is an extended work of the study made by Yadav & Singh (2019) in the sense that here, the partial backlogging concept is being added as the shortages are neither completely backlogged nor completely lost hence, partial backlogging is more appropriate to be considered. The main objective of this article is to find optimal time and optimal total cost per unit time of an inventory system with quadratic demand rate which start from zero and ends at zero with shortages , partial backlogging and **weibull deteriorating rate** . Numerical examples and sensitivity analysis have also been presented.

Following an introductory part rest of the paper is organized as follows. In section 2, the notations and assumptions are described in brief for developing the model. Section 3 gives the mathematical formulations of the model. Section 4 illustrates the developed models through a numerical example while in section 5 sensitive analysis of the optimal solution has been carried out. Finally, the conclusion is given.

2. ASSUMPTIONS AND NOTATIONS:

To develop the model following assumptions and notations have been considered :

1) Replenishment size is constant and the replenishment rate is infinite.
2) Lead time is zero.
3) T is the length of each production cycle;
4) C_1 is the inventory holding cost per unit time;
5) C_2 is Purchase cost per unit;
6) C_3 is the cost of each deteriorated unit;
7) C_4 is Backordered cost per unit;
8) C_5 is Lost sales cost per unit;
9) C(t) is the total inventory cost ;
10) The deterioration rate function θ(t) represents the on-hand inventory deteriorates per unit time and Moreover in the present study the function assumed in the form $\theta(t) = \alpha\beta t^{\beta-1}$; $0 < \alpha < 1$, $\beta > 0$, $t > 0$.

When $\beta = 1$, $\theta(t)$ becomes constant a case of exponential decay. When $\beta < 1$, the rate of deterioration is decreasing with t and when $\beta > 1$, the rate of deterioration is increasing with t.

11) The demand rate starts from zero and ends at zero during the inventory period. It is assumed of the form $D(t) = at(T-t)$ where T is the cycle period

12) During stock out period, the backlogging rate is variable and is depends on the length of the waiting time for next replenishment. So that the backlogging rate for negative inventory is,

$$B(t) = \frac{1}{1+\gamma(T-t)}$$

γ is backlogging parameter and T-t is waiting time and $t_1<t<T$.

3. MATHEMATICAL DESCRIPTION OF THE MODEL:

Assuming an amount S (S>0) as an initial inventory, we find inventory level gradually diminishes due to reasons of market demand and deterioration of the items and ultimately falls to zero at time T . Let I(t) be the on hand inventory at any time t. Clearly, the differential equations which on hand inventory I(t) must satisfy the following two equations, one due to deterioration and demand, second due to shortage and partial backlogging :

$$\frac{dI(t)}{dt} + \theta I(t) = -D(t) \quad , \quad 0 \le t \le t_1 \quad ....(1)$$

$$\frac{dI(t)}{dt} = -\frac{D(t)}{1+\gamma(T-t)} \quad , \quad t_1 \le t \le T \quad ....(2)$$

Using $D(t) = at(T-t)$ the differential equation (1) can be rewritten as

$$\frac{dI(t)}{dt} + \alpha\beta t^{\beta-1} I(t) = -at(T-t) \quad , \quad 0 \le t \le T \quad ....(3)$$

Being standard form of its solution is given by

$$I(t)\, e^{\alpha t^{\beta}} = -\int at(T-t)\, e^{\alpha t^{\beta}}\, dt + C$$

Using the condition $I(0) = S$, we get $C = S$

$$I(t)\, e^{\alpha t^{\beta}} = -a\left(\frac{Tt^2}{2} + \frac{\alpha T t^{\beta+2}}{\beta+2} - \frac{t^3}{3} - \frac{\alpha t^{\beta+3}}{\beta+3}\right) + S \quad ....(4)$$

Using the boundary condition $I(t_1)=0$ we get

$$S = a\left(\frac{Tt_1^2}{2} + \frac{\alpha T t_1^{\beta+2}}{\beta+2} - \frac{t_1^3}{3} - \frac{\alpha t_1^{\beta+3}}{\beta+3}\right). \quad ....(5)$$

Putting the value of S in (4), on simplification we find,

$$I(t)\, e^{\alpha t^{\beta}} = -a\left[\frac{Tt^2}{2} + \frac{\alpha T t^{\beta+2}}{\beta+2} - \frac{t^3}{3} - \frac{\alpha t^{\beta+3}}{\beta+3} - \frac{Tt_1^2}{2} - \frac{\alpha T t_1^{\beta+2}}{\beta+2} + \frac{t_1^3}{3} + \frac{\alpha t_1^{\beta+3}}{\beta+3}\right]$$

$$I(t) = -a\left[\frac{Tt^2}{2} + \frac{\alpha T t^{\beta+2}}{\beta+2} - \frac{t^3}{3} - \frac{\alpha t^{\beta+3}}{\beta+3} - \frac{Tt_1^2}{2} - \frac{\alpha T t_1^{\beta+2}}{\beta+2} + \frac{t_1^3}{3} + \frac{\alpha t_1^{\beta+3}}{\beta+3}\right] e^{-\alpha t^{\beta}}$$

$$I(t) = -a\left[\frac{Tt^2}{2} + \frac{\alpha T t^{\beta+2}}{\beta+2} - \frac{t^3}{3} - \frac{\alpha t^{\beta+3}}{\beta+3} - \frac{Tt_1^2}{2} - \frac{\alpha T t_1^{\beta+2}}{\beta+2} + \frac{t_1^3}{3} + \frac{\alpha t_1^{\beta+3}}{\beta+3}\right](1 - \alpha t^{\beta})$$

$$I(t) = -a\left[\frac{Tt^2}{2} + \frac{\alpha T t^{\beta+2}}{\beta+2} - \frac{t^3}{3} - \frac{\alpha t^{\beta+3}}{\beta+3} - \frac{Tt_1^2}{2} - \frac{\alpha T t_1^{\beta+2}}{\beta+2} + \frac{t_1^3}{3} + \frac{\alpha t_1^{\beta+3}}{\beta+3}\right] + a\alpha\left[\frac{Tt^{\beta+2}}{2} + \frac{\alpha T t^{2\beta+2}}{\beta+2} - \frac{t^{\beta+3}}{3} - \frac{\alpha t^{2\beta+3}}{\beta+3} - \frac{Tt^{\beta}t_1^2}{2} - \frac{\alpha T t^{\beta} t_1^{\beta+2}}{\beta+2} + \frac{t^{\beta} t_1^3}{3} + \frac{\alpha t^{\beta} t_1^{\beta+3}}{\beta+3}\right]$$

$$I(t) = -a\left[\frac{Tt^2}{2}-\frac{(\alpha\beta)Tt^{\beta+2}}{2(\beta+2)}-\frac{t^3}{3}+\frac{(\alpha\beta)t^{\beta+3}}{3(\beta+3)}-\frac{Tt_1^2}{2}-\frac{\alpha Tt_1^{\beta+2}}{\beta+2}+\frac{t_1^3}{3}+\frac{\alpha t_1^{\beta+3}}{\beta+3}\right]+a\alpha\left[\frac{\alpha Tt^{2\beta+2}}{\beta+2}-\frac{\alpha t^{2\beta+3}}{\beta+3}-\frac{Tt^\beta t_1^2}{2}-\frac{\alpha Tt^\beta t_1^{\beta+2}}{\beta+2}+\frac{t^\beta t_1^3}{3}+\frac{\alpha t^\beta t_1^{\beta+3}}{\beta+3}\right]. \quad \text{........(6)}$$

Solution of Equation 2

$$\frac{dI(t)}{dt} = -\frac{at(T-t)}{1+\gamma(T-t)}, \qquad t_1 \le t \le T$$

$$I(t) = -\frac{at^2}{2\gamma}+\frac{at}{\gamma^2}-\frac{a(1+\gamma T)}{\gamma^3}\log(1+\gamma(T-t)) + C_2$$

Using $I(T) = -S_1$ and $I(t_1)=0$, we get

$$S_1 = \frac{a(T^2-t_1^2)}{2\gamma}+\frac{a(t_1-T)}{\gamma^2}-\frac{a(1+\gamma T)(T-t_1)}{\gamma^2} \quad \text{……..(7)}$$

$$I(t) = \frac{a(t_1^2-t^2)}{2\gamma}+\frac{a(t-t_1)}{\gamma^2}-\frac{a(1+\gamma T)(t_1-t)}{\gamma^2} \qquad t_1 \le t \le T \quad \text{……..(8)}$$

Hence total amount of deteriorated units (D)=I(0)−stock loss due to demand $= S - \int_0^{t_1} at(T-t)\ dt$

$$= a\left(\frac{Tt_1^2}{2}+\frac{\alpha Tt_1^{\beta+2}}{\beta+2}-\frac{t_1^3}{3}-\frac{\alpha t_1^{\beta+3}}{\beta+3}\right) - a\left(\frac{Tt_1^2}{2}-\frac{t_1^3}{3}\right)$$

$$= a\left(\frac{\alpha Tt_1^{\beta+2}}{\beta+2}-\frac{\alpha t_1^{\beta+3}}{\beta+3}\right) \quad \text{.........(9)}$$

Total Inventory held $(I_1) = C_1 \int_0^{t_1}(I(t)\ dt$

$$= \int_0^{t_1}\left\{-a\left[\frac{Tt^2}{2}-\frac{(\alpha\beta)Tt^{\beta+2}}{2(\beta+2)}-\frac{t^3}{3}+\frac{(\alpha\beta)t^{\beta+3}}{3(\beta+3)}-\frac{Tt_1^2}{2}-\frac{\alpha Tt_1^{\beta+2}}{\beta+2}+\frac{t_1^3}{3}+\frac{\alpha t_1^{\beta+3}}{\beta+3}\right]+a\alpha\left[\frac{\alpha Tt^{2\beta+2}}{\beta+2}-\frac{\alpha t^{2\beta+3}}{\beta+3}-\frac{Tt^\beta t_1^2}{2}-\frac{\alpha Tt^\beta t_1^{\beta+2}}{\beta+2}+\frac{t^\beta t_1^3}{3}+\frac{\alpha t^\beta t_1^{\beta+3}}{\beta+3}\right]\right\}dt$$

$$I_1 = C_1\left\{-a\left[-\frac{\alpha(\beta^3+4\beta^2+\beta)t_1^{\beta+4}}{3(\beta+3)(\beta+1)}+\frac{\alpha(\beta^3+3\beta^2+\beta)Tt_1^{\beta+3}}{2(\beta+2)(\beta+1)}+\frac{t_1^4}{4}-\frac{Tt_1^3}{3}\right]+a\alpha\left[-\frac{\alpha Tt_1^{2\beta+3}}{(\beta+1)(2\beta+3)}+\frac{\alpha t_1^{2\beta+4}}{(\beta+1)(2\beta+4)}\right]\right\} \quad \text{.........(10)}$$

Cost of deteriorated items = C_3 × total amount of deteriorated units

$$= C_3\ a\left(\frac{\alpha Tt_1^{\beta+2}}{\beta+2}-\frac{\alpha t_1^{\beta+3}}{\beta+3}\right) \quad \text{.........(11)}$$

Backordered cost per cycle $= C_4 \int_{t_1}^{T} -I(t)\ dt$

$$= -C_4 \int_{t_1}^{T}\frac{a(t_1^2-t^2)}{2\gamma}+\frac{a(t-t_1)}{\gamma^2}-\frac{a(1+\gamma T)(t_1-t)}{\gamma^2}\ dt$$

$$= C_4\left\{\frac{a(T^3-t_1^3)}{6\gamma}-\frac{a(T-t_1)t_1^2}{2\gamma}+\frac{a(T-t_1)^2}{2\gamma^2}+\frac{a(1+\gamma T)(t_1-T)^2}{2\gamma^2}\right\}$$

. …….(12)

Lost sales per cycle = $C_5 \int_{t_1}^{T}\left(1-\frac{1}{1+\gamma(T-t)}\right) at(T-t)\ dt$

$$-\frac{at^2}{2\gamma}+\frac{at}{\gamma^2}-\frac{a(1+\gamma T)}{\gamma^3}(T-t)$$

$$= C_5\left\{\frac{a(T^3-Tt_1^2)}{2}-\frac{a(T^3-t_1^3)}{3}-\frac{a(T^2-t_1^2)}{2\gamma}+\frac{a(T-t_1)}{\gamma^2}+\frac{a(1+\gamma T)}{\gamma^2}(T-t_1)\right\}$$

. . …(13)

Purchase cost per cycle = $C_2\ a\left(\frac{Tt_1^2}{2}+\frac{\alpha Tt_1^{\beta+2}}{\beta+2}-\frac{t_1^3}{3}-\frac{\alpha t_1^{\beta+3}}{\beta+3}\right)$ ……(14)

Average total cost per unit time $C(t_1) = \frac{1}{T}$[Total cost per unit time] = $\frac{1}{T}$[Total Inventory held +Cost of deterioration items+ Backordered cost per cycle + Lost sales per cycle + Purchase cost per cycle]

$= \frac{1}{T}\{ C_1 \{ -a \left[-\frac{\alpha(\beta^3+4\beta^2+\beta)t_1^{\beta+4}}{3(\beta+3)(\beta+1)} + \frac{\alpha(\beta^3+3\beta^2+\beta)Tt_1^{\beta+3}}{2(\beta+2)(\beta+1)} + \frac{t_1^4}{4} - \frac{Tt_1^3}{3} \right] + a\alpha \left[- \frac{\alpha T t_1^{2\beta+3}}{(\beta+1)(2\beta+3)} + \frac{\alpha t_1^{2\beta+4}}{(\beta+1)(2\beta+4)} \right] \} + C_3 \quad a(\frac{\alpha T t_1^{\beta+2}}{\beta+2} - \frac{\alpha t_1^{\beta+3}}{\beta+3}) + C_4 \{ \frac{a(T^3-t_1^3)}{6\gamma} - \frac{a(T-t_1)t_1^2}{2\gamma} + \frac{a(T-t_1)^2}{2\gamma^2} + \frac{a(1+\gamma T)(t_1-T)^2}{2\gamma^2} \} + C_5\{ \frac{a(T^3-Tt_1^2)}{2} - \frac{a(T^3-t_1^3)}{3} - \frac{a(T^2-t_1^2)}{2\gamma} + \frac{a(2+\gamma T)}{\gamma^2}(T-t_1)\} + C_2\, a(\frac{Tt_1^2}{2} + \frac{\alpha T t_1^{\beta+2}}{\beta+2} - \frac{t_1^3}{3} - \frac{\alpha t_1^{\beta+3}}{\beta+3})]\}$.

.........(15)

$\frac{dC(t_1)}{dt_1} = \frac{1}{T}\{C_1 \{ -a \left[-\frac{\alpha(\beta^3+4\beta^2+\beta)(\beta+4)t_1^{\beta+3}}{3(\beta+3)(\beta+1)} + \frac{\alpha(\beta^3+3\beta^2+\beta)(\beta+3)Tt_1^{\beta+2}}{2(\beta+2)(\beta+1)} + \frac{t_1^3}{1} - \frac{Tt_1^2}{1} \right] + a\alpha \left[- \frac{\alpha T t_1^{2\beta+2}}{(\beta+1)} + \frac{\alpha t_1^{2\beta+3}}{(\beta+1)} \right] \} + C_3 \quad a(\frac{\alpha T t_1^{\beta+1}}{1} - \frac{\alpha t_1^{\beta+2}}{1}) + C_4 \{ -\frac{a(t_1^2)}{2\gamma} - \frac{a(2T-3t_1)t_1^1}{2\gamma} - \frac{a(T-t_1)^1}{\gamma^2} + \frac{a(1+\gamma T)(t_1-T)^1}{\gamma^2} \} + C_5\{- \frac{a(Tt_1^1)}{1} + \frac{a(t_1^2)}{1} + \frac{a(t_1^1)}{\gamma} - \frac{a(2+\gamma T)}{\gamma^2} \} + C_2\, a(\frac{Tt_1^1}{1} + \frac{\alpha T t_1^{\beta+1}}{1} - \frac{t_1^2}{1} - \frac{\alpha t_1^{\beta+2}}{1})]\}$ ………(16)

$\frac{d^2C}{dt_1} = \frac{1}{T}\{C_1 \{ -a \left[-\frac{\alpha(\beta^3+4\beta^2+\beta)(\beta+4)t_1^{\beta+2}}{3(\beta+1)} + \frac{\alpha(\beta^3+3\beta^2+\beta)(\beta+3)Tt_1^{\beta+1}}{2(\beta+1)} + 3\frac{t_1^2}{1} - \frac{2Tt_1^1}{1} \right] + a\alpha \left[- \frac{2\alpha T t_1^{2\beta+1}}{1} + \frac{\alpha(2\beta+3) t_1^{2\beta+2}}{(\beta+1)} \right] \} \; C_3 \quad a(\frac{\alpha(\beta+1) T t_1^{\beta}}{1} - \frac{\alpha(\beta+2) t_1^{\beta+1}}{1}) + C_4 \{ -\frac{a(t_1^1)}{\gamma} - \frac{a(T-3t_1)}{\gamma} + \frac{a}{\gamma^2} + \frac{a(1+\gamma T)}{\gamma^2} \} + C_5\{- \frac{a(T)}{1} + \frac{a2t_1^1}{1} + \frac{a}{\gamma} \} + C_2\, a(\frac{T}{1} + \frac{\alpha T(\beta+1) t_1^{\beta}}{1} - \frac{2t_1^1}{1} - \frac{\alpha(\beta+2) t_1^{\beta+1}}{1})]\}$ ……..(17)

For minimum $C(t_1)$,the necessary condition is

$\frac{dC(t_1)}{dt_1} = 0$ After solving, we get an equation of odd degree whose last term is negative, then there exists a unique solution $t_1^* \in (0,T)$ can be solved from equation (16) by using MAT Lab. also clearly $\frac{d^2C}{dT^2} > 0$ at $T = t_1^*$

$\therefore$ $C(t_1)$ is minimum at $t_1 = t_1^*$

Hence, optimum value of t_1 is t_1^*

4. NUMERICAL

To illustrate the model we consider following numerical values of the parameters. a = 1000, α = 0.1 , β =2, C_1 = 5₹ ,C_2 = 50₹ , C_3 = 5₹ , C_4 = .9₹ , C_5 = 0.5₹ , γ = 0.5, we obtain for the crisp model Total Cost = 131878.1 ₹ and cycle time t = 0.8388 years. (1 Unit ₹ = ₹ 10,000)

6. SENSITIVITY ANALYSIS

Table-1

Change value		Crisp Model	
		T	C(t)
α	0.01	0.9682	30235.33
	0.02	0.9674	30118.09
	0.03	0.9666	30506.03
	0.04	0.9658	30828.94
	0.05	0.9659	31112.62
	0.06	0.9644	31813.27
	0.07	0.9635	32529.89
	0.08	0.963	33297.25
	0.09	0.9623	34226.7
	0.1	0.9617	35262.2

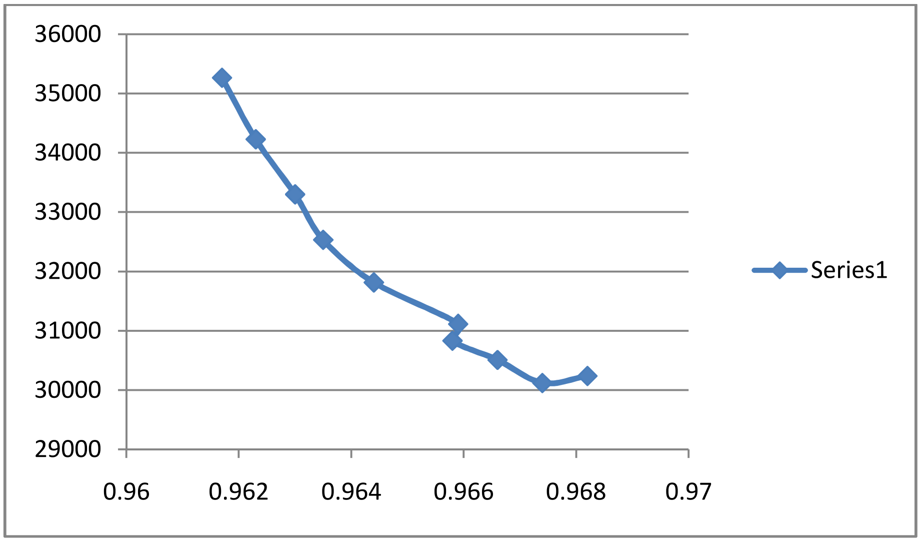
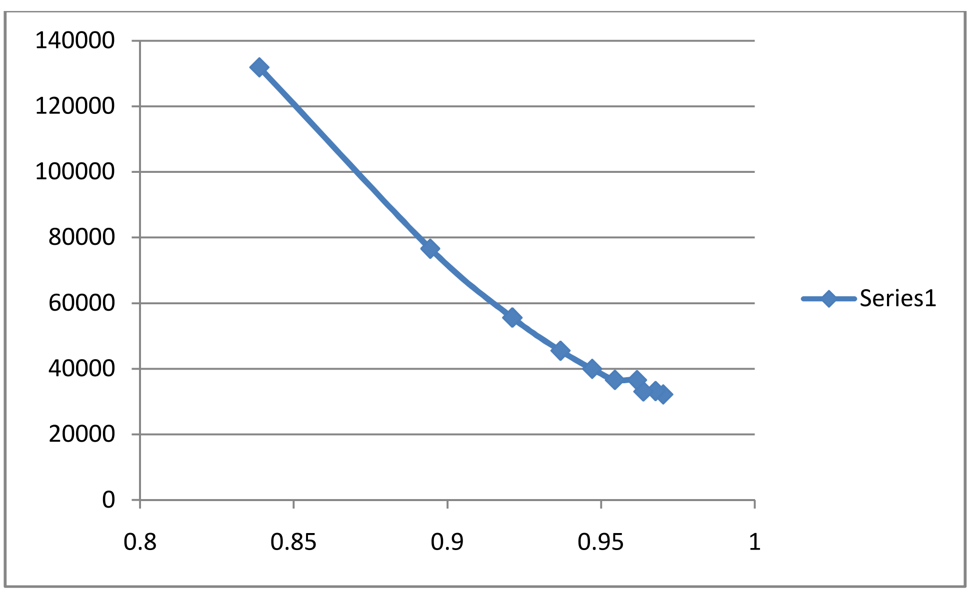


Table-2

Change value		Crisp Model	
		T	C(t)
γ	1.4	0.9702	32222.04
	1.3	0.9677	33250.84
	1.2	0.9638	33106.83
	1.1	0.9617	36567.95
	1	0.9545	36637.09
	0.9	0.9471	39989.97
	0.8	0.9368	45528.82
	0.7	0.9211	55619.26
	0.6	0.8944	76617.74
	0.5	0.8388	131878.1



OBSERVATIONS:

1. When α increases, cycle time decreases and total cost initially decreases and reaches to minimum value and then starts increasing.
2. With the decrease of the parameter γ, cycle time (t_1) decreases, while total cost increases.

7. CONCLUSION:

An Inventory model of deteriorating products in supply chain process has been developed in which demand rate is quadratic function of time while the deterioration follows Weibull Deterioration. Shortages along partial backlogging of items have been discussed. A comparative study of the total inventory cost, obtained by changing parameters, in order to measure the sensitivity analysis has been presented. As the cycle time (t_1) decreases total cost of model increases. The proposed model deals some realistic features likely to be associated with some kind of inventory. The model finds its application in retail business such as of fashionable cloths, domestic goods & electronic component etc.

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GENERALIZED CYCLOTOMIC COSETS MODULO $4p^n$

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ABSTRACT :

Explicit expressions for all the 3(nd+1) cyclotomic cosets modulo $4p^n$ over GF(l),where p, l are distinct odd primes, , $o(l)_{p^n} = \frac{\varphi(p^n)}{d}$, $o(l)_4 = \varphi(4)$, $gcd(\varphi(p^n), \varphi(4))$ = d, p does not divide 3 , are obtained.

Keywords: Cyclotomic cosets, Primitive roots.

MSC(2010): 11A03;15A07;11R09;11T06;11T22;11T71;94B05;94B15

1. INTRODUCTION

Let $F = GF(l)$ be a field of odd prime order l and k ≥ 1 be an integer such that $\gcd(l, k) = 1$. Let $o(l)_k$ denotes the order of l modulo k. Many authors have obtained the complete set of primitive idempotents of the minimal cyclic codes of various lengths. Sahni and Sehgal [1] described the primitive idempotents of minimal cyclic codes of length $p^n q$, p, q are distinct odd primes and $o(l)_{p^n} = \varphi(p^n)$, $o(l)_q = \varphi(q)$, $gcd(\varphi(p^n), \varphi(q))$ = d, p does not divide $q - 1$.

In this paper, we extend the results of Rani S. Singh I.J and Kumar P. [13]. We consider the case when $k = 4p^n$, where p, l are distinct odd primes, $o(l)_{2p^n} = \varphi(2p^n)/d = \varphi(p^n)$, $o(l)_4 = \varphi(4)$, $gcd(\varphi(2p^n), \varphi(4))$ = d, p does not divide 3. In Section 2 (Lemmas 2.1 – 2.4 and Theorem 2.5), we obtain the cyclotomic cosets modulo $4p^n$.

2. CYCLOTOMIC COSETS MODULO $4p^n$

In this section we describe the cyclotomic coset modulo $4p^n$, where p, l are distinct odd primes and $o(l)_{p^n} = \varphi(p^n)$ $o(l)_4 = \varphi(4)$ with $gcd(\varphi(p^n), \varphi(4)) = 2$, p does not divide 3.

Let S = {0, 1, 2, . . . ,$4p^n - 1$ }. For a, b ∈ S, say that a ~ b iff a ≡ bl^i (mod $4p^n$) for some integer i ≥ 0. This defines an equivalence relation on the set S. The equivalence classes due to this relation are called l – cyclostomic cosets modulo $4p^n$. The l – cyclotomic coset containing s ∈ S is denoted by $C_s = \{s, sl, sl^2, . . . , sl^{t_s - 1}\}$, where t_s is the least positive integer such that $sl^{t_s} \equiv s \ (mod\ 4p^n)$ and $|C_s|$ denotes the order of the l – cyclotomic coset C_s.

2.1 Lemma Let p, l be distinct odd primes and n,d ≥ 1 be integers. If $o(l)_{p^{n-j}} = \varphi(p^{n-j})/d$ and $o(l)_4 = \varphi(4)$ with $gcd\left(\varphi\left(p^{n-j}\right), \varphi(4)\right) = d$, p does not divide 3, then $o(l)_{4p^{n-j}} = \varphi(4p^{n-j})/d$, for all 0 ≤ j ≤ n – 1.

Proof. Let γ_j be the order of l modulo $4p^{n-j}$, for 0 ≤ j ≤ n – 1. Then $l^{\gamma_j} \equiv 1 \ (mod\ 4p^{n-j})$.
Therefore, $l^{\gamma_j} \equiv 1 \left(mod\ p^{n-j}\right)$ and $l^{\gamma_j} \equiv 1 \ (mod\ 4)$.
But order of l modulo p^{n-j} is $\varphi(p^{n-j})/d$ and modulo 4 is $\varphi(4) = 2$.
Therefore, $\varphi(p^{n-j})/d$ and $\varphi(4)$ divides γ_j.
Consequently, $lcm\left(\varphi\left(p^{n-j}\right)/d, \varphi(4)\right) = \frac{\varphi(4p^{n-j})}{d} | \gamma_j$.

On the other hand, order of l modulo p^{n-j} is $\varphi(p^{n-j})/d$.
Therefore, $l^{\varphi(p^{n-j})/d} \equiv 1\ (mod\ p^{n-j})$.
Hence, $(l^{\varphi(p^{n-j})/d})^{\frac{\varphi(4)}{2}} \equiv 1\ \left(mod\ p^{n-j}\right)$ i.e. $l^{\frac{\varphi\left(4p^{n-j}\right)}{d}} \equiv 1\ \left(mod\ p^{n-j}\right)$.

Similarly, order of l modulo 4 is $\varphi(4) = 2$. Therefore, $l^{\varphi(4)} \equiv 1\ (mod\ 4)$, hence $\left(l^{\varphi(4)}\right)^{\frac{\varphi\left(p^{n-j}\right)}{d}} \equiv 1\ (mod\ 4)$ i.e. $l^{\frac{\varphi\left(4p^{n-j}\right)}{d}} \equiv 1\ (mod\ 4)$.

So, $l^{\frac{\varphi\left(4p^{n-j}\right)}{d}} \equiv 1\ \left(mod\ 4p^{n-j}\right)$. But order of l modulo $4p^{n-j}$ is γ_j. Therefore, γ_j divides $\frac{\varphi(4p^{n-j})}{d}$. Hence, order of l modulo $4p^{n-j}$ is $\frac{\varphi(4p^{n-j})}{d}$.

2.2 Lemma Let p and l be distinct odd primes such that $\gcd\left(\varphi(p),\ \varphi(4)\right) = 2$ and $o(l)_p = \varphi(p)/d$, $o(l)_4 = \varphi(4)$. Then there exists an integer g, $1 < g < 4p$ such that $o(g)_p = \varphi(p)$ and $o(g)_4 = 1$.

Proof. Let the sets $S_p = \{0, 1, 2, . . . , p-1\}$, $S_4 = \{0, 1, 2,\ 3\}$ and $S_{4p} = \{0, 1, 2, . . . , 4p-1\}$ modulo p, modulo 4 and modulo $4p$ respectively.
Define a mapping $f: S_{4p} \to S_p \times S_4$ given by $f(g) = (r, s)$ where $g \equiv r(mod\ p)$ and $g \equiv s(mod\ 4)$.
Clearly, the mapping f is one-one and onto.
Since $\gcd(p, 4) = 1$, then there exists integers u, v such that $pu + qv = 1$. For a given order pair $(r, s) \in S_p \times S_4$, there exists a unique $g \in S_{4p}$ such that $g \equiv (psu + 4rv)(mod\ 4p)$. Then $f(g) = (r, s)$.
Let us find an integer g such that $1 < g < 4p$ and $o(g)_p = \varphi(p)$ and $o(g)_4 = \frac{\varphi(4)}{2}$, where $\gcd(\varphi(p)$, $\varphi(4) = 2)$. Now l is primitive root modulo p as well as modulo 4.
Choose an integer m_1 such that $l^{m_1}$is also primitive root modulo p. Indeed the integer m_1 is less than and co-prime to $\varphi(p)$. Similarly, choose m_2 such that l^{dm_2} has order $\frac{\varphi(4)}{2}$ modulo 4. Then m_2 satisfying $1 \le m_2 < \varphi(4)$ and $\gcd\left(2m_2,\ \varphi(4)\right) = 1$.
Let $g \in S_{4p}$ be such that $g \equiv (l^{2m_2}pu + l^{m_1}qv)$(mod 4p)
Then, $g \equiv l^{m_1}qv \equiv l^{m_1}(1 - pu) \equiv l^{m_1}(mod\ p)$
and hence g is primitive root modulo p.
Similarly, $g \equiv l^{2m_2}(mod\ 4)$ implies that order of g is $\frac{\varphi(4)}{2}$ modulo 4.

2.3 Lemma There exists a positive integer g, $1 < g < 4p$, such that gcd $(g, 4pl) = 1$ and $o(g)_{2p} = \frac{\varphi(p)}{2}$ or $\varphi(p)$ and $o(g)_4 = \frac{\varphi(4)}{2}$ or $\varphi(4)$, gcd $(\varphi(p), \varphi(4)) = 2$, where $g \notin \left\{1,\ l,\ l^2, . . . , l^{\frac{\varphi(4p)}{d}-1}\right\}$.

Proof. Let us suppose that $\in \left\{1,\ l,\ l^2, . . . , l^{\frac{\varphi(4p)}{d}-1}\right\}$.

Then $g \equiv l^k(mod\ 4p)$ for some k, $1 \le k < \frac{\varphi(4p)}{d}$.
However, $g \equiv (l^{2m_2}pu + l^{m_1}4v)(mod\ 4p)$.
Thus, $l^{im_1} \equiv l^k$(mod p) and $l^{i2m_2} \equiv l^k$(mod 4),
gives that

$$k \equiv im_1(\text{mod}\ \varphi(p))$$

and

$$k \equiv idm_2(\text{mod}\ \varphi(4)).$$

As d divides both $\varphi(2p)$ and $\varphi(4) = 2$, we obtain that $k \equiv m_1 \equiv 2m_2 \equiv 0$(mod 4).
In particular, $m_1 \equiv 0(mod\ 4)$, which is a contradiction as gcd $(m_1, \varphi(p)) = 1$,
Thus, we conclude that $g \notin \left\{1,\ l,\ l^2, . . . , l^{\frac{\varphi(4p)}{d}-1}\right\}$.

2.4 Lemma There exists a positive integer g, $1 < g < 4p$, such that $\gcd(g, 4pl) = 1$ and $g \not\equiv l^k$ (mod $4p$) for any k, $0 \leq k \leq \frac{\varphi(4p)}{d}$. Further, for any j, $1 \leq j < n$, the set

$\{\ 1, l, l^2, . . . \ , l^{\frac{\varphi(4p^{n-j})}{d}-1}, g, gl, gl^2, . . . \ , gl^{\frac{\varphi(4p^{n-j})}{d}-1}, \dots g^{d-1}, lg^{d-1}, l^2g^{d-1}, . . . \ , g^{d-1}l^{\frac{\varphi(4p^{n-j})}{d}-1}\ \}$ form a reduced residue system modulo $4p^{n-j}$.

Proof. Let g be defined in Lemma 2.3, then the set $\{1, l, l^2, . . . \ , l^{\frac{\varphi(4p^{n-j})}{d}-1}$, $g, gl, gl^2, . . . \ , gl^{\frac{\varphi(4p^{n-j})}{d}-1}, \dots, g^{d-1}, lg^{d-1}, l^2g^{d-1}, . . . \ , g^{d-1}l^{\frac{\varphi(4p^{n-j})}{d}-1}\ \}$ has $\varphi(4p^{n-j})$ elements which are co prime to $4p$.
We want to prove that all these elements are incongruent mod $4p^{n-j}$.
Let $gl^k \equiv l^t (\text{mod } 4p^{n-j})$,$0 \leq k, t < \frac{\varphi(4p^{n-j})}{d}$.
Then $g \equiv l^{t-k} (\text{mod } 4p^{n-j})$ implies that $g \equiv l^s (\text{mod } 4p)$ where $s \equiv t - k (\text{mod } \frac{\varphi(4p)}{d})$.
Therefore, $g \in \{1,\ l,\ l^2, . . . \ , l^{\frac{\varphi(4p)}{d}-1}\}$ .Thus, we get $l^k \equiv l^t (\text{mod } 4p^{n-j})$, where $0 \leq k, t < \frac{\varphi(4p^{n-j})}{d}$.
But order of l mod $4p^{n-j}$ is $\frac{\varphi(4p^{n-j})}{d}$ which gives us $k = t$.

2.5 Theorem Let p, l be distinct odd primes and n,$d \geq 1$ be integers. If $o(l)_{p^{n-j}} = \varphi(p^{n-j})/d$ and $o(l)_4 = \varphi(4)$ with $gcd\left(\varphi\left(p^{n-j}\right), \varphi(4)\right) = d$ and p does not divide 3,, then for the integer $n \geq 1$, there are $3(nd + 1)$ cyclotomic cosets (mod $4p^n$) given by
(i) $C_0 = \{0\}$,
(ii) $C_{p^n} = \{p^n, p^n l\}$,
(iii)$C_{2p^n} = \{2p^n l\}$
For $0 \leq j \leq n - 1, 0 \leq k \leq d - 1$,
(iv)$C_{g^kp^j} = \{g^kp^j,\ g^kp^jl,\ g^kp^jl^2,\ . . ., g^kp^jl^{\varphi(4p^{n-j})/d-1}\}$,
(v) $C_{2g^kp^j} = \{2g^kp^j,\ 2g^kp^jl,\ 2g^kp^jl^2,\ . . ., 2g^kp^jl^{\varphi(2p^{n-j})/d-1}\}$,
(vi)$C_{4g^kp^j} = \{4g^kp^j,\ 4g^kp^jl,\ 4g^kp^jl^2,\ . . ., 4g^kp^jl^{\varphi(p^{n-j})/d-1}\}$
Where, g is fixed integer as defined above.
Proof. Case (i) is trivial.
(ii) As $\varphi(4) = 2$, therefore $l^{\varphi(4)} \equiv 1 (\text{mod } 4)$.
Then, $p^n l^{\varphi(4)} \equiv p^n (\text{mod } 4p^n)$. Therefore, the cyclotomic coset containing p^n is $\{p^n,\ p^n l^{\varphi(4)-1}\}$.
(iii) Similar to case (ii)
(iv) As $o(l)_{p^n} = \frac{\varphi(p^n)}{d}$, then $o(l)_{p^{n-j}} = \frac{\varphi(p^{n-j})}{d}$ also; $0 \leq j \leq n - 1$. Therefore, $l^{\frac{\varphi(p^{n-j})}{d}} \equiv 1 (\text{mod } p^{n-j})$.
Then, $4p^j l^{\frac{\varphi(p^{n-j})}{d}} \equiv 4p^j (\text{mod } 4p^n)$.
Hence, the cyclotomic coset containing p^j is $\{p^j,\ p^jl,\ p^jl^2,\ . . ., p^jl^{\frac{\varphi(4p^{n-j})}{d}-1}\}$.
(v) On the similar lines again $o(l)_{2p^{n-j}} = \frac{\varphi(2p^{n-j})}{d}$ for $0 \leq j \leq n - 1$,
Therefore,$l^{\frac{\varphi(2p^{n-j})}{d}} \equiv 1 (\text{mod } 2p^{n-j})$.
Then, $2p^j l^{\frac{\varphi(2p^{n-j})}{d}} \equiv 2p^j (\text{mod } 4p^n)$.

Hence, the cyclotomic coset containing $2p^j$ is $\{2p^j,\ 2p^jl,\ 2p^jl^2,\ \ldots, 2p^jl^{\frac{\varphi(p^{n-j})}{d}-1}\}$,
(vi) By Lemma 2.4, $C_{4g^kp^j}$ and $C_{4g^hp^j}$ are pairwise disjoint whenever $k \neq h$ or $i \neq j$.

For a fixed k, $0 \leq k \leq d-1$ and fixed j, $0 \leq j \leq n-1$, $4g^kp^jl^{\frac{\varphi(p^{n-j})}{d}-1} \equiv 4g^kp^j(\mathrm{mod}\ 4p^n)$.
Hence, the cyclotomic coset containing

$4g^kp^j$ is $\{4g^kp^j,\ 4g^kp^jl,\ 4g^kp^jl^2, \ldots, 4g^kp^jl^{\frac{\varphi(p^{n-j})}{d}-1}\}$.

By the constructions of cyclotomic cosets in (i) – (vi) it follows that:
$|C_0| = |C_{2p^n}| = 1$, $\quad |C_{p^n}| = \varphi(4)$, $\quad |C_{g^kp^j}| = \frac{\varphi(4p^{n-j})}{d}$,
$|C_{2g^kp^j}| = |C_{4g^kp^j}| = \frac{\varphi(p^{n-j})}{d}$
Then, $|C_0| + |C_{p^n}| + |C_{2p^n}| +$

$$+\sum_{k=0}^{d-1}\sum_{j=0}^{n-1}(|C_{g^kp^j}| + |C_{2g^kp^j}| + |C_{4g^kp^j}|) = \sum_{k=0}^{d-1}\sum_{j=0}^{n-1}\left(2\frac{\varphi(p^{n-j})}{d} + \frac{\varphi(4p^{n-j})}{d}\right)$$

$$= 4 + 4(\varphi(p^n) + \varphi(p^{n-1}) + \varphi(p)) = 4 + 4\varphi(p)(p^{n-1} + p^{n-2} + \cdots + 1) = 4 + 4\varphi(p)\frac{(p^n-1)}{p-1}$$

$$= 4 + 4(p^n - 1) = 4p^n.$$

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