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Dr. T.P. Singh
Chief Editor & Professor in Maths & O.R.

# FOREWORD

It gives me an immense pleasure in writing this foreword for **'Aryabhatta Journal of Mathematics & Informatics' (AJMI) Vol.14 issue 1 Jan.-June. 2022** published by Aryans Research & Educational Trust. The first issue of the journal was published in year 2009.Since then & till today the journal publication is regular & well in time. It gives us a great pleasure to put forward before the scholars and researchers that journal from its start is making an effort to produce good quality articles. The credit goes to its reviewer team which review sincerely and furnish valuable suggestions to improve the quality of papers. The Journal covers areas of mathematical and statistical sciences, Operational Research, data based managerial &, economical issues and information sciences.

**AJMI VOL.14 issue 1** is before you. I am pleased to note that research scholars, professors, executives from different parts of country have sent their papers for this issue. The papers are relevant and focus on the futuristic trends and innovations in the related areas. We have received around 34 papers for this issue from which on reviewer's report only 14 have been selected for publication. DOI no. by cross Ref. have been mentioned on each article.

1. Prof. Mallkarjuna, Krishna Kumar et.al. Proposed mathematical study to analyze the thermal pollution using continuity equation.The model results showed graphically the variation of concentration of dispersants with displacement at different time of intervals.
2 Prof. Manimannan G, John Britto et.al. presented an overview of Stochastic model and statistical testing in survival data analysis in reference to biosciences and genetic resemblance.
3. Prof. MadhuDadhwal , Pankaj et.al.studied ternary algebraic structures and introduced the notion of left Euclidean norm on a ternary semiring,
4. John Britto & Prof. Rajagopalan et.al.in their paper exploredpattern recognition and image processing through Bayesian classifier approach.
5. In paper no.5, Dr. Meenu et.al. constructed a queue model with balking and reneging behavior of impatient customers and preemptive priority service.
6. Vandana et.al. discussed a complex semibi-tandem queue model in stochastic environment where transition probabilities of customers moving from one state to other are different.
7. Prof. Birajdar & Sangle introduced new subclass of the function class and established Taylor-Maclaurin coefficients using Catas operator.
8. Sarita et.al. dealt stochastic reliability model of two dissimilar units with the assumption that failure and repair times are correlated.
9. Prof. Rajesh and Preeti Proposed a novel approach using the RGB Luminance Channel for the conversion of a color image into one channel image.
10. In paper 13 A.Krishnaa presented Lattices from Discrete Mathematics in order to achieve a highly safe data transfer between two parties.
11. In paper 14 Surya P. Gautam et.al. discussed cubic non- linear Diophantine equation and obtained positive integral solution for different value of m and n in two different ways.

I would like to thank and felicitate the contributors in this issue and at the same time invite quality papers from academic and research community for Vol.14 issue 2 to be published in Nov. 2022.
Comments, suggestions and feedback from discerning readers, scholars and academicians are always welcome.

**DR. T.P. SINGH**

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# MATHEMATICAL STUDY ON THERMAL POLLUTION WITH REFERENCE TO DIFFUSIONAL TRANSFER OF DISPERSANTS

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**ABSTRACT**

***Mathematical study has been developed to analyze the thermal pollution using continuity equation. The diffusion equation is a partial differential equation consists of thermal dispersants such as tracer concentration C(x, t) and longitudinal mixing coefficient with respect to x, y and z directions at time't'. The diffusion equation represents the dispersants has been solved analytically and the range of dispersion is considered for the range of 10ft to 1000ft. Fully mixed contour defined by equation gives the dispersants surroundings, the area as computed for solving partial differential equation (diffusion equation)***

***Key words: Dispersants, Pollution, Thermal***

## INTRODUCTION

Pollution occurs in the nature in various forms such as: industrial pollution, air pollution, water pollution, Toxic pollution, soil pollution, noise pollution and Thermal pollution. Direct effect of pollution on the surrounding vegetarian and life can always be determined by estimating the extent of pollution in a given industrial domain or land area. Therefore the pollution may be in terms of chemical or biological gases, fumes, particulate matter, polluted land and water resources. To explore the potential threat of thermal pollution which exists in the surrounding environment is studied using mathematical model. Pollution, is an undesirable change in the physical, chemical or biological characteristics of our air, land and water that may or will harmful and affect the living conditions. The thermal pollution problem consists of the larger use of electric energy, growth and construction of new electricity generating stations, and fossil fuels to nuclear reactors. There is also heat addition takes place due to industrial waste heat to waterways which increases in the heat load of the waters moderately. Also municipal discharge and irrigation return waters also causes thermal pollution. Return flows from irrigation fields are also a thermal pollutant. As in any environmental situation the actual rise is a function of initial water temperature, volume of abstraction, velocity of flow, length of time in field, and climatic conditions.

The largest single source of manmade heat (thermal) addition to our rivers and waterways is from electric generating plants. Analysis shows that steam electric cooling discharges indicates an average increase of $15^0$F in water temperature after passing through condensers.

Literature provides that, the temperature rise of water from initial point to discharge point of the sewage treatment plant is slightly dispersed. The dispersion and diffusion of the substances for the same density of the receiving water; but it is little known concerning with density differences exist.

J. M. Capuzzo [1], studied the relative importance physiological features like thermal, mechanical, and biocidal stress due to toxic effect of power plant operation on marine zooplankton. Victor S Kennedy [3], explains about the effect of thermal pollution due to industrial activity on vegetation from stream banks, and of disposal of heated effluents from them. M. Abbaspouret. al [4], studied the impacts of thermal pollution, i.e., disorders in reproduction, nourishment and other biological habits and was modeled using MIKE21 software to avoid a decrease on the power plant efficiency. Adrijana Radoševi´c et.al [6],analyze the thermal pollution due to thermal power plants of the Plomin bay induced by the used cooling water released from Plomin 1 and Plomin 2 and flow simulations and temperature field analysis were made and the measurements of the bay surface temperature field were carried out using 3Dwater flow simulations . Christopher A. Ollson et.al [10], asses the results of a comprehensive human health risk assessment for energy-from-waste thermal treatment facility based on extensive sampling of baseline environmental conditions as well as 87 identified contaminants of potential concern.

From the literature it is concluded that thermal pollution has not been a pervasivethreat in the past but could become prominent unless provision is made for its control.

## FORMULATION

For thermal pollution on the reference for the heated waters, the continuity equation for the transport of a conservative tracer is given by the partial differential equation. The tracer concentration is modeled as,

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial c}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial z}\left(D\frac{\partial c}{\partial x}\right) - \frac{\partial}{\partial x}(v_x c) - \frac{\partial}{\partial y}(v_y c) - \frac{\partial}{\partial z}(v_z c) \quad (1)$$

Where ‘c’ equals the tracer concentration: x, y, and z are the coordinate directions with corresponding velocities$v_x$, $v_y$ and $v_z$; and D includes the effects of both molecular and turbulent diffusivity. The first term explains the unsteady part of the conservation equation, the next three terms explains the diffusional transfer of matter, and the last three terms explains mass transfer by convective motion of the fluid.

We consider the cross-sectional average concentration, the empirical expression for velocity and the Reynolds analogy to define the variation of velocity ($v_x$) and D with cross section. The transport mechanism described by equation (1) which could be represented by the following one dimensional dispersed flow model:

$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2} - v_x\frac{\partial c}{\partial x}$, concentration at time 't' and displacement 'x' across the cross section

$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial y^2} - v_y\frac{\partial c}{\partial y}$, concentration at time 't' and displacement 'y' across the cross section

$\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial z^2} - v_z\frac{\partial c}{\partial z}$ , concentration at time 't' and displacement 'z' across the cross section

Defining $D_L$ as the longitudinal mixing coefficient and $\bar{u}$as the average velocity,

Then we have,

$$\frac{\partial c}{\partial t} = D_L\frac{\partial^2 c}{\partial x^2} - \bar{u}\frac{\partial c}{\partial x} \qquad (2)$$

Where 'c' is the function of time, displacement in x, y and z directions, it can be related to linear form, c(x,y,z,t). In the present study, the profiles of tracer concentration has been analyzed with average values of longitudinal mixing coefficient of diffusion, '$D_L$' with average velocity '$\bar{u}$'.

## ANALYSIS

Solving analytically for $D_L$ using empirical data, then, for tracer concentration c(x,t) in series form with average values of $D_L$ and $\bar{u}$ as,

$$c\,(x,t) = \left[C_1 e^{\left(\frac{\frac{\bar{u}}{D_L} + \sqrt{\left(\frac{\bar{u}}{D_L}\right)^2 + \frac{4\times K}{D_L}}}{2}\right)x} + C_2 e^{\left(\frac{\frac{\bar{u}}{D_L} - \sqrt{\left(\frac{\bar{u}}{D_L}\right)^2 + \frac{4\times K}{D_L}}}{2}\right)x}\right] \times C_3 e^{Kt} \qquad (3)$$

$$c(x,t) = \left[C_4 e^{\left(\frac{\sqrt{\left(\frac{\bar{u}}{D_L}\right)^2 + \frac{4\times K}{D_L}}}{2}\right)x + K\,t} + C_5 e^{-\left(\frac{\sqrt{\left(\frac{\bar{u}}{D_L}\right)^2 + \frac{4\times K}{D_L}}}{2}\right)x + K\,t}\right] \times e^{\alpha x + Kt}$$

$$c(x,t) = \left[1+\alpha x+\frac{(\alpha x)^2}{2!}+\frac{(\alpha x)^3}{3!}+\cdots\right]\times\left[1+2kt+\frac{4(kt)^2}{2!}+\frac{8(kt)^3}{3!}+\cdots\right]\times\left\{\left(C_4\left[1+\left(\frac{\sqrt{\alpha^2+4\beta}}{2}\right)x+\frac{\left[\left(\frac{\sqrt{\alpha^2+4\beta}}{2}\right)x\right]^2}{2!}+\frac{\left[\left(\frac{\sqrt{\alpha^2+4\beta}}{2}\right)x\right]^3}{3!}+\cdots\right]\times\left[1+2kt+\frac{4(kt)^2}{2!}+\frac{8(kt)^3}{3!}+\cdots\right]\right)+\left(C_5\left[1-\left(\frac{\sqrt{\alpha^2+4\beta}}{2}\right)x+\frac{\left[\left(\frac{\sqrt{\alpha^2+4\beta}}{2}\right)x\right]^2}{2!}-\frac{\left[\left(\frac{\sqrt{\alpha^2+4\beta}}{2}\right)x\right]^3}{3!}+\cdots\right]\times\left[1+2kt+\frac{4(kt)^2}{2!}+\frac{8(kt)^3}{3!}+\cdots\right]\right)\right\} \quad (4)$$

Where $C_1C_3 = C_4$ and $C_2C_3 = C_5$, constants have been obtained using the conditions c(1,0) and c(1,1).

For the two-dimensional case, the longitudinal velocity is computed and the surface area (A) enclosing at any specific temperature rise to be as, For the three-dimensional case

$$A = (0.496)\left(\frac{2Q_p\sqrt{Wd}}{\pi^{\frac{3}{2}}(D_yD_z^3)^{\frac{1}{4}}}\right)\left(\frac{Q_p}{Q_r}\right)^{\frac{1}{2}}\left[\frac{S_C}{S_p}\right]^{-\frac{3}{2}} \quad (5)$$

For values of $\frac{S_c}{S_p} < 0.60$. The cross section area of the isothermal contour $A_c$ is,

$$\frac{A_C}{A_R} = \left[\frac{Q_p}{Q_r}+\left(\frac{\pi\sqrt{D_yD_z}L}{Q_r}\right)\right]\left[\ln\left(\frac{S_P}{S_c}\right)\right]\left(\frac{1}{1+\frac{\pi\sqrt{D_yD_z}L}{Q_r}}\right) \quad (6)$$

Where $A_r$ is the cross-sectional area of the river and L is the distance from the discharge point to the cross sectional area of the interest.

The distance to the completely mixed contour is obtained as

$$L_m = \frac{Q_r}{\pi\sqrt{D_yD_z}}\left(1-\frac{Q_p}{Q_r}\right) \quad (7)$$

## RESULTS AND DISCUSSION

The theoretical studyhas been presented in the form of mathematical model. It is found that the dispersants spread over the range 100ft to 1000ft decreases as displacement increases at time‘ t’. The contour consists of dispersants with fully mixed pollutants which can be taken at a constant factor of pollutants in the environment. Model results are compared with [4] and [7]. Fig.(1),(2),(3) and (4) shows that the variation of concentration of dispersants with displacement at time t = 2min., t = 4min., t = 6min. and t = 8min.

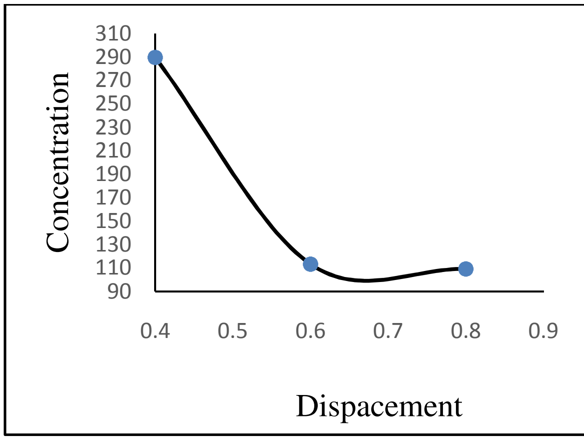


Fig.1: Variation of Concentration vs Displacement at time t = 2min

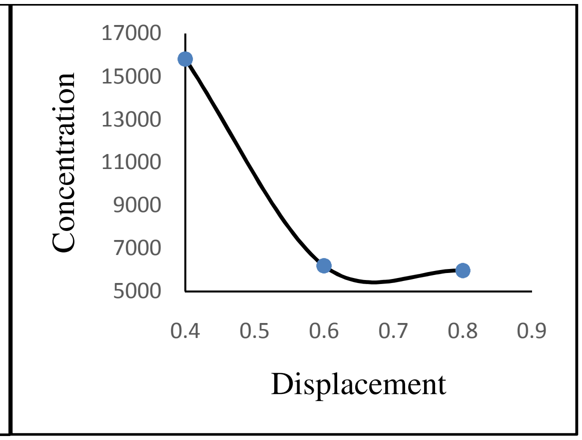


Fig.2: Variation of Concentration vs Displacement at time t = 4min

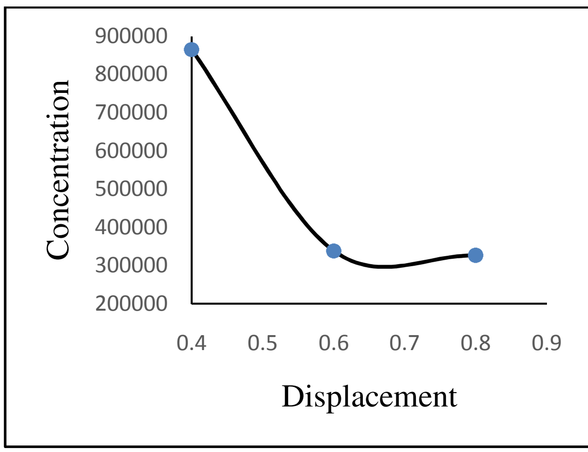


Fig.3: Variation of Concentration vs Displacement at time t = 6min

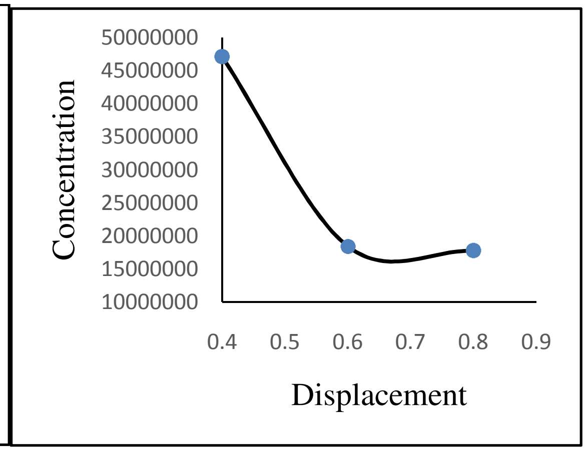


Fig.4: Variation of Concentration vs Displacement at time t = 8min

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# STOCHASTIC MODELING ON INTELLIGENT QUOTIENT: SURVIVAL DATA ANALYSIS A REVIEW

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**ABSTRACT**
***In this study an overview of Stochastic model and statistical testing have been presented. The study is applicable in survival data analysis particular in biosciences and genetic resemblance. The paper throws relevant information in analysis of survival data.***
***Keywords: How state Markovian Chain, Reliability, Parallel Series, Model Wright-Fisher Genetic Model, Weinberg Law, Bayesian Inference, Gamma Distribution, Memory less property of Exponential Distribution, Binet-Standford Intelligent Quotient***

## 1. INTRODUCTION

Recently application of reliability concept in discrete time Kalbfeisch and Prentice (1980) have established the role of discrete models in the analysis of life time data. A word ‘Survival’ known in Human genetics that ‘A’ Dominant allel, ‘a’ recessive allel in monohybrid (including human population).
1.2.1 i.e.: 25:50:25 A:a in applying 2 state Markovian transition probability (tpm), used in Reliability Survival data analysis may be studied. “Gregor Mendel (1986), experimenting with chosen traits in peas, proposed ‘gene’ as the basic unit of inheritance, developed mathematical model for the transmission of inheritable (genetic) traits based on the concept of probability, how organism pass.
On traits to their offspring, in an attempt ‘Dynamic Modelling’ to investigate the growth and interactions of population has been discussed.

### 1.1 Model Specification

i) $Q(x,t)$ under random mating for one pair of genes, the distribution of the genotypes becomes stationary from the first generation, no matter with the original distribution is known as Memory less property.
ii) AAssuming ‘Polya Urn Scheme’, two urns contain a very large number of coins of three Types AA, Aa, aa with u; high IQ, 2v; normal IQ, w; least IQ based on BinetWais and WiscPschyometric measurement assuming Normality.

iii) S.Wright, introduced the following genetical model, Suppose 2N genes of type A and a (the allels) are selected from each generation. The number of A genes is the State of the Markov Chain

iv) Wright-Fisher Model and the transition probabilities are given below; $I = \{0,1,2,...,2N\}$ and $p_{ij} = (2N_j)\left(\frac{i}{2N}\right)^j \left(1-\frac{i}{2N}\right)^{2N-j}$.

Thus if the number of A genes in any generation is equal to I, then we may suppose that there is an infinite pool of both types of genes in which the proportion of A to a is as $I, 2N-I$ and $2N$iindependent drawings are made from it to give the genes of the next generation. We are dealing with 2N independent Bernoullian trials with success probability $i/2N$, which results, the expected number of A genes is $\sum_{j=0}^{2N} P_{ij} = 2N\frac{i}{2N} = i$

v) Poisson Point process (pure Birth Process) Linear Growth process;

This represents with a population whose members can give birth to new members but death cannot occur.

vi) Pure death process: In this process, a single person cannot give birth to a new person but death can occur simultaneously the species get Extinction.

## 2. THE MODEL

Start from Malthusian Model, $P_{tt_1} = \lambda_t$ (Exponential or Geometric (either growth or Decay)

Consider an example a population structure based on Leslie Model, $x(t_1), x(t_2)$ no of individuals with normal IQ, High IA low IQ is considered as right censored problem, Bayesian non-Parametric approach, in many of the Experimental situations, is to be studied.

**Type I censoring,** we have the proportion of observations censored is a

Random variable while the time of censoring is obtained if we stop the observation as soon as a $Y_i = T_i$ if $T_i \le t_0$, $t_0$ if $t_0 < T_i$

A fixed proportion of the sample is covered

**Type II censoring,** A type II censoring is obtained if we stop the observation as soon as a fixed proportion of the sample is covered.

Progressively censored Data; let us assume, constant hereditary on IQ, then density function is

$$f(t) = \lambda e^{-\lambda t},\ \lambda > 0,\ t \ge 0$$

The progressive censoring schemes for the $i^{\text{th}}$ patient $[0, T_i]\ T_{ij_0} \lambda . e^{-\lambda t} .dt = 1 - e^{-\lambda T_i} = \theta_i$ (say)

$u = \frac{1}{\lambda}$ the Mean period of pupil from the successive censored data $(o, T)$

## 2.1 Frailty Model

The term coined by Vaupet, Manton and Stallard (1979). In studies by survival, the hazard function,For each individual may depend on a set of risk factors (Explanatory Variables) i.e., hereditary, this unknown and unobservable risk factor of the hazard function is termed as the individuals Heterogeneity (or) Fraility, which is known as Frailty Model.Weibull Baseline Hazard, discussed by Alanidou and Sinha (1997), Cox-Regression Analysis utilized Genetic level and Category (job)

iv) Now consider the R.V.,

$T = 1$, if the person doesn't survive upto $T_i$ with probability $= 0\left(1-e^{-\lambda T_i}\right)$

If the person survives upto $T_i$ with probability $e^{-\lambda T_i}$

v) To gain insight into how population with distinct age groups we can describe the population through

$x_1(t) =$ no of individuals age 0 through 14 at time $t$

$x_2(t) =$ no of individuals age 15 through 29 at time $t$

$x_3(t) =$ no of individuals age 30 through 44 at time $t$

$x_4(t) =$ no of individuals age 45 through 59 at time $t$

$x_5(t) =$ no of individuals age 60 through 75 at time $t$

Here, $f_\lambda$ denotes a birth rate for parents of $i^{\text{th}}$ age class and $r, i+1$ denotes a survival rate for Those in the $i^{\text{th}}$ age class passing into the $(i+1)$ th. Because of single set of parents, we should

Atribute half of their offspring to each in choosing value of *f*.

vi) The Matrix notation, the model is simply $x_{i+1} = P_{x_t}$

$$x_t = \left(x_1(t), x_2(t), x_3(t), x_4(t), x_5(t)\right)$$

is the column vector sub population sizes at time *t* and

$$P = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \\ r1.2 & 0 & 0 & 0 & 0 \\ 0 & r2.3 & 0 & 0 & 0 \\ 0 & 0 & r3.4 & 0 & 0 \\ 0 & 0 & 0 & r4.5 & 0 \end{bmatrix}$$

$$x_1(t+1) = f_1x_1(t) + f_2x_2(t) + f_3x_3(t) + f_4x_4(t) + f_5x_5(t)$$

$$x_2(t+1) = r_{1.2}x_1(t)$$

$$x_3(t+1) = x_{2,3}(t)$$

$$x_4(t+1) = r_{3,4}x_3(t)$$

$$x_5(t+1) = r_{4,5}x_4(t)$$

As demographer often use 5 year old are likely to give birth than 15 to 30 year old are likely to give.

The top row would have fecundity information, the subdiagonal would have survival information and the rest of the matrix would have entries of 0.

## 3. METHODS AND NOTATIONS

Reliability (Survival) Data Analysis:

Mean Resident life (MRL)

For a Random Variable *X* representing the life time of a component $y = x - t / x > t$ is called the residual life distribution

$$r(t) = E[X - t / x > t]$$

$$\overline{F}(n) = \frac{r(0)}{r(n)} e^{-\int_0^x \frac{1}{r(t)} dt}$$

Is called the mean residual function and this represents the average life remaining for the component given that it has survival upto time *t*.

Muthu (1977) argued that MRLF will serve as a better describe the failure pattern as the failure rate account only for immediate failure at a time point where as MRLF accounts for the Complete failure on the average.

The Mean Residual Life function determines the distribution uniquely. In fact analogous to failure rate, the above result provides a powerful tool to model life time data.

Through a knowledge of the functional form we can determine the distribution uniquely.

A distribution is said to belong to the IMRI, class if $r(x)$ is increasing in *X* and DMRL class.

If $r(x)$ is decreasing in *x* incidentally

IFR → DMRL

DFR → IMRL

But the converse is not true

**3.1 Characterization of Exponential Model: Memory less Property**

i. A continuous non-negative variable X with $E(X) <$ follows, the exponential distribution iff $E(x - t / X > t) = E(X)$ for all $t > 0$

ii. A Relationship of the form (for the random variable X considered above) holds for all $t > 0$ if and only if

iii. X follows the Exponential distribution if $a = 0$

iv. X follows Pareto distribution if $a > 0$

v. X follows finite range distribution if $a < 0$ (Mukerjee & Roy 1987)

## 4. DISCUSSION & CONCLUSION

### Evolutionary Genetics of Race

The heritability is threshold character in which there has three classes in its visible scale, then comparison can be made with the variance of population as well as between the Means. The comparison of variances assuming that the interval between the two threshold is constant from one population another but the intermediate class express the same difference of concentration (or) rate in the two population concerned.

Race is generally used as a synonyms for subspecies, Race and IQ have either hereditarians or culture only. Heritability refers to the genetic contribution to the individual difference (variance) in a particular group, not to the phenotype of a single individual. Hereitability describes what is the genetic contribution to individual in differences in a particular population at a particular time, not what could be.

Francis GGalton (1869) (a) all individuals posses some level of general mental capacity called general intelligence, that, to some degree, influences all cognitive activity (b) the differences between individuals and between groups in general intelligence are largely the result of genetic differences.

Hereditarian heuristics include constructing better tests, developing better techniques for measuring mental abilities and discovering biological correlation (e.g. heritability, inbreeding depression and heterosis, brain size, brains metabolic rate, brain evopked potentials, brain imaging of these tests. The process the involvesexaming the similarities of the scores among people whose varying degrees of genetic resemblance can be predicted from Mendelian Theory (Fishers 1918).

Jensen (1969) reviewd, data sets for 10 categories of evidence, they include the international pattern of IQ scores more and less g-coaded components of tests, heritability, brain size and cognitive-ability relations transgocial apposition, racial admixture, regression to the Mean, the race finding are then used to evaluate the culture-only and hereditarian models in terms of methodology proposes by Lakatofs (1979, 1978), indeed the best trials for identifying subspices are now simply those within the best phylogentic resolution. The variability of the tradition sub-species definition of human races can be addressed by examing the patterns and amount of genetic diversity found within and among human populations, one common method of quantity test statistic of wright 1964) and some of its more modern variants that have been designed specifically for molecular data such as Kst (Handsonet.a. 1992) or N st (Lynch and escase 1990). Fst and related statistics range from 0 (all the genbetic diversity within a species a shared equally by all populations with no genetic differences among populations) to 1 (all the genetic diversity within a specifies is found as fixed differences among population with no genetic diversity within population).

*F* values in humans truly reflect a balance between gene flow versus local drift/selection Wright (1969) ist law model to quanify this balance is Nm, the product gene flow independent of geographical distances $F = 1/4N + 1$ of local effective population size (*N*) with *m*, the migration between dems.

Crow and Kiruma (1970), the population involves as a single evolutionary lineage over a long periods of times. Population genetic theory also indicate that fluctuation around an average non of order one in conductive to the rapid spread of selectively favoured gene throughout the spreads and to local population differences and adaptive (Baston and Rouhani 1993). The assumption that, the heman*F* value arise from the between of gene flow varsus local drift and selection. Nevertheless these populations show, extreme differentiation and local adaptation for the abdomen syhndrome, a complex polygenic suite of phenotypes that affect morphology development time, female fecundity, male sexual maturation and longevity in adaptive significant ways (Holoches and Thempleton 1994) Hollocher et.at. 1992, Templetan et.al 1993, 1989)

**Race differences in Cognitive Ability**

**Mean Race-IQ Differences and Regression to the Mean**

Regression towards the mean provides still another method of testing if the group differences are genetic. Regression towards the mean is seen, an average when individuals with IQ score made an their children show lower screen that their parents. This is because the parents pass on some, but not all, of their genes to trace offspring. The converse happens for low IQ parents they have children random half of their genes to their offspring, they can't pass on a random half of their genes to their offspring. The converse happens for low IQ. Although parents pass on a random half of their genes to their offspring, they can't pass on the particular combination of genes, that cause their own exceptionally. This is analogous to rolling a point of dice and having them come up two b's or two's. The odds are that one the next roll. You will get, some value that is not quite as ghier (or low), physical and psychological trains involving dominant and recessive genes show some regression.

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# ON EUCLIDEAN NORMS AND FACTORIZATION IN TERNARY SEMI-DOMAINS

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**ABSTRACT**
***In this paper, general properties of right divisors and greatest common right divisors in a ternary semiring are considered. The notion of left Euclidean norm on a ternary semiring is introduced to characterize the existence of the greatest common right divisor of a non-empty finite subset in a principal left ideal subtractive ternary semiring. The relationships of left Euclidean ternary semirings with principal left ideal subtractive ternary semirings, greatest common right divisor ternary semidomains, factorization ternary semi-domains and discrete valuation ternary semirings are also established.***
***Keywords: Ternary semirings, Left Euclidean norm, Factorization ternary semidomain, Discrete valuation ternary semirings***
***MSC 2020: 16Y60, 16Y99***

## 1. INTRODUCTION

Ternary algebraic structures occur naturally not only in mathematics, but also in many other domains of sciences. Hestenes [4] had studied ternary algebra with applications in matrices and linear transformations. Cayley introduced the idea of cubic matrices and a generalization of a determinant known as hyperdeterminant related to ternary algebra. Many applications of ternary algebraic systems are observed in supersymmetry and in theoretical & mathematical physics [5]. It also has applications in generalizations of Hopfalgebras [7] which play a huge contribution in quantum group theory.

In 1971, Lister [6] identified some algebraic subsets of rings namely additive subgroups which are also closed under ternary multiplication and referred these algebraic structures as ternary rings. To generalize the notion of ternary rings, Dutta and Kar [2] introduced the notion of ternary semirings. Ternary semirings are also a generalization of semi-rings introduced by Vandiver [13] and studied by various mathematicians (cf. [8]-[12]). A non-empty set $\mathcal{T}$ with a binary addition and a ternary multiplication is called a ternary semiring if $\mathcal{T}$ is an Abelian semigroup under addition along with the following axioms:

$(i)(abc)de = a(bcd)e = ab(cde)$ (Associative Law)

$(ii)(a+b)cd = acd + bcd$ (Right Distributive Law)

$(iii)\ a(b+c)d = abd + acd$ (Lateral Distributive Law)

$(iv)\ ab(c+d) = abc + abd$ (Left Distributive Law), $\forall a,b,c,d,e \in \mathcal{T}$.

It is observed that every ternary semiring may not be a semiring in general but converse is always true. For instance, $\mathbb{Z}_0^-$ the set of non-positive integers and $\mathbb{R}_n^-[x] = \{a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n : a_i \in \mathbb{R}^-\}$ under usual addition and ternary multiplication are commutative ternary semirings which are not semirings. Further, an element $\mathrm{e} \in \mathcal{T}$ satisfying $x\mathrm{ee} = \mathrm{e}x\mathrm{e} = \mathrm{ee}x = x, \forall x \in \mathcal{T}$ is called a unital element. Unital element may not be unique: For example, consider $\mathcal{S} = \{0,1,2,\ldots,n\}$, where $n \geq 1$ a positive integer and $\mathcal{T} = \mathbb{Z} \times \mathcal{S} = \{(x,r) : x \in \mathbb{Z}, r \in \mathcal{S}\}$. Define binary addition $\mp$ and ternary multiplication $\dotplus$ by $(x,r) \mp (y,s) = (x \mp y, \min(r,s))$ and $(x,r) \dotplus (y,s) \dotplus (z,t) = (xyz,$ $\max(r,s,t))$ respectively, $\forall (x,r),(y,s),(z,t) \in \mathcal{T}$. Then, $\mathcal{T}$ is a ternary semiring with additive identity $(0,n)$ and two unital elements $(1,0)$ and $(-1,0)$. In addition, $\mathcal{T}$ is said to be commutative, if $xyz = xzy = yxz = yzx = zxy = zyx$ holds $\forall x,y,z \in \mathcal{T}$. An element $x \in \mathcal{T}$ is called a unit, if $\exists$ an element $y \in \mathcal{T}$ with $xyt = xty = txy = yxt = ytx = tyx = t \, \forall \, t \in \mathcal{T}$. The set of all units in ternary semiring $\mathcal{T}$ is denoted by $U(\mathcal{T})$. In the above example, $U(\mathbb{R}_n^-[x]) = \mathbb{R}^- \backslash \{0\}$. Further, a ternary semiring is yoked, if for any pair of elements $x,y \in \mathcal{T}$, $\exists t \in \mathcal{T}$ such that $x + t = y$ or $y + t = x$ and $\phi \neq \mathcal{S} \subseteq \mathcal{T}$ is subtractive, if $x \in \mathcal{S}$ and $x + y \in \mathcal{S}$ then $y \in \mathcal{S}$. Also, $\mathcal{T}$ is said to be zero divisor free, if $xyz = 0$, for any three elements $x, y$ and $z$ in $\mathcal{T}$, then either $x = 0$ or $y = 0$ or $z = 0$.

In the present paper, we characterize the existence of the greatest common right divisor of a non-empty subset in ternary semi-rings. Further, the relationships of left Euclidean ternary semi-rings with principal left ideal subtractive ternary semi-rings, greatest common right divisor semi-domains, factorization ternary semidomains and discrete valuation ternary semidomains are also established.

In section 2, we prove some results regarding right divisors and ternary left Euclidean norms in ternary semirings which are necessary for the development of this paper. In addition, Proposition 2.9, provides an equivalent condition for an element to become a greatest common right divisor of a non-empty subset in a ternary semiring.

Thereafter, we continue by introducing the notion of left ternary Euclidean norm on a ternary semi-ring which is analogous to the left Euclidean norm defined by Golan [3] as follows:

A left Euclidean norm $\mu$ defined on a ternary semiring $\mathcal{T}$ is a function $\mu : \mathcal{T} \backslash \{0\} \to \mathbb{N}$ such that

$(i)$ $\mu(x) \leq \mu(t_1t_2x) \forall 0 \neq x, t_1, t_2 \in \mathcal{T}$ such that $t_1t_2x \neq 0$;

$(ii)$ If $x, y \in \mathcal{T}$ with $y \neq 0$, then $\exists c, q$ and $r$ elements of $\mathcal{T}$ satisfying $x = cqy + r$ with $r = 0$ or $\mu(r) < \mu(y)$.

Similarly, one can define a right Euclidean norm, except that in condition $(ii)$ we have $x = ycq + r$. Note that for a commutative ternary semiring, the left and right Euclidean norms coincide and $\mathcal{T}$ is called Euclidean ternary semiring.

In Theorem 2.24, we prove that every subtractive left ideal in a ternary semi-ring with a left Euclidean norm is principal. It is also shown that any non-empty subset in a ternary semi-ring in which every left ideal is principal, has a greatest common right divisor (see Theorem 2.25). Therefore, by combining

Theorem 2.24 and 2.25, we conclude that any non-empty subset of a left Euclidean ideal subtractive ternary semiring has a greatest common right divisor.

Further, we introduce the concept of **Ternary ring of differences** analogous to the concept of **Ring of differences in semirings** (see chapter 8 in [3]). Using this construction of the ternary ring of differences, an attempt to answer the question that under what conditions every principal ideal of a ternary semiring with a multiplicative left Euclidean norm on it becomes subtractive is made. It is also shown that an additively cancellative, yoked ternary semiring with a multiplicative left Euclidean norm is a subtractive left Noetherian ternary semiring.

Finally, we establish that any non-empty subset of finite order in a principal left ideal subtractive ternary semiring with a left Euclidean norm on it, possesses a greatest common right divisor.

In section 3, some results regarding factorization in ternary semi-domains are proved. The relationship between left Euclidean ternary semi-ring with principal left ideal ternary semiring which was established in the previous section is further extended to factorization ternary semi-domain. Moreover, we prove that the principal ideal ternary semi-domain is factorization ternary semi-domain.

Section 4 is devoted to study discrete valuations on ternary semi-rings and it is proved that a discrete valuation gives rise to a Euclidean norm in a ternary semiring.

In this paper, $\mathcal{T}$ represents a ternary semiring with a unital element $\mathrm{e}$ in it and $(\gcd)_r$ is a greatest common right divisor.

## 2. EUCLIDEAN TERNARY SEMIRINGS

In this section, the notion of right divisors of an element in a ternary semiring is introduced and prove some results regarding a left Euclidean norm on a ternary semiring. The results proved in this section are frequently used in the sequel.

**Definition 2.1.** An element $y \in \mathcal{T}$ is called a right divisor of $x \in \mathcal{T}$ if $x \in \mathcal{T}\mathcal{T}y$, where $\mathcal{T}\mathcal{T}y = \{t_1t_2y : t_1, t_2 \in \mathcal{T}\}$.

The collection of all right divisors of an element $x$ in $\mathcal{T}$ is $\Re_{\mathcal{D}}(x) = \{y \in \mathcal{T} : x \in \mathcal{T}\mathcal{T}y\}$.

Note that if $\mathcal{T}$ is commutative, then $\Re_{\mathcal{D}}(x)$ becomes the set of divisors of $x \in \mathcal{T}$.

**Example 2.2.** If $\mathbb{Z}_0^-$ is the set of all non-positive integers, then $\mathbb{Z}_0^-$ is a ternary semiring with usual binary addition and ternary multiplication with a unital element $-1$. Consider $-12 \in \mathbb{Z}_0^-$. Then $\Re_{\mathcal{D}}(-12) = \{-1, -2, -3, -4, -6, -12\}$.

**Lemma 2.3.** Let $x, y$ be any two elements of $\mathcal{T}$ and $u \in \mathcal{C}\Re_{\mathcal{D}}(\{x, y\})$. Then the following statements hold:

$(i)\Re_{\mathcal{D}}(x) \subseteq \Re_{\mathcal{D}}(t_1t_2x), \forall t_1, t_2 \in \mathcal{T}$;

$(ii)\ u \in \Re_{\mathcal{D}}(t_1t_2x + t_3t_4y), \forall t_1, t_2, t_3, t_4 \in \mathcal{T}$;

$(iii)\Re_{\mathcal{D}}(u) \subseteq \mathcal{C}\Re_{\mathcal{D}}(\{x, y\})$;

$(iv)$If $\mathcal{T}$ is commutative, then $U(\mathcal{T}) \subseteq \Re_{\mathcal{D}}(\mathrm{e}) \subseteq \Re_{\mathcal{D}}(x)$.

***Proof.*** **Straight forward.** □

**Definition 2.4.** The collection of all common right divisors of a non-empty set $\mathcal{A}$ in $\mathcal{T}$ is $\mathcal{C}\,\Re_{\mathcal{D}}(\mathcal{A}) = \cap \{\Re_{\mathcal{D}}(x) : x \in \mathcal{A}\}$.

**Remark 2.5.** In Example 2.2, let $\mathcal{A} = \{-6, -12, -24\}$ be a subset of $\mathbb{Z}_0^-$, then $\mathcal{C}\Re_{\mathcal{D}}(\{-6, -12, -24\}) = \{-1, -2, -3, -6\}$.

**Lemma 2.6.** For $x, y \in \mathcal{T}$, we have

$(i) \Re_{\mathcal{D}}(x) = \{y \in \mathcal{T} : \mathcal{T}\mathcal{T}x \subseteq \mathcal{T}\mathcal{T}y\}$;

$(ii)\ \mathcal{C}\Re_{\mathcal{D}}(\mathcal{A}) = \{y \in \mathcal{T} : \mathcal{T}\mathcal{T}\mathcal{A} \subseteq \mathcal{T}\mathcal{T}y\}$;

$(iii)\ \mathcal{C}\Re_{\mathcal{D}}(\{x, y\}) \subseteq \mathcal{C}\Re_{\mathcal{D}}(\{x + y, y\})$.

*Proof.* The proofs of $(i)$ $and$ $(ii)$ are quite easy, so we omit the proofs.

$(iii)$ Let $a \in \mathcal{C}\Re_{\mathcal{D}}(\{x, y\})$. Then, by definition there exist $t_1, t_2, t_3$ and $t_4$ such that $x = t_1 t_2 a$ and $y = t_3 t_4 a$. Also, we can write $x + y = t_1 t_2 a + t_3 t_4 a = \mathrm{e}(\mathrm{e}t_1 t_2 + \mathrm{e}t_3 t_4)a \in \mathcal{T}\mathcal{T}a$, as $\mathrm{e} \in \mathcal{T}$. Thus $a \in \Re_{\mathcal{D}}(x + y)$. Therefore, $a \in \Re_{\mathcal{D}}(x + y) \cap \Re_{\mathcal{D}}(y)$, as $a \in \Re_{\mathcal{D}}(y)$. Hence $a \in \mathcal{C}\Re_{\mathcal{D}}(\{x + y, y\})$. □

**Definition 2.7.** An element $b \in \mathcal{C}\Re_{\mathcal{D}}(\mathcal{A})$ is said to be a $(\text{gcd})_r$ of $\mathcal{A}$ iff $\mathcal{C}\Re_{\mathcal{D}}(\mathcal{A}) = \Re_{\mathcal{D}}(b)$.

**Example 2.8.** In Remark 2.5, the $(\text{gcd})_r$ of the set $\{-12, -6, -24\}$ is $-6$, since $\mathcal{C}\Re_{\mathcal{D}}(\{-12, -6, -24\}) = \{-1, -2, -3, -6\} = \Re_{\mathcal{D}}(-6)$.

**Proposition 2.9.** Let $\phi \neq \mathcal{A}$ be a subset of $\mathcal{T}$. Then an element $b \in \mathcal{T}$ is a $(\text{gcd})_r$ of $\mathcal{A}$ iff the following two conditions are satisfied:

$(i)\ \mathcal{T}\mathcal{T}\mathcal{A} \subseteq \mathcal{T}\mathcal{T}b$;

$(ii)$ If $c \in \mathcal{T}$ satisfies $\mathcal{T}\mathcal{T}\mathcal{A} \subseteq \mathcal{T}\mathcal{T}c$, then $\mathcal{T}\mathcal{T}b \subseteq \mathcal{T}\mathcal{T}c$.

*Proof.* Let $b$ be a $(\text{gcd})_r$ of $\mathcal{A}$. Then $b \in \mathcal{C}\Re_{\mathcal{D}}(\mathcal{A})$, so $b \in \Re_{\mathcal{D}}(a)$ for each $a \in \mathcal{A}$. Thus, by Lemma 2.6(i), $\mathcal{T}\mathcal{T}a \subseteq \mathcal{T}\mathcal{T}b$ for each $a \in \mathcal{A}$, gives that $\mathcal{T}\mathcal{T}\mathcal{A} \subseteq \mathcal{T}\mathcal{T}b$. Moreover, if $\mathcal{T}\mathcal{T}\mathcal{A} \subseteq \mathcal{T}\mathcal{T}c$ for some $c \in \mathcal{T}$, then again by Lemma 2.6(ii), $c \in \mathcal{C}\Re_{\mathcal{D}}(\mathcal{A}) = \Re_{\mathcal{D}}(b)$, because $b$ is a $(\text{gcd})_r$ of $\mathcal{A}$. So $c \in \Re_{\mathcal{D}}(b)$ and $\mathcal{T}\mathcal{T}b \subseteq \mathcal{T}\mathcal{T}c$.

Conversely, by given hypothesis and Lemma 2.6(ii), $b \in \mathcal{C}\Re_{\mathcal{D}}(\mathcal{A})$. Then by definition, for $x \in \Re_{\mathcal{D}}(b)$, we have $x \in \mathcal{T}$ and $b \in \mathcal{T}\mathcal{T}x$. This implies that there exist some $t_1$ and $t_2$ such that $b = t_1 t_2 x$, and by condition $(i)$; $\mathcal{T}\mathcal{T}\mathcal{A} \subseteq \mathcal{T}\mathcal{T}b \subseteq \mathcal{T}\mathcal{T}t_1 t_2 x \subseteq \mathcal{T}\mathcal{T}x$. Thus, by Lemma 2.6(ii), $x \in \mathcal{C}\Re_{\mathcal{D}}(\mathcal{A})$, implying $\Re_{\mathcal{D}}(b) \subseteq \mathcal{C}\Re_{\mathcal{D}}(\mathcal{A})$. Also, $b \in \Re_{\mathcal{D}}(b)$, so we have $b \in \mathcal{C}\Re_{\mathcal{D}}(\mathcal{A})$ and hence $\mathcal{T}\mathcal{T}b \subseteq \mathcal{T}\mathcal{T}\mathcal{A}$. Now, by condition $(ii)$, if $c \in \mathcal{C}\Re_{\mathcal{D}}(\mathcal{A})$, then $\mathcal{T}\mathcal{T}\mathcal{A} \subseteq \mathcal{T}\mathcal{T}c$, so we get $\mathcal{T}\mathcal{T}b \subseteq \mathcal{T}\mathcal{T}c$. Therefore, $c \in \Re_{\mathcal{D}}(b)$ and $\mathcal{C}\Re_{\mathcal{D}}(\mathcal{A}) = \Re_{\mathcal{D}}(b)$. □

Next theorem allows us to compute inductively a $(\text{gcd})_r$ of a non-empty subset consisting of more than two elements of $\mathcal{T}$.

**Theorem 2.10.** If $a, b, c \in \mathcal{T}$ and $d_1$ is a $(\text{gcd})_r$ of $\{a, b\}$ and $d_2$ is a $(\text{gcd})_r$ of $\{c, d_1\}$, then $d_2$ is a $(\text{gcd})_r$ of $\{a, b, c\}$.

*Proof.* The result easily holds, by using the definition of a $(\gcd)_r$ of a set in a ternary semiring, since $\mathcal{C}\Re_{\mathcal{D}}(d_2) = \Re_{\mathcal{D}}(d_1) \cap \Re_{\mathcal{D}}(c) = \Re_{\mathcal{D}}(a) \cap \Re_{\mathcal{D}}(b) \cap \Re_{\mathcal{D}}(c) = \mathcal{C}\Re_{\mathcal{D}}(\{a, b, c\})$. □

**Remark 2.11.** A ternary semiring $\mathcal{T}$ with a left (resp. right) Euclidean norm on it is termed as a left (resp. right) Euclidean.

**Example 2.12.** The ternary semiring $\mathbb{Z}_0^-$ is a Euclidean with a Euclidean norm defined by $\mu(x) = x^2$, $\forall x \in \mathbb{Z}_0^-$.

**Proposition 2.13.** A left Euclidean norm $\boldsymbol{\mu}$ on $\mathcal{T}$ induces another left Euclidean norm $\mu^*$ on $\mathcal{T}$ satisfying the conditions:

$(i) \mu^*(x) \leq \mu(x) \forall x \in \mathcal{T}\backslash\{0\}$;

$(ii) \mu^*(y) \leq \mu(t_1 t_2 y) \forall y, t_1, t_2 \in \mathcal{T}$ satisfying $t_1 t_2 y \neq 0$.

*Proof.* For each $0 \neq x \in \mathcal{T}$, define $\mu^*(x) = \min\{\mu(t_1 t_2 x) : t_1 t_2 x \neq 0\}$. Clearly, this new defined function $\mu^*$ satisfies the above stated conditions, so it is enough to show that the Euclidean norm $\mu^*$ is certainly a left Euclidean norm on $\mathcal{T}$. For this, let $x$,$y \in \mathcal{T}$ be two non-zero elements with $\mu^*(x) \geq \mu^*(y)$. Then $\exists\, t_1, t_2 \in \mathcal{T}$ such that $\mu^*(y) = \mu(t_1 t_2 y)$. Thus, $\mu(x) \geq \mu^*(x) \geq \mu^*(y) = \mu(t_1 t_2 y)$, which concludes that $\mu(x) \geq \mu(t_1 t_2 y)$. As $\mathcal{T}$ is a left Euclidean ternary semiring, therefore,$\exists\, c, q$ and $r \in \mathcal{T}$ such that $x = cqt_1 t_2 y + r$, where $r = 0$ or $\mu(r) < \mu(t_1 t_2 y)$. For $r = 0$, the case is trivial. Further, if $\mu(r) < \mu(t_1 t_2 y)$, then we get that $\mu^*(r) \leq \mu(r) < \mu(t_1 t_2 y) = \mu^*(y)$. This completes the proof. □

**Remark 2.14.** If $\mu$ is a left Euclidean norm defined on a ternary semiring $\mathcal{T}$, then without loss of generality, we can consider a function $\mu$ which satisfies the condition $\mu(y) \leq \mu(t_1 t_2 y)$, $\forall 0 \neq y, t_1, t_2 \in \mathcal{T}$ such that $t_1 t_2 y \neq 0$.

**Definition 2.15.** A left Euclidean norm $\mu$ is said to be multiplicative on $\mathcal{T}$, if $\mu(t_1 t_2 y) = \mu(t_1)\mu(t_2)\mu(y)$, $\forall t_1, t_2, y \in \mathcal{T}$.

Note that the left Euclidean norm $\mu$ is multiplicative on $\mathcal{T}$ iff it is a ternary semigroup homomorphism from $(\mathcal{T}\backslash\{0\}, .)$ to $(\mathbb{N}, .)$.

**Example 2.16.** The ternary semiring $\mathcal{T} = \langle \mathbb{Z}^-, \min, + \rangle$ with a Euclidean norm $\mu: \mathcal{T} \to \mathbb{N}$, defined by $\mu(x) = 2^{-x}$, $\forall x \in \mathcal{T}$ is a multiplicative Euclidean norm.

**Proposition 2.17.** Let $\mu: \mathcal{T}\backslash\{0\} \to \mathbb{N}$ be a left Euclidean norm on $\mathcal{T}$ and if $\mathcal{M}_\mu = \{t \in \mathcal{T} : \mu(t) \leq \mu(x)\ \forall\, x \in \mathcal{T}\}$. Then the following statements hold:

$(i)$ (Unital element) $\mathsf{e} \in \mathcal{M}_\mu$;

$(ii)$ If $x \in \mathcal{M}_\mu$, then $\exists\, t_1$ and $t_2$ such that $\mathsf{e} = t_1 t_2 x$;

$(iii)$ $U(\mathcal{T}) \subseteq \mathcal{M}_\mu$. Moreover, if $\mathcal{T}$ is commutative then $U(\mathcal{T}) = \mathcal{M}_\mu$.

***Proof.*** $(i)$ For every non-zero element $a \in \mathcal{T}$, $\mu(\mathsf{e}) \leq \mu(\mathsf{ee}a) = \mu(a)$, as $\mu$ is a left Euclidean norm. Thus $\mathsf{e} \in \mathcal{M}_\mu$.

$(ii)$ Let $x \in \mathcal{M}_\mu$. Then $\exists t_1, t_2$ and $r$ in $\mathcal{T}$ such that $\mathsf{e} = t_1 t_2 x + r$, with $r = 0$ or $\mu(r) < \mu(x)$. Clearly, $\mu(r) < \mu(x)$ is not possible, as $x \in \mathcal{M}_\mu$. Thus $r = 0$ and hence, $\mathsf{e} = t_1 t_2 x$.

$(iii)$ If $x \in U(\mathcal{T})$, then $\exists\, y \in \mathcal{T}$ such that $xyt = xty = txy = t, \forall\, t \in \mathcal{T}$. In particular, $xy\mathrm{e} = x\mathrm{e}y = \mathrm{e}xy = \mathrm{e}$. Now, in view of Remark 2.14, we have $\mu(x) \leq \mu(xy\mathrm{e}) = \mu(\mathrm{e})$and $\mathrm{e} \in \mathcal{M}_\mu$. This concludes that $\mu(x) = \mu(\mathrm{e})$, implying $x \in \mathcal{M}_\mu$. Hence $U(\mathcal{T}) \subseteq \mathcal{M}_\mu$.

Now, assume that $\mathcal{T}$ is commutative and $x \in \mathcal{M}_\mu$. Then by $(ii)$, we have $\mathrm{e} = t_1t_2x$. Further, for all $t \in \mathcal{T}$, $t = \mathrm{ee}t = (t_1t_2x)\mathrm{e}t = x(t_1t_2\mathrm{e})t$. Finally, take $(t_1t_2\mathrm{e}) = y \in \mathcal{T}$ and by the commutativityof$\mathcal{T}$, we have $xyt = xty = txy = t, \forall t \in \mathcal{T}$. Thus $x \in U(\mathcal{T})$. This completes the result. □

The proof of Proposition 2.18 follows verbatim as Proposition 12.13 in [3].

**Proposition 2.18.** Let $\theta: \mathcal{T} \to \mathcal{S}$ be an onto morphism of ternary semirings and $\mu$ a left Euclidean norm defined on$\mathcal{T}$. Then corresponding to $\mu$there is a left Euclidean norm $\mu'$ on$\mathcal{S}$satisfying

$$\mu'(p) = \min\{\mu(x) : x \in \mu^{-1}(p)\}, \forall 0 \neq x \in \mathcal{S}.$$

**Definition 2.19.** If every left ideal of $\mathcal{T}$ is subtractive, then $\mathcal{T}$ is calleda left ideal subtractive (IS) ternary semiring.

**Example 2.20.** The commutative ternary semiring $\mathcal{T} = \langle \mathbb{Z}_0^- \cup \{-\infty\}, \min, \max\rangle$ is an ideal subtractive ternary semiring, as only ideals of $\mathcal{T}$ are $J_n = \{0, -1, -2, -3, \ldots, -n\}$ for some$n \in \mathbb{Z}_0^+$; or $\mathbb{Z}_0^-$ or $\mathcal{T}$ and each ideal in$\mathcal{T}$ is subtractive.

**Remark 2.21.** $\mathcal{T}$is termed as a principal ideal ternary semiring if every ideal of $\mathcal{T}$ is principal.

**Example 2.22.** The ternary semiring $\mathcal{T} = \langle \mathbb{Z}_4^-, +_4, \times_4\rangle$ is commutative and all its ideals $\{0\}$, $\{0, -2\}$ and $\mathbb{Z}_4^-$ are principal. Thus, $\langle \mathbb{Z}_4^-, +_4, \times_4\rangle$ is a principal ideal ternary semiring.

**Definition 2.23.** A ternary semiring $\mathcal{T}$ is said to be a principal left ideal ternary semiring (PLI) if each left ideal of $\mathcal{T}$ is principal.

**Theorem 2.24.** Let $\mathcal{T}$ beleft Euclidean and $\mathcal{I}$ be an arbitrary subtractive left ideal in $\mathcal{T}$. Then $\mathcal{I}$ is principal.

***Proof.*** Assume that $\mathcal{T}$ is left Euclidean with a left Euclidean norm $\mu$ and $\mathcal{I}$ is any arbitrary non-zero subtractive left ideal of $\mathcal{T}$. Consider a set $\mathcal{A} = \{\mu(t) \in \mathbb{N} : t \in \mathcal{I}\backslash\{0\}\}$ and by well ordering principle the set $\mathcal{A}$ possesses a least member, say $\mu(t)$. We claim that $\mathcal{I} =< t >$. Assume that $x \in \mathcal{I}$, then $\exists\, c, q$ and $r$ in $\mathcal{T}$ such that $x = cqt + r$, where $r = 0$ or $\mu(r) < \mu(t)$. If $r \neq 0$, then $r \in \mathcal{I}$, as $\mathcal{I}$ is a subtractive ideal. Since $\mu(t)$ is the least element of$\mathcal{A}$, so $\mu(t) \leq \mu(r)$. This leads to a contradiction and hence $r = 0$. Therefore, $x = cqt \in< t >$, so $\mathcal{I} \subseteq< t >$. But$< t >\subseteq \mathcal{I}$, so $\mathcal{I} =< t >$ and hence the result. □

**Theorem 2.25.** Let every left ideal of $\mathcal{T}$ bea principal ideal. Then every non-empty subset of $\mathcal{T}$ has a $(\text{gcd}\,)_r$.

***Proof.*** Suppose that $\phi \neq \mathcal{A}$ is an arbitrary subset of $\mathcal{T}$. Note that the set $\mathcal{T}\mathcal{T}\mathcal{A} = \{\Sigma_{finite} t_1t_2a :\ t_1, t_2 \in \mathcal{T}, a \in \mathcal{A}\}$, is either $\mathcal{T}$ or some proper left ideal of $\mathcal{T}$. Since, $\mathcal{T}$ is principal therefore one can find an element $b$of $\mathcal{T}$, satisfying $\mathcal{T}\mathcal{T}\mathcal{A} = \mathcal{T}\mathcal{T}b$. Also, if $c \in \mathcal{T}$ with $\mathcal{T}\mathcal{T}\mathcal{A} \subseteq \mathcal{T}\mathcal{T}c$, then $\mathcal{T}\mathcal{T}b \subseteq \mathcal{T}\mathcal{T}c$. Hence, by Proposition 2.9 element $b$ is the $(\text{gcd}\,)_r$ of $\mathcal{A}$. □

The next theorem follows by combining Theorem 2.24 and 2.25.

**Theorem 2.26.** Every non-empty subset in a left Euclidean IS ternary semi-ring has a $(\text{gcd}\,)_r$.

The converse of Theorem 2.24, is true for additively cancellative, yoked ternary semiring with multiplicative left Euclidean norm $\mu$ satisfying the condition $(*)$ given by $\mu(x+y) \geq \mu(x)+\mu(y)$.

**Theorem 2.27.** Let $\mathcal{T}$ be an additively cancellative, yoked ternary semiring with multiplicative left Euclidean norm satisfying the condition $(*)$, then every principal ideal of $\mathcal{T}$ is subtractive.

Before proceeding with the proof of Theorem 2.27, we introduce the notion of ternary ring of differences $\mathcal{T}^{\Delta}$ and prove a basic lemma which is required for the proof of the above theorem.

**Remark 2.28. (The Ternary Ring of Differences)** An ideal $\mathcal{J}$ of a ternary semiring $\mathcal{T}$ defines an equivalence relation $\equiv_{\mathcal{J}}$ on $\mathcal{T}$, called the **Bourne** relation, given by $t \equiv_{\mathcal{J}} t'$ if and only if there exist elements $x, x' \in \mathcal{J}$ satisfying $t + x = t' + x'$. The set of all equivalence classes of elements of ternary semiring $\mathcal{T}$ under this relation is denoted by $\mathcal{T}/\mathcal{J}$ and the equivalence class of an element $t$ of $\mathcal{T}$ denoted by $t/\mathcal{J}$. Let $\mathcal{T}$ be a ternary semiring with a unital element $\mathrm{e}$ and $\mathcal{S} = \mathcal{T} \times \mathcal{T}$. Define the operation of addition and ternary multiplication on $\mathcal{S}$ by $(p,q)+(r,s) = (p+r, q+s)$ and $(p,q)(r,s)(u,v) = (pru+qsu+psv+qrv,\ prv+qsv+psu+qru)$, $\forall p,\ q,\ r,\ s, u, v \in \mathcal{T}$. These operations turn $\mathcal{S}$ into a ternary semiring with additive identity $(0,0)$ and unital element $(\mathrm{e}, 0)$. Clearly, the set $\mathcal{M} = \{(x,x) : x \in \mathcal{T}\}$ is an ideal of $\mathcal{S}$. Thus, $\mathcal{M}$ defines the Bourne relation, given by $(p,q) \equiv_{\mathcal{M}} (r,s)$ iff there exist $(t,t)$ and $(t',t') \in \mathcal{M}$, i.e., $t, t' \in \mathcal{T}$ such that

$$\begin{aligned} (p,q)+(t,t) &= (r,s)+(t',t') \\ \text{or } (p+t, q+t) &= (r+t', s+t') \\ \text{or } p+t = r+t' &\text{ and } q+t = s+t'. \end{aligned}$$

Here $\mathcal{S}/\mathcal{M}$, denotes the set of all equivalence classes of elements of $\mathcal{T}$ under this relation. Moreover, $0/\mathcal{T} \neq \mathcal{T}$ iff $\mathcal{T}$ is non-zeroic. If $\mathcal{T}$ is non-zeroic, then $\mathcal{S}/\mathcal{M}$ is a ternary ring known as the ternary ring of differences of the ternary semiring $\mathcal{T}$. Therefore, in $\mathcal{S}/\mathcal{M}$ we conclude that $(p,q)/\mathcal{M} = (r,s)/\mathcal{M}$ iff there exist $t$ and $t' \in \mathcal{T}$ such that $p+t = r+t'$ and $q+t = s+t'$. It becomes a ternary ring by addition and ternary multiplication according to $(p,q)/\mathcal{M} + (r,s)/\mathcal{M} = (p+r, q+s)/\mathcal{M}$ and $\big((p,q)/\mathcal{M}\big)\big((r,s)/\mathcal{M}\big)\big((u,v)/\mathcal{M}\big) = (pru+qsu+psv+qrv,\ prv+qsv+psu+qru)/\mathcal{M}$ . The zero element of $\mathcal{T}^{\Delta}$ is $(0,0)/\mathcal{M}$ or $(x,x)/\mathcal{M}$ and a unital element is $(\mathrm{e},0)/\mathcal{M}$ and it is denoted by $\mathcal{T}^{\Delta}$.

**Lemma 2.29.** The following conditions hold in $\mathcal{T}$:

$(i)$ $(p,q)/\mathcal{M} = (r,s)/\mathcal{M}$ in $\mathcal{T}^{\Delta}$ if and only if $p+s = q+r$, for $p,q,r,s \in \mathcal{T}$;

$(ii)$ If $\mathcal{T}$ is yoked, then for any $p, q \in \mathcal{T}$, we have either $p-q \in \mathcal{T}$ or $r-s \in \mathcal{T}$.

*Proof.* $(i)$ Let $(p,q)/\mathcal{M} = (r,s)/\mathcal{M}$ in $\mathcal{T}^{\Delta}$. Then there exist $t$ and $t' \in \mathcal{T}$ such that $p+t = r+t'$ and $q+t = s+t'$. This implies that $p+s+t = r+s+t' = q+r+t$. Since $\mathcal{T}$ is additively cancellative, we get $p+s = q+r$. Conversely, let $p+s = q+r$. Take $s = t, q = t'$. Therefore, we get the relation $p+t = r+t'$. Moreover, since $q+s = s+q$, so we also get the relation $q+t = s+t'$. Hence $p-q = r-s$.

$(ii)$ If $\mathcal{T}$ is yoked, then for any $p, q \in \mathcal{T}$, there exists $t \in \mathcal{T}$ such that either $q + t = p$ or $p + t = q$. By using $(i)$, either $p - q = t$ or $q - p = t$. Hence, either $p - q \in \mathcal{T}$or $q - p \in \mathcal{T}$. □

Now, we have all the necessary information required for the proof of Theorem 2.27.

**Proof of Theorem 2.27.** Let $\mathcal{J} = \langle t \rangle$ be an ideal of $\mathcal{T}$ with $t \neq 0$and $\mathcal{J} \neq \mathcal{T}$. Assume that $x + y, y \in \mathcal{J}$with $y \in \mathcal{T}$. Then there exist some elements $t_1, t_2, t_3, t_4 \in \mathcal{T}$ such that $t_1t_2t + y = t_3t_4t$. Therefore, $\mu(t_3t_4t) = \mu(t_1t_2t + y) \geq \mu(t_1t_2t) + \mu(y)$ implying that $\mu(t_3t_4t) \geq \mu(t_1t_2t)$ . Thus, we have $\mu(t_3)\mu(t_4) \geq \mu(t_1)\mu(t_2)$, as $\mu$ is multiplicative. Further, $\mathrm{e}, t_1, t_2, t_3$and $t_4 \in \mathcal{T}$, so $\mathrm{e}t_1t_2$ and $\mathrm{e}t_3t_4 \in \mathcal{T}$. But$\mathcal{T}$ is yoked, so by Lemma 2.29 (ii), either $\mathrm{e}t_1t_2 - \mathrm{e}t_3t_4 \in \mathcal{T}$ or $\mathrm{e}t_3t_4 - \mathrm{e}t_1t_2 \in \mathcal{T}$. If $\mathrm{e}t_1t_2 - \mathrm{e}t_3t_4 \in \mathcal{T}$, then $\mathrm{e}t_1t_2 = \mathrm{e}t_3t_4 + (\mathrm{e}t_1t_2 - \mathrm{e}t_3t_4)$ gives $\mu(\mathrm{e}t_1t_2) \geq \mu(\mathrm{e}t_3t_4) + \mu(\mathrm{e}t_1t_2 - \mathrm{e}t_3t_4)$. Thus, we have $\mu(\mathrm{e}t_1t_2) > \mu(\mathrm{e}t_3t_4)$, which implies that $\mu(t_1)\mu(t_2) > \mu(t_3)\mu(t_4)$, a contradiction. Therefore, $\mathrm{e}t_3t_4 - \mathrm{e}t_1t_2 \in \mathcal{T}$. In this case, $y = t_3t_4t - t_1t_2t = \mathrm{e}(\mathrm{e}t_3t_4 - \mathrm{e}t_1t_2)t \in \langle t \rangle = \mathcal{J}$. Hence $\mathcal{J}$ is a subtractive ideal of$\mathcal{T}$.

**Definition 2.30.** A ternary semiring $\mathcal{T}$ is called subtractive left Noetherian, if for any ascending chain $\mathcal{J}_1 \subseteq \mathcal{J}_1 \subseteq \ldots \subseteq \mathcal{J}_n \subseteq \ldots$ of subtractive left ideals in $\mathcal{T} \exists\ k \in \mathbb{N}$ such that$\mathcal{J}_k = \mathcal{J}_m$, $\forall k \geq m$.

In next theorem, it is proved that after a finite number of steps every ascending chain of subtractive left ideals of an additively cancellative, yoked ternary semiring $\mathcal{T}$ with a multiplicative left Euclidean norm becomes stationary and hence the ternary semiring $\mathcal{T}$ becomes subtractive left Noetherian.

**Theorem 2.31.** An additively cancellative, yoked ternary semiring with a multiplicative left Euclidean norm is subtractive left Noetherian.

***Proof.*** By Theorem 2.24, every subtractive left ideal of a left Euclidean ternary semiring is principal, so let $\mathcal{J}_n = \langle t_n \rangle$ for some $t_n \in \mathcal{T}$. Consider $\mathcal{J} = \cup_{n=1}^{\infty} \mathcal{J}_n$, then $\mathcal{J}$ is a left ideal of $\mathcal{T}$, as $\mathcal{J}_k \subseteq \mathcal{J}_l \forall k \leq l$. Now, weshall show that $\mathcal{J}$ is subtractive. For this, let $a, a + b \in \mathcal{J}$with $b \in \mathcal{T}$. Then $\exists$ integers $k, l$ such that $a \in \mathcal{J}_k$ and $a + b \in \mathcal{J}_l$ and this implies that either $\mathcal{J}_k \subseteq \mathcal{J}_l$ or $\mathcal{J}_l \subseteq \mathcal{J}_k$ but in that case, either $a, a + b \in \mathcal{J}_l$or $a, a + b \in \mathcal{J}_k$. Therefore, either $b \in \mathcal{J}_l \subseteq \mathcal{J}$or $b \in I_k \subseteq \mathcal{J}$, as both $\mathcal{J}_k$ and $\mathcal{J}_l$ are subtractive left ideals. Thus, $\mathcal{J}$ is subtractive and hence a principal left ideal of $\mathcal{T}$. Therefore, $\exists t \in \mathcal{T}$ such that $\mathcal{J} = \langle t \rangle$. Now, $t \in \mathcal{J} = \cup_{n=1}^{\infty} \mathcal{J}_n$, so there exists some integer $n_0$ such that $a \in \mathcal{J}_{n_0}$ which infers that $\mathcal{J}_k = \mathcal{J}_{n_0} = \mathcal{J} \forall k \geq n_0$. □

**Remark 2.32.** Note that $\mathcal{T} = \langle \mathbb{Z}_0^- \cup \{-\infty\}, \min, \max \rangle$ in Example 2.20, is not subtractive Noetherian, because an ascending chain of subtractive ideals $\mathcal{J}_1 \subsetneq \mathcal{J}_2 \subsetneq \mathcal{J}_3 \subsetneq \ldots$ in $\mathcal{T}$ does not terminate. Therefore, we conclude that the conditions imposed on a ternary semiring $\mathcal{T}$ in the above theorem are necessary.

In the next proposition, we investigate a condition for having equality in Lemma 2.6(iii):

**Proposition 2.33.**The following conditions are equivalent in $\mathcal{T}$ :

$(i)$ $\mathcal{CR}_{\mathcal{D}}(\{a + b, b\}) = \mathcal{CR}_{\mathcal{D}}(\{a, b\}) \forall a, b \in \mathcal{T}$;

$(ii)$ Every principal left ideal of$\mathcal{T}$ is subtractive.

*Proof.* $(i) \Rightarrow (ii)$: Let $\mathcal{T}\mathcal{T}x$ be a principal left ideal of $\mathcal{T}$ with $b,\ a+b \in \mathcal{T}\mathcal{T}x$. Then $x \in \mathfrak{R}_{\mathcal{D}}(b)$ and $x \in \mathfrak{R}_{\mathcal{D}}(a+b)$. Therefore, $x \in \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a+b,b\})$. From condition $(i)$, $x \in \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a,b\})$, so we have $a \in \mathcal{T}\mathcal{T}x$. Thus, $\mathcal{T}\mathcal{T}x$ is subtractive.

$(ii) \Rightarrow (i)$: Assume that the condition $(ii)$ holds. Then, by Lemma 2.6(iii), $\mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a,b\}) \subseteq \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a+b,b\})$. Therefore, it is sufficient to prove that $\mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a+b,b\}) \subseteq \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a,b\})$. For this, let $a,b \in \mathcal{T}$ and $x \in \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a+b,b\})$ which infers that $x \in \mathfrak{R}_{\mathcal{D}}(a+b) \cap \mathfrak{R}_{\mathcal{D}}(b)$. This implies that $a+b \in \mathcal{T}\mathcal{T}x$ and $b \in \mathcal{T}\mathcal{T}x$. Now, by our assumption $\mathcal{T}\mathcal{T}x$ is subtractive, therefore, $a \in \mathcal{T}\mathcal{T}x$. Further, by definition, $x \in \mathfrak{R}_{\mathcal{D}}(a)$ implying $x \in \mathfrak{R}_{\mathcal{D}}(a) \cap \mathfrak{R}_{\mathcal{D}}(b) = \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a,b\})$, so we have, $\mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a+b,b\}) \subseteq \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a,b\})$. □

**Definition 2.34.** A ternary semiring $\mathcal{T}$ satisfying equivalent conditions in the above proposition is called a principal left ideal subtractive ternary semiring.

**Example 2.35.** Note that $\mathcal{T} = \langle \mathbb{Z}_4^-, +_4, \times_4 \rangle$ is a principal left ideal subtractive ternary semiring.

**Proposition 2.36.** If $\mathcal{T}$ is a left Euclidean ternary semiring, then the following conditions are equivalent:

$(i)$ $\mathcal{T}$ is a principal left ideal subtractive ternary semiring;

$(ii) \exists \mu'$ (a left Euclidean norm) on $\mathcal{T}$ with $a = tqb + r$ for $r \in \mathcal{T}\backslash\{0\}$ and $\mu(r) < \mu(b)$, then $a \notin \mathfrak{R}_{\mathcal{D}}(b)$.

***Proof.*** $(i) \Rightarrow (ii)$: Let $\mathcal{T}$ be a left Euclidean ternary semiring. By Proposition 2.13, corresponding to a left Euclidean norm on $\mathcal{T}, \exists \mu'$ satisfying the condition that $\mu'(s) \leq \mu'(t_1 t_2 s) \forall t_1, t_2$, and $s \in \mathcal{T}\backslash\{0\}$ such that $t_1 t_2 s \neq 0$. Further, for $a,b \in \mathcal{T}$, we have $a = tqb + r$ for $r \in \mathcal{T}\backslash\{0\}$ and $\mu'(r) < \mu'(b)$. If $a \in \mathcal{T}\mathcal{T}b$, then by $(i)$, we must have $r = t_1 t_2 b$, for some $t_1, t_2 \in \mathcal{T}$ and so $\mu'(r) \geq \mu'(b)$, which is a contradiction. Hence, $a \notin \mathcal{T}\mathcal{T}b$.

$(ii) \Rightarrow (i)$: Let $a,b \in \mathcal{T}$ and $t \in \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a+b,b\})$. Then, by definition $a+b = t_1 t_2 t$ and $b = t_3 t_4 t$ for some elements $t_1, t_2, t_3, t_4 \in \mathcal{T}$. By the choice of $\mu$, we know that $\mu(a) \geq \mu(t)$, therefore either $a = cqt$ or $a = cqt + r$ for some $0 \neq r \in \mathcal{T}$ satisfying $\mu(r) < \mu(t)$. Further, $t_1 t_2 t = a + b = cqt + r + t_3 t_4 t = \mathrm{e}(\mathrm{e}cq + \mathrm{e}t_3 t_4)\, t + r$, which contradicts condition $(ii)$. Therefore, we must have $a = cqt$ and so $t \in \mathfrak{R}_{\mathcal{D}}(a)$. This infers that $t \in \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a,b\})$, as $t \in \mathfrak{R}_{\mathcal{D}}(b)$. Further, by Proposition 2.33, $\mathcal{T}$ is a principal left ideal subtractive ternary semiring. □

Now, we will use the above result to prove the next theorem.

**Theorem 2.37.** Any non-empty finite subset in a left Euclidean principal left ideal subtractive ternary semiring $\mathcal{T}$ has a (gcd $)_r$.

***Proof.*** By Theorem 2.10, it is sufficient to prove the result for a subset $\mathcal{S} = \{a,b\}$ of $\mathcal{T}$. The result is trivial,if $\mathcal{S} = \{0\}$. Therefore, without loss of generality, let $b \neq 0 \in \mathcal{S}$. By Proposition 2.36, one can find a left Euclidean norm $\mu$ such that if $a = tqb + r$ for $r \in \mathcal{T}\backslash\{0\}$ and $\mu(r) < \mu(b)$, then $a \notin \mathcal{T}\mathcal{T}b$, as $\mathcal{T}$ is a principal left ideal subtractive ternary semiring.

After applying left Euclidean norm $\mu$ repeatedly, we can find elements $c_1, c_2, \ldots, c_{n+1}, q_1, q_2, \ldots, q_{n+1}$ and $r_1, r_2, \ldots, r_n$ of $\mathcal{T}\backslash\{0\}$ such that $a = c_1q_1b + r_1$, $b = c_2q_2r_1 + r_2$, …, $r_{n-2} = c_nq_nr_{n-1} + r_n, r_{n-1} = c_{n+1}q_{n+1}r_n$, and $\mu(b) > \mu(r_1) > \ldots > \mu(r_n)$.

We observe that the process of selecting elements $c_i, q_i$, and $r_i$ must terminates aftera finitenumber of steps, as no infinite decreasing sequence of positive integers exists. Further, proceeding backwards, we get that $r_{n-2} = c_nq_nc_{n+1}q_{n+1}r_n + r_n$;

$$r_{n-3} = c_{n-1}q_{n-1}r_{n-2} + r_{n-1} = c_{n-1}q_{n-1}c_nq_nc_{n+1}q_{n+1}r_n + c_{n-1}q_{n-1}r_n + c_{n+1}q_{n+1}r_n;$$

and so on until we express $b$ and $a$ in terms of $r_n$ (i.e., $a, b \in \mathcal{T}\mathcal{T}r_n$, as e $\in \mathcal{T}$). Therefore, by definition, we get that $r_n \in \mathfrak{R}_{\mathcal{D}}(a)$and $r_n \in \mathfrak{R}_{\mathcal{D}}(b)$. Thus $r_n \in \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a, b\})$. Again, by Lemma 2.3(iii), we have $\mathfrak{R}_{\mathcal{D}}(r_n) \subseteq \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a, b\})$. Now, we claim that $r_n$ is the (gcd $)_r$ of $\mathcal{S} = \{a, b\}$. To prove our claim, it suffices to prove that $\mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a, b\}) \subseteq \mathfrak{R}_{\mathcal{D}}(r_n)$. For this, let $d \in \mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a, b\}) = \mathfrak{R}_{\mathcal{D}}(a) \cap \mathfrak{R}_{\mathcal{D}}(b)$.Then we have $a, b \in \mathcal{T}\mathcal{T}d$. Further, by Proposition 2.36, there exists$\mu: \mathcal{T}\backslash\{0\} \rightarrow \mathbb{N}$ with the property: if $r_n = tqd + r$ for $r \in \mathcal{T}\backslash\{0\}$and $\mu(r) < \mu(b)$, then $a \notin \mathcal{T}\mathcal{T}d$, which is impossible. Thus, we have $r = 0$, which implies $r_n \in \mathcal{T}\mathcal{T}d$. This concludes that $d \in \mathfrak{R}_{\mathcal{D}}(r_n)$ which implied that $\mathcal{C}\mathfrak{R}_{\mathcal{D}}(\{a, b\}) = \mathfrak{R}_{\mathcal{D}}(r_n)$. Hence $r_n$ is the (gcd $)_r$ of $\mathcal{S} = \{a, b\}$. □

**Definition 2.38.** A ternary semidomain $\mathcal{T}$ is called a (gcd $)_r$ ternary semidomain (GCRD ternary semidomain) if for any two non-zero elements$a, b \in \mathcal{T}$, a(gcd $)_r$ of $a$, $b$exists in $\mathcal{T}$.

The upcoming theorem follows from Theorem 2.37.

**Theorem 2.39.** A zero divisor free left Euclidean principal left ideal subtractive ternary semiring$\mathcal{T}$ is a GCRD ternary semidomain.

## 3. FACTORIZATION IN TERNARY SEMIDOMAINS

This section is devoted to establishing a relationship between a principal ideal ternary semi-domain and a factorization ternary semi-domain. In the rest of this section, $\mathcal{T}$ represents a commutative ternary semidomain.

**Definition 3.1.** A non-zero non-unit element $\pi$ in a ternary semiring is called irreducible, if it has no proper divisors in$\mathcal{T}$.

Therefore, units and associates are the only divisors of an irreducible element in $\mathcal{T}$. So, whenever we write $\pi = xyz$, then one of the elements $x, y, z$ is an associate to $\pi$ and the remaining two elements are units in $\mathcal{T}$.

**Definition 3.2.** A zero divisor free ternary semi-ring is called a factorization ternary semi-domain if a non-zero, non-unit element of $\mathcal{T}$ can be written as a product of irreducible elements in $\mathcal{T}$.

**Definition 3.3.** An element $y \in \mathfrak{R}_{\mathcal{D}}(x)$ is called a proper divisor of $x$ in a commutative ternary semiring $\mathcal{T}$, if it is neither a unit nor an associate to $x$.

Note that elements $x, y \in \mathcal{T}$ are associates, if $\exists t_1, t_2$ units in $\mathcal{T}$ so that $x = t_1 t_2 y$. Consequently, in this case we have $\Re_{\mathcal{D}}(x) = \Re_{\mathcal{D}}(y)$

**Proposition 3.4.** If a zero divisor free ternary semiring $\mathcal{T}$ contains no infinite sequence $t_1, t_2, t_3$, ... of non-zero elements in $\mathcal{T}$ such that $t_{k+1}$ is a proper divisor of $t_k$ for $k = 1,2, \ldots$ , then $\mathcal{T}$ is a factorization ternary semidomain. Moreover, if $\mathcal{T}$ is a Euclidean ternary semiring with Euclidean norm $\mu$ which assigns a non-negative integer $\mu(t)$ to every non-zero element $t \in \mathcal{T}$, such that $\mu(u) < \mu(v)$, whenever $u$ is a proper divisor of $v$ $(v \neq 0)$, then $\mathcal{T}$ is a factorization ternary semidomain.

*Proof.* If possible, assume that $\mathcal{T}$ is not a factorization ternary semidomain. Then, by definition $\exists\ 0 \neq t_1 \in \mathcal{T}$, which is neither a unit nor a product of irreducible elements i.e., $t_1$ is not an irreducible element and hence has a proper factorization $t_1 = pqr$, where $p, q, r$ are non-units in $\mathcal{T}$. Therefore, at least one of$p$, $q$ and $r$ is not irreducible. Thus, there exists a proper divisor $t_2$ $(= p$ or $q$or$r)$ of $t_1$ which is not irreducible. Again, applying the same argument to $t_2$, we get a proper divisor $t_3$ of $t_2$ which is not irreducible. Continuing in this manner, we get asequence (infinite) $t_1, t_2, t_3$, ... of non-zero elements of $\mathcal{T}$ such that $t_{k+1}$ is a proper divisor of $t_k$ $(k = 1,2, \ldots)$, which is not possible. So we conclude that $\mathcal{T}$ is a factorization ternary semidomain. To prove the second assertion, assume that $t_1, t_2, t_3$, ... is a sequence (infinite) of non-zero elements of $\mathcal{T}$ such that $t_{k+1}$ is a proper divisor of $t_k$ $(k = 1, 2, \ldots)$. Then $\mu(t_1) > \mu(t_2) > \ldots$ is a strictly decreasing sequence of non-negative integers, which is not possible. Therefore, no such sequence $t_1, t_2, t_3$, ... exists and hence by the first part, $\mathcal{T}$ is a factorization ternary semidomain. □

**Theorem 3.5.** Every zero divisor free principal ideal ternary semiring $\mathcal{T}$ is a factorization ternary semidomain.

***Proof.*** Assume that $\mathcal{T}$ is a zero divisor free principal ideal ternary semiring and $t_1, t_2, t_3$, ... be a sequence(infinite) of elements of $\mathcal{T}$ such that $t_{k+1}$ is a proper divisor of $t_k$ for $k = 1, 2, \ldots$ . Then $\langle t_1 \rangle \subsetneq \langle t_2 \rangle \subsetneq \langle t_3 \rangle \subsetneq \ldots$ where $\langle t_i \rangle = \{t' t'' t_i : t', t'' \in \mathcal{T}\}$ is an ideal generated by $t_i$. Consider $\mathcal{I} = \cup \langle t_r \rangle$, $r = 1,2, \ldots$ and claim that $\mathcal{I}$ is an ideal of $\mathcal{T}$. For this, let $x, y \in \mathcal{A}$. Then $\exists\ r, s$ such that $x \in \langle t_r \rangle$ and $y \in \langle t_s \rangle$. Further, both $x$, $y$ are in $\langle t_j \rangle$ with $j = \max(r, s)$. Thus, for all $t', t'' \in \mathcal{T}$, $x + y = t_1 t_2 t_j + t_3 t_4 t_j = \mathrm{e}(\mathrm{e} t_1 t_2 + \mathrm{e} t_3 t_4) t_j$ and $t' t'' x \in \langle t_j \rangle$ and hence in $\mathcal{A}$. Thus, $\mathcal{A}$ is an ideal of $\mathcal{T}$. Now, for an arbitrary ideal $\mathcal{I}$ of $\mathcal{T} \exists\ t \in \mathcal{T}$with $\mathcal{I} = \langle t \rangle$, as $\mathcal{T}$ is a principal ideal semiring. Since $t \in \mathcal{I} = \cup_r \langle t_r \rangle$, so $\exists\ m$ such that $t \in \langle t_m \rangle$. Hence $\mathcal{I} = \langle t \rangle \subset \langle t_m \rangle$. Now, we have $\mathcal{I} \subset \langle t_m \rangle \subset \langle t_{m+1} \rangle \subset \ldots \subset \mathcal{I}$, so that $\langle t_m \rangle = \langle t_{m+1} \rangle = \ldots (= \mathcal{I})$ , which is not possible, as $t_1, t_2, t_3$, ... is a sequence (infinite) of proper divisors in $\mathcal{T}$. Therefore, no infinite sequence $t_1, t_2, t_3$, ... of above kind exists in $\mathcal{T}$, hence by Proposition 3.4, $\mathcal{T}$ is a factorization ternary semidomain. □

By combining Theorem 2.24 and 3.5 we have

**Corollary 3.6.** Every IS Euclidean ternary semidomain is a factorization ternary semi-domain.

## 4. DISCRETE VALUATION TERNARY SEMIRINGS

The main ambition behind the development of this section is to generalize the notion of discrete valuation maps on rings and semirings defined in [1] and [8] respectively. Throughout the rest of this paper $\mathcal{T}$ is commutative and e is a unital element. Now, we begin with a short note on the construction of totally ordered monoid with the maximal element as follows:

Consider $\mathcal{T}$ is a commutative ternary semiring and $(\mathcal{G}, \leq)$ is a totally ordered monoid without the maximal element. Then we can attach a new element '$\infty$' to $\mathcal{G}$ and write $\mathcal{G}_\infty$, to make it a totally ordered monoid with maximal element $\infty$ by extending the relation '$\leq$' to $\mathcal{G}_\infty$ as follows: $g < \infty$ and $g + \infty = \infty + g = \infty$, $\forall g \in \mathcal{G}_\infty$.

Further, we define a valuation map on $\mathcal{T}$ analogous to the valuation map on semirings in [8].

**Definition 4.1.** Let $\mathcal{G}_\infty$ be a totally ordered monoid with the maximal element $\infty$. A $\mathcal{G}$-valuation on $\mathcal{T}$ is a map $\eta: \mathcal{T} \to \mathcal{G}_\infty$ which satisfies the following conditions:

$(i)$ $\eta(t_1 t_2 t_3) = \eta(t_1) + \eta(t_2) + \eta(t_3)$, for $t_1, t_2, t_3 \in \mathcal{T}$;

$(ii)$ $\eta(t_1 + t_2) \geq \min\{\eta(t_1), \eta(t_2)\}$, for $t_1, t_2 \in \mathcal{T}$;

$(iii)$ $\eta(\mathrm{e}) = 0$and$\eta(0) = \infty$.

The proofs of Proposition 4.2 and Theorem 4.3 follows verbatim as Proposition 1.5 and Theorem 1.8 in [8]:

**Proposition 4.2.** $(i)$ The set $\mathcal{T}_\eta = \{x \in \mathcal{T} : \eta(x) \geq 0\}$ is a ternary subsemiring of $\mathcal{T}$ and its subset $\mathcal{P}_\eta = \{x \in \mathcal{T} : \eta(x) > 0\}$ is a prime ideal of $\mathcal{T}_\eta$. Moreover, the set $\eta^{-1}(+\infty)$ is a prime ideal of $\mathcal{T}$.

$(ii)$ If $u \in \mathcal{T}$ is a unit, then $\eta(u^{-1}) = -\eta(u)$. Further, if $u \in \mathcal{T}_\eta$ is a unit, then $\eta(u) = 0$. Therefore $U(\mathcal{T}_\eta) \subseteq \{t \in \mathcal{T} : \eta(t) = 0\}$ .

**Theorem 4.3.** If $\mathcal{T}_\eta$ be a ternary valuation semiring and $\eta$ a $\mathcal{G}$-valuation on it, then:

$(i)$ The set $\mathcal{M}_\eta = \{t \in \mathcal{T} : \eta(t) > 0\}$ is the unique maximal ideal of the ternary semiring $\mathcal{T}_\eta$iff$U(\mathcal{T}_\eta) = \{t \in \mathcal{T} : \eta(t) = 0\}$.

$(ii)$ If $\eta$ is a surjective valuation map, then $\mathcal{T}$ is a ternary semifieldiff$\mathcal{G}$ is Abelian and $U(\mathcal{T}_\eta) = \{t \in \mathcal{T} : \eta(t) = 0\}$.

**Definition 4.4.** $\mathcal{T}$is said to be a valuation ternary semiring if there exists a surjective $G$-valuation map $\eta$ on some ternary semifield of $k$ such that $\mathcal{T} = k_\eta = \{x \in k : \eta(x) \geq 0\}$.

**Note that a valuation is called discrete valuation if $\mathcal{G} = \mathbb{Z}$.**

It is also observed that in above definition, $\mathcal{G}$ should be an Abelian group and $\mathcal{T}$ needs to be a multiplicatively cancellative ternary semiring, since it is a ternary subsemiring of $k$.

**Proposition 4.5.** Let $\mathcal{T}$ be a multiplicatively cancellativeNoetherian ternary semiring. Then $t \in \mathcal{T}$ is a non-unit if and only if $\cap_{n=0}^{\infty} (t^{2n+1}) = (0)$.

*Proof.* Straight forward. □

**Theorem 4.6.** Any discrete valuation ternary semiring is Euclidean.

*Proof.* Assume that $\mathcal{T}$ is a discrete valuation ternary semiring. Therefore, by definition there is a surjective $\mathbb{Z}$-valuation $\eta$ on a ternary semifield $k$. Define a Euclidean norm $\mu: \mathcal{T}\backslash\{0\} \to \mathbb{N}$ on $\mathcal{T}$by $(t) = \eta(t) \forall\, t \in \mathcal{T}$. Now, for $a, b \in \mathcal{T}$ with $b \neq 0$, if $\eta(a) < \eta(b)$, we have $a = 0qb + a$ and if $\eta(a) \geq \eta(b)$, then clearly $y = eab^{-1} \in \mathcal{T}$, satisfying $a = eyb + 0$. Hence, any discrete valuation ternary semiringis Euclidean. □

**Definition 4.7.** A ternary semiring $\mathcal{T}$ is called local if $\mathcal{T}$ possesses a unique maximal ideal.

**Theorem 4.8.** If $\mathcal{T}$ is a discrete valuation ternary semiring, then $\mathcal{T}$ is a local principal ideal ternary semiring.

***Proof.*** Clearly, by definition there exists a surjective $\mathbb{Z}$-valuation map $\eta$ on a ternary semifield $k$ such that $\mathcal{T} = k_\eta = \{x \in k : \eta(x) \geq 0\}$. It is obvious that the set $\mathcal{M}_\eta = \{x \in \mathcal{T} : \eta(x) > 0\} \subseteq \mathcal{T}_\eta$ is the unique maximal ideal, so $\mathcal{T}$ is local. Further, assume that$\mathcal{J} \neq 0$ is an ideal of $\mathcal{T}$ and $n = \min\{\eta(t) : t \in \mathcal{J}\}$. Then $\exists\ 0 \neq a \in \mathcal{J}$ such that $\eta(a) = n$ and clearly $\eta(b) \geq n = \eta(a)$, for any $b \in \mathcal{J}$. Now, $\eta(bea^{-1}) = \eta(b) + \eta(e) - \eta(a) \geq 0$. So, $bea^{-1} \in \mathcal{T}$, which infers that $b \in \langle a \rangle$.Thus $\mathcal{J} \subseteq \langle a \rangle$ and hence$\mathcal{J} = \langle a \rangle$. □

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# PROBABILITY MODELS IN IMAGE PROCESSING : BAYESIAN INFERENCES

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**ABSTRACT**

***The present paper explores pattern recognition and image processing through Bayesian classifier approach. We have discussed the theoretical Continuous time Markov Chain (CMC) signal processing system along with Hidden Markov Chain (HMC) using well known probability distribution of Gibbs Sampling.***

***Keywords: Pattern, Bayesian Classifier, Gibbs Sampling Procedure, Jeffreys, Priori Probability***

## I. INTRODUCTION

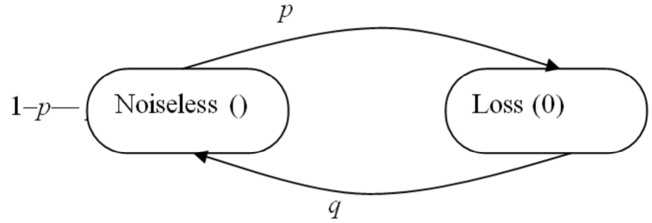
Dong Guo.et. Al (2003) IEEE. Monte Carlo Bayesian methods, the parameters of two state Gilbert Burst model consistent with a Markovian property, which assume multivariate random variables Families such as Poisson, Gamma and Multivariate normal. The signals can be discrete or continuous also signal sources stationary, the theoretical description of signal processing system, which can be used to process the signal so as to provide a desired output Gaussian process, Poisson process, Markov process, Hidden Markov process have also been discussed.

## 2. SYSTEM DESCRIPTION

In this section, reviewed about Gilbert Burst Model, is assumes Hyper-geometric distribution, Delay of Image (Signal) is based on Exponential distribution is discussed.

### 2.1 Image Loss Model: Gilbert Burst Model

Consider TCP protocol (a) Link Loss Model is the simple Bernoulli model, the dependency can be characterized by the Gilbert model, it is two state Markov model as shown in the figure where p is the probability that next packet is lost, given the current packet is delivered, we have . If hen the Gilbert burst model reduces to a Bernoulli model.




Gilbert Burst Model

The stationary probability $\pi$ is given by the solution

$$\begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix}\begin{pmatrix} 1-\pi \\ \pi \end{pmatrix} \to \pi = \frac{p}{p+q} \quad \ldots (1)$$

The probability *pk* of a burst loss of *k* consecutive packets given at the occurrence of a loss is given by

$$P_k = fk / \sum_{k=1}^{\infty} fk = (1-q)^{k-1} q \ldots (2)$$

Where $f = p(1-q)^{k-1} q$. Hence, the burst loss probability $\{p\}$ follows geometric distribution.

### 2.2 Image Delay Model: Exponential Distribution

Because of total time delay for a point-to-point transmission of signals can be written

As $y = \sum_{i=1}^{n} x_i$ where *x* is the time delay incurred on the $i^{\text{th}}$ link, the pdf of the image link delay *x* is given by

$$p(x_i) = \begin{cases} \lambda e^{-\lambda}(x_\lambda - dt), & x_l \geq dt \\ 0, & x_l < dt \end{cases} \quad \ldots. (3)$$

Where, $\lambda_e$ is the image reaching delay parameter.

### 2.3 Hidden Markov Models (HMM)

Consider a system which may be described at any time as being in one of a set *N* distinct states $S_1, S_2, \ldots S_N$ as regularly spaced discrete time, the system undergoes a change of state (possibly back to the same state) according to a set of probabilities associated with the state, denote the actual state at $t = 1, 2, \ldots$ and we denote to be the actual state at time *t* as *q*. A full probabilities description of the above would in general require specification of the current start (at time *t*) as well as the predecess states. For the special case of discrete. First order Markov chain probabilistic description is truncated to just to convert and the predecessor state

$$p(q_c = s_i / qt = 1 = s_i,\ qt - 3 = st \ldots)$$
$$= P(qt = st / qt - 1 = si) \ldots (4)$$

Right hand side of (4) is independent of time, thereby leads to the set of state transition probabilities $a_{ij}$ of the form

$$a_{ij} = p\left(q_t = s_j / q_{t-1} = s_i\right),\ 1 \le i, j < N \ \ldots (5)$$

with the state transition coefficient having properties $a_{ij} > 0 \ldots$ (6a)

$$\sum_{j=1}^{N} a_{ij} = 1 \ \ldots (6b)$$

This is the standard stochastic constant

## 3. METHODS AND MATERIALS

### 3.1 Gaussian Bayesian Network

Gaussian Bayesian Network is (GPN), the associated network which has 2 nodes, 2 links
Assuming priori distribution, the threshold and the window size $W_i t_k\left(t^*\right) = \left(W\left(t/2\right)\right)$ and $W\left(t^*\right) = 1$. The throughput of a TCP connection is defined as $B = \lim No/t$
Where $N_t$ is the number of packets transmitted during time interval $\left(0, t\right)$.
The probability $\left(I, W\right)$ that after the first packet lost in an epoch, a total of I packets, including the first loss, are lost in the following window size *w* is given by

$$P\left(I, w\right) = \begin{cases} w-1 & yi-1\left(1-y\right)m \\ i-1 & \end{cases}$$

Then the probability $Q\left(W\right)$ that the timeout will take place by

$$Q\left(w\right) = \begin{cases} 1, & \text{if } w < 4 \\ 1 - P\left(1, w\right), & \text{if } w < 10 \\ 1 - P\left(1, w\right) - P\left(2, W\right), & \text{otherwise} \end{cases}$$

### 3.2 Markov Chain Monte Carlo (MCMC)

Monte Carlo sampling procedure including Gibbs sampling and the Metropolis algorithm are discussed in this section.

Consider a system which may be described at any time as being in one of a set of *N* distinct states, $S_1, S_2, ...., S_N$ of $N = 2$, Gilbert Burst Model, we denote the actual state at time *tq*. A full specification of the current state is (at time) as well as the predecessor states. For the special cases of discrete is truncated to just the current and the predecessor state.

$$P\left(q_i = s_j\right) / q_{i-1} = s_j \ \ldots (7)$$

This leads to the set of transmission probabilities $a_{ij}$

$$A_{ij} = p\left(q_j = s_j / q_{j-1} = s_j\right),\ 1 < i, j \leq W \ \ldots (8)$$

With the state transition co-efficient using the properties $a_{ij} > 0 \ \ldots (9)$

$$a_{ij} < 1 \ldots (10)$$

The probability can be evaluated as

$$P\left(q / \text{Model}\right) = P\left(s_1, s_2 / \text{Model}\right)$$

$$= P\left(s_1\right) P\left(s_2 / s_1\right)\left(Ps_3\right) / PS_3 / s_2 P\left(s_4\right) / P\left(s_3, s_2, s_1\right)$$

The probability of the observation sequence is sequence

$$P\left(0 / \text{Model}, q_i = s_i\right) = \left(a_{ij}\right) d - 1\left(1 - a_{ij}\right) = P_i\left(d\right) \ \ldots (11)$$

The Quantity $p_1\left(d\right)$ is the discrete probability density function of deviation *d* in state *j*.

### 3.3 Exponential Distribution: Memory less Property

The exponential distribution density is characteristics of the state duration in a Markov chain, based on $p\left(d\right)$ we can readily calculate the expected number of observation derivations in a state, conditional on state in the state as

$$D_i = dp_i\left(d\right) \ldots (12)$$

$$= d\left(\left(d_i\right)\right) s - 1\left(1 - s_{ij}\right) = 1/1 - a_{ij} \ldots (13)$$

The model is a memory less process and there is a degenerate use of Markovian Model.

Hidden Markov Model (HMM)

## 4. CONCLUSION

### 4.1 Bayesian Classifier

Probability consideration becomes important in pattern recognition because of the randomness under which pattern classes normally are generated. The probability that a particular pattern '*x*' comes from class $w_i$ is denoted $p\left(w_i / x\right)$. If the pattern classifier decides that *x* comes from $w_j$. When it actually came from $w_i$, it incurs a loss denoted $L_{ij}$. As pattern *x* may belong to any one of *w* class under consideration, the average loss incurred in assigning *x* to class $w_j$ is

$$r_j(x) = \sum_{k=1}^{w} Lk_j P(w_r / \pi)$$

This equation is called the conditional average risk or loss is decision theory terminology

Using this expression, we write equation

$$r_j(x) = 1/p(x) \sum_{k=1}^{w} Lk_j P(x/w_k) Pw_k$$

Where $P(r/w_k)$ the probability density is function of the pattern from $w_k$ and $p(w_k)$ is the probability of occurrence of each $w_k$

### 4.2 Object Recognition (Pattern Recognition)

The probability that particular pattern '*x*' comes from class $w_i$, $p(x_i/x)$ the pattern classifies decides that *x* came from $w_j$ the average loss incurred is assigning *x* to class $w_j$ is

$$R_j(x) = Lk_j P(W_p / \pi) \text{ .........(8)}$$

This equation is called the conditiona9l average risk (loss)

$$R_j(x) = 1/p(x) lk_j P(x/w_k) pw_k \text{ ... (9)}$$

The expression for the average loss that reduces to $r_j(x) = Lk_j p(x/w_k) p(w_k)$

### 4.3 Markovian Probability Models

In this method introduce the variable order Bayesian Signal Network (VCBN) includes Hidden Markov Model (HMM). For instance if all the classes are equally likely $p(w_j) = 1/M$ . Even if their condition Posterior is not known but priori probability is known if compute $r_i(x) r_2(x) ... r_w(x)$ for each pattern *x* and assigns the patter to the class with the smallest class. While the Bayesian classifier is contains hidden variable estimation, the likelihood (MLE) which the function of the joint marginal distribution of priori distribution leads to Gamma, Beta distribution.

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# ANALYSIS OF A QUEUING PROCESS WITH BALKING, RENEGING AND PRE-EMPTIVE PRIORITY

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**ABSTRACT**

***Waiting lines or queues are omnipresent. Irregularities in arrival pattern or service mechanism of the system generate waiting lines. Since irregularities of the system depend upon chance, so feasibility of probability theory was realised in the study of queues. Cobham (1954), Kesten and Runnenberg (1957), Stephan (1958), White and Christie (1958), Cox and Smith (1961), studied the queuing models in the steady-state having Poisson arrivals, exponential service times and priority queue discipline. Vikram (1996) formulated the queuing model where a customer may leave the system with or without getting service and pre-emptive priority is the queue discipline for service.***

***We construct the queuing model having balking and reneging where the customers arrive in Poisson stream, service time distribution is negative exponential and queue discipline for service is pre-emptive priority. It has been considered here that the customers may leave the system with or without getting service and may not join the queue due to large number of customers already present there. We write its difference-differential equations and steady-state equations. The Steady-state solutions are found by iterative method. The performance of the system is projected by calculating the mean queue length for infinite waiting space.***

***Keywords & phrases: Probability, Difference-differential equations, Steady-state, Poisson arrival, Exponential service.***

## 1. INTRODUCTION:

After world war II, number of mathematicians, engineers and economists have shown their keen interest in the field of queuing theory because of its practical utility in different fields and projects and developed variety of queuing models under different types of queue disciplines and customer's behaviour for limited and unlimited capacity. In the context of the problems of waiting lines, O'Brien (1954), Jackson (1954), Hunt (1956) and Finch (1959) constructed the queuing models and derived their steady-state solutions under the assumptions that the customer would join the initial stage of the serial channels from outside and must go through each service channel before leaving the system. Barrer (1955) introduced reneging in the study of single channel queuing model. It has been mentioned in the beginning that Cobham (1954), Kesten and Runnenberg (1957), Stephan (1958), White and Christie (1958), Cox and Smith (1961), studied the queuing models in the steady-state having Poisson arrivals, exponential service times and priority queue discipline. It is worth mentioning here that the customers are not allowed to renege in the studies mentioned above. In order to bring the subject closer to practical situations, Singh (1984), Sigh and Umed (1994) introduced reneging in their studies where the customers may leave the system at any stage without getting service when the queue discipline are SIRO (Service in random order) and FCFS (First come first serve). Vikram (1996) used priority queue discipline for deriving the steady-state solution of a waiting line problem.

The single channel queuing model discussed in our work is governed by the following characteristics.

- The input process is Poisson and depends upon the queue size.
- The service time distribution is exponential.
- The customer may renege and can leave the system without getting service.
- The queue discipline is priority selection and random selection.
- Waiting space is infinite.

## 2. APPLICATION OF THE MODEL:

Real life applications of this queuing model are observed in many situations where it is desirable and even justified to give priority to certain customers over others either in the interest of Institutions for reducing the cost, waiting time and mean queue length or for the welfare of the society. As in the banks, senior citizens and disabled persons are served on priority in the comparison of other customers Similarly VIP and serious patients are treated on priority in hospitals as compared to other patients. In many institutions, scientists are served on priority basis everywhere in the administrative set up because of their dedication and contributions in academic fields.

## 3. FORMULATION OF THE MODEL:

Proposed model is a single channel model with balking, reneging and pre-emptive priority in which a customer of low priority at the service channel is displaced by the customer of higher priority immediately on its arrival. Two classes of customers (class I- customers with high priority and class II – customers with low priority) are considered with independent random arrival rates $\lambda_1, \lambda_2$ and exponential service rates $\mu_1, \mu_2$ and number of customers m, n respectively.

Due to Balking the arrival rates $\lambda_1$ and $\lambda_2$ would become $\frac{\lambda_1}{m+1}$ and $\frac{\lambda_2}{n+1}$. Reneging rates of class I-customers and class II- customers after a wait of certain time $T_1$ and $T_2$ are defined as $C_{1m} = \frac{\mu_1 e^{\frac{-\mu_1 T_1}{m}}}{\left(1-e^{\frac{-\mu_1 T_1}{m}}\right)}$ and $C_{2n} = \frac{\mu_2 e^{\frac{-\mu_2 T_2}{n}}}{\left(1-e^{\frac{-\mu_2 T_2}{n}}\right)}$ respectively. It is assumed that the service commences instantaneously when the customer arrives at an empty channel.

## 4. FORMULATION OF EQUATIONS:

Define $P(m,n;t) =$ The probability that at time 't' there are m class I- customers and n class II-customers waiting (may balk or renege) in the service channel.

Elementary probability reasoning leads to the following difference- differential equations

$$\begin{aligned}\frac{d}{dt}P(m,n;t) = &-\left[\frac{\lambda_1}{m+1}+\frac{\lambda_2}{n+1}+\delta(m)(\mu_1+C_{1m})+\{(1-\delta(m))(\mu_2+C_{2n})\}\delta(n)\right]P(m,n;t)\\ &+\frac{\lambda_1}{m}P(m-1,n;t)+\frac{\lambda_2}{n}P(m,n-1;t)+(\mu_1+C_{1(m+1)})P(m+1,n;t)\\ &+\{(1-\delta(m))(\mu_2+C_{2(n+1)})\}P(m,n+1;t) \quad \forall m\geq 0; n\geq 0 \text{.................}(1)\end{aligned}$$

Where $P(m,n;t)=0$ if any of the arguments is zero,

$$\delta(m)=\begin{Bmatrix}1; m\neq 0\\ 0; m=0\end{Bmatrix} \ \& \ \delta(n)=\begin{Bmatrix}1; n\neq 0\\ 0; n=0\end{Bmatrix}$$

Steady-state equations:

The steady-state equations of the system are derived by equating the time-derivatives to zero in the equation (1). The resulting equilibrium equations are as under

$$\begin{aligned}&\left[\frac{\lambda_1}{m+1}+\frac{\lambda_2}{n+1}+\delta(m)(\mu_1+C_{1m})+\{(1-\delta(m))(\mu_2+C_{2n})\}\delta(n)\right]P(m,n)=\\ &\frac{\lambda_1}{m}P(m-1,n)+\frac{\lambda_2}{n}P(m,n-1)+(\mu_1+C_{1(m+1)})P(m+1,n)\\ &+\{(1-\delta(m))(\mu_2+C_{2(n+1)})\}P(m,n+1) \quad \forall m\geq 0; n\geq 0 \text{.................}(2)\end{aligned}$$

There are three cases-

Case-I

When m=0=n, then the steady-state equation (2) reduces to

$$(\lambda_1+\lambda_2)P(0,0)=(\mu_1+C_{11})P(1,0)+(\mu_2+C_{21})P(0,1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Equation (3) is satisfied when $P(1,0)=\left(\frac{\lambda_1}{\mu_1+C_{11}}\right)P(0,0);\quad P(0,1)=\left(\frac{\lambda_2}{\mu_2+C_{21}}\right)P(0,0)$

Case-II

When m=0, n>0, then the equation (2) reduces to

$$\left(\lambda_1+\frac{\lambda_2}{n+1}+\mu_2+C_{2n}\right)P(0,n)=\frac{\lambda_2}{n}P(0,n-1)+(\mu_1+C_{11})P(1,n)+(\mu_2+C_{2(n+1)})P(0,n+1)\ldots\ldots(4)$$

Equation(4) is satisfied by

$$P(1,n)=\frac{\lambda_1}{\mu_1+C_{11}}P(0,n)$$

Where $P(0,n)=\frac{1}{\lfloor n}\left(\frac{(\lambda_2)^n}{\prod_{j=1}^{n}(\mu_2+C_{2j})}\right)P(0,0)$

Case-III

When m>0 and n>0; the equation (2) will be reduced to

$$\left[\frac{\lambda_1}{m+1}+\frac{\lambda_2}{n+1}+(\mu_1+C_{1m})\right]P(m,n)=$$

$$\frac{\lambda_1}{m}P(m-1,n)+\frac{\lambda_2}{n}P(m,n-1)+(\mu_1+C_{1(m+1)})P(m+1,n)\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

The steady-state solutions of the equation (5) can be verified to be

$$P(m,n)=\left[\frac{1}{\lfloor m}\frac{(\lambda_1)^m}{\prod_{j=1}^{m}2(\mu_1+C_{1j})}\right]\left[\frac{1}{\lfloor n}\frac{(2\lambda_2)^n}{\prod_{j=1}^{n}\left\{2\left(\frac{\lambda_2}{j+1}-(\mu_1+C_{1m})\right)+\frac{\lambda_1}{m+1}\right\}}\right]P(0,0)$$

The compact solution of the steady-state equation (2) for all the three cases derived above, is summarized as under

$$P(m,n)=\left[\frac{1}{\lfloor m}\frac{(\lambda_1)^m}{\prod_{j=1}^{m}(1+\delta(m))(\mu_1+C_{1j})}\right]$$

$$\left[\frac{1}{\lfloor n}\frac{(2\lambda_2)^n}{\left[\prod_{j=1}^{n}(1-\delta(m))2(\mu_2+C_{2j})\right]\left[\prod_{j=1}^{n}\left\{2\left(\frac{\lambda_2}{j+1}-(\mu_1+C_{1m})\right)+\frac{\lambda_1}{m+1}\right\}\right]\delta(m)}\right]P(0,0).......(6)$$

Rewriting the above result (6) as under:

$$P(m,n)=P(0,0)\left[\frac{1}{\lfloor m}(x_1)^m\right]\left[\frac{1}{\lfloor n}(x_2)^n\right] for \quad m\geq 0, n\geq 0...............(7)$$

Where

$$(x_1)^m=\frac{(\lambda_1)^m}{\prod_{j=1}^{m}(1+\delta(m))(\mu_1+C_{1j})}$$

$$(x_2)^n=\frac{(2\lambda_2)^n}{\left[\prod_{j=1}^{n}(1-\delta(m))2(\mu_2+C_{2j})\right]\left[\prod_{j=1}^{n}\left\{2\left(\frac{\lambda_2}{j+1}-(\mu_1+C_{1m})\right)+\frac{\lambda_1}{m+1}\right\}\right]\delta(m)}$$

From result (7), we obtain P(0,0)by applying the normalizing condition $\sum_{m=0,n=0}^{\infty}P(m,n)=1$ with the restriction that the utilization factor of the system is less than unity.

$$1=P(0,0)\left[\sum_{m=0}^{\infty}\frac{1}{\lfloor m}(x_1)^m\right]\left[\sum_{n=0}^{\infty}\frac{1}{\lfloor n}(x_2)^n\right]$$

$$=P(0,0)e^{x_1}e^{x_2}=P(0,0)e^{x_1+x_2}$$

$$\left[P(0,0)\right]^{-1}=e^{x_1+x_2}.................(8)$$

Steady-state Marginal Probabilities:

The steady-state marginal probability of m I-customers of the system denoted by P(m) is calculated as under

$$P(m)=\sum_{n=0}^{\infty}P(m,n)=P(0,0)\sum_{n=0}^{\infty}\frac{1}{\lfloor m}(x_1)^m\frac{1}{\lfloor n}(x_2)^n=P(0,0)\frac{1}{\lfloor m}(x_1)^m e^{x_2}..............(9)$$

Using result (8) we get $P(m)=\frac{1}{e^{x_1}}\frac{1}{e^{x_2}}\frac{1}{\lfloor m}(x_1)^m e^{x_2}=\frac{1}{\lfloor m}(x_1)^m\frac{1}{e^{x_1}}$

Similarly, the steady-state marginal probability of n II- customers is as

$$P(n)=\frac{1}{e^{x_1}}\frac{1}{e^{x_2}}\frac{1}{\lfloor n}(x_2)^n e^{x_1}=\frac{1}{\lfloor n}(x_2)^n\frac{1}{e^{x_2}}$$

**Mean Queue Length:**

The Marginal mean queue length of m class I- customers and n class II customers in the system are determined by

$$L_1=\sum_{m=0}^{\infty} mP(m)=\sum_{m=0}^{\infty} m\frac{1}{\lfloor m}(x_1)^m\frac{1}{e^{x_1}}=\frac{1}{e^{x_1}}\sum_{m=0}^{\infty}\frac{m}{\lfloor m}(x_1)^m=\frac{1}{e^{x_1}}\left[x_1+\frac{1}{\lfloor 1}x_1^{\,2}+\frac{1}{\lfloor 2}x_1^{\,3}+\frac{1}{\lfloor 3}x_1^{\,4}+.........\infty\right]$$

$$=\frac{1}{e^{x_1}}x_1\left[1+x_1+\frac{1}{\lfloor 2}x_1^{\,2}+\frac{1}{\lfloor 3}x_1^{\,3}+.........\infty\right]=\frac{1}{e^{x_1}}x_1\left[e^{x_1}\right]=x_1$$

*Similarly*

$$L_2=x_2$$

**Mean Queue Length** = $L_1+L_2=x_1+x_2$

**Particular Case:**

If there is no balking and no reneging in the system, then the steady-state equation (2) reduces to

$$\left[\lambda_1+\lambda_2+\delta(m)\mu_1+\left(1-\delta(m)\right)\left(\delta(n)\mu_2\right)\right]P(m,n)=$$
$$\lambda_1P(m-1,n)+\lambda_2P(m,n-1)+\mu_1P(m+1,n)+\left(1-\delta(m)\right)\mu_2P(m,n+1)\,for\quad m\geq 0,n\geq 0............(10)$$

Equation (10) is in conformity with the corresponding equation (44) due to Barry (1956) as given in “Queues” by Cox and Smith (1961).

**Case (i)**

When $m=0=n$ the equation (10) reduces to

$$(\lambda_1+\lambda_2)P(0,0)=\mu_1P(1,0)+\mu_2P(0,1)..........(11)$$

The above equation is satisfied by

$$P(1,0)=P(0,0)\frac{\lambda_1}{\mu_1}\,and\quad P(0,1)=P(0,0)\frac{\lambda_2}{\mu_2}$$

**Case (ii)**

When $m=0,n>0,$ the equation (10) reduces to

$$(\lambda_1+\lambda_2+\mu_2)P(0,n)=\lambda_2P(0,n-1)+\mu_1P(1,n)+\mu_2P(0,n+1)$$

This equation is satisfied when $P(1,n)=\frac{\lambda_1}{\mu_1}P(0,n)$

Where

$$P(0,n) = P(0,0)\left(\frac{\lambda_2}{\mu_2}\right)^n$$

**Case (iii)**

When $m > 0 \,\&\, n > 0$, the equation (10) reduces to

$$(\lambda_1 + \lambda_2 + \mu_1)P(m,n) = \lambda_1 P(m-1,n) + \lambda_2 P(m,n-1) + \mu_1 P(m+1,n)$$

Its solution would be

$$P(m,n) = P(0,0)(x_1)^m (x_2)^n$$

Where

$$x_1 = \frac{\lambda_1}{2\mu_1} \quad \& \quad x_2 = \frac{2\lambda_2}{2(\lambda_2 - \mu_1) + \lambda_1}$$

The solution of the equation (10) for all the three cases are combined together as under

$$P(m,n) = P(0,0)(x_1)^m (x_2)^n \quad for \quad m \geq 0, n \geq 0$$

Where

$$x_1 = \frac{\lambda_1}{(1+\delta(m))\,\mu_1}; x_2 = \frac{2\lambda_2}{2\mu_2\,(1-\delta(m)) + [2(\lambda_2 - \mu_1) + \lambda_1]\,\delta(m)}$$

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# ANALYTICAL STUDY OF A COMPLEX FEED BACK SEMI BI-TANDEM QUEUE SYSTEM DIFFER IN TRANSITION PROBABILITIES

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**ABSTRACT**
***In this paper we consider a complex queue network comprised of bi-serial and parallel service channels. The model consists of three subsystems in which one subsystem has two bi-serial sub channels and other consisting two parallel sub channels. The subsystem is a single service channel which is commonly connected in series with these two subsystems. The concept of feedback is introduced that means a customer can revisit the system at most once if he is unsatisfied. The queue model is analyzed in steady state under Poisson stream with the help of differential difference equations, probability generating function technique, L'Hospital rule and law of calculus. Partial queue lengths and mean queue length is derived by using classical formula under steady state behaviour of the model. Behavioral analysis is done graphically by giving particular values to all variables. A particular case is also discussed to check the validity of the model. The model is applicable to different real life situations like banking service system, supermarkets, business managements, industries etc.***
***Key words: Complex feedback queue network, parallel service channels, bi-serial service channels, transient behaviour, queue characteristics***

## 1. INTRODUCTION:

In the development of queuing theory, many mathematicians and researchers contribute their work. Jackson R.R.P (1954) studied the queuing system with phase type service. Maggu (1970) introduced the bi-tandem concept in queuing with applicability in production. Singh T.P & his research scholars did a remarkable work in modeling the feedback queue system and developed many complex queue networks. Gupta Deepak, Singh T.P.(2007) analysed queue model comprised of parallel and bi-serial service channels. Kusum (2020) & Vandana Saini et al (2021) developed a queue model in which a customer is allowed to revisit the service channel one time. This paper is further an extended work of Kumar & G. Taneja (2017) in the sense that the analysis of a complex semi bi-tandem queue model is discussed and the transition probabilities of customer from moving one state to other in second visit are taken different. The model is analyzed in steady state and all queue performance measures are obtained in stochastic environment.

## 2. MODEL DESCRIPTION:

The model shown in figure 1 given below is comprised of three subsystems $C_1$ , $C_2$ , $C_3$. The subsystem $C_1$ has two bi-serial service channels $C_{11}$ & $C_{12}$ and the subsystem $C_2$ has two parallel service channels $C_{21}$ & $C_{22}$. The subsystems $C_1$ and $C_2$ are commonly linked in series with a service channel $C_3$. At first the customer will arrive at subsystem $C_1$ and $C_2$in front of service channels $C_{11}$, $C_{12}$ & $C_{21}$, $C_{22}$ with Poisson arrival rates $\lambda_1$, $\lambda_2$ &$\lambda'_1$, $\lambda'_2$ respectively. After that he will move to the subsystem $C_3$ for service. He can revisit any of the servers at most once.

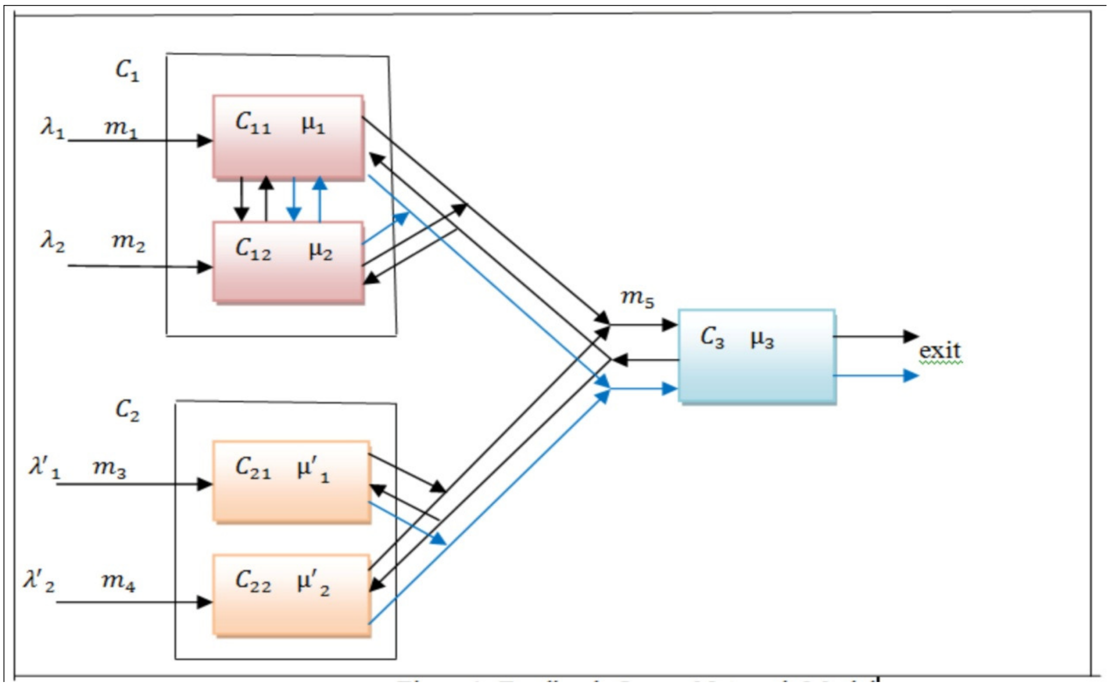
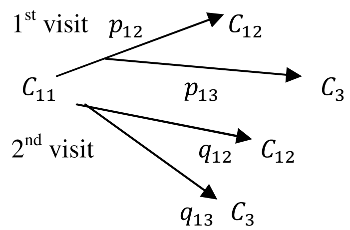



**Figure1: Feedback Queue Network Model**

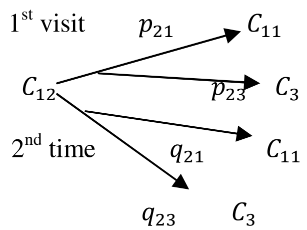
## NOTATIONS:

| Service Channels | | $C_{11}$ | $C_{12}$ | $C_{21}$ | $C_{22}$ | $C_3$ |
|---|---|---|---|---|---|---|
| Service Rate | | $\mu_1$ | $\mu_2$ | $\mu'_1$ | $\mu'_2$ | $\mu_3$ |
| No. of customers | | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ |
| Probability of customers moving from one service Channel to another | 1st visit | $C_{11} \rightarrow C_{12}$ $p_{12}$ $C_{11} \rightarrow C_3$ $p_{13}$ | $C_{12} \rightarrow C_3$ $p_{23}$ $C_{12} \rightarrow C_{11}$ $p_{21}$ | $C_{21} \rightarrow C_3$ $p'_{13}$ | $C_{22} \rightarrow C_3$ $p'_{23}$ | $C_3 \rightarrow C_{11}$ $p_{31}$ $C_3 \rightarrow C_{12}$ $p_{32}$ $C_3 \rightarrow C_{21}$ $p'_{31}$ $C_3 \rightarrow C_{22}$ $p'_{32}$ $C_3 \rightarrow exit$ $p_3$ |
| | 2nd visit | $C_{11} \rightarrow C_{12}$ $q_{12}$ $C_{11} \rightarrow C_3$ $q_{13}$ | $C_{12} \rightarrow C_3$ $q_{23}$ $C_{12} \rightarrow C_{11}$ $q_{21}$ | $C_{21} \rightarrow C_3$ $q'_{13}$ | $C_{22} \rightarrow C_3$ $q'_{13}$ | $C_3 \rightarrow exit$ $q_3$ |
| Probability of leaving the service Channel | 1st visit | a | b | c | d | e |
| | 2nd visit | $a_1$ | $b_1$ | $c_1$ | $d_1$ | $e_1$ |

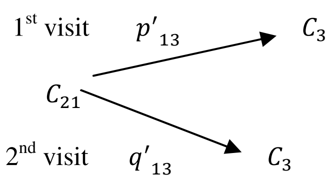
Possibilities of leaving the server $C_{11}$:



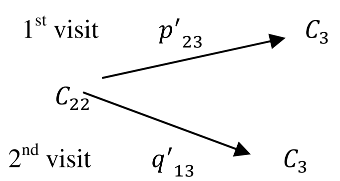

Possibilities of leaving the server $C_{12}$:



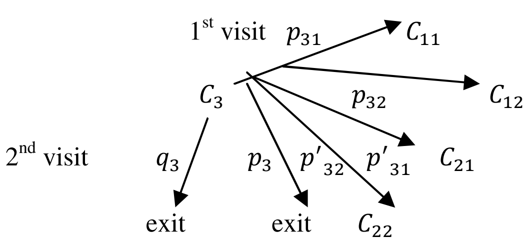

Possibilities of leaving the server $C_{21}$:




Possibilities of leaving the server $C_{22}$:




Possibilities of leaving the server $C_3$:




After taking service at $C_{11}$ then with leaving probability a the customer will visit either the service channel $C_{12}$ or server $C_3$ with probabilities $p_{12}$ and $p_{13}$ such that $p_{12} + p_{13} = 1$ .If the customer revisit the service channel $C_{11}$, in any case then after leaving the service channel with probability $a_1$, he can visit either service channel $C_{12}$ or server $C_3$ for service with probability $q_{12}$ and $q_{13}$ such that $q_{12} + q_{13} = 1$ and $ap_{12} + a\, p_{13} + a_1 q_{12} + a_1 q_{13} = 1$ . After taking service from service channel $C_{12}$ the customer may either visit to server $C_{11}$ or server $C_3$ with probabilities $p_{21}, p_{23}$ and with probabilities $q_{21}, q_{23}$ in case of revisit such that $p_{21} + p_{23} = 1, q_{21} + q_{23} = 1$ and $bp_{21} + b\,p_{23} + b_1 q_{21} + b_1 q_{23} = 1$

From service channels $C_{21}, C_{22}$ the customer can only moves to service channel $C_3$ such that for service channel $C_{21}$, $c\,p'_{13} + c_1 q'_{13} = 1$ and for service channel $C_{22}$, $d\,p'_{23} + d_1 q'_{23} = 1$

From service channel $C_3$, a customer can either visit to any of the service channels $C_{11}, C_{12}, C_{21}, C_{22}$ with probabilities $p_{31}, p_{32}, p'_{31}, p'_{32}$ or may exit the system with probability $p_3$ at first time and with probability $q_3$ in second time exit, such that $p_{31} + p_{31} + p'_{31} + p'_{32} + p_3 = 1$ also in this case the condition $ep_{31} + e\,p_{31} + ep'_{31} + ep'_{32} + e\,p_3 + e_1 q_3 = 1$, will hold.

## MATHEMATICAL MODELING:

Let us suppose that $P_{m_1,m_2,m_3,m_4,m_5}$ (t) denotes the joint probability of $m_1, m_2, m_3, m_4, m_5$ customers in front of the service channels $C_{11}, C_{12}, C_{21}, C_{22}, C_3$ respectively, where $m_1, m_2, m_3, m_4, m_5 \geq 0$ .The differential difference equation of the complex queue network model in transient state is as follows:

$$
\begin{aligned}
P_{m_1,m_2,m_3,m_4,m_5}(t) = &-\left(\lambda_1 + \lambda_2 + \lambda'_1 + \lambda'_2 + \mu_1 + \mu_2 + \mu'_1 + \mu'_2 + \mu_3\right) P_{m_1,m_2,m_3,m_4,m_5} \\
&+ \lambda_1 P_{m_1-1,m_2,m_3,m_4,m_5} + \lambda_2 P_{m_1,m_2-1,m_3,m_4,m_5} + \lambda'_1 P_{m_1,m_2,m_3-1,m_4,m_5} + \lambda'_2 P_{m_1,m_2,m_3,m_4-1,m_5} \\
&+ \mu_1 (a\,p_{12} + a_1 q_{12}) P_{m_1+1,m_2-1,m_3,m_4,m_5} + \mu_1 (a\,p_{13} + a_1 q_{13}) P_{m_1+1,m_2,m_3,m_4,m_5-1} \\
&+ \mu_2 (b\,p_{21} + b_1 q_{21}) P_{m_1-1,m_2+1,m_3,m_4,m_5} + \mu_2 (b\,p_{23} + b_1 q_{23}) P_{m_1,m_2+1,m_3,m_4,m_5-1} \\
&+ \mu'_1 (c\,p'_{13} + c_1 q'_{13}) P_{m_1,m_2,m_3+1,m_4,m_5-1} + \mu'_2 (d\,p'_{23} + d_1 q'_{23}) P_{m_1,m_2,m_3,m_4+1,m_5-1} \\
&+ \mu_3 (p_{31}) P_{m_1-1,m_2,m_3,m_4,m_5+1} + \mu_3 (p_{32}) P_{m_1,m_2-1,m_3,m_4,m_5+1} + \mu_3 (p'_{31}) P_{m_1,m_2,m_3-1,m_4,m_5+1} \\
&+ \mu_3 (p'_{32}) P_{m_1,m_2,m_3,m_4-1,m_5+1} + \mu_3 (e\,p_3 + e_1 q_3) P_{m_1,m_2,m_3,m_4,m_5+1}
\end{aligned}
$$

The steady state equation for the model as $t \to \infty$, is as follows:

$$
\begin{aligned}
\left(\lambda_1 + \lambda_2 + \lambda'_1 + \lambda'_2 + \mu_1 + \mu_2 + \mu'_1 + \mu'_2 + \mu_3\right) P_{m_1,m_2,m_3,m_4,m_5} = &\ \lambda_1 P_{m_1-1,m_2,m_3,m_4,m_5} \\
&+ \lambda_2 P_{m_1,m_2-1,m_3,m_4,m_5} + \lambda'_1 P_{m_1,m_2,m_3-1,m_4,m_5} + \lambda'_2 P_{m_1,m_2,m_3,m_4-1,m_5} \\
&+ \mu_1 (a\,p_{12} + a_1 q_{12}) P_{m_1+1,m_2-1,m_3,m_4,m_5} + \mu_1 (a\,p_{13} + a_1 q_{13}) P_{m_1+1,m_2,m_3,m_4,m_5-1} \\
&+ \mu_2 (b\,p_{21} + b_1 q_{21}) P_{m_1-1,m_2+1,m_3,m_4,m_5} + \mu_2 (b\,p_{23} + b_1 q_{23}) P_{m_1,m_2+1,m_3,m_4,m_5-1} \\
&+ \mu'_1 (c\,p'_{13} + c_1 q'_{13}) P_{m_1,m_2,m_3+1,m_4,m_5-1} + \mu'_2 (d\,p'_{23} + d_1 q'_{23}) P_{m_1,m_2,m_3,m_4+1,m_5-1} \\
&+ \mu_3 (p_{31}) P_{m_1-1,m_2,m_3,m_4,m_5+1} + \mu_3 (p_{32}) P_{m_1,m_2-1,m_3,m_4,m_5+1} + \mu_3 (p'_{31}) P_{m_1,m_2,m_3-1,m_4,m_5+1} \\
&+ \mu_3 (p'_{32}) P_{m_1,m_2,m_3,m_4-1,m_5+1} + \mu_3 (e\,p_3 + e_1 q_3) P_{m_1,m_2,m_3,m_4,m_5+1} \quad \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots(1)
\end{aligned}
$$

For all possible values of $m_1, m_2, m_3, m_4, m_5$ we get 31 more equations like equation (1).

For solving these equations we will use probability generating function technique. We will define generating function as,

$$G(X,Y,Z,R,S)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{m_3=0}^{\infty}\sum_{m_4=0}^{\infty}\sum_{m_5=0}^{\infty}P_{m1,m2,m3,m4,m5}(X)^{m_1}(Y)^{m_2}(Z)^{m_3}(R)^{m_4}(S)^{m_5},$$

$$|X|=1,|Y|=1,|Z|=1,|R|=1,|S|=1$$

and partial generating functions are,

$$G_{m_2,m_3,m_4,m_5}(X)=\sum_{m_1=0}^{\infty}P_{m_1,m_2,m_3,m_4,m_5}(X)^{m_1}$$

$$G_{m_3,m_4,m_5}(X,Y)=\sum_{m_2=0}^{\infty}G_{m_2,m_3,m_4,m_5}(X)(Y)^{m_2}$$

$$G_{m_4,m_5}(X,Y,Z)=\sum_{m_3=0}^{\infty}G_{m_3,m_4,m_5}(X,Y)(Z)^{m_3}$$

$$G_{m_5}(X,Y,Z,R)=\sum_{m_4=0}^{\infty}G_{m_3,m_4,m_5}(X,Y,Z)(R)^{m_4}$$

$$G(X,Y,Z,R,S)=\sum_{m_5=0}^{\infty}G_{m_4}(X,Y,Z,R)(S)^{m_5}$$

When we use above defined generating function and solve the differential difference equations, we find the solution as,

$$G(X,Y,Z,R,S)=\frac{\begin{aligned}&=\mu_1\left[1-\frac{(a\,p_{12}+a_1q_{12})Y}{X}-\frac{(a\,p_{13}+a_1q_{13})S}{X}\right]G_1\\&+\mu_2\left[1-\frac{(b\,p_{21}+b_1q_{21})X}{Y}-\frac{(b\,p_{23}+b_1q_{23})S}{Y}\right]G_2\\&+\mu_1'\left[1-\frac{(c\,p_{13}'+c_1q_{13}')S}{Z}\right]G_3+\mu_2'\left[1-\frac{(d\,p_{23}'+d_1q_{23}')S}{R}\right]G_4\\&+\mu_3\left[1-\frac{(p_{31})X}{S}-\frac{(p_{32})Y}{S}-\frac{(p_{31}')Z}{S}-\frac{(p_{32}')R}{S}-\frac{(e\,p_3+e_1q_3)}{S}\right]G_5\end{aligned}}{\begin{aligned}&=\lambda_1(1-X)+\lambda_2(1-Y)+\lambda_1'(1-Z)+\lambda_2'(1-R)\\&+\mu_1\left[1-\frac{(a\,p_{12}+a_1q_{12})Y}{X}-\frac{(a\,p_{13}+a_1q_{13})S}{X}\right]\\&+\mu_2\left[1-\frac{(b\,p_{21}+b_1q_{21})X}{Y}-\frac{(b\,p_{23}+b_1q_{23})S}{Y}\right]\\&+\mu_1'\left[1-\frac{(c\,p_{13}'+c_1q_{13}')S}{Z}\right]+\mu_2'\left[1-\frac{(d\,p_{23}'+d_1q_{23}')S}{R}\right]\\&+\mu_3\left[1-\frac{(p_{31})X}{S}-\frac{(p_{32})Y}{S}-\frac{(p_{31}')Z}{S}-\frac{(p_{32}')R}{S}-\frac{(e\,p_3+e_1q_3)}{S}\right]\end{aligned}}$$

Where, $G_1=G_0(Y,Z,R,S), G_2=G_0(X,Z,R,S), G_3=G_0(X,Y,R,S), G_4=G_0(X,Y,Z,S),$
$G_5=G_0(X,Y,Z,R)$

Let us take G(X,Y,Z,R,S)=1,at X=Y=Z=R=S=1, and above equation reduces to indeterminate form. Using L'Hospital rule for indeterminate form and taking limit as X,Y,Z,R,S tends to 1, one by one in above equation then we get following equations,

$$\mu_1 G_1 - \mu_2 G_2 (b p_{21} + b_1 q_{21}) - \mu_3 (p_{31}) G_5 = -\lambda_1 + \mu_1 - \mu_2 (b p_{21} + b_1 q_{21}) - \mu_3 (p_{31}) \text{.............}(I)$$

$$-\mu_1 (a p_{12} + a_1 q_{12}) G_1 + \mu_2 G_2 - \mu_3 (p_{32}) G_5 = -\lambda_2 + \mu_2 - \mu_1 (a p_{12} + a_1 q_{12}) - \mu_3 (p_{32}) \text{...........}(II)$$

$$\mu_1^{'} (c p_{13}^{'} + c_1 q_{13}^{'}) G_3 - \mu_3 (p_{31}^{'}) G_5 = -\lambda_1^{'} + \mu_1^{'} (c p_{13}^{'} + c_1 q_{13}^{'}) - \mu_3 (p_{31}^{'}) \text{.................................}(III)$$

$$\mu_2^{'} (c p_{23}^{'} + c_1 q_{23}^{'}) G_4 - \mu_3 (p_{32}^{'}) G_5 = -\lambda_2^{'} + \mu_2^{'} (d p_{23}^{'} + d_1 q_{23}^{'}) - \mu_3 (p_{32}^{'}) \text{..............................}(IV)$$

$$-\mu_1 G_1 (a p_{13} + a_1 q_{13}) - \mu_2 G_2 (b p_{23} + b_1 q_{23}) - \mu_1^{'} G_3 - \mu_2^{'} G_4 + \mu_3 G_5 = -\mu_1 (a p_{13} + a_1 q_{13})$$
$$-\mu_2 (b p_{23} + b_1 q_{23}) - \mu_1^{'} - \mu_2^{'} + \mu_3 \text{..........................................................................................}(V)$$

Solving above equations for $G_1, G_2, G_3, G_4, G_5$ we obtain the values as follows,

$$G_1 = 1 - \left[ \frac{(\lambda_1 + C\lambda_2)(K) + (HC + L)\lambda}{\mu_1 (1 - AC)(K)} \right]$$

$$G_2 = 1 - \left[ \frac{(\lambda_2 + A\lambda_1)(K) + (H + LA)\lambda}{\mu_2 (1 - AC)(K)} \right]$$

$$G_3 = 1 - \left[ \frac{(K)\lambda_1^{'} + I\lambda}{(K)E\mu_1^{'}} \right]$$

$$G_4 = 1 - \left[ \frac{(K)\lambda_2^{'} + J\lambda}{(K)F\mu_2^{'}} \right]$$

$$G_5 = 1 - \left[ \frac{\lambda}{(K)\mu_3} \right]$$

here, $A = (a p_{12} + a_1 q_{12}), B = (a p_{13} + a_1 q_{13}), C = (b p_{21} + b_1 q_{21}), D = (b p_{23} + b_1 q_{23})$
$E = (c p_{13}^{'} + c_1 q_{13}^{'}), F = (c p_{23}^{'} + c_1 q_{23}^{'}), H = p_{32}, I = p_{31}^{'}, J = p_{32}^{'}, K = (e p_3 + e_1 q_3), L = p_{31}$
$\lambda = \lambda_1 + \lambda_2 + \lambda_1^{'} + \lambda_2^{'}$

To find utilization factor of all the servers we will use the classical formula as,
$\rho_1 = 1 - G_1, \rho_2 = 1 - G_2, \rho_3 = 1 - G_3, \rho_4 = 1 - G_4, \rho_5 = 1 - G_5$......................(*)

Hence the steady-state solution for the model will be,

$P_{m_1, m_2, m_3, m_4, m_5} = (1-\rho_1)^{m_1} (1-\rho_2)^{m_2} (1-\rho_3)^{m_3} (1-\rho_4)^{m_4} (1-\rho_5)^{m_5} \rho_1 \rho_2 \rho_3 \rho_4 \rho_5$ Solution for the model will exists only if $\rho_1, \rho_2, \rho_3, \rho_4, \rho_5 < 0$

**QUEUE PERFORMANCE MEASURES:**

1. Mean Queue Length(L) = $L_1 + L_2 + L_3 + L_4 + L_5 = \frac{\rho_1}{1-\rho_1} + \frac{\rho_2}{1-\rho_2} + \frac{\rho_3}{1-\rho_3} + \frac{\rho_4}{1-\rho_4} + \frac{\rho_5}{1-\rho_5}$
2. Variance ($V_{ar}$) = $\frac{\rho_1}{(1-\rho_1)^2} + \frac{\rho_2}{(1-\rho_2)^2} + \frac{\rho_3}{(1-\rho_3)^2} + \frac{\rho_4}{(1-\rho_4)^2} + \frac{\rho_5}{(1-\rho_5)^2}$
3. Expected waiting time for customers= $\frac{L}{\lambda}$

**BEHAVIOURAL ANALYSIS OF THE MODEL:**

In this section we will discuss the behavior of partial queue length and average queue length with the change of service rates and arrival rates in following manner:

(i) Behaviour of $L_1, L_2, L_3, L_4, L_5, L$ for different values of $\lambda_1, \lambda_2, \lambda_1', \lambda_2'$

(ii) Behaviour of $L_1, L_2, L_3, L_4, L_5, L$ for different values of $\mu_1, \mu_2, \mu_1', \mu_2', \mu_3$

(iii) Graphical analysis of partial queue lengths and mean queue length of the system for different values of $\lambda_1, \lambda_2, \lambda_1', \lambda_2'$ and $\mu_1, \mu_2, \mu_1', \mu_2', \mu_3$

Table1: partial queue lengths, mean queue length of the system with respect to $\lambda_1$

$p_{12} = 0.5, p_{13} = 0.5, q_{12} = 0.4, q_{13} = 0.6, p_{21} = 0.3, p_{23} = 0.7, q_{21} = 0.4, q_{23} = 0.6, p_{31} = 0.2,$
$p_{32} = 0.1, p_{31}' = 0.2, p_{32}' = 0.1, p_3 = 0.4, p_{13}' = 1, q_{13}' = 1, p_{23}' = 1 q_{23}' = 1, q_3 = 1,$
$a = 0.5, a_1 = 0.5, b = 0.4, b_1 = 0.6, c = 0.7, c_1 = 0.3, d = 0.5, d_1 = 0.5, e = 0.7, e_1 = 0.3,$
$\mu_1 = 10, \mu_2 = 11, \mu_1' = 7, \mu_2' = 6, \mu_3 = 20, \lambda_2 = 3, \lambda_1' = 2, \lambda_2' = 3$

| $\lambda_1$ | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | L |
|---|---|---|---|---|---|---|
| 2 | 5.807352 | 3.514673 | 3.5106 | 3.701457 | 6.251632 | 22.78571 |
| 2.2 | 7.826125 | 3.885198 | 3.721435 | 3.833253 | 7.285004 | 26.55102 |
| 2.4 | 11.53133 | 4.324814 | 3.950495 | 3.970179 | 8.661836 | 32.43865 |
| 2.6 | 20.64502 | 4.847953 | 4.205622 | 4.117707 | 10.60093 | 44.41723 |
| 2.8 | 78.36508 | 5.489293 | 4.485464 | 4.271481 | 13.49275 | 106.1041 |

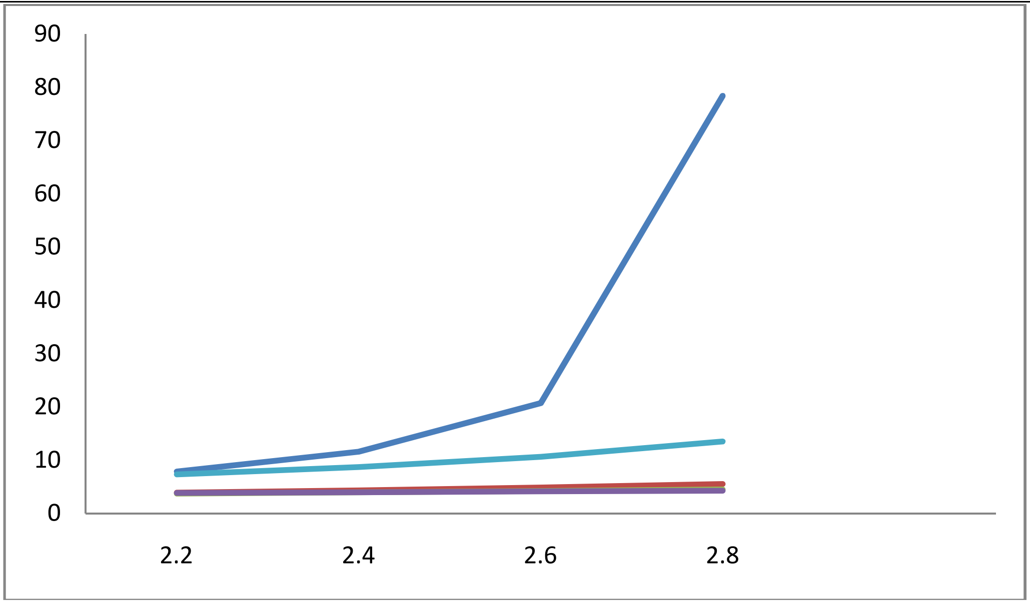



**Figure2: $L_1, L_2, L_3, L_4, L_5$ vs. $\lambda_1$**

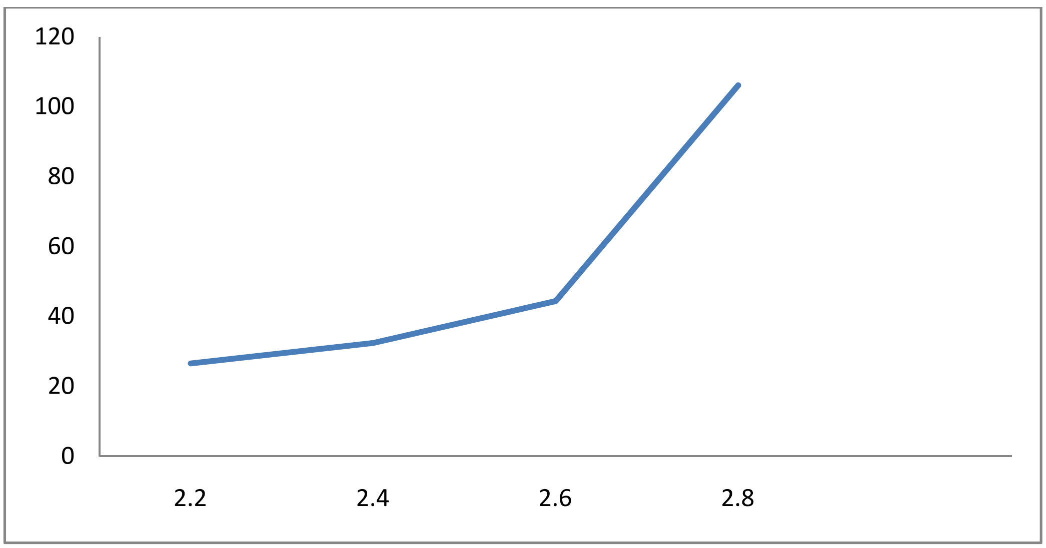



**Figure3: L vs. $\lambda_1$**

**Table2: partial queue lengths, mean queue length of the system with respect to $\lambda_2$**

| $\lambda_2$ | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | L |
|---|---|---|---|---|---|---|
| 3 | 5.807352 | 3.514673 | 3.5106 | 3.701457 | 6.251632 | 22.78571 |
| 3.2 | 6.77605 | 7.928571 | 3.721435 | 3.833253 | 7.285004 | 29.54431 |
| 3.4 | 8.066183 | 11.4533 | 3.950495 | 3.970179 | 8.661836 | 36.10199 |
| 3.6 | 9.869565 | 16.69912 | 4.205622 | 4.117707 | 10.60093 | 45.49294 |
| 3.8 | 12.56852 | 57.82353 | 4.485464 | 4.271481 | 13.49275 | 92.64175 |

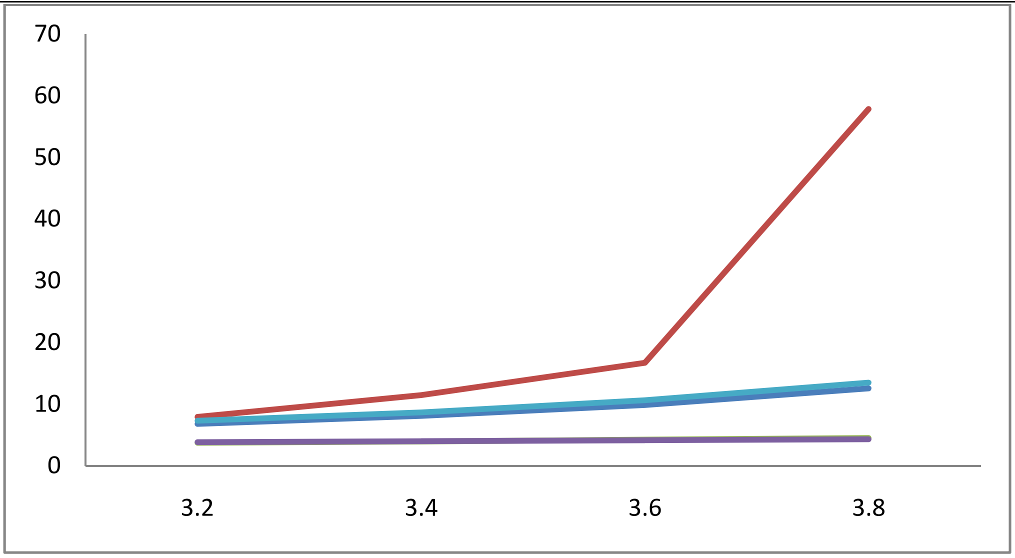



**Figure4: $L_1, L_2, L_3, L_4, L_5$ vs. $\lambda_2$**

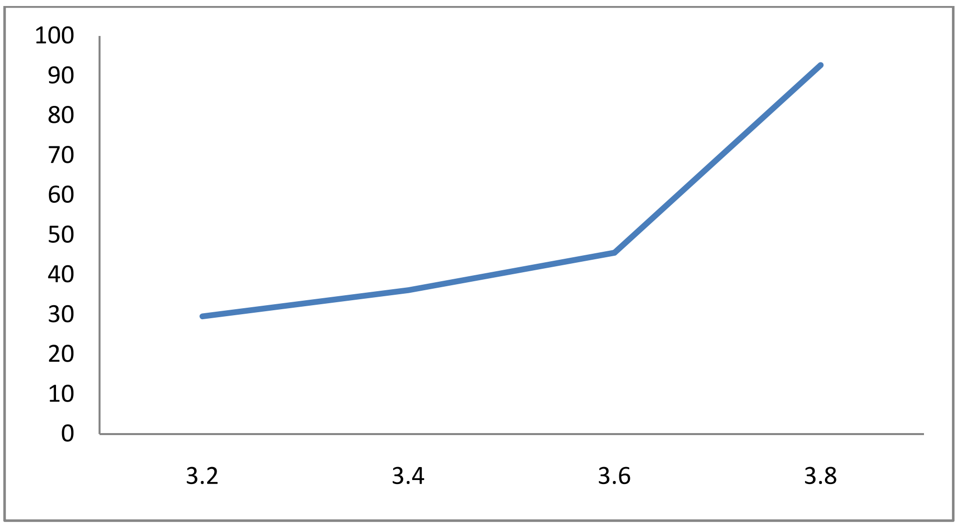



**Figure5:L vs. $\lambda_2$**

**Table3: partial queue lengths, mean queue length of the system with respect to $\lambda_1^{'}$**

| $\lambda_1^{'}$ | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | L |
|---|---|---|---|---|---|---|
| 2 | 5.807352 | 3.514673 | 3.5106 | 3.701457 | 6.251632 | 22.78571 |
| 2.2 | 6.28863 | 6.358352 | 4.455537 | 3.833253 | 7.285004 | 28.22078 |
| 2.4 | 6.843137 | 6.806401 | 5.906077 | 3.970179 | 8.661836 | 32.18763 |
| 2.6 | 7.488964 | 7.319468 | 8.398496 | 4.117707 | 10.60093 | 37.92556 |
| 2.8 | 8.250694 | 7.896797 | 13.70588 | 4.271481 | 13.49275 | 47.61761 |
| 3 | 9.162602 | 8.560229 | 32.78378 | 4.434783 | 18.34236 | 73.28376 |

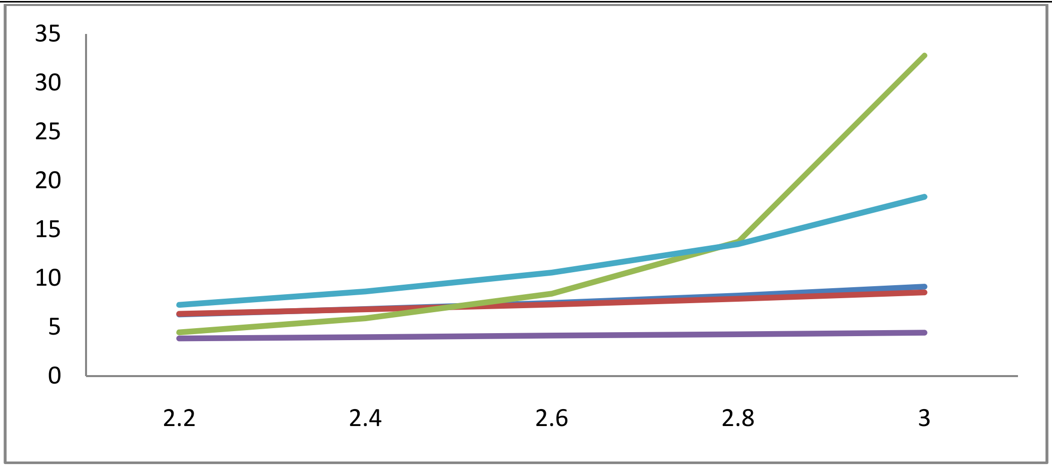



**Figure6: $L_1, L_2, L_3, L_4, L_5$ vs. .$\lambda'_1$**

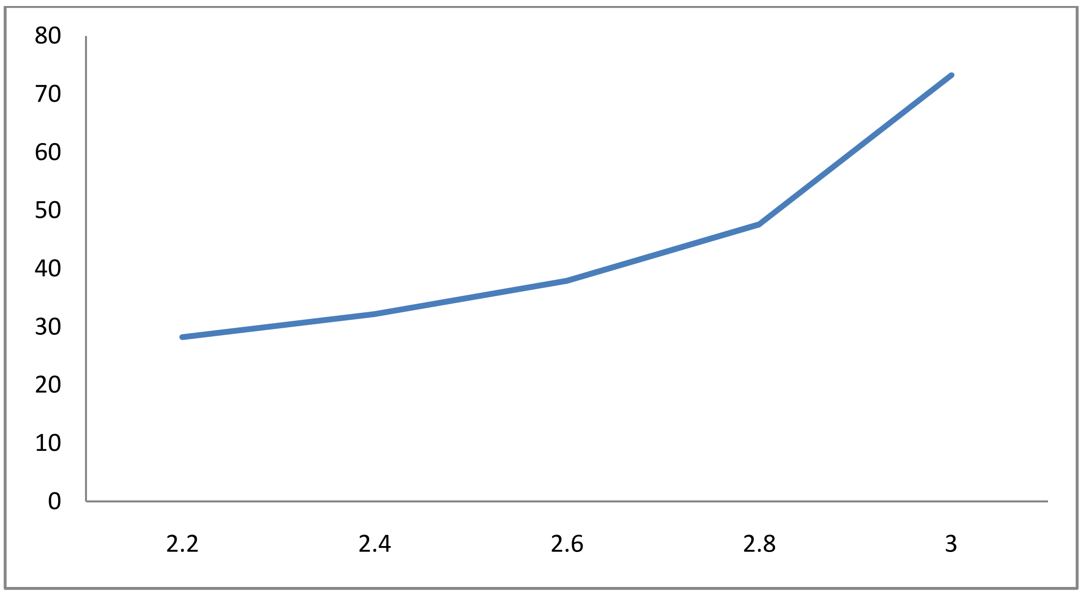



**Figure7:L vs.$\lambda'_1$**

**Table4: partial queue lengths, mean queue length of the system with respect to.$\lambda'_2$**

| $\lambda'_2$ | $L_1$ | $L_2$ | $L_3$ | $L_4$ | $L_5$ | L |
|---|---|---|---|---|---|---|
| 3 | 5.807352 | 3.514673 | 3.5106 | 3.701457 | 6.251632 | 22.78571 |
| 3.2 | 6.28863 | 6.358352 | 3.98008 | 4.760369 | 7.285004 | 28.67243 |
| 3.4 | 6.843137 | 6.806401 | 3.950495 | 6.434944 | 8.661836 | 32.69681 |
| 3.6 | 7.488964 | 7.319468 | 4.205622 | 9.471204 | 10.60093 | 39.08619 |
| 3.8 | 8.250694 | 7.896797 | 4.485464 | 16.76199 | 13.49275 | 50.8877 |
| 4 | 9.162602 | 8.560229 | 4.800464 | 57.13953 | 18.34236 | 98.00519 |

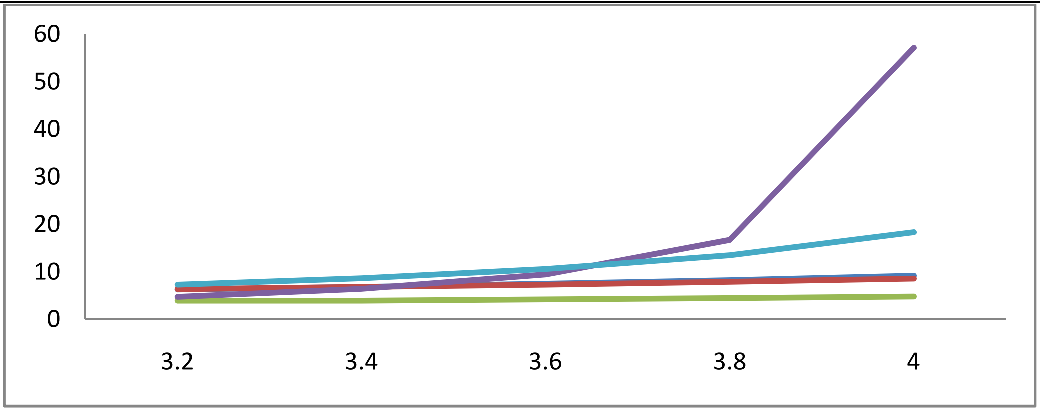



**Figure8: $L_1, L_2, L_3, L_4, L_5$ vs. $\lambda'_2$**

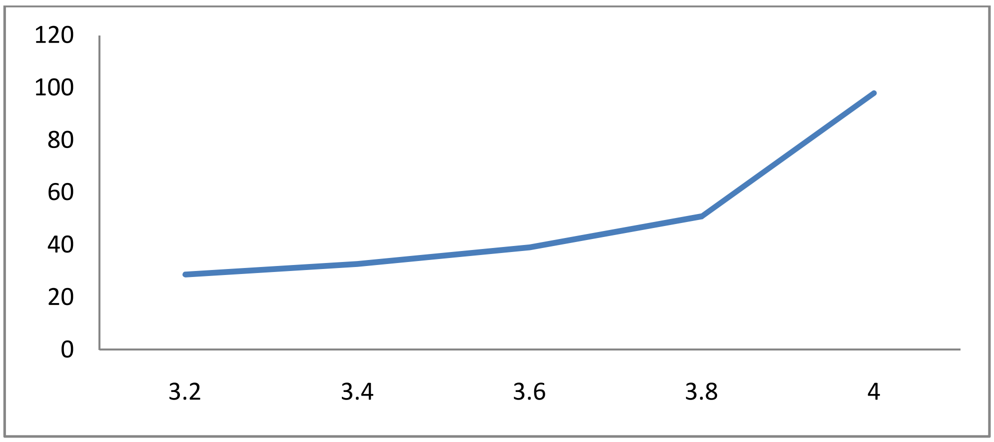



**Figure9: L vs. $\lambda'_2$**

**Table 5: mean queue length of the system with respect to $\mu_1, \mu_2, \mu'_1, \mu'_2, \mu_3$**

$p_{12}=0.5,\ p_{13}=0.5, q_{12}=0.4, q_{13}=0.6,\ p_{21}=0.3,\ p_{23}=0.7, q_{21}=0.4, q_{23}=0.6,\ p_{31}=0.2,$
$p_{32}=0.1,\ p'_{31}=0.2,\ p'_{32}=0.1,\ p_3=0.4,\ p'_{13}=1, q'_{13}=1,\ p'_{23}=1 q'_{23}=1, q_3=1,$
$a=0.5, a_1=0.5, b=0.4, b_1=0.6, c=0.7, c_1=0.3, d=0.5, d_1=0.5, e=0.7, e_1=0.3,$
$\lambda_1=2, \lambda_2=3, \lambda'_1=2, \lambda'_2=3$

| $\mu_1$ | L | $\mu_2$ | L | $\mu'_1$ | L | $\mu'_2$ | L | $\mu_3$ | L |
|---|---|---|---|---|---|---|---|---|---|
| 10 | 22.78571 | 11 | 22.78571 | 7 | 22.78571 | 6 | 22.78571 | 20 | 22.78571 |
| 10.5 | 21.3117 | 11.5 | 22.18647 | 7.5 | 21.93009 | 6.5 | 21.74458 | 20.5 | 21.82339 |
| 11 | 20.4327 | 12 | 21.76266 | 8 | 21.40991 | 7 | 21.16023 | 21 | 21.12067 |
| 11.5 | 19.85133 | 12.5 | 21.44564 | 8.5 | 21.06063 | 7.5 | 20.78623 | 21.5 | 20.58204 |
| 12 | 19.43737 | 13 | 21.20102 | 9 | 20.80933 | 8 | 20.52626 | 22 | 20.15729 |
| 12.5 | 19.12797 | 13.5 | 20.98769 | 9.5 | 20.61978 | 8.5 | 20.33499 | 22.5 | 19.81307 |
| 13 | 18.88703 | 14 | 20.83121 | 10 | 20.48897 | 9 | 20.18908 | 23 | 19.52769 |

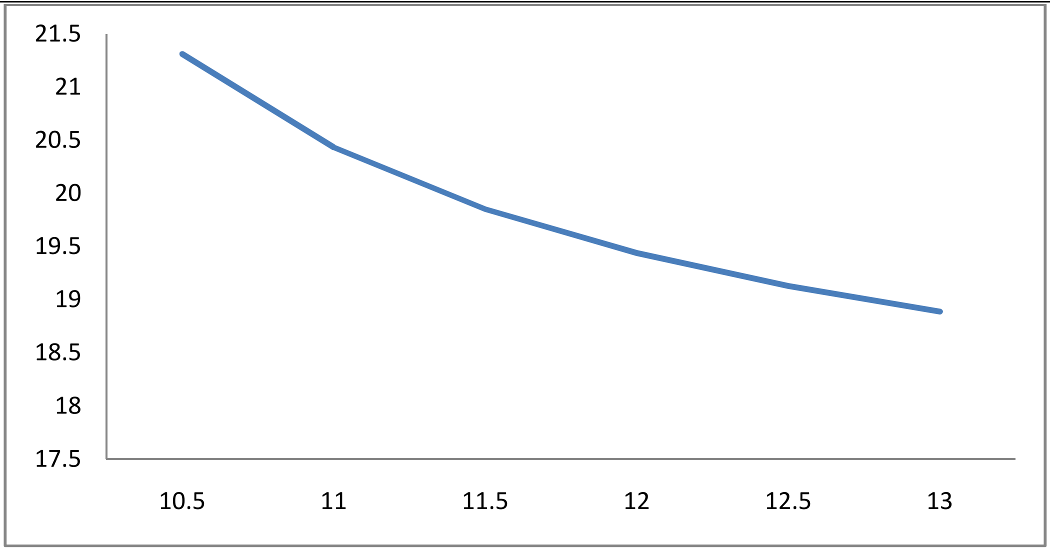



**Figure10: L vs. $\mu_1$**

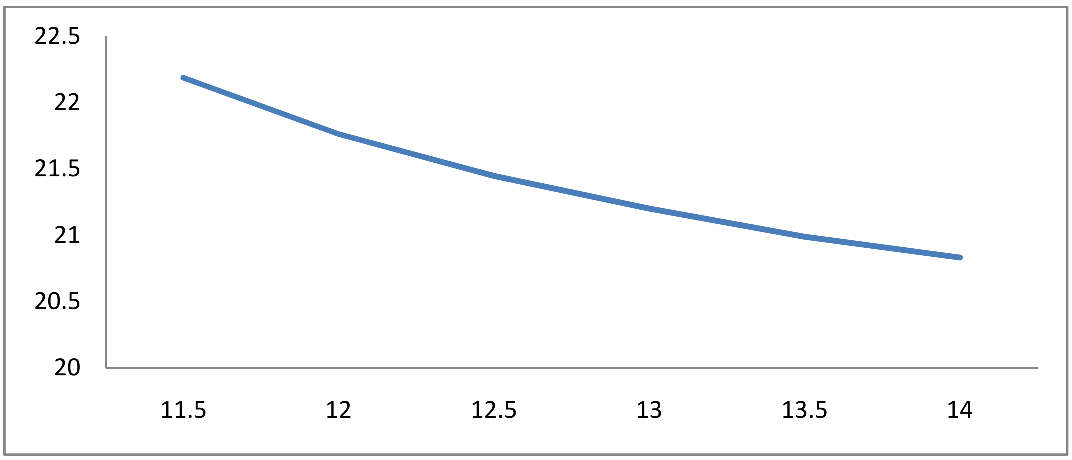



**Figure11: L vs. $\mu_2$**

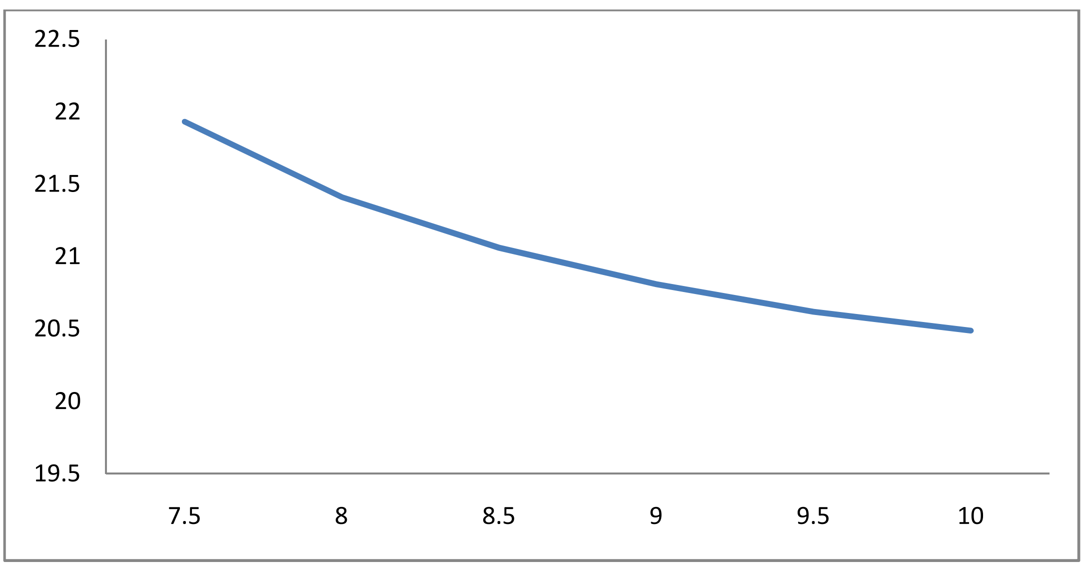


**Figure12: L vs. $\mu'_1$**

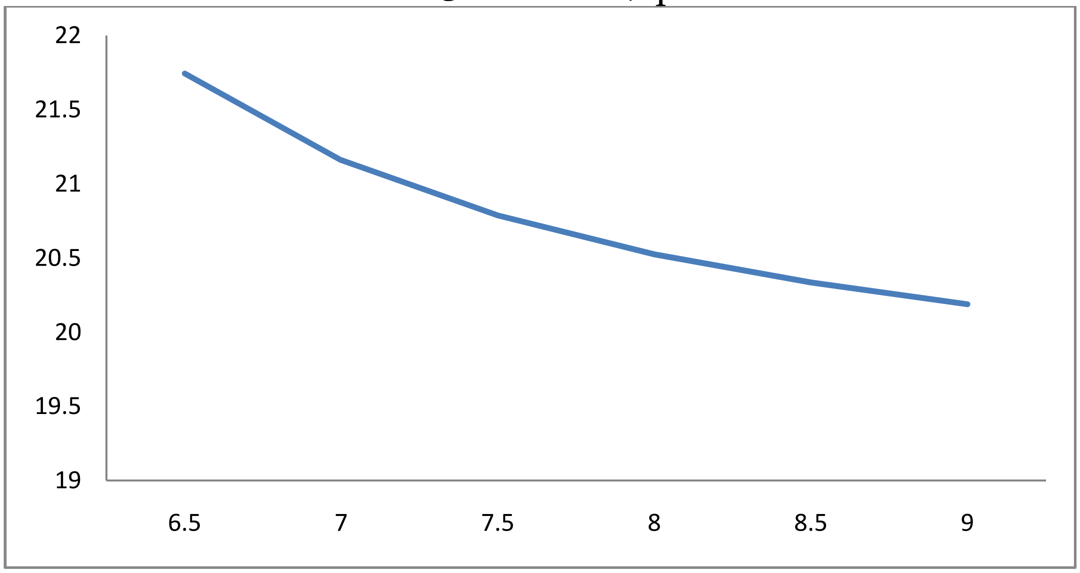



**Figure13: L vs. $\mu'_2$**

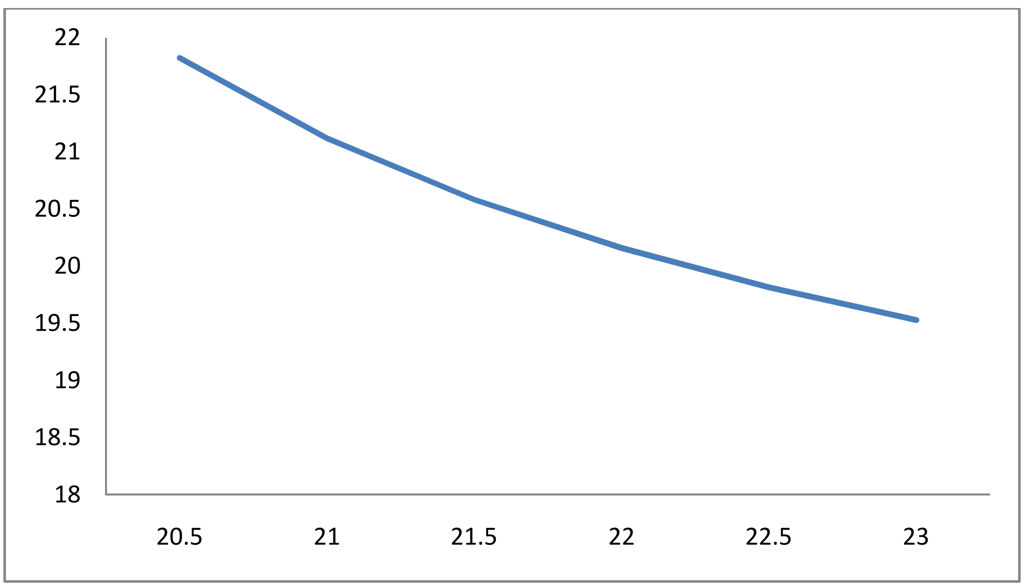



**Figure14: L vs. $\mu_3$**

## RESULT & DISCUSSION:

- From table 1 and from figure2 and figure3 we can observe that as arrival rate at $C_{11}$ increases from 2 to 2.4, partial queue lengths and mean queue length are increasing slowly. But suddenly at $\lambda_1$=2.6 $L_1, L_5$ and L are increasing faster than before. At $\lambda_1$=2.8 all queue lengths except $L_3, L_4$ are at maximum level of increasing.
- From table 2 and from figure4 and figure5, it is clear that when arrival rate at server $C_{12}$ increases from 3 to 3.4, partial queue lengths and mean queue length are increasing at a constant rate. A sudden increase can be seen in $L_1, L_2, L_5$ and L at $\lambda_2$=3.6 and at $\lambda_2$=3.8 increasing rate is much faster than before. Other partial queue lengths are constantly increasing at all points.

- Table 3 and figures 6 & 7 shows that at $\lambda'_1$=2.8, $L_1, L_3, L_5$ and L are increasing at faster speed than before and increasing rate becomes much high at $\lambda'_1$=3, but other partial queue lengths $L_2, L_4$ are slowly increasing.
- Similar results can be seen in table 4 and from figures 8 & 9 for queue lengths except $L_3$ at $\lambda'_2$=3.8 and $\lambda'_2$=4.
- Table 5 and figures from 10 to 14 shows that as service rate increases, Mean queue length is decreasing but after some time it is decreasing slowly at constant rate.

## PARTICULAR CASE:

1. If we remove the concept of feedback, then the results tally with Gupta Deepak [5].

## CONCLUSION:

While analysing this complex feedback queue network with bi-serial and parallel servers we can conclude that as arrival rate increases the queue length increases but after some time it increases faster and faster. With the increase in service rate queue length is decreasing and after some time it disperse with constant rate. When we will apply this model to real life situations then this analysis will be very helpful to understand the system and to redesign the system to minimize congestion and to increase customer's satisfaction level.

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# COEFFICIENT ESTIMATION OF CERTAIN SUBCLASS OF BI-UNIVALENT FUNCTIONS DEFINED BY CATAS OPERATOR

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**ABSTRACT**

***The purpose of the present paper is to introduce new subclass of the function class $\sum$ . Furthermore, we establish Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ defined in the open unit disc $U = \{z \in C: |z| < 1\}$ using Catas operator.***

***2000 Mathematics Subject Classification: 30C45, 30C50.***

***Keywords: Analytic, Univalent, Bi-univalent function, Catas operator***

## 1. INTRODUCTION

Let $A$ denote the class of the function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0 \tag{1.1}$$

which are analytic and univalent in the unit disc $U = \{z \in C: |z| < 1\}$ and satisfy the normalization condition $f(0) = 0$ ,$f'(0) = 1$.

Let $S$ denote the subclass of $A$ consisting functions of the form (1.1) which are univalent in $U$.

A function $f \in S$ is said to be starlike of order $\alpha(0 \leq \alpha < 1)$ if and only if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U \tag{1.2}$$

and convex of order $\alpha$ if and only if

$$Re\left(1 + \frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U \tag{1.3}$$

Denote these classes respectively $S^*(\alpha)$and $K(\alpha)$.

The Koebe one quarter theorem [8] asserts that the image of $U$ under univalent function $f \in S$ contains a disc of radius $\frac{1}{4}$.Thus every function $f \in S$ has an inverse $f^{-1}$defined by $f^{-1}[f(z)] = z$. $(z \in U)$ and $f^{-1}[f(w)] = w$ , $(|w| < r_0(f) \geq \frac{1}{4}$.

where

$$g(w) = w - a_2 w + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots \tag{1.4}$$

A function $f \in S$ is said to be bi-univalent in $U$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $U$.

Let $\sum$ denote the class of bi-univalent functions in $U$ given by (1.1).Some functions in the class $\sum$ are as below (see Srivastava et.al.[10]);

$$\frac{z}{1-z}, -log(1-z), \frac{1}{2}(\frac{1+z}{1-z}). \quad (1.5)$$

The class $\sum$ of bi-univalent functions was first investigated by Lewin [8] and it was shown that $|a_2| < 1.51$. Brannan and Clunnie [1] improved Lewin's result ad conjectured that $|a_2| \leq \sqrt{2}$. Later, Netanyahu [6] showed that max $f \in \sum$ $|a_2| = \frac{4}{3}$. Brannan and Taha [2] , also introduced certain subclasses of bi-univalent function and estimates for their initial coefficients.

Recently ,Srivastava et.al.[10], Frasin and Aouf [7], Srivastava and Bansal [11], SourabhPorwal and M.Darus [9], Sangle et.al.[12] are also introduced and investigated the various subclasses of bi-univalent functions and obtained bound for the coefficients $|a_2|$ and $|a_3|$ .

Motivating with their work, we introduce new subclass of the function class $\sum$ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for the functions in these new subclass of the function class $\sum$ employing the techniques used earlier by Srivastava et.al.[10] and Sourabh Porwal and M.Darus [9].

In order to prove our main results, we require the following lemma due to [3].

**Lemma 1.1** *If $h \in P$ then $|c_n| \leq 2$ for each $n$ ,where $P$ is the family of all functions $h$ analytic in $U$ for which Re $\{h(z)\} > 0$,*

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots for z \in U. \quad (1.6)$$

## 2 COEFFICIENT BOUNDS FOR THE FUNCTION CLASs $B_{\Sigma}^{\lambda}(\delta, n, \alpha, l)$

**Definition 2.1** *A function $f(z)$ given by (1.1) is said to be in the class $B_{\Sigma}^{\lambda}(\delta, n, \alpha, l)$ if he following conditions are satisfied:*

$$f \in \sum \quad and\ arg\left|\left(\frac{(1-\delta)I^n(\lambda,l)f(z)+\delta I^{n+1}(\lambda,l)f(z)}{z}\right)\right| < \frac{\alpha\pi}{2} \quad (2.1)$$

$(0 < \alpha \leq 1), \lambda \geq 0, \delta \geq 1, l \geq 0, n \in N_0, z \in U)$.

$$f \in \sum \quad and\ arg\left|\left(\frac{(1-\delta)I^n(\lambda,l)f(w)+\delta I^{n+1}(\lambda,l)f(w)}{w}\right)\right| < \frac{\alpha\pi}{2} \quad (2.2)$$

$(0 < \alpha \leq 1), \lambda \geq 0, \delta \geq 1, l \geq 0, n \in N_0, w \in U)$.

Where $I^n(\lambda, l)f(z)$ stands for Catas operator introduced by A Catas et.al. [4] and$g$ is extension of $f^{-1}$ to $U$ and is given by (1.4).

**Theorem 2.1** *Let $f(z)$ given by (1.1) be in the class $B_{\Sigma}^{\lambda}(\delta, n, \alpha, l)$ then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left(\frac{1+2\lambda+l}{1+l}\right)^n + \left(\frac{1+2\delta\lambda+l}{1+l}\right)\alpha + (1-\alpha)\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\delta\lambda+l}{1+l}\right)\right]^2}} \quad (2.3)$$

and

$$|a_3| \le \frac{2\alpha}{\left(\frac{1+2\lambda+l}{1+l}\right)^n\left(\frac{1+2\delta\lambda+l}{1+l}\right)} + \frac{4\alpha^2}{\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\delta\lambda+l}{1+l}\right)\right]^2}. \qquad (2.4)$$

**Proof:** It follows from (2.1) and (2.2) that

$$\frac{(1-\delta)I^n(\lambda,l)f(z)+\delta I^{n+1}(\lambda,l)f(z)}{z} = [p(z)]^\alpha \qquad (2.5)$$

and

$$\frac{(1-\delta)I^n(\lambda,l)g(w)+\delta I^{n+1}(\lambda,l)g(w)}{w} = [q(z)]^\alpha \qquad (2.6)$$

where $p(z)$ and $q(w)$ in $P$ and have he forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \ldots$$

Now, equating the coefficients in (2.5) and (2.6),we obtain

$$\left(\frac{1+\lambda+l}{1+l}\right)^n\left[\frac{1+\lambda\delta+l}{1+l}\right]a_2 = \alpha p_1 \qquad (2.7)$$

$$\left(\frac{1+2\lambda+l}{1+l}\right)^n\left[\frac{1+2\lambda\delta+l}{1+l}\right]a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2 \qquad (2.8)$$

$$\left(\frac{1+\lambda+l}{1+l}\right)^n\left[\frac{1+\lambda\delta+l}{1+l}\right]a_2 = -\alpha q_1 \qquad (2.9)$$

$$\left(\frac{1+2\lambda+l}{1+l}\right)^n\left[\frac{1+2\lambda\delta+l}{1+l}\right](2a_2^2 - a_3) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2 \qquad (2.10)$$

From (2.7) and (2.9) ,we get

$$p_1 = -q_1 \qquad (2.11)$$

Squaring and adding (2.7) and (2.9) ,we obtain

$$\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\lambda\delta+l}{1+l}\right)\right]^2 2a_2^2 = \alpha^2(p_1^2 + q_1^2) \qquad (2.12)$$

Now adding equation (2.8) and (2.10), we get

$$\left(\frac{1+2\lambda+l}{1+l}\right)^n\left(\frac{1+2\lambda\delta+l}{1+l}\right)2a_2^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)$$

By using (2.12), we obtain

$$\left(\frac{1+2\lambda+l}{1+l}\right)^n\left(\frac{1+2\lambda\delta+l}{1+l}\right)2a_2^2 = \alpha(p_2+q_2) + \frac{(\alpha-1)}{\alpha}\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\lambda\delta+l}{1+l}\right)\right]^2 2a_2^2$$

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{\left(\frac{1+2\lambda+l}{1+l}\right)^n\left(\frac{1+2\delta\lambda+l}{1+l}\right)\alpha + (1-\alpha)\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\delta\lambda+l}{1+l}\right)\right]^2}$$

Applying lemma (1.1) for the coefficients $p_2$ and $q_2$ , we have

$$|a_2| \le \frac{2\alpha}{\sqrt{\left(\frac{1+2\lambda+l}{1+l}\right)^n\left(\frac{1+2\lambda+l}{1+l}\right)\alpha + (1-\alpha)\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\lambda+l}{1+l}\right)\right]^2}}.$$

This gives bound on $|a_2|$.

Next, in order to find the bound on $|a_3|$ ,by subtracting equation (2.10) from equation (2.8) ,we obtain

$$2(\frac{1+2\lambda+l}{1+l})^n(\frac{1+2\lambda\delta+l}{1+l})(a_3-a_2^2)=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2)$$

From (2.11), we get $p_1^2=q_1^2$ and also using (2.12), we have

$$2a_3(\frac{1+2\lambda+l}{1+l})^n(\frac{1+2\lambda\delta+l}{1+l})=\alpha(p_2-q_2)+2a_2^2(\frac{1+2\lambda+l}{1+l})^n(\frac{1+2\lambda\delta+l}{1+l})$$

$$a_3=\frac{\alpha(p_2-q_2)}{2\left(\frac{1+2\lambda+l}{1+l}\right)^n\left(\frac{1+2\delta\lambda+l}{1+l}\right)}+\frac{\alpha^2(p_1^2+q_1^2)}{2\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\delta\lambda+l}{1+l}\right)\right]^2}.$$

Applying lemma (1.1) once again for the coefficients $p_1, q_1, p_2$and $q_2$,we obtain

$$|a_3|\le\frac{2\alpha}{\left(\frac{1+2\lambda+l}{1+l}\right)^n\left(\frac{1+2\delta\lambda+l}{1+l}\right)}+\frac{4\alpha^2}{\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\delta\lambda+l}{1+l}\right)\right]^2}.$$

This completes the proof of Theorem 2.1.

## 3. COEFFICIENT BOUND FOR THE FUNCTION CLASS $B_\Sigma^\lambda(\delta,n,\beta,l)$

**Definition 3.1** *A function $f(z)$ given by (1.1) is said to be in the class $B_\Sigma^\lambda(\delta,n,\beta,l)$ if the following conditions are satisfied:*

$$f\in\Sigma and Re[\frac{(1-\delta)I^n(\lambda,l)f(z)+\delta I^{n+1}(\lambda,l)f(z)}{z}]>\beta \tag{3.1}$$

$$(0<\beta\le 1),\lambda\ge 0,\delta\ge 1,l\ge 0,n\in N_0,z\in U)$$

$$f\in\sum \quad and Re[\frac{(1-\delta)I^n(\lambda,l)g(w)+\delta I^{n+1}(\lambda,l)g(w)}{w}]>\beta \tag{3.2}$$

$(0<\beta\le 1),\lambda\ge 0,\delta\ge 1,l\ge 0,n\in N_0,w\in U)$,

where$g$ is given by (1.4).

**Theorem 3.1** *Let $f(z)$ given by (1.1) be in the class $B_\Sigma^\lambda(\delta,n,\beta,l)$ then*

$$|a_2|\le\sqrt{\frac{2(1-\beta)}{\left(\frac{1+2\lambda+l}{1+l}\right)^n\left[\frac{1+2\delta\lambda+l}{1+l}\right]}}$$

and

$$|a_3|\le\frac{4(1-\beta)^2}{\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\lambda\delta+l}{1+l}\right)\right]^2}+\frac{2(1-\beta)}{\left(\frac{1+2\lambda+l}{1+l}\right)^n\left[\frac{1+2\lambda\delta+l}{1+l}\right]}.$$

**Proof:** It follows from (3.1) and (3.2) that there exist $p(z)$ and $q(z)$ in $P$ such that

$$\frac{(1-\delta)I^n(\lambda,l)f(z)+\delta I^{n+1}(\lambda,l)f(z)}{z}=\beta+(1-\beta)p(z) \tag{3.3}$$

and

$$\frac{(1-\delta)I^n(\lambda,l)g(w)+\delta I^{n+1}(\lambda,l)g(w)}{w}=\beta+(1-\beta)q(z) \tag{3.4}$$

Clearly,

$$\beta+(1-\beta)p(z)=1+(1-\beta)p_1z+(1-\beta)p_2z^2+\ldots$$

$$\beta+(1-\beta)q(w)=1+(1-\beta)q_1w+(1-\beta)q_2w^2+\ldots$$

Now equating coefficients in (3.3) and (3.4) , we get

$$\left(\frac{1+\lambda+l}{1+l}\right)^n\left[\frac{1+\delta\lambda+l}{1+l}\right]a_2=(1-\beta)p_1 \tag{3.5}$$

$$\left(\frac{1+2\lambda+l}{1+l}\right)^n\left[\frac{1+2\delta\lambda+l}{1+l}\right]a_3=(1-\beta)p_2 \tag{3.6}$$

$$\left(\frac{1+\lambda+l}{1+l}\right)^n\left[\frac{1+\delta\lambda+l}{1+l}\right]a_2=-(1-\beta)q_1 \tag{3.7}$$

$$\left(\frac{1+2\lambda+l}{1+l}\right)^n\left[\frac{1+2\delta\lambda+l}{1+l}\right](2a^2-a_3)=(1-\beta)q_2 \tag{3.8}$$

From (3.5) and (3.7) , we have

$$p_1=-q_1 \tag{3.9}$$

Squaring and adding (3.5) and (3.7) , we get

$$[\left(\frac{1+\lambda+l}{1+l}\right)^n\left[\frac{1+\delta\lambda+l}{1+l}\right]]^2 2a_2^2=(1-\beta)^2(p_1^2+q_1^2) \tag{3.10}$$

Now adding equation (3.6) and (3.8) , we get

$$\left(\frac{1+2\lambda+l}{1+l}\right)^n\left[\frac{1+2\delta\lambda+l}{1+l}\right]2a_2^2=(1-\beta)(p_2+q_2) \tag{3.11}$$

$$a_2^2=\frac{(1-\beta)(p_2+q_2)}{2\left(\frac{1+\lambda+l}{1+l}\right)^n\left[\frac{1+\delta\lambda+l}{1+l}\right]}.$$

By applying lemma (1.1) for the coefficients $p_2$and $q_2$ , we obtain

$$|a_2|\le\sqrt{\frac{2(1-\beta)}{\left(\frac{1+2\lambda+l}{1+l}\right)^n\left[\frac{1+2\delta\lambda+l}{1+l}\right]}}$$

Which is bound on $|a_2|$.

In order to find $|a_3|$ by subtracting equation (3.8) from (3.6) , we get

$$2\left(\frac{1+2\lambda+l}{1+l}\right)^n\left[\frac{1+2\delta\lambda+l}{1+l}\right](a_3-a_2^2)=(1-\beta)(p_2-q_2) \tag{3.12}$$

$$a_3=a_2^2+\frac{(1-\beta)(p_2-q_2)}{2\left(\frac{1+2\lambda+l}{1+l}\right)^n\left[\frac{1+2\delta\lambda+l}{1+l}\right]}.$$

Using equation (3.10) ,we obtain

$$a_3=\frac{(1-\beta)^2(p_1^2+q_1^2)}{2\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\delta\lambda+l}{1+l}\right)\right]^2}+\frac{(1-\beta)(p_2-q_2)}{2\left(\frac{1+2\lambda+l}{1+l}\right)^n\left(\frac{1+2\delta\lambda+l}{1+l}\right)}.$$

Applying once again Lemma (1.1), we obtain

$$|a_3| \leq \frac{4(1-\beta)^2}{\left[\left(\frac{1+\lambda+l}{1+l}\right)^n\left(\frac{1+\delta\lambda+l}{1+l}\right)\right]^2} + \frac{2(1-\beta)}{\left(\frac{1+2\lambda+l}{1+l}\right)^n\left(\frac{1+2\delta\lambda+l}{1+l}\right)}.$$

This completes the proof of Theorem 3.1.

**Remarks:** On specializing the parameters, we obtain well known results as follows.

1. If we put $l = 0, \lambda = 1$ in Theorems 2.1 and 3.1, we obtain the corresponding results due to Porwal and Darus [9].
2. If we put $n = 0, l = 0, \lambda = 1$ In Theorems 2.1 and 3.1, we obtain the corresponding results due to Frasin and Aouf [7].
3. If we put $n = 0, \lambda = 1, l = 1, \delta = 1$ In Theorems 2.1 and 3.1, we obtain the corresponding results due to Srivastava et al.[10].

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# ANALYSIS OF A STOCHASTIC MODEL OF A STANDBY SYSTEM WITH SUBSTANDARD UNIT AND CORRELATED LIFE TIME

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**ABSTRACT**

***This model deals with a stochastic model of two dissimilar units, one is of good quality and other substandard one. The substandard unit may be repaired or may be replaced by the other substandard unit depends upon the cost of replacement. If a unit becomes operative, it remains operative till it fails. On failure of a unit, a repairman comes immediately to repair it and lifetimes are correlated. Various measures of performances of the system are obtained by Semi-Markov process and Regenerative point techniques. Numerical results and graphs pertaining to a particular case also included.***

***Key words:- Stochastic process, Regenerative point, Semi-Markov process***

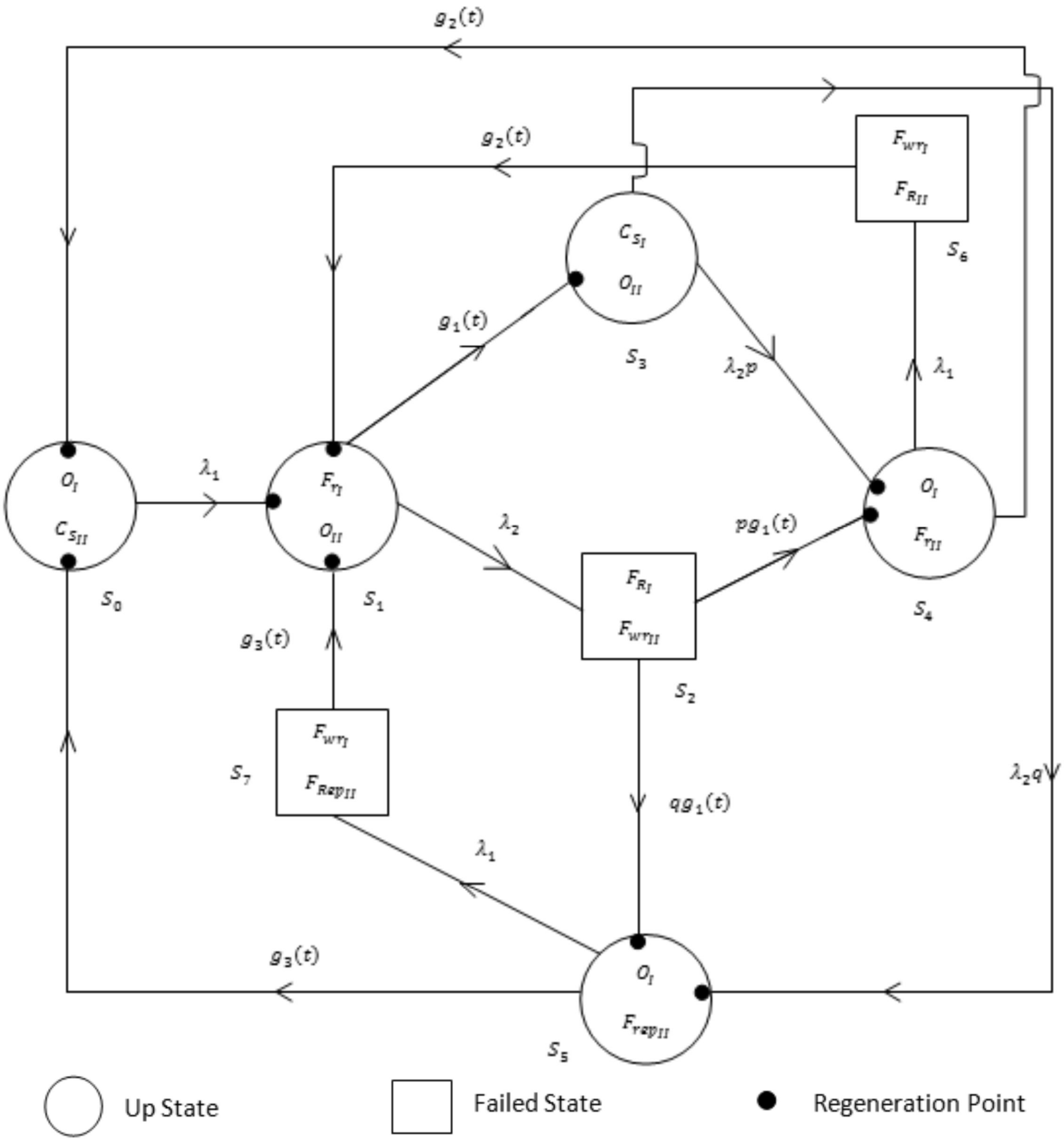
## 1. INTRODUCTION:

Many researchers including Garg and Kumar [1], Goel et al [5], Taneja et al [6], Kumar and Bhatia [11], Kumar and Kumar [13] etc. have contributed many fold in the area of Reliability modeling. A large number of researchers have discussed many models under various assumptions including the assumption that if a unit fails, it has to repaired/replaced. Various concepts under different situations have been analyzed. In this paper we deal with a stochastic model of two dissimilar units, one of which is good quality and other is substandard one. The substandard unit may be repaired or replaced depending upon the cost of replacement are higher or less than that of repair. Assuming the condition that if a unit becomes operative, it will remain operative till it fails. On the failure of a unit, a repairman comes immediately to repair or replace it. With the assumption that failure and repair /replacement are correlated to each other and joint distribution of failure and repair / replacement times is taken as bi-variate exponential. This model is analyzed by making use of semi-Markov process and regenerating point technique.

## DESCRIPTION OF MODEL AND ASSUMPTIONS :-

1. The system consists of two dissimilar units. Initially one unit is operative and other is cold standby.
2. Upon failure of an operative unit, the cold standby unit becomes operative instantaneously and failed unit goes under repair.

3. If a unit is under repair, it does not work for the system.
4. When both the unit fails, the system becomes inoperable.
5. The system is good as new after each repair and replacement.
6. Single service man facility is provided to the system for inspection, repair and replacement of the components.
7. Time distributions of various failures are exponential.



**NOTATION:-**

$\lambda_1 / \lambda_2$ = constant failure rate of first and second unit

p = probability that unit 2 is repairable on failure

q = probability that unit 2 is not repairable on failure

$G_1$ (t), $g_1$ (t) = c.d.f and p.d.f of the repair time of unit 1

$G_2$ (t), $g_2$ (t) = c.d.f and p.d.f of the repair time of unit 2

$G_3$ (t), $g_3$ (t) = c.d.f and p.d.f of the replacement time of unit 2

F (x, y) = Joint pdf of (x,y)

$= \mu\lambda \ (1\text{-}r)\, e^{-\lambda x - \mu y}\, l_o(2\sqrt{\lambda\mu r x y})$

Where x = a random variable denoting failure time of a unit

y = a random variable denoting repair time of a unit

r = Corr (x, y) , $\mu, \lambda, x, y > 0$ and $|r| < 1$

$l_o(z) = \sum (z/2)^{2k} / (k\,!)^2$

**SYMBOLS FOR THE STATES OF THE SYSTEM:-**

$S_i$ = State number i, i = 1,2,3,4,5,6,7

$O_I$, $O_{II}$ = Operating state of first and second unit respectively.

$C_{SI}$, $C_{SII}$ = Cold standby state of first and second unit respectively.

$F_{rI}$, $F_{rII}$ = Repair state of first and second unit respectively.

$F_{RI}$, $F_{RII}$ = Repair is continuing from previous state of first and second unit respectively.

$F_{wrI}$,$F_{wrII}$ = waiting for repair of first and second unit respectively.

$F_{repII}$ = replacement of second unit

$F_{RepII}$ = replacement of second unit from previous state model.

**TRANSITION PROBABILITIES:-**

$q_{0,1}\ (t) = [\,1 - e^{-\lambda 1(1-r1)t}]$

$$q_{1,3}\,/x = \ g_1 e^{-\lambda_1 r_1 x} \sum_{J=0}^{\infty} \frac{(\lambda_1\, g_1\, r_1 x)^j}{|\, j\,|^2} \int_0^t e^{-g_1 v}\, e^{-\lambda_2(1-r_2)v} v^j\, dv$$

$$q_{1,2}/x = \ g_1 e^{-\lambda_1 r_1 x} \sum_{J=0}^{\infty} \frac{(\lambda_1\, g_1\, r_1 x)^j}{|\, j\,|^2} \int_0^t e^{-g_1 v}\ (1 - \ e^{-\lambda_2(1-r_2)v}\,) v^j\, dv$$

$$q_{1,4}^2\,/\,x = \ p g_1 e^{-\lambda_1 r_1 x} \sum_{J=0}^{\infty} \frac{(\lambda_1\, g_1\, r_1 x)^j}{|\, j\,|^2} \int_0^t e^{-g_1 v}\ (1 - \ e^{-[g_1+\lambda_2(1-r_2)v]}\,) v^j\, dv$$

$$q_{1,5}^2 \,/\, x = \; q g_1 e^{-\lambda_1 r_1 x} \sum_{J=0}^{\infty} \frac{(\lambda_1 \, g_1 \, r_1 x)^j}{|\, j \,|^2} \int_0^t e^{-g_1 v} \;(1 - \; e^{-[g_1+\lambda_2(1-r_2)v]}\;) v^j \, dv$$

$q_{3,4} \,/x =$p$\int_0^t \lambda_2(1-r_2)\, e^{-\lambda_2(1-r_2)u}$ du

$q_{3,5} \,/x =$q$\lambda_2(1-r_2) \int_0^t \; e^{-\lambda_2(1-r_2)u}$ du

$$q_{4,6}/x = \; g_2 e^{-\lambda_2 r_2 x} \sum_{J=0}^{\infty} \frac{(\lambda_2 \, g_2 \, r_2 x)^j}{|\, j \,|^2} \int_0^t (1 - \; e^{-\lambda_1(1-r_1)v}\;) e^{-g_2 v} v^j \, dv$$

$$q_{4,0}/x = \; g_2 e^{-\lambda_2 r_2 x} \sum_{J=0}^{\infty} \frac{(\lambda_2 \, g_2 \, r_2 x)^j}{|\, j \,|^2} \int_0^t \left(e^{-[g_2+\lambda_1(1-r_1)v]}\right) v^j \, dv$$

$$q_{4,1}^6 \,/\, x = g_2 e^{-\lambda_2 r_2 x} \sum_{J=0}^{\infty} \frac{(\lambda_2 \, g_2 \, r_2 x)^j}{|\, j \,|^2} \int_0^t \left(e^{-[g_2+\lambda_1(1-r_1)v]}\right) v^j \, dv$$

$$q_{5,0}/x = \; g_3 e^{-\lambda_1 r_3 x} \sum_{J=0}^{\infty} \frac{(\lambda_1 \, g_3 \, r_3 x)^j}{|\, j \,|^2} \int_0^t \left(e^{-[g_3+\lambda_1(1-r_1)v]}\right) v^j \, dv$$

$$q_{5,1}^7 \,/\, x = g_3 e^{-\lambda_1 r_3 x} \sum_{J=0}^{\infty} \frac{(\lambda_1 \, g_3 \, r_3 x)^j}{|\, j \,|^2} \int_0^t \left(e^{-[g_3+\lambda_1(1-r_1)v]}\right) v^j \, dv$$

**Conditional Transitional Probability :-**

$P_{0,1} = 1$

$P_{1,2}/x = [\, 1 - g_1^{'} \, e^{-\lambda_1 r_1 (1-g_1^{'})x}]$

$P_{1,3}/x = g_1^{'} e^{-\lambda_1 r_1 (1-g_1^{'})x}$

$P_{1,4}^2 \,/x =$ p$[1 - g_1^{'} \, e^{-\lambda_1 r_1 (1-g_1^{'})x}]$

$P_{1,5}^2 \,/x =$ q$[1 - g_1^{'} \, e^{-\lambda_1 r_1 (1-g_1^{'})x}]$

$$P_{3,4}/x = p$$

$$P_{3,5}/x = q$$

$P_{4,6}/x = [\, 1 - g_2^{'} \, e^{-\lambda_2 r_2 (1-g_2^{'})x}]$

$P_{4,0}/x = g_2^{'} \, e^{-\lambda_2 r_2 (1-g_2^{'})x}$

$$P_{4,1}^6/x = 1 - g_2^{'} \, e^{-\lambda_2 r_2 (1-g_2^{'})x}$$

$P_{5,0}/x = g_3^{'} \, e^{-\lambda_1 r_3 (1-g_3^{'})x}$

$$P_{5,1}^7/x = 1 - g_3^{'} \, e^{-\lambda_1 r_3 (1-g_3^{'})x}$$

**Transitional Probabilities are:-**

$P_{0,1} = 1$

$P_{1,2} = 1 - \frac{g_1^{'}(1-r_1)}{1-r_1 g_1^{'}}$

$P_{1,3} = \frac{g_1^{'}(1-r_1)}{1-r_1 g_1^{'}}$

$P_{4,0} = \frac{g_2^{'}(1-r_2)}{1-r_2 g_2^{'}}$

$P_{4,6} = 1 - \frac{g_2^{'}(1-r_2)}{1-r_2 g_2^{'}}$

$P_{4,1}^6 = 1 - \frac{g_2^{'}(1-r_2)}{1-r_2 g_2^{'}}$

$P_{1,4}^{2} = p[1 - \frac{g_1^{'}(1-r_1)}{1-r_1 g_1^{'}}]$ $\quad$ $P_{1,5}^{2} = q[1 - \frac{g_1^{'}(1-r_1)}{1-r_1 g_1^{'}}]$

$P_{4,0} = \frac{g_3^{'}(1-r_3)}{1-r_3 g_3^{'}}$ $\quad$ $P_{5,1}^{7} = 1 - \frac{g_3^{'}(1-r_3)}{1-r_3 g_3^{'}}$

Now

i. $P_{0,1} = 1$

ii. $P_{1,2} + P_{1,3} = 1$

iii. $P_{3,4} + P_{3,5} = 1$

iv. $P_{1,3} + P_{1,4}^{2} + P_{1,5}^{2} = 1$

v. $P_{4,0} + P_{4,6} = 1$

vi. $P_{4,0} + P_{4,1}^{6} = 1$

vii. $P_{5,0} + P_{5,1}^{7} = 1$

$$\mu_0 = \frac{1}{\lambda_1(1-r_1)}$$

$$\mu_1 = \frac{1}{\lambda_2(1-r_2)}[1 - \frac{g_1^{'}(1-r_1)}{1-r_1 g_1^{'}}]$$

$$\mu_3 = \frac{1}{\lambda_2(1-r_2)}$$

$$\mu_4 = \frac{1}{\lambda_1(1-r_1)}[1 - \frac{g_2^{'}(1-r_2)}{1-r_2 g_2^{'}}]$$

$$\mu_5 = \frac{1}{\lambda_3(1-r_3)}[1 - \frac{g_3^{'}(1-r_3)}{1-r_3 g_3^{'}}]$$

**The unconditional mean time taken by the system to translate from any regenerative state J, it is counted from the epoch of entrance into the state i**

$m_{0,1} = \mu_0$

$m_{1,2} + m_{1,3} = \mu_1$

$m_{1,3} + m_{1,4}^{2} + m_{1,5}^{2} = k_1$

$m_{3,4} + m_{3,5} = \mu_3$

$m_{4,0} + m_{4,6} = \mu_4$

$$m_{5,0} + m_{5,1}^{7} = k_3$$

$m_{4,0} + m_{4,1}^{6} = k_2$

**ANALYSIS OF CHARACTERISTICS:-**

**MTSF (Mean time to system failure):-** To determine the MTSF of the system, we regard the failed state of the system as absorbing state, by probabilistic argument, we get

$${}_0(t) = Q_{01}(t)©_1(t)$$

$${}_1(t) = Q_{13}(t)©_3(t) + Q_{12}(t)$$

$$_3(t) = Q_{34}(t)©_4(t) + Q_{35}(t)©_5(t)$$

$$_4(t) = Q_{40}(t)©_0(t) + Q_{46}(t)$$

$$_5(t) = Q_{50}(t)©_0(t) + Q_{57}(t)$$

Taking Laplace Stieltjes transforms of these relations and solving for $_0^{**}(s)$

$$_0^{**}(s) = \frac{N(s)}{D(s)}$$

Where N(s) = $\mu_0 + \mu_1 + p_{13}[\mu_3 + p_{34}\mu_4 + p_{35}\mu_5]$

D(s) = 1-$p_{13}[p_{34}p_{40} + p_{35}p_{50}]$

**Availability Analysis:-**

Let $A_i$ (t) be the probability that the system is in upstate at instant t given that the system entered regenerative state i at t=0. Using the arguments of the theory of a regenerative process, the point wise availability $A_i$ (t) is seen to satisfy the following recursive relations:-

$$A_0(t) = M_0(t) + q_{01}(t)©A_1(t)$$

$$A_1(t) = M_1(t) + q_{13}(t)©A_3(t) + q_{14}^2(t)©A_4(t) + q_{15}^2(t)©A_5(t)$$

$$A_3(t) = M_3(t) + q_{34}(t)©A_4(t) + q_{35}(t)©A_5(t)$$

$$A_4(t) = M_4(t) + q_{40}(t)©A_0(t) + q_{41}^6(t)©A_1(t)$$

$$A_5(t) = M_5(t) + q_{50}(t)©A_0(t) + q_{51}^7(t)©A_1(t)$$

Where

$M_0(t) = e^{-\lambda_1(1-r_1)t}$ $\quad M_1(t) = e^{-\lambda_2(1-r_2)t}G_1(t)$

$M_3(t) = e^{-\lambda_2(1-r_2)t}$ $\quad M_4(t) = e^{-\lambda_1(1-r_1)t}G_2(t)$

$M_5(t) = e^{-\lambda_1(1-r_3)t}G_3(t)$

Now taking Laplace Transform of these equations and solving for $A_0^*(s)$, we get

$$A_0^*(s) = \frac{N_1(s)}{D_1(s)}$$

The steady state availability is $A_0 = \lim_{s=0}[sA_0^*(s)] = \frac{N_1}{D_1}$

Where

$N_1 =$

$\mu_0[1 - p_{15}^2p_{51}^7 - p_{14}^2p_{41}^6 - p_{13}(p_{34}p_{41}^6 + p_{35}p_{51}^7)] + \mu_1 + p_{13}\mu_3 +$
$(p_{14}^2 +$ $p_{13}p_{34})\mu_4 + (p_{15}^2 + p_{13}p_{35})\mu_5$

$D_1 =$

$K_1 + p_{13}\mu_3 + (K_2 + p_{40}\mu_0)(p_{14}^2 + p_{13}p_{34})$ +$(K_3 + p_{50}\mu_0)(p_{15}^2 +$ $p_{13}p_{35})$

**Busy Period Analysis of the Repairman:-**

Let $B_i(t)$ be the probability that the repairman is busy at instant t, given that the system entered in regenerative state i at t=0. By probabilistic arguments, we have the following recursive relations for $B_i(t)$

$B_0(t) = q_{01}(t) ©B_1(t)$

$B_1(t) = W_1(t) + q_{13}(t)©B_3(t) + q_{14}^2(t)©B_4(t) + q_{15}^2(t)©B_5(t)$

$B_3(t) = q_{34}(t)©B_4(t) + q_{35}(t)©B_5(t)$

$B_4(t) = W_4(t) + q_{40}(t)©B_0(t) + q_{41}^6(t)©B_1(t)$

$B_5(t) = q_{50}(t)©B_0(t) + q_{51}^7(t)©B_1(t)$

Where $W_1(t) = e^{-\lambda_2(1-r_2)t}G_1(t) + \left[1 - e^{-\lambda_2(1-r_2)t}\right]G_1(t) = G_1(t)$

$W_4(t) = G_2(t)$

Taking Laplace Transform of the equations of busy period analysis and solving them for $B_0^*(s),$ we get

$B_0^*(s) = \frac{N_2(s)}{D_1(s)}$

In steady state

$$B_0 = \lim_{s=0}[sB_0^*(s)] = \frac{N_2}{D_1}$$

Where $N_2 = K_1 + K_2[p_{13} + p_{14}^2 p_{34}]$

$D_1$is already specified.

**Expected Numbers of visits by the Repairman:-**

We defined as the expected number of visits by the repairman in (0, t), given that the system initially starts from regeneration state $S_i$

By probabilistic arguments, we have the following recursive relation for $V_i(t)$

$V_0(t) = Q_{01}(t)©[1 + V_1(t)]$

$V_1(t) = Q_{13}(t)©V_3(t) + Q_{14}^2(t)©V_4(t) + Q_{15}^2(t)©V_5(t)$

$V_3(t) = Q_{34}(t)©[1 + V_4(t)] + Q_{35}(t)©V_5(t)$

$V_4(t) = Q_{40}(t)©V_0(t) + Q_{41}^6(t)©V_1(t)$

$V_5(t) = Q_{50}(t)©V_0(t) + Q_{51}^7(t)©V_1(t)$

Taking Laplace stieltjes transform of these equations of expected number of visits and solving them for $V_0^{**}(s)$, we get

$$V_0 = \lim_{s=0}[sV_0^*(s)] = \frac{N_3}{D_1}$$

Where $N_1 = 1 - p_{15}^2 p_{51}^7 - p_{14}^2 p_{41}^6 + p_{13}p_{34}p_{40} - p_{13}p_{35}p_{51}^7$

And $D_1$ is already specified.

**Expected Number of Replacements in the System:-**

Let $R_i(t)$ be the expected number of replacement in (0,t), given that the system started from the regenerative state i at t=0. By probabilistic arguments, we have the following recursive statements:-

$R_0(t) = Q_{01}(t)©R_1(t)$

$R_1(t) = Q_{13}(t)©R_3(t) + Q_{15}^2(t)©[1 + R_5(t)] + Q_{14}^2(t)©R_4(t)$

$R_3(t) = Q_{34}(t)©R_4(t) + Q_{35}(t)©[1 + R_5(t)]$

$R_4(t) = Q_{40}(t)©R_0(t) + Q_{41}^6(t)©R_1(t)$

$R_5(t) = Q_{50}(t)©R_0(t) + Q_{51}^7(t)©R_1(t)$

Taking Laplace stieltjes transform of these equations and solving them for$R_0^{**}(s)$, we get

$$R_0^{**}(s) = \frac{N_4(s)}{D_1(s)}$$

In steady state

$$R_0 = \lim_{s=0}[sR_0^{**}(s)] = \frac{N_4}{D_1}$$

Where $N_4 = p_{15}^2 + p_{13}p_{35}$ and $D_1$ is already specified.

**COST-BENEFIT ANALYSIS:-**

The expected total profit incurred to the system in steady state is given by

$P_2 = C_0A_0 - C_1B_0 - C_2V_0 - C_3R_0$

$C_0$ =Revenue per unit up time of the system.

$C_1$ = Cost per unit for which the repairman is busy.

$C_2$ = Cost per visit of the repairman.

$C_3$ = Cost per replacement in the system.

**MTSF Vs Failure rate for different value of repair rate and Correlation Coefficient:**

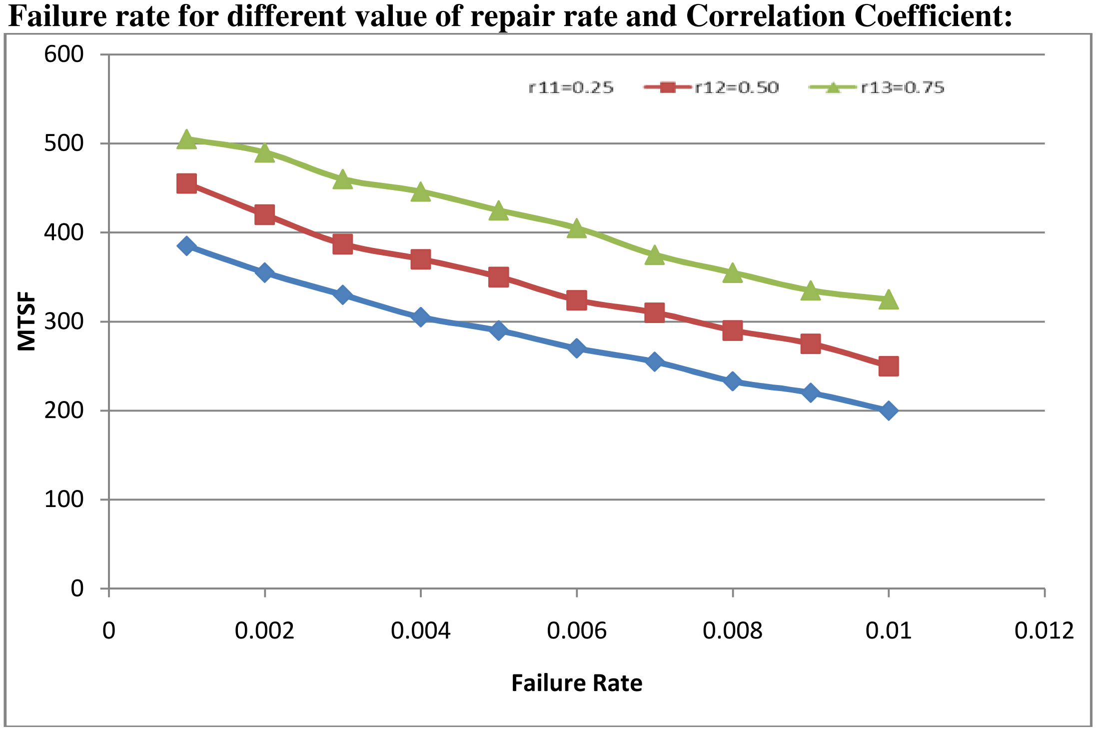



**Fig. 1 :MTSF VsFailure rate**

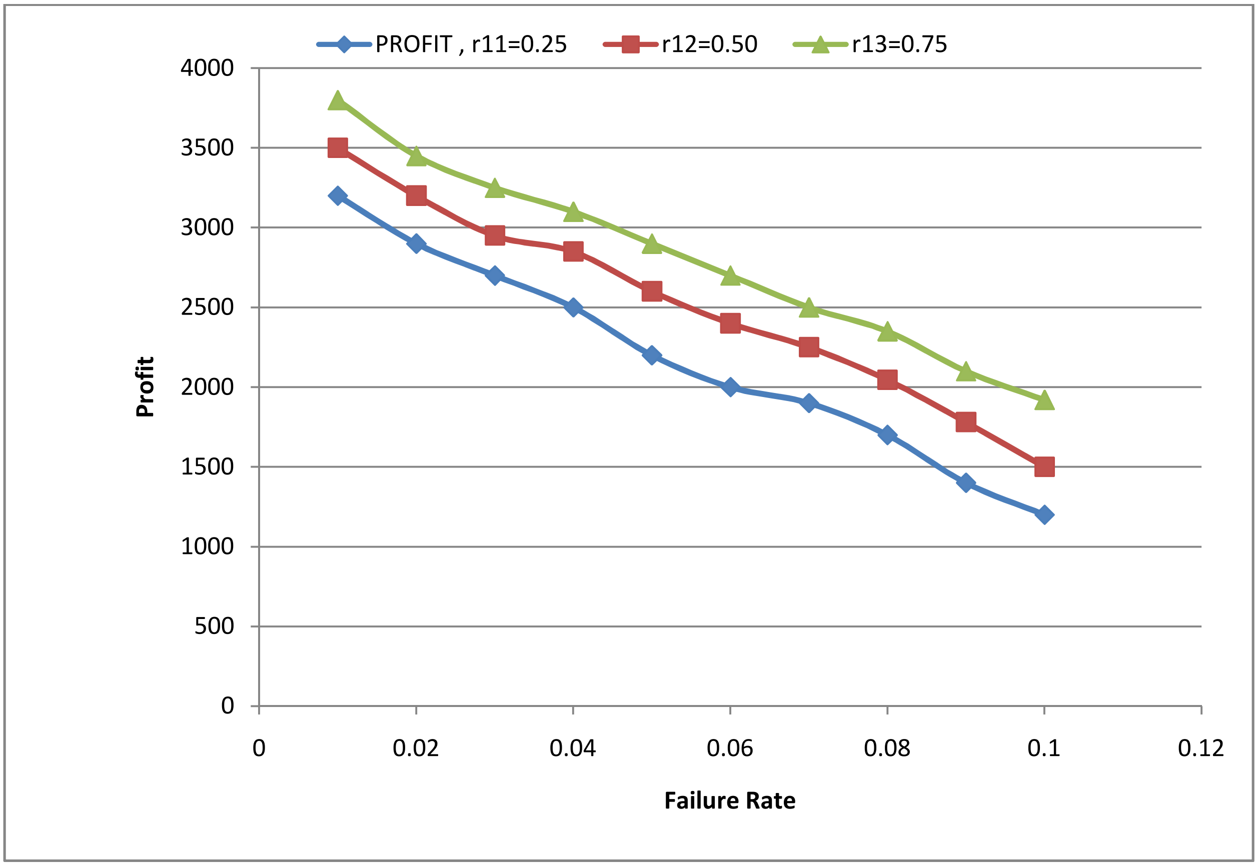



**Fig. 2 : Profit Vs Failure rate**

**CONCLUSION :**

For a more clear view of the system characteristics w.r.t. various parameters involved, we plots the curve for MTSF vs failure rate and it was observed that MTSF decreases as failure rate increases. We also observe that as coefficient of correlation increases expected life of the system also increases(fig1).From (fig2) we observe that profit decreases as failure rate increases also observe that higher for higher correlation. So we concluded that high correlation between failure and repair of the system yields the better performance of the system.

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# A NOVEL FUZZY APPROACH FOR ENHANCEMENT OF UNEVEN ILLUMINATED IMAGES

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**ABSTRACT**

***Uneven illumination occurs due to the non-uniform lighting directions so the captured images have some dark areas as well as bright areas. The darker areas must be brightened and bright areas should be a little darkened to make the image uniform. The color image is converted into RGB luminance. RGB Luminance is converted into the fuzzy plane and the fuzzy entropy is used to quantify the level of enhancement. The RGB color channels are modified as per the modification ratio of RGB luminance. The proposed scheme is compared with the methods which work in the spatial domain as well as in the fuzzy domain. The supremacy of the proposed approach is proved by evaluating the visual image quality.***

***Keywords: Unevenly illuminated image, Color image, RGB Luminance, Fuzzy entropy, Fuzzy image enhancement.***

## 1. INTRODUCTION

Images captured by imaging devices are prone to uneven illumination due to different lighting sources or insufficient lighting. Unevenly illuminated images may also have low contrast and contain dark areas as well as bright areas. Contrast enhancement techniques are used to enhance the visual quality of the image by changing the intensity of the pixels.

Histogram equalization and its variations such as Adaptive Histogram Equalization (AHE), Dynamic Histogram Equalization (DHE), Brightness Preserving Dynamic Histogram Equalization (BPDHE), Contrast Limited Adaptive Histogram Equalization (CLAHE), etc. are popular among researchers because they are easy to implement and straightforward. These techniques were proposed for grayscale images and usually applied on all three channels of RGB images, which produces artifacts in the image [20]. Kong & Ibrahim (2008) applied the BPDHE approach on different channels such as green channel, Y channel of YCbCr model, V channel of HSV color space, etc. for color images by preserving color information, so the color image needs to be converted into one channel image for the enhancement.

The second approach is to use the retinex theory wherein the grayscale image is considered as the blend of reflectance and illumination. Ng & Wang (2011), Wang et al. (2013), and Gupta & Agarwal (2017) proposed different techniques using retinex theory for the enhancement of non-uniformly illuminated color images. Retinex algorithms are also susceptible to artifacts like halos, local blur, and over-enhancement.

The concept of fuzzy set theory was presented by Zadeh (1965) and used by Pal & King (1980) in the enhancement of grayscale image because the pixel intensities present in the image are uncertain and ambiguity/uncertainty is best dealt by it. The INT operator was used by [2] to change the membership values and, as a result, improve the image contrast. Later, [4-7, 9, 12-14, 16, 18, 21] used different fuzzy operators to modify the membership values, and the parameters used are optimized by different optimization methods. Other approaches such as fuzzy filtering, fuzzy relaxation, fuzzy logic, etc. are also used by researchers in the past. Hasikin & Isa (2014) and Hanmandlu et al. (2016) focused on the issue of non-uniform illumination using fuzzy image enhancement.

As the Histogram Equalization techniques are very easy and popular among the researchers, Sheet et al. (2010) proposed BPDFHE which applies brightness preserving dynamic histogram equalization technique on fuzzy histogram computed using triangular membership function. Later, [17, 22] also applied variations of histogram equalization technique on the fuzzy histogram. These techniques are better than HE techniques but still have the disadvantages of HE techniques.

The proposed approach used the RGB Luminance channel for the conversion of a color image into one channel image because of less computationally intensive and cubic function for the modification of membership values using the maximum entropy principle. The color image is reconstructed based on the proportional change in Luminance. This paper consists of four sections. Section 2 describes the proposed method with RGB Luminance enhancement and color image reconstruction. Experiment and results are provided in Section 3. Section 4 presents the conclusion of the paper.

## 2. PROPOSED METHOD

The proposed method includes three subsections: Contrast Stretching, RGB Luminance Enhancement, and Color Image Reconstruction. Usually, the RGB image has a small range of pixel intensities, so contrast stretching is performed to spread the intensities in the whole dynamic range. Let I be RGB color image with global minimum $I_{min}$ and global maximum $I_{max}$ among all three color channels. The expanded image I' can be computed as follows:

$$I'_c(i,j) = (G-1)\frac{I_c(i,j) - I_{min}}{I_{max} - I_{min}} \quad (1)$$

Where, c represents the different color channels of image I with intensity levels G for every channel. After getting the expanded color image, it would be converted into one channel, RGB Luminance image L as:

$$L = 0.299 \times R + 0.587 \times G + 0.114 \times B \quad (2)$$

Where, R, G, and B respectively represent the red, green, and blue channels in the image. RGB Luminance image L is enhanced using fuzzy image enhancement and then RGB color image is reconstructed based on the changes made in RGB Luminance. This is explained in the forthcoming subsections.

### 2.1 Fuzzification, Defuzzification

The image L of size M x N with the intensity levels G is fuzzified using membership function associated with pixel intensities of the image. A very simple function is used for fuzzification:

$$\mu = \frac{L - L_{min}}{L_{max} - L_{min}} \tag{3}$$

The membership values are modified to enhance the luminance based on some criteria. Using the modified membership values, the image is defuzzified or converted back to spatial plane from the fuzzy plane using the following function:

$$L' = (G - 1)\mu' \tag{4}$$

### 2.2 RGB Luminance Enhancement

For the contrast enhancement of unevenly illuminated images, the dark area should be made brighter and the brighter area should be made a little darker. The image is divided into two areas. So the pixel intensities in the darker area should increase and in the bright area should decrease. The membership enhancement curve is shown in figure 1. The image is split into two parts at membership value T and it is calculated as:

$$T = (1 - \mu_{mean}) \tag{5}$$

$$\mu_{mean} = \frac{\sum\sum \mu\,(i,j)}{MN} \tag{6}$$

The image is divided into two parts - the dark area with μ≤T and the bright area with the membership values μ>T.

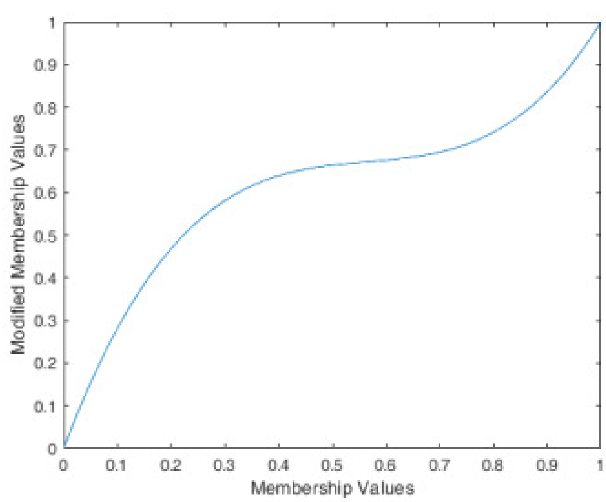



**Figure 1: Curve used for enhancement**

The following equation is used for the curve shown in figure 1:

$$\mu' = B_1\mu^3 + B_2\mu^2 + B_3\mu + B_4 \quad (7)$$

The above equation is non-linearly fitted using the nlinfit() function of MATLAB. It requires four points to fit this equation. The first and last points would be (0, 0) and (1, 1) because the desired curve is covering the whole dynamic range. The third point would be (T, T) because the membership value of the pixels in the dark area will be up to T and in the bright area will start from T. Fourth point would be based on a parameter α and defined as:

$$(x_4, y_4) = \begin{cases} (0.1, \alpha), & T \geq 0.5 \\ (0.9, 1-\alpha), & T < 0.5 \end{cases} \quad (8)$$

The maximum Fuzzy entropy principle is used to optimize the parameter α, whose value is in the range 0-1. The Shannon entropy function is used to calculate the fuzzy entropy. The proposed scheme for the enhancement of RGB Luminance is as

- RGB Luminance (L) is converted into the fuzzy plane using equation 3.
- The value of α is optimized to modify the membership values using fuzzy entropy.
- The optimized modified membership value is calculated using equation 7.
- The Luminance channel is converted back to the spatial plane from the fuzzy plane using equation 4 and the output is enhanced RGB luminance (L').

### 2.3 Color Image Reconstruction

After getting enhanced RGB Luminance, the enhanced color image needs to be constructed. The pixel intensities in the enhanced-color image should not go out of the gamut range, so the following function is used to reconstruct the color image from enhanced RGB Luminance as given in [21]:

$$I_c''(i,j) = I_c'(i,j)\left[\frac{L'(i,j)}{L(i,j)}\right]^{1-\sqrt{\left(I_c'(i,j)\right)}} \quad (9)$$

## 3. EXPERIMENT AND RESULTS

The experiment is conducted in an Intel i5 laptop with 2.50 GHz speed and 4GB RAM on MATLAB R2017a software and applied on around 100 images from different databases. The proposed scheme is compared with Histogram Equalization (HE), BPDHE [8], CLAHE [3], NPEA [15], BGupta2017 [19], Pal1980 [2], BPDFHE [10], Hasikin2014 [16], Hanmandlu2016 [18]. The approaches belong to different categories - histogram modification, retinex, fuzzy histogram modification, and fuzzy image enhancement. The results of these approaches on color images are shown in figures 2-6.


**Figure 2: (a) Original Image; Enhanced Images by (b) HE, (c) BPDHE, (d) CLAHE, (e) NPEA, (f) BGupta2017, (g) Pal1980, (h) BPDFHE, (i) Hanamandlu2016 and (j) Proposed Method**

Figure 2(a) shows the image of a roadside with sky and clouds. The lower part of the image is dark and the upper part is bright. The image enhanced by HE is shown in figure 2(b) and doesn't have a clear sky. Images shown in figure 2(d, g, h, i) are darker on the road and the color of the sky is not appropriate. Figure 2(c) is an enhanced image by BPDHE and has staircase effects.

Image enhanced by the BGupta2017 approach is very bright at the sky and shown in figure 2(f). The image displayed in figure 2(e) is looking good but the parts on the left side of the image are not visually clear. The image enhanced by the proposed method is shown in figure 2(j) and all parts of the image are visually good in comparison to other images. The image shown in figure 3(a) is of a fort built in a hilly area with the sky. The enhanced images have the almost same type of effects as shown in the enhanced images of figure 2.

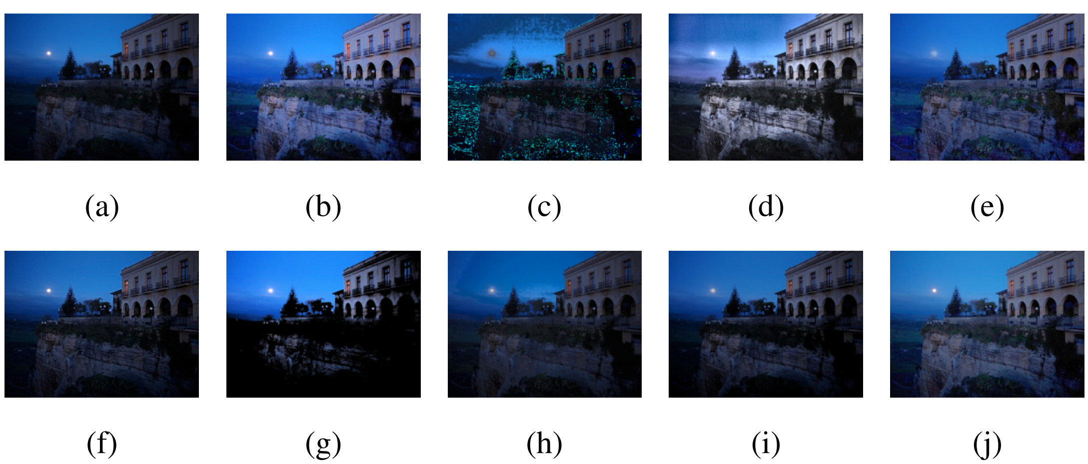


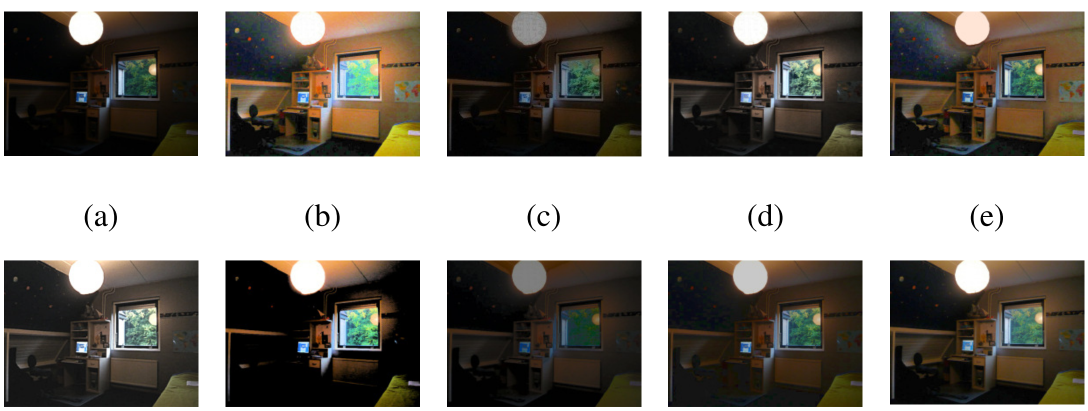

**Figure 3: (a) Original Image; Enhanced Images by (b) HE, (c) BPDHE, (d) CLAHE, (e) NPEA, (f) BGupta2017, (g) Pal1980, (h) BPDFHE, (i) Hanamandlu2016 and (j) Proposed Method**

(a) (b) (c) (d) (e)

(f) (g) (h) (i) (j)

**Figure 4: (a) Original Image; Enhanced Images by (b) HE, (c) BPDHE, (d) CLAHE, (e) NPEA, (f) BGupta2017, (g) Pal1980, (h) BPDFHE, (i) Hanamandlu2016 and (j) Proposed Method**

Figure 4 (a) shows the original image of a building near a water body with a dark sky. Upper part of the building is not properly visible due to the dark sky area. The area near the building is brightly illuminated. In figures 4 (b, e), the over enhancement can be seen in the sky; which is looking unnatural. Image enhanced by Pal1980 method become darker in dark areas. The color of the building seems to be fader than the original image in figure 4(d, f). The upper parts of the buildings in figure 4(c, h) become darker than the original image. The good visual quality of the image can be seen in figure 4(i, j).



(a) (b) (c) (d) (e)

(f) (g) (h) (i) (j)

**Figure 5: (a) Original Image; Enhanced Images by (b) HE, (c) BPDHE, (d) CLAHE, (e) NPEA, (f) BGupta2017, (g) Pal1980, (h) BPDFHE, (i) Hanamandlu2016 and (j) Proposed Method**

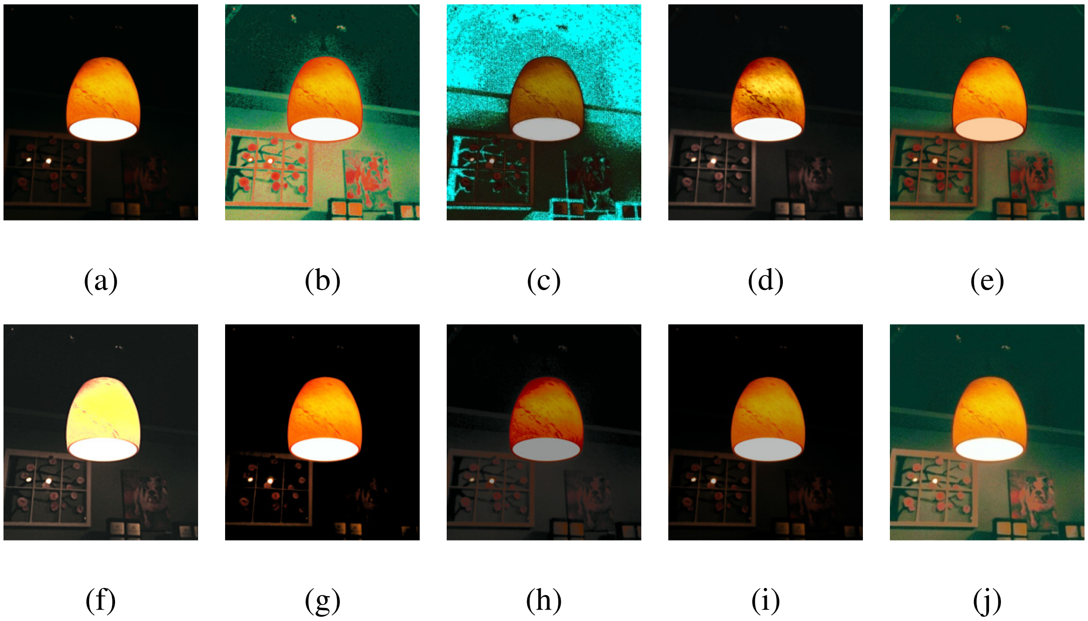



**Figure 6: (a) Original Image; Enhanced Images by (b) HE, (c) BPDHE, (d) CLAHE, (e) NPEA, (f) BGupta2017, (g) Pal1980, (h) BPDFHE, (i) Hanamandlu2016 and (j) Proposed Method**

Figure 5(a) contains the original image of a dark room with one light source. Images in Figure 5(c, d, g, i) are still dark on the left wall side. Figure 5(b) has a brighter image but due to over enhancement, the upper part of the image has white patches. The color of the light source in figure 5(e) becomes a little bit pink. The color of figure 5(f) has changed in comparison to the original image. The image shown in figure 5(j) has no color cast, white patches.

Figure 6(a) displays the original image of a dark room with a lamp at the center. Artifacts and the washing effect can be seen in figure 6(b, c, f, h). The bright area at the lamp turns darker in figure 6(d). The dark areas become darker in figure 6(g). Figures 6(e, j) are displaying good quality images.

## 4. CONCLUSION

Unevenly illuminated images consist of both dark and bright areas. Either increasing the intensities of the dark area or decreasing the intensities of the bright area doesn't work here. Both areas need to be altered for good visual quality. The proposed method enhanced the image by brightening the dark areas and lightening the bright area, which also increases the image contrast. The proposed scheme is compared with the approaches using the techniques - histogram modification, retinex, fuzzy image enhancement and fuzzy histogram modification. Results from the experiment have proved that the proposed method is superior to the state-of-the-art methods.

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# STOCHASTIC EPIDEMIC MODEL SEVERE ACUTE RESPIRATORY SYNDROME CORONOVIRUS 2 (SARS-COV-19)

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**ABSTRACT**

***Severe Acute Respiratory Syndrome Corono virus 2 (Sars-Cov 19) its origin, transmission, pathogenesis, mitigation by vaccines, the recapture and non paramcutical intervention its diagnosis, etiognosis and progonosis how disease evolves is statistically understanding using Poisson Birth Process, elementary Survival Data Analysis and Epidemiological parameters is reviewed.***

***Keywords: (I) Test positive rate (TPR) (ii) Case Fatality Rate (CFR), RT-PCR test, R0 Reproductive number, Diagnosis Protocol, Bath Tube Curve.***

## INTRODUCTION

### 1.1 Origin of SARS-Cov 2

According joint Who-China reported phenomena of a unknown cause in Wuhan on 30th December 2019, Sars related Corona Virus in a (horsehoe) bat 96.2% genome similarity is less caused in other epidemics such as MERS (Middle East Respiratory Syndrome), consists of 4 amino acids in its spike protein, that makes more efficient than the virus causing virus. According Kristian G. Andersan et. All

In Nature (March 17, 2020) the natural selection on a human or human like host appeared to facilitate in the spike protein of the novel corona virus with ACE2 receptor the genetic difference between the virus, while the RaTg13 in bats is extremely similar to SARS-Cov-2, the receptor binding domain (RBD) of the spike the RBD protein binds to the ACE2 receptor actually is divergent for the two with the former appearing less efficient

### 1.2 Molecular view of SARS-COV-2

Protein is a linear chain of linked amino acids, 20 each contains between 10 and 27 atoms, amino acids. Positive, Negative, Neutral, fe regions of the chain of amino acids interaction between proteins important for assembling a structure for binding the receptor to antibodies, codes such as E484K, D614G the number takes as basis to linear chain of amino acids in a protein. Which in spike protein initiate infection, it is attracted to and binds to a receptor molecules that lies on the surface of cell in our Lungs and other tissues. The protein molecule is a chain of 1,273 amino acids and three individual molecules locks together form the familiar 'Spike' shape. The 484 is the position in the chain it lies within birds to the host receptor E is shorthand for guletamate, an amino acids, with a negative change which is now mutate to K (Sysie) an amino acids with a positive changes this mutuationis found in the Beta and Gamma Variate, infectivity of this particular variant of the virus seems to be enhanced. It also appears to make this variant less recognizable to some antibodies generated against the virus. The Delta in change and polar. The second mutation is L452R, which is also lies with the receptor-binding of spike. L is levcine, an unchanged. 'Sticky' amino acid and R the positivity changed Arginince,

**1.3 Variants of Concern (VOC)**

The Alpha variant has a total of 23 mutations, 9 of these are on other parts of the spike protein, B.1.117, B.1.617, B.1.618 the changes of this sort a replacement of two in a molecular can be modelled in competitive environment different transmission efficiency can be modelled Statsitically.

Duy considering (i) transmission efficiency (ii) disease severity (iii) escape from immunity cover.

B.1.617.2 'Delta' variant of the corona virus, which has 12 mutations in its spike protein comparted to SARS-Cov-2, that appeared to the more ACE2 enzymes aiding the entry of corona viruses around these mutations L452R and T4784K that characterised the variant, the variant was also characterized by high transmissibility, an accelerated surge in infection, prior infection, high seropositivity and vaccination.

**1.4 Nomenclature Scheme**

(1) Phylogenetic Assignment of Global Outbreak Lineages (PAN-GOLIN), that uses a hierarchal system based on genetic relatedness an invaluable tool of genomic surveillance. It uses alphabets (A,B,C,P) and numerals starting with 1 variant lineages are at the emerging edge of the pandemic in different geographics.

(2) Lineage B

The Linage B is the most profit the variants in circulation are B.1; B.1.1; B.1.7; B.1.167; B,1.77; B.1.351, B1.427 and B.1.429,

(3) Lineage B

The lineage B is deviated from the original B.

The three most frequent ones are named by their geography of origin U.K.Variant B.1.1.7 (defected in 2020, September (UK), B.1.357 (detected in October 2020 (South Africa), P.1 (detected in December (Brazil), Wu.Hu.1 (Wuhan Virus) original pandemic variant, variant D614G emerged and become globally dominant 'Genetic Script' encoded in DNA or RNA are copied repeatedly for virus replication error do occur. RNA virus are more error prone than DNA viruses, SARS-Cov-2 genome is single stranded RNA and error is multiply in host cell, therefore studying the host genome is paramount to studying both susceptibility(s) and protection against the virus in a given population.

The studies published in NEJM, nature and MedRxiv, provide clues how certain host genome confer an increased risk of developing the severe disease SARS-Cov-2 is new in humanand as it spreads, mutations are frequent, emerging variants with higher transmission efficiency become dominant, tending to replace other variants in different geographic communities epidemic spreading widely. Studying the genomes of individual in particular genetic population group, a region on host chromosome 3 acts as a significant genetic risk factors towards getting seriously III and at the same time, a group of genes on chromosomes 6, 12, 19 and 21 protect us against virus.

**1.5 Indian Sars Cov-2 Genome Consortium (INSACOG)**

INSACOG, has conducted genome sequencing of over 15,000 sample, the proportion of the known variants of concern (VOC)'s namely the U.K, South Africa and Brazil variants was to be around 1% of the samples collected, which is about 1% of Indian corona virus case load, the Indian variants was that they possessed two mutations E484 and L245R – that together would increase the likelihood of wider range of antibodies being unable to counter the virus. In 27 instances of break through infection analysis, the scientists found that two linages dominated 8.1.617 (kappa) composed 8% Delta was 76% and the remaining to the variants that belonged to broader 8.1. Lineage. The new mutation in Delta called T478K that the scientist believe has a role to play in allowing the corona virus to better intrude human cells. The data indicates B.1.617.2 shows high transmissible and surges without increase in the case of case fatality rate (CFR).

**1.6 RT-PCR, Reverse Transcription Mediated PCR**

Amplification of double standard DNA can also be applied to produce double standard cDNA also, which is synthesized from mRNA using the enzyme reverse transcription, based on this principle RT-PCR was developed. RT-PCR includes a single application combining the process of cDWA synthesis (by transcription) and PCR amplification using this principle one can isolate very rare mRNA transcripts from cell through RT-PCR, the thermos table enzyme rTtn uses RNa template from cDNA synthesis and thus allows single enzyme RT-PCR viral reverse transciptse (RTase) from avianmmurine virus (AMV RTase) or mamaellian leukemia virus (MLV RTase) can also be used fro cDNA synthesis. Both the enzymes RTase and rTtn DNA polymerease are used alternatively in the RT-PCR.

RAPD makers have been successfully used for (i) construction of genetic maps e.g. Arabidopsis, plune, Hellanthus etc. (ii) Mapping of traits (iii) Analysis of genetic structure of population (iv) finger printing of individuals (v) identification of hybrids.

### 1.7 Host Shout off Factor

Researchers Dr.Rajamish Gin, Asst. Prof in Biotechnology, IIT, Mandi, Himachal Pradesh, the findings in 'Current Research in Virological Science' have revealed the part of structure of a key protein in Covid-19 virus which helps in understanding its mode of action, its relates in the speed and severity of the disease and development of antiviral therapies and understand the viral disease and develop drugs that are effective against the virus. The virus has 16 non-structured proteins (NSP1-NSP16) of which the NSP! Plays a vital role in the pathogen city (ability to cause disease) of the virus. The NSPI disrupts the proteins of the host cell and suppress its immune functions is called 'host shutoff factor'.

## 2. METHODS AND MATERIALS

### 2.1 Terminology and Notation

Event: The patients oxygen Saturation level of more than 94% and respiratory rate less than 24/mns asymptom (mild)

T = Survival time (T>0); if developed fever and breathlessness at to

Where *T* is 'Random Variable', to specific value (duration) for *T*.

Treatment Protocol: stay at Covd Care Centre (CCC)/Home isolation (3 days)

T = 1, if cured

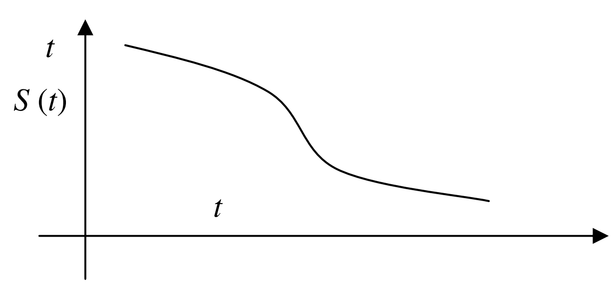
0, otherwise

### 2.2 Survival Function for three days

$S(t) = P(T > t)$

| $T$ | $S(t)$ |
|---|---|
| 1 | $s(1) = P(T > 1)$ |
| 2 | $s(2) = P(T > 2)$ |
| 3 | $s(3) = P(T > 3)$ |

The survival function, $S(t)$ gives the Probability that the probability that a person survives longer than specified time 3 days. $S(t)$ given the probability that the random variable, *T* exceeds the specified time



### 2.3 Poisson birth (infection) process

Since $N(t)$ has Poisson distribution $\Pr(N(t)=k)=\frac{\lambda e^{-\lambda t}}{k!};\ k=0,1,2,...$

And $E(nt)=E(N(t))=\lambda t$

Observing the excess life at time t exceeds x, if and only if there are no infection (renewal) in the interval $(t,t+x)$ this event has the same probability as that of no renewals in the interval $(0,x)$ since a Poisson process has stationary leads to the Memoryless property of Exponential distribution

## ANALYSIS

A Stochastic model to estimate the expected time to sero conversion is discussed using the threshold distribution as are which satisfies so called setting the clock back to zero (SCBZ) property. This property has been discussed by Rey-Rev and Talwaliker (1990)

### 3.1 Assumption

- There is transmission of SARS-COV-19 in successive sexual contacts between the infected and uninfected.
- There is contribution to antigenic diversity of SARS-COV-19 due to successive contacts.
- The breakdown of the immune system occurs as and when the total antigenic diversity accumulation crosses the threshold level.

### 3.2 Mathematical Model for SARS-COV-19

1. Deterministic models
2. Stochastic models
3. Stochastic models using direct exploration
4. Kalman filter models.

Kannan, et. Al (2011) have discussed a stochastic model to derive the expected time to sero conversion and its variance. The person is about in the probability. ‘P’ so that the transmission avoided, in a person is as adult than the transmission occurs with probability B, the threshold is assumed to the random variable which follows mixed exponential with parameters $\theta_1$ and $\theta_2$.

The random variables *Y*, which denotes the antigenic diversity threshold have the p.d.f

$n_1(y)=\theta_1\ e^{-\theta_1 y};\ y\le y_0$

$n_2(y) = \theta_2 \ e^{-\theta_2 y}; \ e^{y_0}(\theta_2 - \theta_1)$ if $y > y_0$

When $y_0$ is the truncation point

**Notation**

$X_i$ = random variables denotify of the amount of contribution to antigenic diversity due to successive contacts. $i = 1, 2, K$ and $X_i$ has p.d.f (.), c.d.f, G(%)

$y =$ The random variable which denotes the threshold level and $y$ has p.d.f $h(.)$ and c.d.f. $H(.)$

$v_i =$The random variable denotify the inter arrivals time between contacts and $u_i$ has p.d.f, $f(.)$ and c.d.f. $f(.)$ $g(.), f(.)$ are the laplace transform of $g(.)$ and $f(.)$ respectively

$F_k(.) = K$ convolution of $F(.)$

$T =$ a random variable denotify the time to zero conversion.

**Formulation of the Model**

A T-4 cell can be thought to be in any one of the following two phases. In the first phase, the T-4 cell, is an uninfected (no attainment is an SARS-COV-19). The second phase, begins, when an SARS-COV-19 attaches itself to the receptors of an uninfected T-4 cells that is this uninfected cell has now got converted to an infected cell. At any time *t*,

1. An infected T-4 cell undergoes its normal differentiation. With probability $e^{-ut}$
2. $\lambda$ is the probability of an SARS-COV-19 to an infected T-4 cell
3. $\alpha$ is the probability of lysis to occur let $x(t)$

**Method**

**2.1 Stochastic Model:** non-homogeneous (heterogeneity) poisson process-consider the population individuals with atleast one partner ina given a time period. Suppose that the number of additional partners *k* – 1, that the person has is the time period follows a Poissson distribution, with expected value $\lambda$, i.e., $P(\lambda)$. A poisson distribution with Gamma-distribution rate parameters $\lambda$ is a classical hierarchical model, with marginal distribution of negative binomial (Johnson et.al) 1992. Statistical analysis and modify have contributes greatly to our data longitudinal study refers to an investigation where participants outcomes are collected at multiple variate morns 1997), using epidemiological/models.

**2.2 Statistical inference**

Likelihood framework, to estimate the model parameters and compare the different models against each other, MLE; is approximately normally distributes (Johnson et. Al. 1992). Mixed parameters $c$ likelihood approach (Groeneboom and Wellener, 1992). Define $K_{\min}$ as the degree above which the parametric model (e.g. the yale) is fit and denote.

$P(K=k)=\pi_k$ for $K \le K_{\min}$, given '$n$' total observation with observed frequencies $n_0, n_1, \ldots$ an network deforce… $K=0,1,\ldots$ the log-likelihood of the data is

$$L(n;\theta / K_1 = K_1,\ldots,K_n) = {}^{K_{\min}}\sum_{m=0} nr_n \log(r_n) + \left(n - {}^{K_{\min}}\sum_{m=0} n_m\right)\log\left(1 - {}^{K\min}\sum_{m=0}\pi + 1\right)$$
$$+\sum_{m=k_{\min}+1}^{n} \log\left(p(k=m) K > K_{\min}\right)$$

**2.3 Multiple measurement on the pathogenesis of SARS-COV-19-1 infurious**

Stochastic represents the full data log-likelihood for the models. We verify that the maximum likelihood estimates $\pi_m$ is simply the sample proportion of $m$, $m_m / n$

The MLE is given by $\dfrac{\partial L(\pi, \theta / k_1 = k_1, \ldots, k_n = k_n)}{\partial \pi_k} = \dfrac{nk}{\pi_k} = n - \dfrac{\sum_{m=0}^{k\min} nm}{1 - \sum_{m=0}^{k_{\min}} \pi_m}$

Which is sero, only, is $\pi \; m/n \; \forall \; m, 0, \ldots$ This partial is not defined for $T_k = 0$ or ${}^{n}\sum_{m=0}^{k\min} \pi_m = 1$. However in these special case the likelihood is maximized (with probability) by the sample proportion.

The Henian Matrix of equation

$$\frac{\partial \alpha(\pi, \theta / k_1 = k_1, \ldots, k_n = k_n)}{\partial \pi k \partial \pi j}$$

is negative definite indicating that the MLE is the unique global maximum.

**2.4 Model Selection**

In general two different approaches. (1) The Akaivk information extension (AIC) Akaike, 1974 Burnhman and Anderson, 2002) (2) The Bayesian information criterion (BIC) (Reftery, 1995)

The Models with more parameters may be expected to fit the likelihood $L$ provides a measure of the goodness of fit, of a model to the data that

For a simple random sample of $n$ people with data $K_1, K_2, \ldots, K_n$ the AIC is defined as $AIC = -2L(\partial) K_1, \ldots. K_n = K_n$ the AIC is defined as $AIC = -2L(\theta) K_1, \ldots, K_n = K_n$ ) and $AIC = 2L(\theta) K_1, \ldots, K_n = K_n) + \log(n) d$

## 3. ANALYSIS

### 3.1 Data Augmentation

We present a data augmentation Marker Chain Mante Carlo (MCMC) framework for Bayesian estimation of Stochastic epidemic model, paramentation Stochastic epidermic models (SEM) describe the dynamic of an apidemic as a disease spreads through a population. SEM are classic tools for modelling the spread of infectious disease. A SEM represents the time $t_0$ of an epidemicin terms of the disease histories of individuals is important when the disease. Prevalance is low or high the population size is small. In both cases, the stochastic variability is the evolution of an epidemic greatly influences. The probability and severity of an outbreak, along with the conclusion we draw its dynamics, (Keeling and Rohan, 2008, Allen 2008, Moreover may questions e.g., what is the outbreak size distribution). What is the probability that a disease has been eradicated? Can't be answered using deterministic methods (Britton, 2010). The Date augmentation Algorithm for an SIR Model; Bayesian data Augmentation (BDA), Markov Chain Monte Carlo (MCMC). Hidden Markov Model (HMM) continuous-Time Markov Chain (CTMC). **Susceptible-Exposed-infectual-Received (SEIR)** epidemic (binomial distribution) process a Measurement Process and Data. Our data $y = ...y_1...y_2$) are disease prevalence counts accorded at time $t_1,...,t_2 = ....t_1,t_2$) Let $S_r, L_r$ and $R_r...$ denote the **Total Susceptible**, infected and recovered people at time *t*. We model the observation prevalence a binomial sample, with constant **detection**. Probability *p*, *q* the true prevalence of each observation time. **Thus,** $y_i / I_{t,\alpha} \rightarrow$ Binomial $\left(I_{t\alpha}P\right)\ldots$ (1)

### 3.2 Latest Epidemic Process

The data are sampled from a latest epidemic process $x = x_1,...,x_n$ that evolves continuously in time, as individual becomes infected and recover. The state space of the process $S = \left(S, I, R\right)^N$ the Cartesian product of state levels taking $\left(S, I, R\right)$. The State space of triangle subject, $X_j$ is $S_r = \left(S, I, R\right)$ and a subject-path is of the form

$$X_j\left(T\right) = \begin{cases} S,\ T < T_1^{(j)} \\ I,\ T_1^{(j)} \le T < T_R^{(j)} \\ R,\ T_R^{(j)} \le T \end{cases} \ldots (2)$$

Where $T_I^{(j)}$ and $T_R^{(j)}$ are the infection and recover times for subject *j* (though subject *j* may also never become infected or recover or may become infected or recover outside of the observation period $\left(t_1, t_2\right)$.

We write the Configuration of $X$ at time $\tau$ as $X(t)=\left(X_1(\tau)...X_N(\tau)\right)$ and derived quantities e.g. $I_r$ depend on the configuration first before $\tau$. We use $\tau^+$ for quantities ecated first after a particular time. The waiting time between transition events are taken to be exponents distributed and we denotes by $\beta$ and $\mu$. The per-contact infelicity and recovery rates. Thus the latest epidemic process according to a time-homogeneous CTMC, with transition rate from configuration $X$ to $X^1$ given by

$$\lambda_{X_c X^1}=\begin{cases}\beta \text{ if } X \text{ and } X^1 \text{ differ only in subject j with } X_j=S \text{ and } X_j^1=I\\ \mu \text{ if } X \text{ and } X^1 \text{ denotes only in subject } j_i \text{ with } X_j^1=I \text{ and } X_j^1=R \; \ldots (3)\\ 0, \text{ if } x \text{ is all other configurations } X \text{ and } X^1\end{cases}$$

with $x_j=I$ and $x_j^1=k$.

0 for all other configuration $x$ and $x^1$...

Existing approximate to titling SEM, with likelihood have largely fallen into (4) groups. (1) Martingales Method (2) Approximation Methods (3) Simulation based methods and (4) Data augmentation (DA) methods (O'Neill 2010). At the first observation time, we let $X(t_1)(P_{t-1})$ categorical $\left(\{S,I,R\}P_{t_1}\right)$ where $P_{t_1}=\left(P_s,P_t,P_k\right)$ are the probabilities that an individual is susceptible, infected or recovered. Let $\tau=\left(\tau_0,\tau_1,\tau_{k+1}\right)$ when $t_1=\tau_0$ and $t_1=t_{k+1}$ be the ordered set of $k$-infection and recovery timing of all individuals along with the end points of the observation period $(t_1,t_2)$. Let $\pi(T_k\in I)$ and $I(T_k\in R)$ individual whether $T_k$ is as infection or recovery time and let $\theta=(\beta,\mu,p,p_k)$ denote the vector of unknown parameters. The complete data likelihood.

$$L(x,y/\theta)=\Pr(y/x,p)\times\Pr(x_{t_1})(P_{t_1})\times\pi\left(X\left(X(t_1),\beta,\mu\right)=^L \pi_{I=1}\left(It\alpha_{y_1}\right)P^{1/1}(1-p)lt\alpha-x_1\right)$$

$$\times\left(P_s^{s_{t_1}}PI_1^{lt}\right)=_x \pi_{k=1}^k\left(BI_{rk}\times I(t_k \square I)\mu\times\pi(T_k \square R)\right)$$

$$\exp\left(-(\tau_k-\tau_{rk})\beta I_{\tau k}S_{\tau k}+\mu I_{rk}\right)\times\exp\left[-(t_1-I_k)(\beta I_{rk}+S_{rk}+\mu I_{rk+})\right]$$

SIR dynamics is a population of five subjects. Infection event S,I,I,RI,S,I,I,R Rate $2\beta$ current configuration 2βS,S,I,I,R,S,S,R,I,R $\mu$ recover μS,S,I,R, Recovery events subject each of these configuration at rate $\mu$.

**Subject-path Proposal framework**

However, the Markov process is the canonical model, is a jumping of our process with respect to the portion induced by aggregating the individuals in each model compartment. Therefore, inference mode on the full subject-level statespace will exactly match influence based on the canonical model. The observed data Lilulihood is the posterior

$n(\theta / y) \alpha \pi(y / \theta) \pi(\theta) = \{ ... (y / x\theta), \pi(x / \theta) \pi(\theta) d\pi(x) \}$ … (5)

When $\pi(\beta), \pi(\mu), \pi(0)$ and $\pi(P_{t_1})$ are priori densities … MCMC. Targets the irst posterior distribution (5) is

$\pi(\theta, x / y) \alpha \Pr(y / x, \pi) \times \pi(x / x(t_1), \beta, \mu) \times \Pr(t_1) Pt_1 \times \pi(\beta) / \pi(\mu) \pi(R) \pi / pt_1$ …

as we alternate between …

$x / \theta,\ y$ and $\theta / x,\ y$

In MCMC Algorithm, we propose each new subject-level path, conditional on the data using a time is homogeneous continuous time Marker process (CTMP) with rates determined of the inclusion his … of the individual the particle Marginal metropolis attaching algorithm of … et. Al (2010) is been a final method of given the of subject-paths, $X$ we propose $X$ by sampling the path of a single subject $X\partial$. Conditionally on the data using a time-homogeneous (TMC with state space $S_j$ and the rates conditioned as the collection of disease histories of other individuals. $X(-j) = x_1, x_2 - x_{j-1}, x_{j+1}, ..., x_n$ the proposed collection of paths is accepted or in a metropolis attaching step. Let $\pi^{(\partial)} = T_j^{(\partial)}, T_{jk}^{(\partial)}$ the (possibly empty) set of infection and recovery times for subject $\partial$ and define. $T(-\partial) = t_1, t_2 uT\left(T^{(-\partial)}\right) = T_{1j}^{(-\partial)}, T_{1j}^{(-\partial)}, ...Tm^{(-\partial)}, T_{m+1}^{(-\partial)}$. Where $t_1 \equiv T_0^{(-\partial)}$ and $t_1 \equiv t_2 \equiv \tau^{(-\partial)}$ to be the set of MCM (…) times at which other subjects become infected or recover along with $t_1$ and $t_2$. Let $I = I_1, ..., I_{m+1}$ be the intervals that partition $(t_1, t_2)$ i.e., $T_1 =$ along with $t_1$ and $t_2$ $ut\ I = I_1, ...I_{m+1}$) the intervals that partition $[t_1, t_2]$ i.e. $I_1 = \left[T_i^{(-\partial)}, T_1^{(-\partial)}\right],\ I\alpha = \left[T_i^{(-\partial)}, T_2^{(-\partial)}\right],$

$= T_{m+1} = \left[T_m^{(-\partial)} T_{m+1}^{(-\partial)}\right]$ Let $I_T^{(-\partial)} = \sum_t \mp_\partial I$

$(x_i / T = I)$

We the prevalence at time $\tau$, excluding subject *j* where for $m = 1, ..., m+1$. The rate for subject *j* is

$$\operatorname{Im}^{(-\partial)}(\theta) = \begin{matrix} {}^{S}I_R \\ I \\ R \end{matrix} \begin{pmatrix} {}^{s}-\beta I_{T_m}^{(-\partial)} & \beta I \beta_{rm-\mu_0}^{(-\partial)} & {}^{\beta}0_\mu \\ 0 & -\mu & \mu \\ 0 & 0 & 0 \end{pmatrix}$$

We can construct the transition probability for interval in as $P^{(-\partial)}(T_{m+1}, T_m) = P^{(\partial)}(T_{m-1}, t_m)$...

## CONCLUSION

Testing got gradually expanded to include different population subgroups during the pre-peak the pre-peak phase of both first and the second wave. The second wave saw daily new cases growing 3 to 4 times. During first wave 11.4 tests/detected case, the average daily test positivity rate (TPR) was only 8.5%.

In Second wave 95 days (Total out of 10.6 Crore test 130 lakh more cases) WHO recommends test positive rate below 5% level. The growth of the testing should for out pace the growth of the case may leads to epidemic curve $R < 1, R = 1, R > 1$.

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# ON THE DIOPHANTINE EQUATION $n(x-y)=xy, n=pq$, p AND q ARE CONSECUTIVE PRIMES

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**ABSTRACT**

***In this paper, the Diophantine equation $n(x-y)=xy$, ($n=pq$, p and q are consecutive primes) has been discussed.***

***Key words : Diophantine Equation and Integral Solution***

## 1. INTRODUCTION:

**Brian Miceli** (2021) proposed open problem given below:

In how many ways can we write $\frac{1}{2021}=\frac{1}{x}-\frac{1}{y}$ where *x, y* are positive integers.

In this paper, the Diophantine equation $n(x-y)=xy$, ($n=pq$, p and q are consecutive primes) has been discussed. This is some sort of generalization of the above problem.

## 2. ANALYSIS:

The given Diophantine equation can be written as

$$y=\frac{nx}{n+x}. \quad \ldots(1)$$

(i) First we consider the case when $n=pq$ and difference of $p$ and $q$ is one. There is only one such case with $p$=2 and $q$=3. Therefore $n$=6. So from (1) we have

$y=\frac{6x}{6+x}$. This is satisfied for $x$=3 which gives $y$=2. Thus the given Diophantine equation is satisfied by $n$=6, $x$=3 and $y$=2.

In another way, the given Diophantine equation can be written as

$$\frac{1}{y}-\frac{1}{x}=\frac{1}{n}=\frac{q-p}{pq}=\frac{1}{p}-\frac{1}{q}=\frac{1}{2}-\frac{1}{3}.$$

This gives x=3 and y=2.

(ii) In the second case, we consider the twin primes. Here difference of $p$ and $q$ is two, i.e. $q-p=2$. The given Diophantine equation can be written as

$$\frac{1}{y}-\frac{1}{x}=\frac{1}{n}=\frac{1}{pq}=\frac{1}{(q-p)}\left(\frac{1}{p}-\frac{1}{q}\right)=\frac{1}{(q-p)p}-\frac{1}{(q-p)q}.$$

This gives $x = (q-p)q$ and $y = (q-p)p$. Few solutions are given below:

| p | q | n | x | y |
|---|---|---|---|---|
| 3 | 5 | 15 | 10 | 6 |
| 5 | 7 | 35 | 14 | 10 |
| 11 | 13 | 143 | 26 | 22 |
| 17 | 19 | 323 | 38 | 34 |
| 29 | 31 | 899 | 62 | 58 |
| 41 | 43 | 1763 | 86 | 82 |
| 59 | 61 | 3599 | 122 | 118 |
| 71 | 73 | 5183 | 146 | 142 |
| 101 | 103 | 10403 | 206 | 202 |

(iii) In the third case, we consider the primes $p$ and $q$ whose difference is four, i.e. $q - p = 4$. The given Diophantine equation can be written as

$$\frac{1}{y} - \frac{1}{x} = \frac{1}{n} = \frac{1}{pq} = \frac{1}{(q-p)}\left(\frac{1}{p} - \frac{1}{q}\right) = \frac{1}{(q-p)p} - \frac{1}{(q-p)q}.$$

This gives $x = (q-p)q$ and $y = (q-p)p$. Few solutions are given below:

| p | q | n | x | y |
|---|---|---|---|---|
| 7 | 11 | 77 | 44 | 28 |
| 13 | 17 | 221 | 68 | 52 |
| 19 | 23 | 437 | 92 | 76 |
| 37 | 41 | 1517 | 164 | 148 |
| 43 | 47 | 2021 | 188 | 172 |
| 67 | 71 | 4757 | 284 | 268 |
| 79 | 83 | 6557 | 332 | 316 |
| 97 | 101 | 9797 | 404 | 388 |
| 103 | 107 | 11021 | 428 | 412 |

(iv) In the fourth case, we consider the primes $p$ and $q$ whose difference is six, i.e. $q - p = 6$. The given Diophantine equation can be written as

$$\frac{1}{y} - \frac{1}{x} = \frac{1}{n} = \frac{1}{pq} = \frac{1}{(q-p)}\left(\frac{1}{p} - \frac{1}{q}\right) = \frac{1}{(q-p)p} - \frac{1}{(q-p)q}.$$

This gives $x = (q-p)q$ and $y = (q-p)p$. Few solutions are given below:

| p | q | n | x | y |
|---|---|---|---|---|
| 23 | 29 | 667 | 174 | 138 |
| 31 | 37 | 527 | 222 | 186 |
| 53 | 59 | 3127 | 354 | 318 |
| 61 | 67 | 4087 | 402 | 366 |
| 73 | 79 | 5767 | 474 | 438 |
| 83 | 89 | 7387 | 534 | 498 |
| 131 | 137 | 17947 | 822 | 786 |
| 151 | 157 | 23707 | 942 | 906 |
| 157 | 163 | 25591 | 978 | 942 |

(v) In the fifth case, we consider the primes $p$ and $q$ whose difference is eight, i.e. $q-p=8$. The given Diophantine equation can be written as

$$\frac{1}{y}-\frac{1}{x}=\frac{1}{n}=\frac{1}{pq}=\frac{1}{(q-p)}\left(\frac{1}{p}-\frac{1}{q}\right)=\frac{1}{(q-p)p}-\frac{1}{(q-p)q}.$$

This gives $x=(q-p)q$ and $y=(q-p)p$. Few solutions are given below:

| p | q | n | x | y |
|---|---|---|---|---|
| 89 | 97 | 8633 | 776 | 712 |
| 359 | 367 | 131753 | 2936 | 2872 |
| 389 | 397 | 154433 | 3176 | 3112 |
| 401 | 409 | 164009 | 3272 | 3208 |
| 449 | 457 | 205193 | 3656 | 3592 |
| 479 | 487 | 233273 | 3896 | 3832 |
| 491 | 499 | 245009 | 3992 | 3928 |
| 683 | 691 | 471953 | 5528 | 5464 |
| 701 | 709 | 497009 | 5672 | 5608 |

(vi) In the sixth case, we consider the primes $p$ and $q$ whose difference is ten, i.e. $q-p=10$. The given Diophantine equation can be written as

$$\frac{1}{y}-\frac{1}{x}=\frac{1}{n}=\frac{1}{pq}=\frac{1}{(q-p)}\left(\frac{1}{p}-\frac{1}{q}\right)=\frac{1}{(q-p)p}-\frac{1}{(q-p)q}.$$

This gives $x=(q-p)q$ and $y=(q-p)p$. Few solutions are given below:

| p | q | n | x | y |
|---|---|---|---|---|
| 181 | 191 | 34571 | 1910 | 1810 |
| 241 | 251 | 60491 | 2510 | 2410 |
| 283 | 293 | 82919 | 2930 | 2830 |
| 337 | 347 | 116939 | 3470 | 3370 |
| 409 | 419 | 171371 | 4190 | 4090 |
| 547 | 557 | 304679 | 5570 | 5470 |
| 577 | 587 | 338699 | 5870 | 5770 |
| 631 | 641 | 404471 | 6410 | 6310 |
| 709 | 719 | 509771 | 7190 | 7090 |

(vii) In the seventh case, we consider the primes $p$ and $q$ whose difference is twelve, i.e. $q-p=12$. The given Diophantine equation can be written as

$$\frac{1}{y}-\frac{1}{x}=\frac{1}{n}=\frac{1}{pq}=\frac{1}{(q-p)}\left(\frac{1}{p}-\frac{1}{q}\right)=\frac{1}{(q-p)p}-\frac{1}{(q-p)q}.$$

This gives $x=(q-p)q$ and $y=(q-p)p$. Few solutions are given below:

| p | q | n | x | y |
|---|---|---|---|---|
| 199 | 211 | 41989 | 2532 | 2388 |
| 211 | 223 | 47053 | 2676 | 2532 |
| 467 | 479 | 223693 | 5748 | 5604 |
| 509 | 521 | 265189 | 6252 | 6108 |
| 619 | 631 | 390589 | 7572 | 7428 |
| 661 | 673 | 444853 | 8076 | 7932 |
| 797 | 809 | 644773 | 9708 | 9564 |

## 3. CONCLUDING REMARKS:

Here the Diophantine equation $n(x-y) = xy$ where $n=pq$, $p$ and $q$ are consecutive primes, has been discussed for $q-p = 1, 2, 4, 6, 8, 10$ and 12. The problem can further be extended for other values of $q-p$.

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# A NOTE ON STATISTICAL POPULATION GENETICS WITH WRIGHT FISHER MODEL

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**ABSTRACT**

***The famous Hardy Weinberg Law (1908) gave fundamental approach to many calculations in population genetics. Besides it, the statistics in the classical work done by Fisher, Haldons and Wright proved great advantageous to theoretical understanding of evolution process. In this direction the stochastic process with nonlinear differential, difference equations and algebra yield mathematics with genetics. The pioneer work of all this synthesis was done by S.N.Bernestein in his series of papers (1971-79). In this paper we have made detail review on Statistical methods applied successfully to population genetics.***

***Keywords: Hardy Weinberg Law, Panmix population, inheritance and survival co-efficients, Mean life time, Survivorship curve, Bayesian inference.***

## 1. INTRODUCTION

### 1.1 Elements of Classical Genetics

The fundamental concept of classical genetics, the gene was introduced by Mendal under the name constant character to explain the observed status of inheritance. Each cell carries in its nucleus an enormous number of genes. The set of these genes in genotype of the cell. The genotype of all the cells of an organism except the sex cells or gametes and a few other specialized cell type is identical to the genotype of the zygote the fertilized egg from which the organism develop. This uniformity is secured by the mechanism of mitosis a process of cell division in which each gene is duplicated and the true copies depend to the two daughter cells.

### 1.2 Genotypic Differentiation

Genotypic differentiation is the process of population into genotypes for a given locus or system of loci.

### 1.3 Evolution or Dynamics

Evolution of a population comprises a determined change of state in the generation as a result of reproduction and selection. The evolution of a population is a random process.

### 1.4 Selection

Selection occurs through a process of organism environment interrelation and internal process affecting the probabilities of survival before the reproductive period.

## 2. MODEL

### General Evolutionary Equation

### 2.1 Assumption 1:

Assuming the population is bisexual. We suppose that the set of females can be partioned into finitely many different types indexed by $\{1,2,...,n\}$ and similarly that the male types are indexed by $\{1,2,...,v\}$ the number $n+v$ is called the dimension of the population.

The population is described by its state vector $(x,y)$in $\Delta^{n-1}\times\Delta^{v-1}$. The product of the unit simplifies in $R^n$ and $R^v$ respectively, *x* and *y* are the probability distribution of the females and males over the possible types.

$$x_i \geq 0,\ \sum_{i=1}^{n} x_i = 1 \ \ y_k \geq 0,\ \sum_{k=1}^{y} y_k = 1$$

Then the state space $S=\Delta^{n-1}\times\Delta^{v-1}$ is a compact, convex, subset of $R^{n+v}$

We call the partition into types hereditary if for each possible state $z=(x,y)$ in *S*describing $x_i$. Current generation, the state $Z'=(x',y')$ is uniquely defined describing the next generation obtained by reproduction and selection.

This means that the association $Z\rightarrow Z'$ defines a map $V:S\rightarrow s$ called the evolutionary operation and $Z'=V(Z)$ $(Z\in S)$ is called the evolutionary equation (or equation of evolution) for the system.

### Assumption 2:

The set $w(z)$ consist entirely of equilibrium points if and only if $\lim_{t\to\infty} d\left(z^{(t+1)},z^t\right)=0$ in that case, $w(z)$ is connected set.

### Assumption 3: Random mating or Panmixic

The forces of evoluationary operator depends on assumptions about the reproductive and selection process, Such that mating partners are randomly selected from the population with no dependence on types.

**Explanation:**

Formally, if $(x, y)$ is the population state than the probability that a female of type $i$ mates a male of type $k$ is given by the product $x_i y_k$.

**The free single locus population: Hardy Weinberg Law**

With two allels$A$ and $a$ at a locus the Zygote are $AA$, $aa$, $Aa$

| Male | Female | | |
|---|---|---|---|
| | ***AA*** | ***aa*** | ***Aa*** |
| ***AA*** | $AA$ | $Aa$ | $\frac{1}{2}AA+\frac{1}{2}Aa$ |
| ***aa*** | $Aa$ | $aa$ | $\frac{1}{2}aa+\frac{1}{2}Aa$ |
| ***Aa*** | $\frac{1}{2}AA+\frac{1}{2}Aa$ | $\frac{1}{2}aa+\frac{1}{2}Aa$ | $\frac{1}{4}AA+\frac{1}{4}aa+\frac{1}{2}Aa$ |

**Mendel's dialleliczogotic algebra (Mendel's First Law)**

With the two allels$A$ and $a$a locus the zygote $AA, aa, Aa$ the table presents multiplication in corresponding evolutionary algebra. It reflects the fact during meiosis.

$$AA \leftrightarrow A, aa \leftrightarrow a, Aa \leftrightarrow \frac{1}{2}A+\frac{1}{2}a \ \dots (1)$$

The coefficients at the gammates are equal to the probabilities of their from a given zygote and during fertilization fathers and mothers gamates are chosen at random and independently,

$$AA \times Aa = A\left(\frac{1}{2}A+\frac{1}{2}a\right)=\frac{1}{2}AA+\frac{1}{2}Aa \ \dots (2)$$

$$AA \times Aa = \left(\frac{1}{2}A+\frac{1}{2}a\right)^2=\frac{1}{4}AA+\frac{1}{4}aa+\frac{1}{2}Aa$$

In particular the evolutionary operator defined in coordinate form,

$$x'j=\sum_{l,k=1}^{n} p_{ik,j}x_i x_k \left(1 \le j \le n\right) \ \dots (3)$$

In some generation the state is $x = x_1 AA + x_2 aa + x_3 Aa$ then in the next generation $x' = x^2 = \left(x_1 AA + x_2 AA + x_3 AA\right)^2$ using in multiplication table

$$x_1^{'} = x_1^2 + x_1 x_3 + \frac{1}{4}x_3^2$$

$$x_2^{'} = x_2^2 + x_2 x_3 + \frac{1}{4}x_3^2$$

$$x_3^{'} = 2x_1 x_2 + x_1 x_3 + x_2 x_3 + \frac{1}{2}x_3^2 \ \dots (3)$$

This is the evolutionary operator of the population equation (1) is averaged in corresponding with the current state of the population of zygotes.

The two level character follows

$$x_1^{'} = p^2,\ x_2^{'} = q^2,\ x_3^{'} = 2pq \ \dots (4)$$

$$\text{where } p = x_1 + \frac{1}{2}x_3 \quad q = x_2 + \frac{1}{2}x_3 \ \dots (5)$$

The map $\mu : \Delta^2 \to \Delta'$ defined by equation (5) is the meiosis operator and the map $\varphi : \Delta' = \Delta^2$ defined by equation (4) is the fertilization operator (that $\text{lm}\mu = \Delta'$ is obvious: $p+q=1,\ p \geq 0,\ q \geq 0$ and any point $(p,q) \leq y'$ enters $\text{lm}\mu$ being image of $(p,q,0) \in \Delta^2$ .

Equation (5) corresponding to converging of (1) with respect to state *x* with *p* and *q* the probabilities of gamates*A*, *a* in the gamete pool of the entire population in a state *x*.

Equation (4), corresponds to the independent combination of parent gamates chosen at from the gamete code in the state $(p,q)$. The coefficients in the last formula is connected with distinction between father and mother gamates, as there are two ways of generating the hetrozygote*Aa*, i.e. by using the fathers *A* with mothers *a* and viz versa.

Let us compute the evolutionary operators of the gamete code. It is equal to $U_q = \mu_q$

i.e., $p' = p^2 + \frac{1}{2}2pq = p(p+q) = p$

$q' = q^2 + \frac{1}{2}2pq = q(p+q) = q \ \dots (6)$

The operator hence to be identify such that all states of the gamete pool are in equilibrium.

**2.2 Departure from Panmixia**

Departure from panmixia do occur in natural population. The most important are (i) assortative mating (preference for similar types) (ii) inbreeding (preference for relatives) (iii) self sterility (incapability of like types)

**2.3 Inheritance Coefficients**

$p_{ik,j}^{(t)}$ and $p_{ik,l}^{(m)}$ are defined as the probability that a female offspring is type *j* and respectively that a male offspring is of type *l*, when the parental pairs $ik\left(i, j = 1,\dots,n \text{ and } k = 1,\dots,v\right)$

$$p_{ik,j}^{(f)} \geq 0,\quad \sum_{j=1}^{n} p_{ik,j}^{(f)} = 1 \quad p_{ik,l}^{(m)} \geq 0,\quad \sum_{l=1}^{v} p_{ik,l}^{(n)} = 1$$

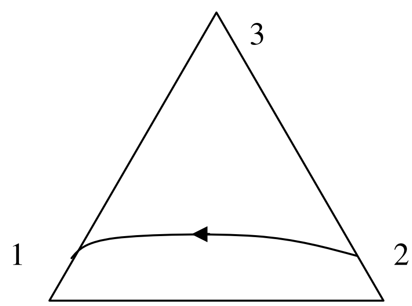
The inheritance coefficients include the effects of recombination, mutation, gamete selection, variation of fertility.

### 2.4 Survival Coefficients

$\lambda_j^{(f)}$ and $\lambda_l^{(f)}$ are defined as the probability that a female individual of type *j* and that a male individual of type *l* will survive to reproductive maturity. The complementary probabilities $1-\lambda_j^{(f)}$ and $1-\lambda_l^{(m)}$ are mortality coefficients for each type.

We will assume that the coefficients are constants and independent of time and the current state of the population.

*Q*




The figure shows the simplest case $m=2$ here $E_\xi$ is an arc of the parabola. $x_1 = p_2,\ x_2 = q^2,\ x_3 = 2pq\left(0 \le p \le 1,\ q = 1-p\right)$ at the point of $O\left(\frac{1}{2},\frac{1}{4},\frac{1}{2}\right)$ which corresponds to the equilibrium observed in Mendels experiments.

A dominate over the equilibrium ratio of phenotypes at this point is 3:1 and at an arbitrary point $\left(p,q\right)$ is equals to $\left(1-q^2\right): p^2$.

## 3. ANALYSIS

### 3.1 Stochastic Treatment: Discrete Series Wright – Fisher Model

Consider a diploid population having in each generation exactly *N* individuals, so that there are 2N genes altogether at the locus in question. Suppose that in some way to be chosen at our discretion, the population reproduces itself to form daughter generation. Suppose also that once the daughter. Generation is formed, no further reproduction is possible for the parent generation, which may be statistical purpose be considered as dying as soon as the daughter generation appears(Wright S. 1937).

If there is no mutation, selection or any other disturbing factor gene frequencies tend to remain steady. It is reasonable to assume in our model that if, in generation *t*, the number of $A_1$ genes is $X\left(t\right)$, then the number $X(t+1)$ of such genes in the following generation is a random variables with mean value $x(t)$ with this constraint, there are several possible models which would reasonably be desired, the most frequently used one and one we shall consider at some length is implicitly to Fisher (1930) and explicitly to Fisher (1930) under this model $X(t+1)$ is a binomial variate with index $2N$ and parameter $X(t)/2N$. Explicitly, given that $X(t)=1$, the probability $p_{ij}$ that $x(t+1)=j$ is given by the equation

$$p_{ij}=\binom{2N}{j}\left\{(i/2N)^j\left(1-(i/2N)\right)\right\}^{2N-j} \quad \dots (7)$$

Clearly $X(i)$ a Markovian variable with transition probability Matrix $P=\{P_{ij}\}$

While under the model (1) there is no tendency for any directed change in gene frequency, random sampling will ensure that eventually one of the two absorbing state, namely $X(0)=0$ and $X(i)=2N$ will be reached. If the eigen values of *P* are $\lambda_i\,(i=0,1,2,...,2N)$ it is clear that, because there are two absorbing state $\lambda_0=\lambda_1=1$ and that if the remaining eigen values are distinct, *p* can be written in the spectral form

$$p^n=C+\lambda_2^n P+....\lambda_{2N}^n \quad \dots (8)$$

Here *C* is a Matrix having positive entries only in the first and last columns, these entries are absorption probabilities in $X(i)=0$ and $X(i)=2N$ respectively for the values of $X(0)$, if $|\lambda_2|>|\lambda_3|>...>|2N|$. Then the rate at which $p^n$ converges to the limiting matrix *C* is governed to a large extent by the value of $\lambda_2$. If $\lambda_2$ is very close to unity, this rate of approach will be lost slowly if it is moderate or small, then rate will be quite rapid. There are several methods for finding by itself, but we shall consider initially a slightly more involved analysis which provides all eigen values of *P* simultaneously.

Suppose that a non-singular matrix *Z* and an upper triangular matrix *A* can be found that

$$PZ=ZA \quad \dots (9)$$

Since these equationimplies $P=ZAZ^{-1}$, the eigenvalues of*P* will be identified to those of *A*, which become of the special nature of *A* are its diagonal elements. A matrix *Z* for which an equation of the form (9) holds is

$$Z=\begin{pmatrix}1 & 0 & 0 & ... & 0 & 0\\ 1 & 1 & 1^3 & ... & 0 & 1^{2N}\\ 1 & 2^2 & 2^3 & ... & 0 & 1\\ 1 & 1 & 3^3 & ... & 0 & 2^{2N}\\ 1 & 2N & (2N)^3 & ... & 0 & (2N)^{2N}\end{pmatrix} \quad \dots (10)$$

With this definition of *Z*, the *i*, *j*th element in *PZ* is $\sum_l Pil^j$ … (11)

Which can be written, $E\left\{X\left(t+1\right)\right\}=j/X\left(t\right)=i$ …(12)

Now, the $ij^{th}$ element of *ZA* is of the form, $a_{ij}$ …(13)

Equation (6) and (7) taken together, show that if, for each *j*, $\left(j=0,1,2,...,2N\right)$ can write

$$E\left\{X\left(t+1\right)\right\}^{j}X\left(t\right)=a_{0j}+a_{ij}^{x}\left(x(t)\right)+a_{2j}\left(x(t)\right)2+a_{jj}\left(x(t)\right)^{j} \quad \text{… (14)}$$

Then all $a_{ij}$ are the eigen values of *P*

For small values of *j*, it is possible to check directly whenever an expression of the form (14) can be obtained. This is obviously possible for $j=0$ since we may choose $a_{00}=1$. Similarly since $E\left\{X\left(t+1\right)/X\left(t\right)\right\}=X\left(t\right)$ an equation of the form (14) is obtained by putting $a_{01}=0, a_{11}=1,$ when $j=2$, $E\left\{X\left(t+1\right)/X\left(t\right)\right\}=X\left(t\right)+\left\{1-\left(2N\right)^{-1}\right\}\left\{x\left(t\right)\right\}^{2}$ which is of the form (8) with $a_{22}=1-\left(2N\right)^{-1}$

By using well known properties of the binomial distribution, it is found generally that an expression of the form (14) is possible for all *j* and that

$$a_{jj}=2N\left(2N-1\right)\left(2N-2\right)...\left(2N-j+1\right)\left(2N\right)^{j},\ j=1,2,...,2N$$

Thus the largest non-uniteigen value of *P* is $a_{22}$ and for large population, this is extremely close to unity.

We conclude that, while in the model (7) genetically variation must eventually be lost through random loss of one or otheralleles, the rate of such loss is extremely slow, so that genetic variation will be preserved for a long time. This conclusion which will be reconsidered and supported from several different viewpoints later may be thought of as the stochastic extension of ‘Variation-preserving’

### 3.2 Deviation of the Hardy-Weinberg theorem:

Before going any further, it is useful to consider a theorem which can be used to find $\lambda_2$ by itself and which gives values quite readily of $\lambda_2$ in more complex situations (Hardy, G. Hand Weinberg W.1908).

**Theorem (1)**Let $Y\left(t\right)$ be a random variable which is zero for $X\left(t\right)=0, X\left(t\right)=2N,$ and positive otherwise if

$$E\left\{Y\left(t+1\right)/Y\left(t\right)\right\}=CY\left(t\right)$$

Where $C$ is a constant, then $\lambda_2=C.$The proof of the theorem results follows directly from (8) and by noting that the above equation implies $E\left\{Y\left(t+1\right)/Y\left(0\right)\right\}=C^{t}Y\left(0\right)$

For all *t*, indeed the various values of $Y(.)$ as $X(.)$ assumes in true the values of $0,1,2,...,2N$ will continue.

The right eigenvector of *P*. Corresponding to

For the model (7), the simplest function satisfying the conditions required is

$Y(t) = X(t)(2N - X(t))$

Clearly $E\{Y(t+1)\} = 2NE\{X(t+1)\} - E\{X(t+1)\}^2$

$= \{1-(2N)\}^{-1} Y(t)$ it follows directly from the theorem that $= 1-(2N)^{-1}$.

As was found previously.

3.3 A second approach, which shows that variation is lost slowly is to find the mean time to absorption at $X(i) = 2N$, given the value of $X(0)$, unfortunately, exactly result seem to be impossible to find and consideration of this mean absorption time, which is a more easily interpreted and probably more relevant quantity than the leading eigen value.

**3.4 Generalization of the Effective Population Size**

We have seen in the previous selection, that even when no directed disturbing forces act, variation ends to be lost very slowly by random sampling (i.e., chance) effects. The largest non-unit eigenvalues of (7), namely $1-(2N)^{-1}$ can be taken as a measure of the rate of distribution of variation. Suppose that in more general models, the largest non-unit eigen values (when no directed for as act) of relevant transition matrix is

$$\lambda = 1-(2N)^{-1} \tag{15}$$

For some constant $N_e$. By analogy with equating the constant *N* will be called the 'effective population size'.

For the model under consideration the effective population size so defined shouldn't be attributed significance beyond that deriving from the definition of above,

There are other definitions of effective population (Crow and Kiruma 1963, Waterson (1964), which for other purpose may be more relevant than the one just given, to evaluate *N* in specific case the following theorem often useful.

**Theorem (2)**

Let $X_1(.), X_2(.), ..., X_n(.)$ be a set of jointly markovian random variables with transition matrix

$$P = \{p_{ij}\}$$

Where $P_{ij} = \text{prob}\{x_1(t+1) = j_1 ... X_n(t+1) = j / X(t) = j_1, ..., X_n(t) = i_n\}$

Suppose that $x_t(.) \geq 0$ and $\sum_i Xt(.) \leq 2N$. Let $Y(.)...Y(j)$ be $r(r < 2N+1)$ function of the $Xt(.)$ such that

$$E\begin{pmatrix} Y_1(t+1) \\ Y_2(t+1) \\ . \\ . \\ Y_n(t+1) \end{pmatrix} = M\begin{pmatrix} Y_1(t)+1 \\ Y_2(t)+1 \\ . \\ . \\ Y_n(t)+1 \end{pmatrix}$$

Where *M* is a matrix of constant *s* and *E* denotes expected conditional on given values at generation *t*. Then the eigenvalues of *M* are included amongst the eigenvalues of *M*. Further, if each $Yt(.)$ is zero for a absorbing states of *P*, is non-negative otherwise and positive for at least one state of the vector $\{X_1(.),...X_n(r)\}$ then the largest non-unit eigenvalues of *M* is identical to the largest non-unit eigen values of *P*.

Quantitative Genetics is the science of variation deals with trait that vary continuously in the population, based on measures of individuals within a population organism. Most quantitative traits involves the contribution of many different genes, which are influenced by environmental factor the work of British Mathematician and Genetist, R.A. Fisher Founded Modern Quantitative Genetics (Biometric) followed by Karl Pearson and Francis Galton.

**3.5 Statistical Estimation of Recombination Frequencies**

The human genetics, mapping human genetic loci *n*(location of genes, makers, phenotypes). Thus by the binomial expansion theorem, in a given with *n* exchanges, the frequency of chromatids with *k* crossovers will be

$$K = \{n(n-1)(n-2)...(n-k+1)/k(k-1)(k-2)...1\}(0.5)^n$$

$a_n =$ the frequency of chrotomatids with crossovers

$a_2 =$ the frequency of double crossover

$E_n =$ the frequency of treads with *n* exchanges

Mathematically $a_2 = (0.5)^2 E_2 = 0.25(E_2)$

$a_1 = (0.5)(E_1) + 0.5(E_2)$

$a_0 = E_0 + (0.5)E_1 + (0.5)^2 E_2$

$= E_0 + (0.5)E_1 + 0.25(E_2)$

We can then use simple algebra to solve the system for the values of *E*

$E_2 = 4a^2$

$E_1 = 2a_1 - 4a_2$

$E_0 = a_0 - a_1 + a_2$

$$a_0 = E_0 + 0.5(E_1) + 0.25(E_2) + 0.125(E_3) + 0.0625(E_4)...P_n, 0\left(0.5^n\right)E_n$$

$$a_1 = 0.5(E_1) + 0.5(E_2) + 0.375(E_3) + 0.25(E_4)...P_n, 1\left(0.5^n\right)E_n$$

$$a_2 = 0.25(E_2) + 0.375(E_3) + 0.375(E_4) + ...P_n, 2(0.5)E_n$$

$$a_3 = 0.125(E_3) + 0.25(E_4) + ...P_n, 3(0.5)E_n$$

$$a_4 = 0.0625(E_4) + ...n, 3\left(0.5^n\right)E_n$$

In general,

$$a_m = p_m, m(0.5m)E_m + P_{m+1}, m(0.5)_{m+1} E_{m+1} P_{m+2}, m(0.5\ m+2)E_{m+2} + ..., P_n, m(0.5)^n E_n$$

Here, up to $n$ crossovers are observed. The value $P_{n,k}$ is given by the binomial co-efficient formula

$$(\text{Pascal's formula})\ \left[n(n-1)(n-2)...(n-k+1)/\{k(k-1)(k-2)...1\}\right]$$

When these equation are solved our final equation for estimating the frequency of a tetral with $r$ crossovers and no snisterchromosid exchange becomes

$$E_1 = 2r\left[a_r - a_{(r+1)} + a_{(r+2)} - a_{(r+3)} + ... + (-1)^n a_n\right]$$

**Statistical estimation of recombination frequencies**

$$\text{LoD} = \log_{10} \frac{\{\text{probability of obtaining the observed data if the two genes are linked}\} \text{ with a recombination frequency } Q_q}{\text{Probability of obtaining the observed data if the two genes are unlinked } (Q = 0.5)}$$

The odds ratio in this case compares the odds (or) probability that the marker and locus are linked (close together on the same chromosome) to the odds that the marker and locus are unlinked (on separate chromosomes or from part on the same chromosome). In doing this comparison, we have to test specific distances between the items, such as testing for whether they are so close together

That 1% that 20% of the meiostic events shows recombination $Q = 0.20$

$Z = n\log(z)$ when$n$ is the number of informative matrix

$Z = n\log(Z) + k\log(Q) + (n-k)\log(1-k)$

Where $n$ is the number of recombination individuals

$K$ the number of recombination individuals

**3.6 Multidimensional Change of Variables**

Let $X_1, X_2, ..., X_n$ be continuous random variables having joint density *f*. Let $Y_1, Y_2, ..., Y_n$ be random variable defined in terms of the *x*'s. In this section we will discuss a method for finding the joint density of the *Y*'s in terms of *f*. We will consider mainly the case when the Y's are defined as linear function of the *X*'s, suppose that the $Y = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} X_j;\ i = 1, 2, ..., n,\ j = 1, 2, ..., m$

The constant co-efficients $a_{ij}$ determine an $nX_n$ matrix $A = \left[ a_{ij} \right] = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ a_{n1} & \dots & a_{nm} \end{bmatrix}$

Associated with such a matrix is its determinants $\det A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ a_{n1} & \dots & a_{nm} \end{bmatrix}$

In $\det A = 0,$ there is a unique inverse matrix $B = \left[ b_{ij} \right]$ such that $BA = 1$ or equivalently

$$\sum_{k=1}^{n} b_{ik} a_{kj} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

The constants $b_{ij}$ can be obtained by solving for each $i$ the system (1) of $n$ equation in the $n$unknowns $b_{11}, b_{12}, ..., b_{1n}$. Alternatively, other constants $b_{ij}$ are uniquely defined by regarding that the equation. In recent years there has been an even increasing inference in the study of system which vary in time in a random manner. Mathematical models such systems are known as stochastic process.

**Theorem 4**

Let $X_1, X_2, ... X_n$ be continuous random variable having joint density $f$ and let random variables $Y_1, ..., Y_n$ be defined by

$$Y_i = \sum_{j=1}^{n} a_{ij} x_j \quad i = 1, 2, ..., n$$

Where the matrix $A = \left[ a_{ij} \right]$ has non zero determinant $\det A,$ then $Y_1, ..., Y_n$ have joint density $\left( f \;\; y_1, ..., y_n \right)$ given by

$$fy_1, ..., Y_n \left\{ y_1, ..., y_n \right\} = 1 / \left| \text{Det}\, A \right| f\left( x_1, ..., x_n \right)$$

Where the $x_i$'s are defined in term of the $y_i$ by equation (2) or as the unique solution to the equations $y_i = \sum_{j=1}^{n} a_{ij} X_j$. This theorem, which are willn't prove here, is equivalent theorem proved in Jacobian Transformation. From the general results proved in advance calculus. We can attend the above theorem to nonlinear changes of variables. We will describe this extension briefly although it will not be needed later. Let the $Y$'s be defined in terms of $X$'s by $Y_i = G_i\left( X_1, ... X_n \right) \;\; i = 1, ..., n$

Suppose that these equation define the $r_i$'s unique in terms of the $y_i$'s that the partial derivatives $\partial y_i / dx_i$ exist and the continuous and that the jacobian,

$$J\left( x_1, ..., x_n \right) = \begin{vmatrix} \partial y_1 / dx_1 & \dots & \partial y_1 / dx_n \\ \dots & \dots & \dots \\ \partial y_n / dx_1 & \dots & \partial y_n / dx_n \end{vmatrix}$$

Is everywhere non zero. Then the random variable $y_1, ..., y_n$ are continuous and have a joint density given by $fy_1, ..., y_n \left( y_1, ..., y_n \right) = 1 / J\left( x_1, ..., x_n \right)^y \; f\left( x_1, ..., x_n \right)$

Where the $x's$ is defined implicitly in terms of the $y_i$'s by equation (4). This change of variable formula can extended still further by requiring that the function $y_i$ be defined only on some open subset $S$ of $R$ such that $P(x_1,...,x_n)=1$

In the special case when $y_j = \sum_{j=1}^{n} a_{ij}x_{ij}$ we see that $dy_i / dx_i = a_{ij}$ and $J(x_1,...,x_n)$ is just the const.

$\det A = \det[a_{ij}]$, so it is clear the equation (14) respect to (9) in linear case.

Let $X_1,...,X_n$ be independent random variables each having an exponential density with parameter. Define $Y_1,Y_2,...,Y_n$ by $Y_i = X_i + .... + X_n$ , $1 \le i \le n$

Find the joint density of $Y_1,...,Y_n$

The Matrix $(a_{ij})$ is $\begin{bmatrix} 1 & 0 & ... & 0 \\ ... & ... & ... & ... \\ 1 & 0 & ... & 1 \end{bmatrix}$

It determinant is clearly 1, the equations

$Y_i = x_1 + x_n,\ i = 1,2,...,x_n$

Have the solution,

$X_1 = Y_1$

$X_i = y_i - y_i - 1,\ i = 2,...,n$

The joint density of $x_1,x_2,...,x_n$ is given

$$F(x_1,...,x_n) = \begin{cases} \lambda^n e^{-\lambda} e(x_1 + x_2 + ... + x_n),\ x_1,x_2,x_n > 0 \\ 0,\ \text{elsewhere} \end{cases}$$

Given, the joint density $fy_1,..., y_n$ is given by

$$Y_1 = \sum_{i=1}^{n} a_{ij}x_j, i = 1,2,...,n$$

Have solution $x_i = b_{ij}y_i;\ m = 1,...,n$

### 3.7 Selection of a Genotype

A given cell possess two chromosomes. A chromosome can be viewed as a string of genes, each gene being at a specific location in the class

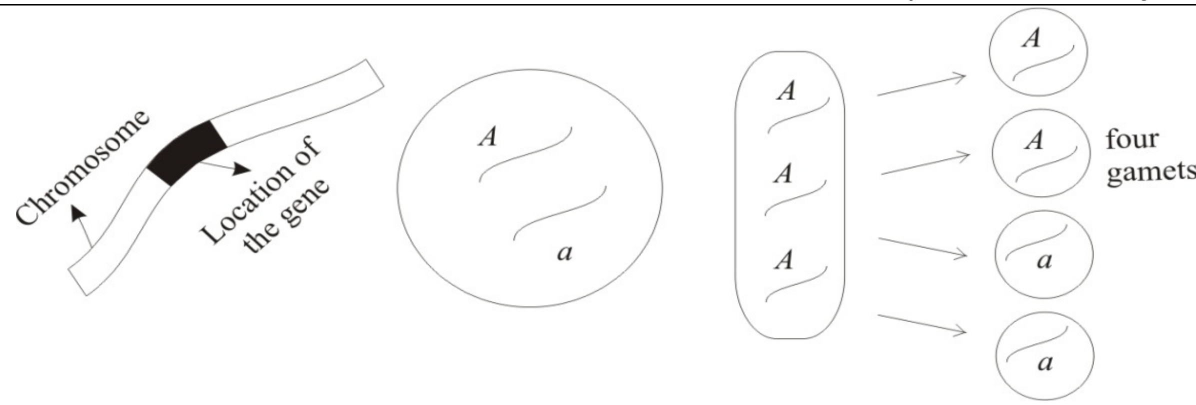



**Diagram 1**
**(Here under the form, or allel, A)**

A schematic representation of a chromosome so the chromosome doubles and fourwards are formed for every chromosome

Let us start from an indefinitely infinite population, where the genotypes are found in the following proportion

$$AA : Aa : aa$$
$$X, \quad 2z, \quad y$$

Here, $x$, $y$ and $z$ are number before 0 and 1

$$x + 2z + y = 1$$

### 3.8 Galton Watson Branching Process

The English Statistician Galton's initiated the theory of Branching Process, he studied the transition of family names through generation and we particularly interested in estimating Survival probability of a given branch in a generalized tree. A particularly simple model of the situation be investigated is the following one. All the individuals of a given colony (eg the male population of a noble family) give birth in their life time to a random number of descendants (eg. The male descendant in the particular example of interest is to Galton. Each individual of the colony process independently.

All other members of the colony, if $X_n$ denotes the size of the $n^{\text{th}}$ generation

$$X_{n+1} = \begin{cases} \sum_{i=1}^{n} Z_n & \text{if } X_n \geq 1 \\ 0 & \text{if } x_n \geq 0 \end{cases}$$

Where $Z_n, i > 0, n > 0$ are identically independent random variables integers valued with common generating function $z(s) = \sum_{n=0}^{\infty} P(z = n) S^n$

With finite mean $m$ and finite variance $\sigma^2$, in this model the number $x_0$ of ancestor is naturally supposed to be independent of the random variable $(z_n, i \geq 0, n \geq 0)$. This simple model is particular case of branching process

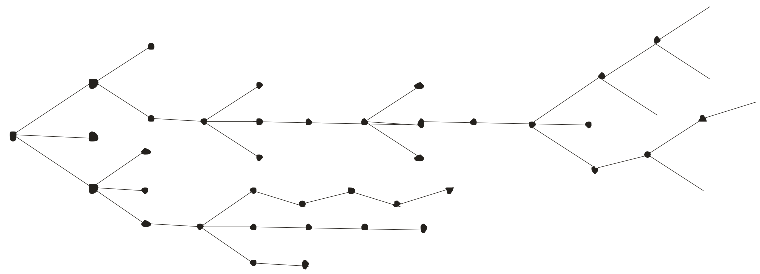


**Diagram 2**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $x_0 = 1$ | $x_1 = 3$ | $x_2 = 3$ | $x_3 = 5$ | $x_4 = 6$ | $x_5 = 6$ | $x_6 = 5$ | $x_7 = 5$ | $x_8 = 2$ |
| $x_9 = 7$ | $x_{10} = 10$ | $x_{11} = 8$ | $x_{12} = 10$ | $x_{13} = 9$ | $x_{15} = 3$ | $x_{16} = 1$ | $x_{17} = 2$ | $x_{18} = 0$ |

A typical realization giving rise to a genological tree when there is one ancestor $x_0(w) = 1$. In this example there is extinction of the family name at the $n^{\text{th}}$ generation

Watson motivates the terminology and gave the following analysis. Denote by $Q_n$, the generating function of $X_n$ $\emptyset_n(S) = \sum_{k=0}^{\infty} P(X_n = k) S_k$

Showing that $\emptyset n+1(s) = \emptyset n(gz(s))$

Deduce from the above relation that if $x_0 = 1$ (one ancestor)

$$Qn+1(s) = g_2(Q_n(s))$$

To find $P_e$ we must therefore study the equation $x = g_2(x),\ x \in [0,1]$, clearly

$g_2^{'}(x)\ b = \sum_{n=0}^{\infty} np(z=n)^{n-1}\ x \geq 0$

For $x \in [0,1]$ we exclude the trivial case when $P(z=0) = 1$

$g_2^{'}$ strictly positive in $(0,1)$

Also $g'z(0) = P(Z=1)$ and $g'z(1) = E(Z) = m$

Differentiating once more, we see that $g''z(x) = \sum_{n=2}^{\infty} n(n-1) P(z=n) x^{n-2}$

$$\geq 0 \text{ in } [0,1]$$

So that $g_2$ is a $u$-convex function

We have been absorbing the linearity property having used it for initiate given (Lebsegue derivative convergence theorem). In view of the independent assumption.

$E\left[\{T=n\} S^{x_1+x_2\ldots,x_n}\right] = E\left[1\{T=n\}\right] E\left[S^{x_1+x_2\ldots x_n}\right]$

$= E\left[1\{T=n\}\right] E\left[S^{x_1}\right] \ldots \left[S^{x_n}\right]$

$= P[T = n]\, g_x(s)^n$

Finally, $E\left[S^Y\right] = \sum_{n=0}^{\infty} P(T = n)\, S_X(S)^n$

That is $S_y(s) = S_y(s) = S_T(g_x(s))$

## CONCLUSION

Let $\left(\overline{x}, \overline{y}\right)$ denote the state of the offspring population at the birth stage, this is obtained from the inheritance co-efficient by using the assumption of panmixia.

$$\overline{x_j} = \sum_{i,k=1}^{n,v} p_{ik,j}^{(f)} x_i y_k ; \quad \overline{y_l} = \sum_{i,k=1}^{n,v} p_{ik,l}^{(f)} x_i j_k \ \dots (1)$$

From the birth stage to the reproductive stage of the offspring generation the survival co-efficient describe the effects of selection due to differential viability

We have by formula of Bayes Theorem

$$x_j^{'} = \frac{\lambda_j^{(f)} \overline{x_j}}{\sum_{r=1}^{n} \lambda_r^{(f)} \overline{x_r}} \quad y_j^{'} = \frac{\lambda_l^{(m)} \overline{y_l}}{\sum_{p=1}^{v} \lambda_e^{(m)} \overline{y_p}} \ \dots (2)$$

Combining equation (1) and (2) we get the evolutionary equations

$$x_j^{'} = \frac{1}{w^{(f)}(x, y)} \sum_{i,k=1}^{n,v} w_{ik,j}^{(f)} x_i y_k \quad y_l^{'} = \frac{1}{w^{(m)}(x, y)} \sum_{i,k=1}^{n,v} w_{ik,j}^{(m)} x_i y_k$$

$$w_{ik,j}^{(f)} = p_{ik,j}^{(f)}, \lambda_j^{(f)} \text{ and } w_{ik,l}^{(n)} = p_{ik,l}^{(m)},\ \lambda_l^{(m)} \ \dots (3)$$

These are fitness coefficients (corresponding to the male and female portion of the population) with mean values

$$w^{(f)}(x, y) = \sum_{i,k,j=1}^{n,v,n} w_{jk,j}^{(f)} x_i y_k$$

$$w^{(n)}(x, y) = \sum_{i,k,j=1}^{n,v,v} w_{jk,j}^{(m)} x_i y_k$$

The bilinear forms are called the mean fitness for male and female gender when the population state is $(x, y)$

If a population is autosomal then the male and female types are identical and in particular $n = v$ the inheritance and survival co-efficients are the same for male and female offspring, defining

$$p_{ik,j} \equiv p_{ik,j}^{(f)} = p_{jk,j}^{(m)}; \quad \lambda_j^{(f)} \equiv \lambda_j^{(f)} = \lambda_j^{(m)} \ \dots (4)$$

Observed that with the assumption of (4) the offspring distribution for male and female are the same both at the birth size $\overline{x} = \overline{y}$ for

$$\overline{x_j} = \sum_{i,k=1}^{n,v} p_{ik,j}^{(f)} x_i y_k ; \quad \overline{y} = \sum_{i,k=1}^{n,v} p_{ik,j}^{(m)} x_i y_k$$

and at adult stage.

A specific problem in population biology an allel may have pleitropic effects (see perseet. Al and references there in this volume) one of the leading theories for the evolution of references involves "antagonistic plentropy" such that alleles that decrease survivorship of olden age classes can increases in frequency when rare, provided they have a pleotropic effect boosting early fitness (generally frequently at younger age). An important question is how natural selection shapes the genetic correlation between early and late age traits associated with fitness.

The simplest argument is that relation fixed alleles whose effects are positive (increase fitness) on both early and latefitness components.The selection foundation for life history has been widely studied and classify fitness is selected to the intrinsic rate of increase of a population. If a genotype confers a population specific schedule of survivorship $(l_m)$ and fecundity $(m_n)$ then the intrinsic rate of increase of a population composed exclusively of that type is obtained from the Euler equation

$$1 = \int e^{-rn} l_n m_n d_x$$

A convenient way to envisage the age specific survivorship and fecundity is to use parametric function.Fecundity schedules often shows a unimodel distribution and a Gaussian distribution on a logarithmic age scale fits may experimental data

There is a good fit to the cumulative normal distribution.The possibilities of evolution of offspring sizes in terms of death rate $d$ and the operative growth rate of a function of the offspring size $\sigma$ with a fixed is $g$.Consider for simplicity suppose it breeds by random mating at a discrete time terminating each generation.

This assumption produces models of mixed genetic simplicity as everybody breeds at the same time. However, for ecological consideration of course, the organism has a continuous development during each generation.The evolutionary importance consequence of natural selection that the genetic composition and hence the typical phenotype of population change. Therefore by natural selection often studied by considering the selection induced on the genotypic variation as a result of the connection between the genotype and phenotype.

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# LATTICES IN CRYPTOLOGY

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**ABSTRACT**

***The use of mathematical concepts in Cryptology applications can result in safer data transfer between two parties and can hide the original message quite effectively. It is the novelty and skillful use of the mathematical ideas in hiding of the original messages which results in a safe data transfer. This paper presents Lattices from Discrete Mathematics in order to achieve a highly safe data transfer between two parties and presents the techniques for encryption and decryption for the same.***

***Keywords : Lattice, Lattice Complement Matrix, Lattice Adjacency Matrix, , Complement, Hasse diagram.***

## 1. INTRODUCTION

[7] defines the lattice-based cryptography as an area of research for construction of tools and protocols for safe data transfer based on hard lattice problems. [4] also discusses lattice based cryptography talking about geometric aspect , modern formalism (SIS –LWE) and uses the definition of Lattice L as a discrete subgroup of a finite-dimensional vector space.[2]discusses the Lattice Reduction and the LLL Algorithm, NTRU public key cryptosystem and NTRU Lattices etc. and uses the definition of lattice L as of having dimension n and being a maximal discrete subgroup of Rand equivalently a lattice is the Z-linear span of a set of n linearly independent vectors. Hard mathematical problems and class of finding closest and shortest vectors in lattices is also discussed. [3] have discussed about the post-quantum cryptography using lattices using concepts of hash function, random oracle model etc. - the focus in this work is mainly on the practical aspects of lattice-based cryptography and less on the methods used to establish their security. [5] gives directions of using graphs and which translate in using Lattices also for similar applications in hiding of the messages. Graph labellings such as *inner magic* and *inner antimagic* discovered in [1] also have been applied in cryptographic application in [6] and labeled graph have been used in cryptographic applications in [8] employing some particular graphs which could also be used for similar mathematically structures, possibly Lattices. Lattices of Discrete Mathematics can be found in [9]. In Geometry and Group Theory, a Lattice in $R^n$ is a subgroup of the additive group $R^n$ which is isomorphic to the additive group $Z_n$, and which spans the real vector space $R^n$. Based on this view-point,

Lattices have applications in Lie Algebras, Number Theory, Group Theory. In Cryptography, Lattices have applications due to conjectured computational hardness of Lattice problems and used in Physics etc. In the present work, the Lattice concept from Discrete Mathematics has been used as compared to the ones used so far discussed above, and is from a different point of view and can result in effectively safe data transfer. Section 2 gives the fundamental definitions and defines certain new concepts developed in this work. Section 3 presents the main results and section 3.1 presents three aspects of the same cube Lattice whereas section 3.2 presents a different Lattice. In both the sections, encryption and decryption algorithms have been presented.

## 2. FUNDAMENTAL DEFINITIONS AND SOME NEW CONCEPTS

In Discrete Mathematics, a Lattice has been defined as follows :

**Lattice :**A Lattice is defined as a partially ordered set (poset), $(L,\leq)$ in which every subset $\{a,b\}$ consisting of two elements has a *least upper bound* (LUB) and a *greatest lower bound* (GLB). WE denote LUB($\{a,b\}$) by $a \vee b$ and call it *join* of a and b. similarly we denote GLB($\{a,b\}$) by $a \cdot b$ and call it *meet* of a and b. Thus, a Lattice is a mathematical structure with two binary operations, *join* and *meet*.

**Lattice Complement Matrix :** the matrix entry in this matrix $a_{ij} = 1$ if j is complement of i otherwise $a_{ij} = 0$.

The idea behind the Lattice Adjacency Matrix is that of the adjacency matrix used for graphs in Graph Theory and has been used in this work similarly for Lattices.

**Lattice Adjacency Matrix :**just like the adjacency matrix for a graph, a matrix entry $a_{ij} = 1$ if there is an edge existing between vertices i and j otherwise $a_{ij} = 0$.

## 3. MAIN RESULTS

It may be noted that in this work, whenever letters of alphabet appear, they can be represented in the following manner to enhance hiding of the original message :

1. Letter itself (ASCII code)
2. Position number of the letter in going from left to right (forward manner) in the alphabet.
3. Position number of the letter in going from right to left (reverse) in the alphabet.

### 3.1 Different Facets of the Lattice in the shape of a Cube

Let us consider the lattice in the shape of a cube with bits as its vertex labels such that adjacent bits differ in exactly one bit. In decimal, the numbers are 0, 1, 2, 3, 4, 5, 6, 7. This cube is shown in **Figure 1**.

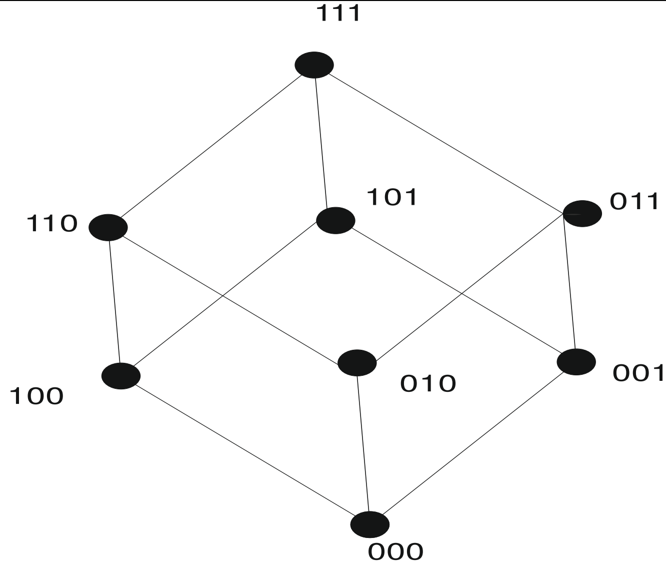


**Figure 1 :**Lattice with bits, adjacent bits differing in exactly one bit

**Illustration :**

In **Figure 1**, the decimal of 110 (binary) = $0 \times 2^0 + 1 \times 2^1 + 1 \times 2^2 = 6$

Similarly decimal of 111 = 7, decimal of 101 = 5, decimal of 100 = 4, decimal of 011 = 3, decimal of 010 = 2, decimal of 001 = 1, decimal of 000 = 0.

Consider the plain-text : THEORIES whose letters are assigned to the vertices of this lattice as follows :

T = 0, H = 4, E = 1, O = 2, R = 6, I = 5, E = 3, S = 7

Calculations of cipher-text are by 3 methods shown below:

1. **Addition operation** : adding the decimal equivalent of the binary numbers to the letters of the plain-text as follows :

T + 0 = T, H + 4 = L, E + 1 = F, O + 2 = Q, R + 6 = X, I + 5 = N, E + 3 = H, S + 7 = Z

Therefore, the cipher-text is :**T L F Q X N H Z**

Another cipher-text in reverse order :**Z H N X Q F L T**

2. **Subtraction operation** :

T – 0 = T, H – 4 = D, E – 1 = D, O – 2 = M, R – 6 = L, I – 5 = D, E – 3 = B, S – 7 = L

Therefore, the cipher-text is **T D D M L D B L**

Another cipher-text in reverse order is **L B D L M D D T**

3. **Modified operation :**addition and multiplication mod 8 has been taken for the 8 vertices of lattice assigned to 8 characters of the plain-text.

$(T\oplus_8 0)\oplus_8(T\odot_8 0) = 0 = T$ (here T = 0)

$(H \oplus_8 4)\oplus_8(H \odot_8 4) = 0 = H$ (H = 4)

$(E\oplus_8 1)\oplus_8(E\odot_8 1) = 3 = H$ ( E = 1)

$(O\oplus_8 2)\oplus_8(O\odot_8 2) = 0 = O$ ( O = 2)

$(R\oplus_8 0)\oplus_8(R\odot_8 5) = 4 = V$ (R = 6)

$(I\oplus_8 0)\oplus_8(I\odot_8 5) = 6 = O$ (I = 5)

$(E\oplus_8 0)\oplus_8(E\odot_8 5) = 2 = G$ (E = 3)

$(S\oplus_8 0)\oplus_8(S\odot_8 5) = 2 = U$ ( S = 7)

Therefore, the cipher-text is :**T H H O V O G U**

Another cipher-text in reverse order is **U G O V O H H T**

The letters of the plain-text can be assigned differently to yield other distinct cipher-texts.

**Encryption** :

1. After calculating the cipher-text, make the following Encryption-Matrixcorresponding to the addition operation. For the other twooperations, we can proceed similarly :

**Encryption-Matrix**

| **Permuted Cipher-text** *For explanation purpose only* - the associated letters of the original Plain-text are written in parentheses | **Ordered pair (D,P)** where D stands for Decimal numberof the vertex label and P stands for the associated letter of the Plain-text | **Vertex label** in binary and decimal (*optional column and for explanation purpose only*) |
|---|---|---|
| Z (S) | (7,8) | 111 (7) |
| X (R) | (6,5) | 110 (6) |
| N (I) | (5,6) | 101 (5) |
| H (E) | (3,7) | 011(3) |
| L (H) | (4,2) | 100 (4) |
| Q (O) | (2,4) | 010 (2) |
| F (E) | (1,3) | 001 (1) |
| T (T) | (0,1) | 000 (0) |

2. Send the kind of operation : addition, subtraction or modified operation along with the Encryption-Matrix and Lattice Adjacency Matrix to the receiver.

**Lattice Adjacency Matrix for Lattice in Figure 1**
**Vertex (in decimal)**

| Vertex (in decimal) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| **0** | **0** | **1** | **1** | **0** | **1** | **0** | **0** | **0** |
| **1** | **1** | **0** | **0** | **1** | **0** | **1** | **0** | **0** |
| **2** | **1** | **0** | **0** | **1** | **0** | **0** | **1** | **0** |
| **3** | **0** | **1** | **1** | **0** | **0** | **0** | **0** | **1** |
| **4** | **1** | **0** | **0** | **0** | **0** | **1** | **1** | **0** |
| **5** | **0** | **1** | **0** | **0** | **1** | **0** | **0** | **1** |
| **6** | **0** | **0** | **1** | **0** | **1** | **0** | **0** | **1** |
| **7** | **0** | **0** | **0** | **1** | **0** | **1** | **1** | **0** |

It is to be noted that the Lattice Adjacency Matrix for all the lattices with the same diagram will remain the same even though the labels(names) of the vertices may be different.

**Decryption**

1. Prepare the Decryption-Matrix by arranging the second elements corresponding to P in the ordered pair (D,P) in ascending order.
2. Note that the 1$^{st}$column in this matrix corresponds to the first elements in theordered pair (P,D) and the 3rd column corresponds to the second elements of the ordered pair (P,D).
3. The original plain-text THEORIES is obtained along with the diagram of the Lattice with the assignments of the vertices of the Lattice to the letters of the Plain-text.
4. To draw the original Lattice, join the vertices by edges (lines) which differ in exactly one bit or by the Lattice Adjacency Matrix.

**Decryption-Matrix**

| **Original Plain-text** | **(P,D), P in ascending order** | **Vertex label** |
|---|---|---|
| T | (1,0) | 0 (000) |
| H | (2,4) | 4 (100) |
| E | (3,1) | 1 (001) |
| O | (4,2) | 2 (010) |
| R | (5,6) | 6 (110) |
| I | (6,5) | 5 (101) |
| E | (7,3) | 3 (011) |
| S | (8,7) | 7 (111) |

Similarly, for the other kinds of operations also, the Encryption-Matrix and Decryption-Matrix can be created.

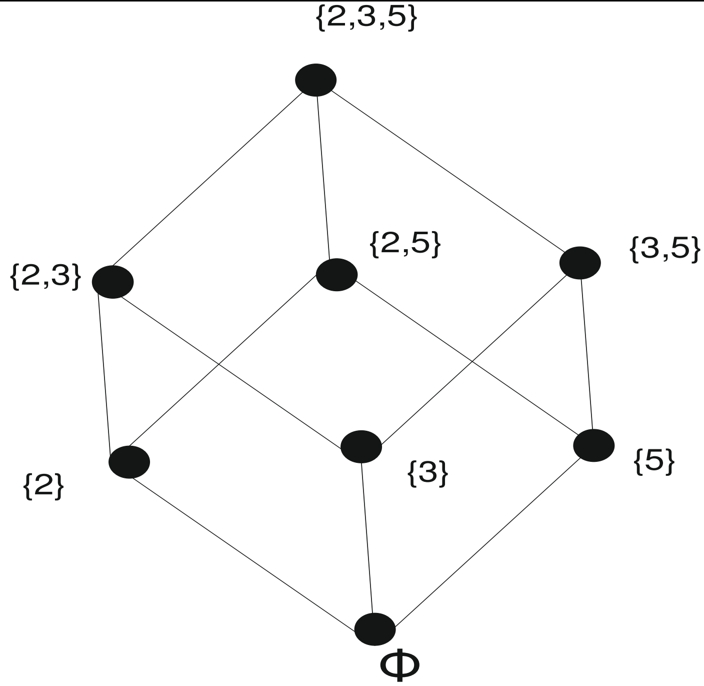


**Figure 2** : Lattice with elements of the Power Set of 3 elements.

(P(S),≤) is a Lattice where P(S) is the Power Set consisting of all the subsets of a set S. For example, in the set S= {a,b,c}, P(S) = {{Φ}, {a}, {b}, {c}, {a,b}, {b,c}, {a,b,c}} and the relation "≤" is that of set inclusion (subset). The *meet* operation is the set *intersection* and the *join* operation is the set *union*. This is also in the shape of cube and is shown in **Figure 2**.

**Illustration :**

The threeoperations demonstrated in the above method can be applied here also and for illustration purpose one of these three operations, namely addition is being shown below :

The letters of the plain-text THEORIES are assigned respectively to T = 0 (empty set), H = 2, E = 5, O = 3, R = 2 + 3 = 5, I = 2 + 5 = 7. E = 3 + 5 = 8, S = 2 + 3 + 5 = 10.

Employing the addition scheme of method we get :

T + 0 = T, H + 2 = J, E + 5 = J, O + 3 = R, R + 5 = X, I + 7 = P, E + 8 = M, S + 10 = C

Thus, the cipher-text is : **T J J R X P M C**

Another cipher-text in reverse order : **C M P X R J J T**

**Encryption:**

1. After calculating the cipher-text, make the following Encryption-Matrix corresponding to the addition operation. For the other two operations, we can proceed similarly :

**Encryption-Matrix**

| **Permuted Cipher-text** *For explanation purpose only* - the associated letters of the original Plain-text are written in parentheses | **Ordered pair (S,P)** where S stands for the Sum of the elements of the subset of the particular vertexand P stands for the associated letter of the Plain-text | **Vertex label**being the subsets of the set {2,3,5} (*optional column and for explanation purpose only*) |
|---|---|---|
| C (S) | (10,8) or (K,I) | {2,3,5} |
| X (R) | (5,5) or (F,F) | {2,3} |
| P (I) | (7,6) or (H,G) | {2,5} |
| M (E) | (8,7) or (I,H) | {3,5} |
| J (H) | (2,2) or (C,C) | {2} |
| R (O) | (3,4) or (D,E) | {3} |
| J (E) | (5,3) or (F,D) | {5} |
| T (T) | (0,1) or (A,B) in this column, starting at A = 0 | {Φ} |

1. Send the Encryption-Matrix, kind of operation : addition, subtraction, modified operation and Lattice Adjacency Matrix to the receiver.

In the second column of the Encryption-Matrix, another option of ordered pairs of equivalent letters is also provided with the alphabets starting at A=0 and B=1, C=2 etc.

**Decryption :**

1. Prepare the Decryption-Matrix by arranging the second elements corresponding to P in the ordered pair (S,P) in ascending order.
2. Note that the 1st column in this matrix corresponds to the first elements in theordered pair (P,S) and the 3rd column corresponds to the second elements of the ordered pair (P,S). The Sum corresponds to the subset, for instance S = 7 corresponds to the subset {2,5} where 2 + 5 =7.
3. The original plain-text THEORIES is obtained along with the diagram of the Lattice with the assignments of the vertices of the Lattice to the letters of the Plain-text.
4. To draw the original Lattice, join the vertices representing subsets by the relationship of subset or set inclusion and by rules of drawing Hasse diagram or by Lattice Adjacency Matrix. For example, {2,3} is a subset of {2,3,5} so join these two vertices.

**Decryption-Matrix**

| **Original Plain-text** | **(P,S), P in ascending order** | **Vertex label**being the subset |
|---|---|---|
| T | (1,0) or B | {Φ} |
| H | (2,2) or C | {2} |
| E | (3,5) or D | {5} |
| O | (4,3) or E | {3} |
| R | (5,5) or F | {2,3} |
| I | (6,7) or G | {2,5} |
| E | (7,8) or H | {3,5} |
| S | (8,10) or I | {2,3,5} |

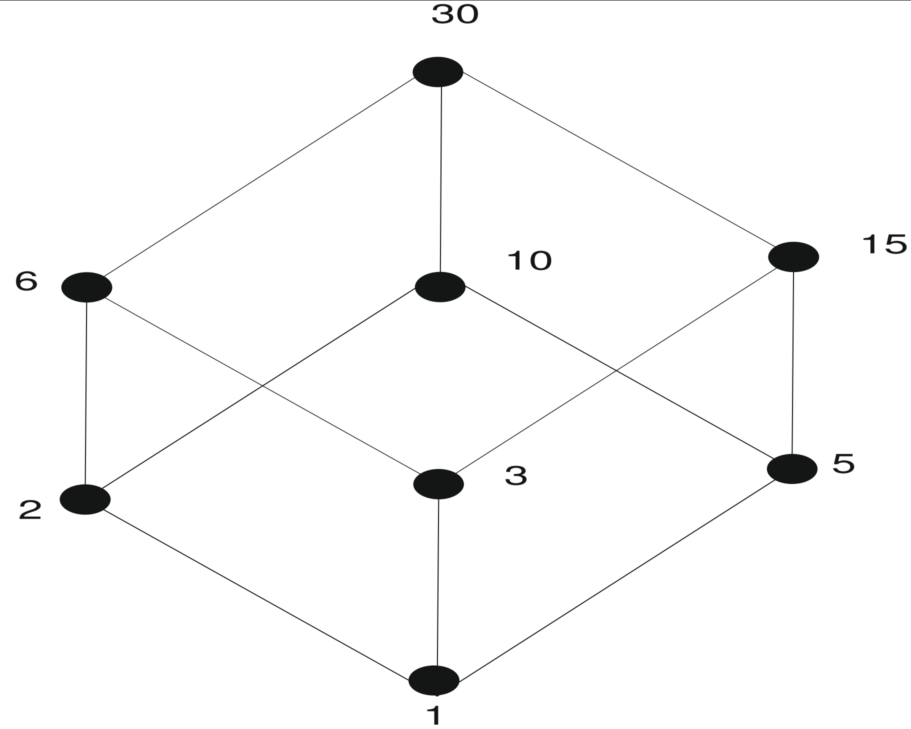


**Figure 3 :**$D_{30}$ (positive integer divisors of 30)

Now, we consider the Lattice $D_{30}$ with positive integer divisors of 30. This is also in the shape of cube and is shown in **Figure 3**.

**Illustration :**

In this Lattice, for illustration purpose, the following assignment of the letters of the plain-text have been made to the vertices with the integer divisors :

T =1, H = 2, E = 5, O = 3, R = 6, I = 10, E = 15, S = 30

The 3 operatios : addition, subtraction and modified operation can be used. For illustration purpose, addition is being used here.

Calculation of cipher-text using addition operation is as follows :

T + 1 = U, H + 2 = J, E + 5 = J, O + 3 = Q, R + 6, I +10 = S, E + 15 = T, S + 30 = W

Therefore, the cipher-text is : **U J J Q X S T W**

Another cipher-text in reverse order is : **W T S X Q J J U**

**Encryption :**

1. After calculating the cipher-text, make the following Encryption-Matrix corresponding to the addition operation. For the other two operations, we can proceed similarly :

**Encryption-Matrix**

| **Permuted Cipher-text** *For explanation purpose only*, the associated letters of the original Plain-text are written in parentheses | **Ordered pair (D,S)** where D stands for Divisor of 30 and S stands for the associated Sequence number of the letter of the Plain-text | **Vertex label** (divisor of 30) (*optional column and for explanation purpose only*) |
|---|---|---|
| W (S) | (30,8) | 30 |
| X (R) | (6,5) | 6 |
| S (I) | (10,6) | 10 |
| T (E) | (15,7) | 15 |
| J (H) | (2,2) | 2 |
| Q (O) | (3,4) | 3 |
| J (E) | (5,3) | 5 |
| U (T) | (1,1) | 1 |

1. Send the Encryption-Matrix, kind of operation : addition, subtraction, modified operation and Lattice Adjacency Matrix to the receiver.

**Decryption :**

1. Prepare the Decryption-Matrix by arranging the second elements corresponding to S in the ordered pair (D,S) in ascending order.
2. Note that the 1st column in this matrix corresponds to the first elements in the ordered pair (S,D) and the 3rd column corresponds to the second elements of the ordered pair (S,D).
3. The original plain-text THEORIES is obtained along with the diagram of the Lattice with the assignments of the vertices of the Lattice to the letters of the Plain-text.
4. To draw the original Lattice, join the vertices representing the least common multiple (LCM) of two vertices or by Lattice Adjacency Matrix. For example, since 6 is LCM of 2 and 3 so join the vertices labeled 2 to 6 and 3 to 6.

**Decryption-Matrix**

| **Original Plain-text** | **(S,D), S in ascending order** | **Vertex label** being the divisor of 30 |
|---|---|---|
| T | (1,1) | 1 |
| H | (2,2) | 2 |
| E | (3,5) | 5 |
| O | (4,3) | 3 |
| R | (5,6) | 6 |
| I | (6,10) | 10 |
| E | (7,15) | 15 |
| S | (8,30) | 30 |

**Depending on the assignment of letters of plain-text THEORIES to the 8 vertices, factorial 8 permutations are possible and as many and more cipher-texts can be obtained using the 3 kinds of operations.**

Thus, we see that with the same lattice diagram and different sets of vertex element assignment, numerous cipher-texts can be obtained. Moreover, these encryption and decryption techniques can be applied to other Lattices also.

### 3.2 Using Complement of an elementin a Lattice

**Complements of an element in a Lattice :** Let L be a bounded Lattice with *greatest element* I and the *least element* 0 and let a be an element in L. Then an element a’ in L is called a *complement* of a if a▪ a’ = I and a ▪ a’ = 0 and note that 0’ = I and I’ = 0.

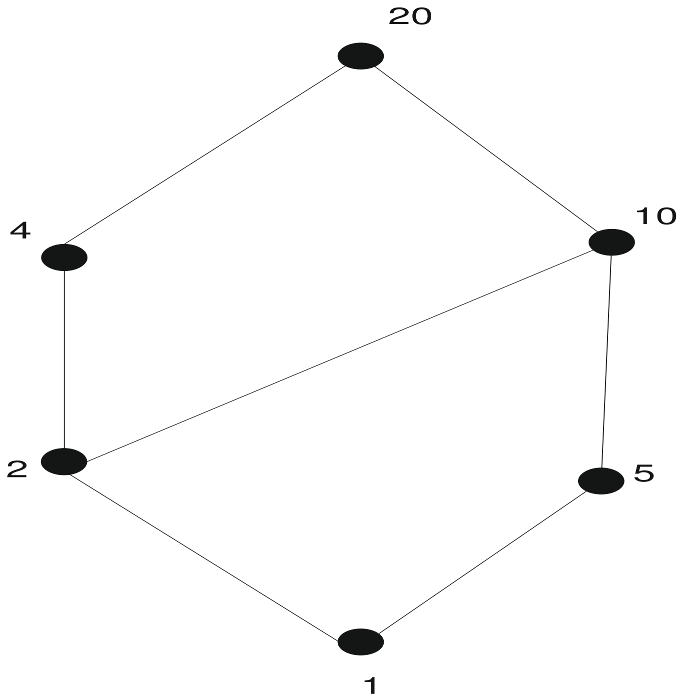


**Figure 4** :Lattice $D_{20}$ (positive integer divisors of 20)

### Illustration :

Let us take the complements of the elements of positive integer divisors of 20, the elements of this lattice $D_{20}$ = {1, 2, 4, 5, 10, 20}. Here, 1’ = 20, 20’ = 1, 4’ = 5, 5’ = 4; 2 and 10 have no complements.

The letters of the Plain-text GRAPHS are assigned with the vertex labels as follows : G = 4, R = 5, A = 2, P = 1, H = 10, S = 20.

For illustration purpose, taking addition as an operation, the Cipher-text is calculated by adding the complement to the letter of the Plain-text. thus,

G + 5 = L (the complement of G = 4 is 5)
R + 4 = V (the complement of R = 5 is 4)
A + 0 = A (the complement of A = 2 is 0)
P + 20 = J (the complement of P = 1 is 20)
H + 0 = H (the complement of H = 10 is 0 )
S + 1 = T (the complement of S = 20 is 1)

Therefore, the Cipher-text obtained is : **L V A J H T**
Another Cipher-text in reverse order is : **T H J A V L**

Prepare the Lattice Complement Matrix and the Lattice Adjacency Matrix and two kinds of Encryption-Matrix are obtained : Encryption-Matrix-1 and Encryption-Matrix-2 being presented one by one as follows :

**Lattice Complement Matrix**

| vertex | 1 | 2 | 5 | 4 | 10 | 20 |
|---|---|---|---|---|---|---|
| **1** | **0** | **0** | **0** | **0** | **0** | **1** |
| **2** | **0** | **0** | **0** | **0** | **0** | **0** |
| **5** | **0** | **0** | **0** | **1** | **0** | **0** |
| **4** | **0** | **0** | **1** | **0** | **0** | **0** |
| **10** | **0** | **0** | **0** | **0** | **0** | **0** |
| **20** | **1** | **0** | **0** | **0** | **0** | **0** |

**Lattice Adjacency Matrix**

| vertex | 1 | 2 | 5 | 4 | 10 | 20 |
|---|---|---|---|---|---|---|
| **1** | **0** | **1** | **1** | **0** | **0** | **0** |
| **2** | **1** | **0** | **0** | **1** | **1** | **0** |
| **5** | **1** | **0** | **0** | **0** | **1** | **0** |
| **4** | **0** | **1** | **0** | **0** | **0** | **1** |
| **10** | **0** | **1** | **1** | **0** | **0** | **0** |
| **20** | **0** | **0** | **0** | **1** | **1** | **0** |

**Encryption 1:**

1. Prepare the Encryption-Matrix-1 as follows :

**Encryption-Matrix-1**

| **Cipher-text,** ***for explanation purpose only*** **- letters of the Plain-text are written in parentheses** | **Complement** |
|---|---|
| L(G) | 5 |
| V(R) | 4 |
| A(A) | 0 |
| J(P) | 20 |
| H(H) | 0 |
| T(S) | 1 |

2. Send the Encryption-Matrix-1, Cipher-text, Lattice Adjacency Matrix and the Lattice Complement matrix to the receiver.

**Decryption 1:**

1. If the operation is addition then do the reverse operation i.e., subtract as following

   L - 5 = G

   V - 4 = R

   A – 0 = A

   J – 20 = P

   H – 0 = H

   T – 1 = S

2. The original Plain-text GRAPHS is obtained

**Encryption 2**

1. The second Encryption-Matrix-2 is obtained as follows :

**Encryption-Matrix-2**

| **Permuted Cipher-text,** ***for explanation purpose only*** **- letters of the Plain-text are written in parentheses** | **Complement pair** |
|---|---|
| A(A) | (3,0) or (D,A) |
| J(P) | (4,20) or (E,U) |
| L(G) | (1,5) or (B,F) |
| T(S) | (6,1) or (G,B) |
| V(R) | (2,4) or (C,E) |
| H(H) | (5,0)or (F,A) |

2. In the Encryption-Matrix-2, a Permuted Cipher-text is included and a Complement pair where the first element in the ordered pair denotes the sequence number of the letters of the original Plain-text and the second element denotes the associated complement presented in both numerals as well as alphabets starting from A = 0.

2. Send the Encryption-Matrix-2, Lattice Complement matrix and Lattice Adjacency Matrix to the receiver.

**Decryption 2:**

1. Arrange the Complement pair with the first elements of the ordered pair in ascending order.
2. Then from the Encryption-Matrix-2, if the operation is addition then do the reverse operation i.e., subtract as following :

   L – 5 = G

   V – 4 = R

   A – 0 = A

   J – 20 = P

   H – 0 = H

   T – 1 = S

3. The original Plain-text GRAPHS is obtained.

**Modified Operation :**

Let us take the position number of the alphabets as follows of the Plain-text GRAPHS :

G = 7, R = 18, A = 1, P = 16, H = 8, S = 1.

In this modified operation, 6 has been taken due to the 6 vertices of the $D_{20}$ Lattice.

$$(7 \oplus_6 1) \oplus_6 (7 \odot_6 1) = 3 = C$$

$$(18 \oplus_6 2) \oplus_6 (18 \odot_6 2) = 2 = B$$

$$(18 \oplus_6 5) \oplus_6 (18 \odot_6 5) = 5 = E$$

$$(16 \oplus_6 4) \oplus_6 (16 \odot_6 4) = 4 = D$$

$$(8 \oplus_6 10) \oplus_6 (8 \odot_6 10) = 2 = B$$

$$(19 \oplus_6 20) \oplus_6 (19 \odot_6 20) = 5 = E$$

Thus, the Cipher-text is **: C B E D B E**

Another Cipher-text in reverse order **: E B D E B C**

Similar to the above case of $D_{20}$, encryption and decryption can be carried out for all the 3 operations : addition, subtraction and modified operation.

**CONCLUSION**

In this work, Lattices have been used from a different definition and view-point, that of from Discrete Mathematics, then done so far as per the available literature. The cube Lattice has been presented with 3 different kinds of vertex elements, each method producing various Cipher-texts and promising still more Cipher-texts with different permutations of the vertex elements. Newly developed concepts like the Lattice Complement Matrix also have been presented. Three different kinds of operations have been presented as well to hide the message still further. Lattice $D_{20}$ making use of the concept of complement of elements also has been presented which can make use of the 3 different kinds of operations as well.

Encryption and decryption techniques for all these methods have been presented with an enhancement of the secrecy in hiding of the message thus leading to a much safer data transfer.

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# ON THE DIOPHANTINE EQUATION $\prod_{i=1}^{3} x_i - n^3 = m\prod_{i=1}^{3}(x_i - n)$; $m$ AND $n$ ARE POSITIVE INTEGERS

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**ABSTRACT**

***The Diophantine equation under consideration is a cubic non-linear Diophantine equation. We have to obtain its positive integral solutions for different values of m and n. For $n = 1$, let $x_1 = a$, $x_2 = b$and $x_3 = c$. Then $abc - 1 = m(a-1)(b-1)(c-1)$. It means we have to find the values of a, b and c such that $abc - 1$ is divisible by $(a-1)(b-1)(c-1)$ where $1 < a < b < c$. This problem was posed initially in April 2017 in math.stakexchange.com and by American University, Sharjah in March 2021. The solutions obtained are (a,b,c)= (2,4,8) and (3,5,15). In this paper, the problem has been discussed for other values of n.***

## INTRODUCTION & ANALYTICAL STUDY :

**Case 1:** For $n = 2$, the problem becomes $abc - 8 = m(a-2)(b-2)(c-2)$ i.e. we have to find the values of *a, b* and *c* such that $abc - 8$ is divisible by $(a-2)(b-2)(c-2)$ where $2 < a < b < c$. This can be done in the following two ways:

We have, $m = \frac{abc-8}{(a-2)(b-2)(c-2)}$

$$= \frac{a}{(a-2)}.\left(1 + \frac{2}{(b-2)} + \frac{2}{(c-2)}\right) + \frac{4}{(b-2)(c-2)}. \qquad \ldots(1)$$

Expression (1) implies, $m > \frac{a}{(a-2)}$. …(2)

Let $a = 3$. Then from (1) and (2), we have

$$3 < m \le 3\left(1 + \frac{2}{(b-2)} + \frac{2}{(c-2)}\right) + \frac{4}{(b-2)(c-2)}. \qquad \ldots(3)$$

Since $b < c$ we have $b - 1 \le c - 2$. Therefore from (3), we have

$$3 < m \le 3\left(1 + \frac{2}{(b-2)} + \frac{2}{(b-1)}\right) + \frac{4}{(b-2)(b-1)}. \qquad \ldots(4)$$

If we take $b = 4$ then from (4), we have

$$3 < m \le 3\left(1 + \frac{2}{2} + \frac{2}{3}\right) + \frac{4}{2.3} = 8\frac{2}{3}.$$

If we take $b = 13$ then from (4), we have

$$3 < m \le 3\left(1 + \frac{2}{11} + \frac{2}{12}\right) + \frac{4}{11.12} = 4\frac{5}{66}. \qquad \ldots(5)$$

If we take $b = 14$ then from (4), we have

$$3 < m \leq 3\left(1 + \frac{2}{12} + \frac{2}{13}\right) + \frac{4}{12.13} = 3\frac{77}{78}. \quad \ldots(6)$$

From expression (6), we see that there exists no integral value of *m*. It means if *a*=3 and *b*=14 then there exists no solution of (1). Therefore for *a*=3, *b* may be 4, 5, 6, 7, 8, 9, 10, 11, 12 or 13. c may take the values 5, 6, 7, 8, 9, 10, 11, 12, 13 or 14. By inspection may get *b*=4 and *c*=6 and 10. Thus we have $(a, b, c) = (3,4,6)$ and $(3,4,10)$.

Let $a = 4$. Then from (1) and (2), we have

$$2 < m \leq 2\left(1 + \frac{2}{(b-2)} + \frac{2}{(c-2)}\right) + \frac{4}{(b-2)(c-2)}. \quad \ldots(7)$$

Since $b < c$ we have $b - 1 \leq c - 2$. Therefore from (7), we have

$$2 < m \leq 2\left(1 + \frac{2}{(b-2)} + \frac{2}{(b-1)}\right) + \frac{4}{(b-2)(b-1)}. \quad \ldots(8)$$

If we take $b = 14$ then from (8), we have

$$2 < m \leq 2\left(1 + \frac{2}{12} + \frac{2}{13}\right) + \frac{4}{12.13} = 2\frac{52}{78}. \quad \ldots(9)$$

From expression (9), we see that there exists no integral value of *m*. It means if *a*=4 and *b*=14 then there exists no solution of (1). Therefore for *a*=4, *b* may be 5, 6, 7, 8, 9, 10, 11, 12 or 13. c may take the values 6, 7, 8, 9, 10, 11, 12, 13, 14 or 15. By inspection it may be checked that for *a*=4, *b*=5, 6 and 10 the corresponding *c*=10, 7 and 11 will satisfy the condition that $abc - 8$ is divisible by $(a-2)(b-2)(c-2)$. Thus $(a, b, c) = (4, 5, 10), (4, 6, 7)$ and $(4, 10, 11)$.

Let $a = 5$. Then from (1) and (2), we have

$$\frac{5}{3} < m \leq \frac{5}{3}\left(1 + \frac{2}{(b-2)} + \frac{2}{(c-2)}\right) + \frac{4}{(b-2)(c-2)}. \quad \ldots(10)$$

Since $b < c$ we have $b - 1 \leq c - 2$. Therefore from (10), we have

$$\frac{5}{3} < m \leq \frac{5}{3}\left(1 + \frac{2}{(b-2)} + \frac{2}{(b-1)}\right) + \frac{4}{(b-2)(b-1)}. \quad \ldots(11)$$

If we take $b = 6$ then from (11), we have

$$\frac{5}{3} < m \leq \frac{5}{3}\left(1 + \frac{2}{4} + \frac{2}{5}\right) + \frac{4}{4.5} = 3\frac{11}{30}$$

If we take $b = 14$ then from (11), we have

$$\frac{5}{3} < m \leq \frac{5}{3}\left(1 + \frac{2}{12} + \frac{2}{13}\right) + \frac{4}{12.13} = 2\frac{53}{234}$$

If we take $b = 28$ then from (11), we have

$$\frac{5}{3} < m \le \frac{5}{3}\left(1 + \frac{2}{26} + \frac{2}{27}\right) + \frac{4}{26.27} = 1\frac{973}{1053}. \qquad \ldots(12)$$

From expression (12), we see that there exists no integral value of *m*. It means if *a*=5 and *b*=28 then there exists no solution of (1). Therefore for *a*=5, *b* may be 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26 or 27. c may take the values 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27 or 28. It will have no solution .

Let $a = 6$. Then from (1) and (2), we have

$$\frac{3}{2} < m \le \frac{3}{2}\left(1 + \frac{2}{(b-2)} + \frac{2}{(c-2)}\right) + \frac{4}{(b-2)(c-2)}. \qquad \ldots(13)$$

Since $b < c$ we have $b - 1 \le c - 2$. Therefore from (13), we have

$$\frac{3}{2} < m \le \frac{3}{2}\left(1 + \frac{2}{(b-2)} + \frac{2}{(b-1)}\right) + \frac{4}{(b-2)(b-1)}.$$

If we take $b = 23$ then from (14), we have

$$\frac{3}{2} < m \le \frac{3}{2}\left(1 + \frac{2}{21} + \frac{2}{22}\right) + \frac{4}{21.22} = \frac{1652}{924} = 1\frac{182}{231}. \qquad \ldots(14)$$

From expression (14), we see that there exists no integral value of *m*. It means if *a*=6 and *b*=23 then there exists no solution of (1). Therefore for *a*=6, *b* may be 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21 or 22 and c may take the values 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21. It will have no solution

.

**Another way:** If we take $a - 2 = x$, $b - 2 = y$ and $c - 2 = z$ then we have

$$\frac{abc-8}{(a-2)(b-2)(c-2)} = \frac{(x+2)(y+2)(z+2)-8}{xyz} = 1 + \frac{2(xy+yz+zx)+4(x+y+z)}{xyz}. \qquad \ldots(15)$$

Also we have

$$0 < 4(x+y+z) + 2(xy+yz+zx) < 4xyz + 2xyz + 2xyz = 8xyz.$$

This implies, $2(x+y+z) + (xy+yz+zx) < 4xyz$.

Therefore $2(x+y+z) + (xy+yz+zx) = xyz$ or $2xyz$ or $3xyz$. …(16)

If $2(x+y+z) + (xy+yz+zx) = xyz$ and $x = 1$ then we have

$$3(y+z) = -2. \qquad \ldots(17)$$

Not possible. Equation (17) does not provide the required values of *y* and *z*.

If $2(x+y+z) + (xy+yz+zx) = xyz$ and $x = 2$ then we have

$$4(y+z) + 4 = yz. \qquad \ldots(18)$$

Equation (18) provides $z = \frac{4y+4}{y-4}$. …(19)

From equation (19), we see that y=5 provides z=24, y=6 provides 14, y=8 provides z=9 and y=9 provides z=8 (not possible). Therefore we have

$(a, b, c) = (4,7,26), (4,8,16)$ and $(4,10,11)$.

If $2(x + y + z) + (xy + yz + zx) = xyz$ and $x = 3$ then we have

$5(y + z) + 6 = 2yz$. …(20)

Equation (20) provides $z = \frac{5y+6}{2y-5}$. …(21)

From equation (21), we see that it provides no appropriate value of *y* and *z*.

If $2(x + y + z) + (xy + yz + zx) = 2xyz$ and $x = 1$ then we have

$3(y + z) + 2 = yz$. …(22)

Equation (22) provides $z = \frac{3y+2}{y-3}$. …(23)

From equation (23), we see that y=4 provides z=14. Therefore we have

$(a, b, c) = (3,6,16)$.

If $2(x + y + z) + (xy + yz + zx) = 2xyz$ and $x = 2$ then we have

$y + z + 1 = yz$. …(24)

Equation (24) provides $z = \frac{y+1}{y-1}$. …(25)

From equation (25), we see that y=3 provides z=2 (not satisfies the given condition).

If $2(x + y + z) + (xy + yz + zx) = 2xyz$ and $x = 3$ then we have

$5(y + z) + 6 = 5yz$. …(26)

Equation (26) provides $z = \frac{5y+1}{5(y-1)}$. …(27)

From equation (27), we see that it provides no appropriate value of *y* and *z*.

If $2(x + y + z) + (xy + yz + zx) = 3xyz$ and $x = 1$ then we have

$3(y + z) = 2yz$.

This implies that

$z = \frac{3y}{2y-3}$. …(28)

From equation (28), we see that y=2 provides z=6 and y=3 provides z=3 (not satisfies the given condition). Therefore we have

$(a, b, c) = (3,4,8).$

If $2(x + y + z) + (xy + yz + zx) = 3xyz$ and $x = 2$ then we have

$4(y + z) + 4 = 5yz.$

This implies that

$$z = \frac{4(y+1)}{5y-4}. \qquad \text{…(29)}$$

From equation (29), we see that it provides no appropriate value of *y* and *z*.

If $2(x + y + z) + (xy + yz + zx) = 3xyz$ and $x = 3$ then we have

$5(y + z) + 6 = 8yz.$

This implies that

$$z = \frac{5y+6}{8y-5}. \qquad \text{…(30)}$$

From equation (30), we see that it provides no appropriate value of *y* and *z*.

Thus we have the following solutions:

$(a, b, c) = (3,6,16), (4,7,26), (4,8,16), (4, 5, 10), (4, 6, 7),$

$(3,4,6), (4, 10, 11)$ and $(3,4,10)$.

**Case 2:** For $n = 3$, the problem becomes $abc - 27 = m(a - 3)(b - 3)(c - 3)$ i.e. we have to find the values of *a, b* and *c* such that $abc - 27$ is divisible by $(a - 3)(b - 3)(c - 3)$ where $3 < a < b < c$. This can be done in the following two ways:

We have

$$m = \frac{abc-27}{(a-3)(b-3)(c-3)}$$
$$= \frac{a}{(a-3)}.\left(1 + \frac{3}{(b-3)} + \frac{3}{(c-3)}\right) + \frac{9}{(b-3)(c-3)}. \qquad \text{…(31)}$$

Expression (31) implies

$$m > \frac{a}{(a-3)}. \qquad \text{…(32)}$$

Let $a = 4$. Then from (31) and (32), we have

$$4 < m \le 4\left(1 + \frac{3}{(b-3)} + \frac{3}{(c-3)}\right) + \frac{9}{(b-3)(c-3)}. \qquad \text{…(33)}$$

Since $b < c$ we have $b - 2 \leq c - 3$. Therefore from (33), we have

$$4 < m \leq 4\left(1 + \frac{3}{(b-3)} + \frac{3}{(b-2)}\right) + \frac{9}{(b-3)(b-2)} . \quad \ldots(34)$$

If we take $b = 5$ then from (34), we have

$$4 < m \leq 4\left(1 + \frac{3}{2} + \frac{3}{3}\right) + \frac{9}{2.3} = 15\frac{1}{2} . \quad \ldots(35)$$

If we take $b = 26$ then from (34), we have

$$4 < m \leq 4\left(1 + \frac{3}{23} + \frac{3}{24}\right) + \frac{9}{23.24} = 5\frac{9}{564} . \quad \ldots(36)$$

If we take $b = 27$ then from (34), we have

$$4 < m \leq 4\left(1 + \frac{3}{24} + \frac{3}{25}\right) + \frac{9}{24.25} = 4\frac{597}{600} . \quad \ldots(37)$$

From expression (37), we see that there exists no integral value of *m*. It means if *a*=4 and *b*=27 then there exists no solution of above equation. Therefore for *a*=4, *b* may be 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 22, 23, 24, 25 or 26. c may take the values 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 22, 23, 24, 25, 26 or 27. By inspection it may be checked that for *a*=4, *b*=6 the corresponding *c*=18 will satisfy the condition that $abc - 27$ is divisible by $(a-3)(b-3)(c-3)$. Thus $(a, b, c) = (4, 6, 18)$

Let $a = 5$. Then from (31) and (32), we have

$$\frac{5}{2} < m \leq \frac{5}{2}\left(1 + \frac{3}{(b-3)} + \frac{3}{(c-3)}\right) + \frac{9}{(b-3)(c-3)}. \quad \ldots(38)$$

Since $b < c$ we have $b - 2 \leq c - 3$. Therefore from (38), we have

$$\frac{5}{2} < m \leq \frac{5}{2}\left(1 + \frac{3}{(b-3)} + \frac{3}{(b-2)}\right) + \frac{9}{(b-3)(b-2)} . \quad \ldots(39)$$

If we take $b = 6$ then from (39), we have

$$\frac{5}{2} < m \leq \frac{5}{2}\left(1 + \frac{3}{3} + \frac{3}{4}\right) + \frac{9}{3.4} = 7\frac{1}{4} . \quad \ldots(40)$$

If we take $b = 103$ then from (39), we have

$$\frac{5}{2} < m \leq \frac{5}{2}\left(1 + \frac{3}{100} + \frac{3}{101}\right) + \frac{9}{100.101} = \frac{53533}{20200} = 2\frac{13133}{20200} . \quad \ldots(41)$$

From expression (41), we see that there exists no integral value of *m*. It means if *a*=5 and *b*=103 then there exists no solution of above equation. Therefore for *a*=5, *b* may be 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 22, 23, 24, 25,...102, c may take the values 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 22, 23, 24, 25, 26, 27,…104. It will have no solution.

**Another way:** If we take $a-3=x$, $b-3=y$ and $c-3=z$ then we have

$$\frac{abc-27}{(a-3)(b-3)(c-3)} = \frac{(x+3)(y+3)(z+3)-27}{xyz} = 1 + \frac{3(xy+yz+zx)+9(x+y+z)}{xyz}. \quad \ldots(42)$$

Also we have

$$0 < 9(x+y+z) + 3(xy+yz+zx) < 9xyz + 3xyz + 3xyz = 15xyz.$$

This implies $3(x+y+z) + (xy+yz+zx) < 5xyz$.

Therefore $3(x+y+z) + (xy+yz+zx) = xyz$ or $2xyz$ or $3xyz$ or $4xyz$ or 3 .

If $3(x+y+z) + (xy+yz+zx) = xyz$ and $x=1$ then we have

$$4(y+z) = -2. \quad \ldots(43)$$

From equation (43), we see that it provides no appropriate value of $y$ and $z$.

If $3(x+y+z) + (xy+yz+zx) = xyz$ and $x=2$ then we have

$$5(y+z) + 6 = yz. \quad \ldots(44)$$

Equation (44) implies that $= \frac{5y+6}{y-5}$ . ...(45)

From equation (45) we see that y=6 provides z=36. Therefore we have

$$(a,b,c) = (5,9,39).$$

If $3(x+y+z) + (xy+yz+zx) = xyz$ and $x=3$ then we have

$$6(y+z) + 9 = 2yz.$$

This implies that

$$z = \frac{6y+9}{2y-6}. \quad \ldots(46)$$

Equation (46) provides no appropriate values.

If $3(x+y+z) + (xy+yz+zx) = xyz$ and $x=4$ then we have

$7(y+z) + 12 = 3yz$. This implies that

$$z = \frac{7y+12}{3y-7}. \quad \ldots(47)$$

Equation (47) provides no appropriate values.

If $3(x+y+z) + (xy+yz+zx) = 2xyz$ and $x=1$ then we have

$$4(y+z) + 3 = yz.$$

This implies that

$$z = \frac{4y+3}{y-4}. \quad \ldots(48)$$

From equation (48), we see that y=5 provides z=23. Therefore we have

$(a, b, c) = (4,8,26)$.

If $3(x + y + z) + (xy + yz + zx) = 2xyz$ and $x = 2$ then we have

$5(y + z) + 6 = 3yz$. This implies that

$$z = \frac{5y+6}{3y-5}. \quad \ldots(49)$$

Equation (49) provides no appropriate values.

If $3(x + y + z) + (xy + yz + zx) = 2xyz$ and $x = 3$ then we have

$6(y + z) + 9 = 5yz$. This implies that

$$z = \frac{6y+9}{5y-6}. \quad \ldots(50)$$

Equation (50) provides no appropriate values.

If $3(x + y + z) + (xy + yz + zx) = 3xyz$ and $x = 1$ then we have

$4(y + z) + 3 = 2yz$.

This implies that

$$z = \frac{4y+3}{2y-4}. \quad \ldots(51)$$

Equation (51) provides no appropriate values.

If $3(x + y + z) + (xy + yz + zx) = 3xyz$ and $x = 2$ then we have

$5(y + z) + 6 = 5yz$.

This implies that

$$z = \frac{5y+6}{5y-5}. \quad \ldots(52)$$

Equation (52) provides no appropriate values.

If $3(x + y + z) + (xy + yz + zx) = 3xyz$ and $x = 3$ then we have

$6(y + z) + 9 = 8yz$.

This implies that

$$z = \frac{6y+9}{8y-6}. \quad \ldots(53)$$

Equation (53) provides no appropriate values.

Thus we have the following solutions:

$(a, b, c) = (4,6,18), (4,8,26)$ and $(5,9,39)$.

**Case 3:** For general $n$, the problem becomes $abc - n^3 = m(a-n)(b-n)(c-n)$ i.e. we have to find the values of $a$, $b$ and $c$ such that $abc - n^3$ is divisible by $(a-n)(b-n)(c-n)$ where $n < a < b < c$. This can be done in the following two ways:

We have

$$m = \frac{abc-n^3}{(a-n)(b-n)(c-n)}$$

$$= \frac{a}{(a-n)}.\left(1 + \frac{n}{(b-n)} + \frac{n}{(c-n)}\right) + \frac{n^2}{(b-2)(c-2)}. \qquad \ldots(54)$$

Expression (54) implies

$$m > \frac{a}{(a-n)}. \qquad \ldots(55)$$

If we take $a = n + 1$ then (54) and (55) implies that

$$n + 1 < m \leq (n+1).\left(1 + \frac{n}{(b-n)} + \frac{n}{(c-n)}\right) + \frac{n^2}{(b-n)(c-n)}. \qquad \ldots(56)$$

Since $b < c$ we have $b - (n-1) \leq c - n$. Therefore from (56), we have

$$n + 1 < m \leq (n+1).\left(1 + \frac{n}{(b-n)} + \frac{n}{(b-n+1)}\right) + \frac{n^2}{(b-n)(b-n+1)}. \qquad \ldots(57)$$

If we take $b = n + 2$ then from (57), we have

$$n + 1 < m \leq (n+1).\left(1 + \frac{n}{2} + \frac{n}{3}\right) + \frac{n^2}{2.3}$$

$$= \frac{(n+1)(5n+6)}{6} + \frac{n^2}{6} = \frac{6n^2+11n+6}{6} = (n+1)^2 - \frac{n}{6}.$$

By giving different values of n we obtained the required solutions.

**Another way:** If we take $a - n = x$, $b - n = y$ and $c - n = z$ then we have

$$\frac{abc-27}{(a-n)(b-n)(c-n)} = \frac{(x+n)(y+n)(z+n)-n^3}{xyz} = 1 + \frac{n(xy+yz+zx)+n^2(x+y+z)}{xyz}. \qquad \ldots(58)$$

Also we have

$$0 < n^2(x+y+z) + n(xy+yz+zx) < n^2xyz + nxyz + nxyz$$

$$= n(n+2)xyz. \qquad \ldots(59)$$

If we put $x = n + 1$ in (59) then we have

$$\left(n^2 + n(n+1)\right)y + n^2(n+1) = \left(n(n+2)(n+1) - n(n+1)\right)z$$

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This implies

$$z = \frac{(2n^2+n)y+n^2(n+1)}{n(n+1)^2}.$$

By giving different values of *n,* we may obtain the required solutions.

**3 CONCLUDING REMARKS:** Here the Diophantine equation $\prod_{i=1}^{3} x_i - n^3 = m\prod_{i=1}^{3}(x_i - n)$ has been discussed for *n*=2 and 3. The results have been obtained by two different ways. For *n*=2, the solutions are given by

$(a, b, c) = (3,6,16), (4,7,26), (4,8,16), (4, 5, 10), (4, 6, 7), (3,4,6), (4, 10, 11)$ and $(3,4,10)$. For *n*=3, the solutions are given by $(a, b, c) = (4,6,18), (4,8,26)$ and $(5,9,39)$. The problem can be extended further for other values of *n*.

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