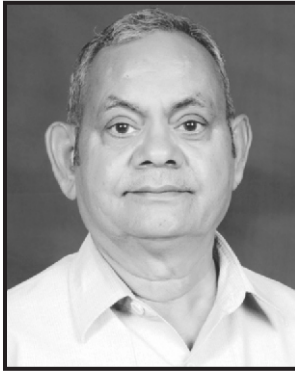


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**Dr. T.P. Singh**

Professor in Maths & O.R.  
Chief Editor

# FOREWORD

It gives me an immense pleasure in writing this foreword for '**Aryabhatta Journal of Mathematics & Informatics**' (AJMI) Vol.12 issue 2 ( July-Dec., 2020) published by Aryans Research & Educational Trust. The first issue of the journal was published in year 2009. Since then & till today the journal publication is regular & well in time. It gives us a great pleasure to put forward before the scholars and academicians that journal from its start is making an effort to produce good quality articles. The credit goes to its reviewer team which review sincerely and furnish valuable suggestions to improve the quality of papers.

Aryabhatta Journal of Mathematics & Informatics mainly covers areas of mathematical and statistical sciences, Operational Research, data based managerial & economical issues and information sciences. Mathematical science is an increasing component of many disciplines including bio-sciences, medical sciences, business, advanced design, climate studies etc. Being an interdisciplinary approach, it is the core of computer simulation, physical sciences, quantitative analysis of financial issues, animation, numerical weather predictions above all it is a key of key technologies of our times . The information sciences, the fastest growing segment due to industrial liberation, change in business environment, globalization and the trends in world economic scenario has posed an increasing challenges for the organization to be competitive and productive. The statistics reveal that today no organization or individual is without communicating or information device. The information science has become the dominant force in our society and culture and will continue to transform the key issues and the world we live in. Mathematical models reveal the dynamic nature of societies which presents the instability of large complex society in a predictable & uncertain phenomenon through random & fuzzy variables. The current problem **COVID-19** a danger for the existence of humanity and **drastic climate** change has posed a challenge before the mathematical community and researchers to explore such models which can represent aspects of reality and behavior of the system in befitting situation.

AJMI VOL.12 issue 2 is before you. I am pleased to note that research groups from different parts of country have sent their papers for this issue. The papers are significant and focus on the futuristic trends and innovations in the relevant areas. I would like to thank and felicitate the contributors in this issue and at the same time invite quality papers from academic and research community for vol.13 issue 1 to be published in May. 2021.

Comments, suggestions and feedback from discerning readers, scholars and academicians are always welcome.

**DR. T.P. SINGH**



## INFORMATION AND COMMUNICATION TECHNOLOGIES IN MATHEMATICS TEACHING

**Dragan Obradovic<sup>\*</sup>, Lakshmi Narayan Mishra<sup>\*\*</sup>**

<sup>\*</sup>Elementary school "Jovan Cvijic", Kostolac-Pozarevac, teacher of Mathematics, Serbia

<sup>\*\*</sup>Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University,  
Vellore 632 014, Tamil Nadu, India

E-mail: dragishaobradovic@yahoo.com, lakshminarayanmishra04@gmail.com

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### ABSTRACT :

*The goal of a modern school is to raise the level of quality of educational work, positive impact on children's development, development of digital competencies of children and teachers, developing digital culture and literacy, informing about safe use of the Internet. Modern teaching aids represent a significant resource in expansion and upgrading learning system. A student in teaching assisted by educational computer software is more responsible for their learning, uses a multitude of resources, has constant feedback and strives for higher achievements. Systematic and continuous teacher training is a prerequisite is for good teaching.*

*Keywords: modern teaching, ICT, student, teacher*

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### 1. INTRODUCTION

Constant improvement and modernization of educational work is one of the biggest and the most important priorities of the school and society because the time in which we live is characteristic of digitization and rapid technological progress. The way of teaching in all subjects over time, it has changed and adapted to the modern concept of acquiring knowledge and development abilities in students. Offer of designed, multimedia, interactive, electronic educational forms opens the possibility for teachers and students to change educational process. Digital technology is not an end in itself, but a tool for achieving it educational goals and tasks.

The path to better education leads through interested, trained and motivated teachers developed interdisciplinary competencies. By applying modern teaching resources and educational software changes and modifies the role and position of teachers in teaching process. The development and realization of the seminar raises the professional capacity employees, increases motivation and enables the development of positive attitudes about importance new technologies.

### 2. ICT IN MATHEMATICS TEACHING

When planning and implementing a lesson, the teacher uses a number of lessons means and aids. One of the teaching aids he can use is ICT technology involving computers (desktops/ laptops), tablets or mobile devices, interactive whiteboards ... Varošaneć (2011) points out that a teacher should decide how, when and where to use ICT in math teaching, and it depends on a few factors:

- whether ICT improves the existing teaching process,
- whether the use of ICT enables more efficient achievement of goals,

- Whether the use of ICT achieves better results in learning and teaching.

Also, Varošaneć (2011) points out that ICT significantly improves the teaching process mathematics in determining the teaching content, learning through pictures, development abilities and skills related to mathematical modeling, discovery learning, by experimenting and working with data. Using information and communication technology students can solve a problem faster and easier and they can do more concentrate on the given problem, rather than on the algorithm and procedure. Glasnović Gracin (2008) discusses several advantages of using ICT in mathematics teaching, namely:

1. Different forms of representation and dynamics in the representation,
2. Modularity,
3. Experimental work,
4. Elementalization of mathematical procedures.

Different forms of presentation are important for teaching mathematics. Given that in mathematics appear different numerical data, pictorial representations, ICT allows fast data transfer from one form to another. The dynamics in the display are related to contents from geometry. If students want to change, for example, using different programs the size of a geometric figure, the basic properties of that figure remain the same, which is strong important for understanding the basic characteristics of individual geometric figures. Using modularities can be linked to different commands (regardless of their record).

An important opportunity provided by ICT in mathematics teaching is experimental work. The new paradigm of schooling is focused on the student and his research and active participation in the teaching process, and ICT enables just that in teaching mathematics. Students can thus solve various mathematical tasks, mathematical problems and through research to reach a solution to a given problem. In addition to those mentioned characteristics of ICT, must not be forgotten and the elementalization of mathematical procedures because with the help of ICT- different tasks can be solved regardless of their weight and complexity. Inder K. Rana (2013) talks about how students use ICT in teaching mathematicians can think critically, analyze information, develop collaboratively learning, students can ask a variety of questions and seek answers to them. Likewise students are active participants in the educational process and collaborative learning they want to get the information and answers they are looking for. Brown, Cadman, Cain et al. (2004) state that the use of ICT in mathematics teaching is useful because it is for students enables research, analysis and development and improvement of one's own mathematical knowledge and skills. They point out how ICT can be taught in mathematics use to demonstrate certain teaching contents, solving different ones problem situations and to introduce interactivity and dynamism into the teaching process. According to Kostić Kovačević, Lukač, Gavrilović and Đurović (2014) information and communication technology in mathematics teaching allows for visual and dynamic explaining complex mathematical phenomena and problems.

This is very important because they are mostly teaching contents in mathematics are abstract and are concretized in this way and visualize. Sivakova, Kochoska, Ristevska and Gramatkovski (2017) stand out five main possibilities provided by ICT in mathematics teaching:

1. Learning through feedback,
2. Observing patterns and noticing connections,
3. Giving commands to the computer,
4. Data research,
5. Developing visual representations.



By solving various tasks via computer, students can quickly get feedback information on how and to what extent they have adopted certain teaching content. Getting fast feedback is important because it allows students to self-evaluate. Solving tasks with the help of ICT students can connect different formulas and the corresponding ones graphic representations. This improves student generalization. The importance of ICT in teaching mathematics is also in developing logical thinking. To solve some mathematical task, students need to first compose it i.e. assign it to a computer to make it could carry out their order. Thus students are encouraged to think logically, a thereby perfecting their mathematical competencies. The great advantage of ICT in Teaching mathematics is also an opportunity to display the results of certain tasks on different ways. The fifth advantage of using ICT in the mathematics teaching they mention Sivakova, Kochoska, Ristevska and Gramatkovski (2017) is developing visuals display. Different visual representations (which can be easily changed and adjusted tasks) respects the principle of visibility, which is important in the teaching of mathematics, especially in classroom teaching.

### **3. PRINCIPLES IN TEACHING MATHEMATICS**

According to Markovac (2001), the principles in the teaching of mathematics are the ideas by which and on which realizes the educational process. The purpose of the principle in teaching mathematics is improving the learning and teaching process. Markovac (2001) states six principles, namely are: the principle of adequacy, the principle of gradualness, the principle of vigilance, the principle of activity, the principle of individualization, the principle of objective reality. These principles are interrelated connected and complement each other. Nor can it be stated which principle it is most importantly because each principle has its role and application in the teaching process. Cause of it is the task of the teacher to use as many principles as possible during his educational work, ie during the teaching of mathematics. Teaching in which everyone is respected the principle is quality teaching. In addition to these six principles stated by Markovac, Kurnik (2002 and 2008) highlights two more principles that are important for teaching mathematics. These are the principle of problem and the principle of scientificity.

#### **3.1. The principle of adequacy**

The principle of adequacy is reflected in the degree of ease / difficulty that students master certain mathematical content. Too easy or too difficult tasks in teaching mathematics are not good because they underestimate or overestimate student abilities. In class math is good when the tasks are a little above the student's abilities. By solving tasks that are a little above their abilities, students develop and improve their mathematical competencies. Student abilities during schooling is changing and improving and should be accordingly devise mathematical problems. Given that they are mathematical concepts mostly abstract, in the initial teaching of mathematics it is good to take such notions explain using different activities. In addition to verbal explanation, to students we certainly need to explain a certain term visually, which helps us significantly ICT technology. Using different graphic or pictorial representations makes it easy for students can represent and visualize a particular mathematical concept. Example for visual display using ICT is the direction. Direction is a basic mathematical term that is not defined, it is only described. The direction is a straight, unbounded line or the direction is straight on both sides unbounded line. A big problem arises when drawing it. Then students often they know how to ask the question: "What is the length of the given direction?" This question cannot be answer edive the correct answer, because the direction reaches infinity so it is often said "draw direction of arbitrary length ". The

very notion of infinity in this example is abstract and can give students a number of problems. In such examples, it would be desirable that the teacher uses ICT technology to help students with dynamic education software, applications and programs explained the term "infinity", and in this example which means that the direction is infinite. (Markovac, 2001)

### **3.2. The principle of gradualness**

According to Markovac (2001), the principle of gradualness encompasses the gradual learning of teachers content, and it is not possible if not all the necessary content has been previously adopted. This the principle is extremely important in mathematical contents that are interconnected as which are addition and multiplication. Before learning multiplication, students need to master arithmetic the addition operation because the computational multiplication operation is first introduced as sequential adding the same adders. The principle of gradualness teaches mathematical teaching content from the simple to the more complex, from the known to the unknown, and from the concrete according to the abstract. Example of learning teaching content from concrete to abstract is the teaching content of geometry. Eg. first graders learn first which are flat and curved surfaces and geometric bodies, and after these are adopted teaching contents teach about point and geometric figures. Students first gradually they adopt mathematical concepts, and finally mathematical symbols and signs.

### **3.3. The principle of vigilance**

The principle of vigilance tells how abstract mathematical concepts and contents are reshaped into perceptual teaching content. This is to concretize all the abstract mathematical concepts and contents for students to understand. On visibility in teaching mathematics is influenced by two factors, namely: the level of intellectual development of students and the mathematical content itself. Since the mathematical contents are abstract, concretization took place by itself. Another factor that is important to the principle visibility is the level of intellectual development of students. Given that students in In the initial teaching of mathematics, all mathematical content is understood at the level of perception, it is necessary to concretize it. Concretization is possible through various activities, such as: merging, assembling, disassembling or subtracting certain sets.

In addition to the manipulation of specific objects, concretization at the level of composing or disassembly of sets is possible with the help of ICT. Counts dynamic tools and programs enable work with sets, and their functioning will be explained in the following

Chapters (Markovac, 2001).

### **3.4. Principle of activity**

According to Markovac (2001) in order to adopt abstract mathematical contents and concepts, student activity is also important. In doing so, in the initial teaching of mathematics there are two ways of student activity: individually and in groups. Individual activities students include all student activities by which he wishes to adopt a particular teaching content. Student group activity involves the relationship between teacher and student on occasion processing new teaching content, practicing or repeating, and analyzing different ones tasks and data. Comparing these two activities, in primary school intellectual activities include inferences, analyzes and syntheses, and attention.

Verbal activities include the transformation of material processes into thought, and manual activities include working and handling students with specific didactics material. Graphic activities include drawing sets, geometric figures and bodies, various diagrams and graphs, and ICT helps a lot in drawing them technology.

### **3.5. The principle of individualization**

One of the important principles in the initial teaching of mathematics is the principle of individualization. Individualization in teaching enables the adaptation of all teaching contents student abilities and capabilities. The subjective is important for individualization condition of students and objective conditions of mathematics. Subjective factors include intellectual abilities of the student and his previous knowledge. Which is before students' knowledge in the initial teaching of mathematics is higher, the procedures of individualization are decrease. To make individualization in mathematics teaching possible, great the teacher is also important. He needs to possess the knowledge and skills to recognize individual differences between students and to enable needed students mathematics a more important role is played by the individual activity of students. Student activity in teaching mathematics can achieve all the goals (outcomes) and all the tasks of educational processes (educational, functional and educational). Student activities in the initial teaching of mathematics are: verbal, graphic, manual and in tellectual learning and teaching according to their abilities. Individualization in the initial teaching mathematics is realized in three ways: with worksheets, differentiation teaching and differentiation by presenting the teaching contents.

The worksheet is needed contain different tasks (numerical and textual), and their creation today does not would be possible without ICT technology. Differentiation of teaching is possible by setting the different amount and difficulty of the tasks assigned to students. So students who possess better mathematical competencies and abilities can solve more complex tasks, and students with fewer competencies can solve simpler ones tasks. The third factor of individualization in elementary mathematics teaching is differentiation of teaching content presentation. It includes the principle of vigilance, exposure with the help of concrete didactic material and their gradual abolition. (Markovac, 2001)

### **3.6. The principle of objective reality**

Respecting the principle of objective reality, students can apply the acquired knowledge from mathematics in everyday life. Eg. learned teaching content related to addition and subtracting numbers up to 20, students can apply in everyday life when going to the store. This principle also enables the connection of teaching contents related to the notation of numbers (and arithmetic operations) and their meaning in real life world. (Markovac, 2001)

### **3.7. The principle of problem**

According to Kurnik (2002) the principle of difficulty emphasizes how important it is for students to be themselves come to a conclusion. In order for students to come to a conclusion, it is important that the teacher asks open-ended questions and has a heuristic conversation with students. Heuristic conversation allows students to come to a conclusion on their own, i.e. to solutions to a question or given problem.

### 3.8. The principle of science

According to Kurnik (2008) all teaching contents taught in the subject Mathematics should be in accordance with the laws of mathematics. According to the principle science teacher should teach those contents that are scientifically validated. The teaching of mathematics must be such as to enable further deepening and expanding the teaching content and the natural continuation of mathematical education at higher level. The basic methods on which the principle of science is based are: analysis and synthesis, analogy, abstraction and concretization, generalization and specialization induction and deduction (Kurnik, 2002, p. 319).

## 4. CLASSIFICATION OF AVAILABLE MATHEMATICAL SOFTWARE TOOLS

There are a large number of concrete examples of mathematical software tools in the world today. They point to the complexity of the scientific field itself, and equally testify to the continuous development of these tools. Given their current popularity, as well as their increasingly frequent implementation for educational purposes and beyond, the trend of their modification, upgrading and development of new examples is expected to continue in the future. Mathematical software tools are being developed for all areas of mathematics and for all forms of problem tasks. According to which area they cover, it is possible to perform their more detailed classification. Furthermore, the same simplifies the approach to their search, as well as their use in practice.

In principle, it is possible to talk about general and specific mathematical software tools. The general ones cover several areas of this science, while the specific ones are intended for a specific area or teaching (scientific) unit. A more detailed presentation of groups of specific mathematical software tools can be presented as follows (Mathematical Software, 2019):

- direct linear solvers;
- sharing and ordering charts;
- iterative linear solvers;
- eigen-values and polynomial roots;
- other matrix packages;
- non-linear solvers;
- numerical integration;
- computing with functions;
- ordinary differential equations;
- differential-algebraic equations;
- stochastic differential equations;
- partial differential equations;
- network generation;
- special functions;
- computer algebra.

Considering that within each of these groups there are several concrete examples of these tools, it is possible to confirm their complexity and development in modern times. Equally, it is pointed out that they are characterized by high specialization, which makes an additional contribution to the study and mastering of mathematical knowledge and skills and more.

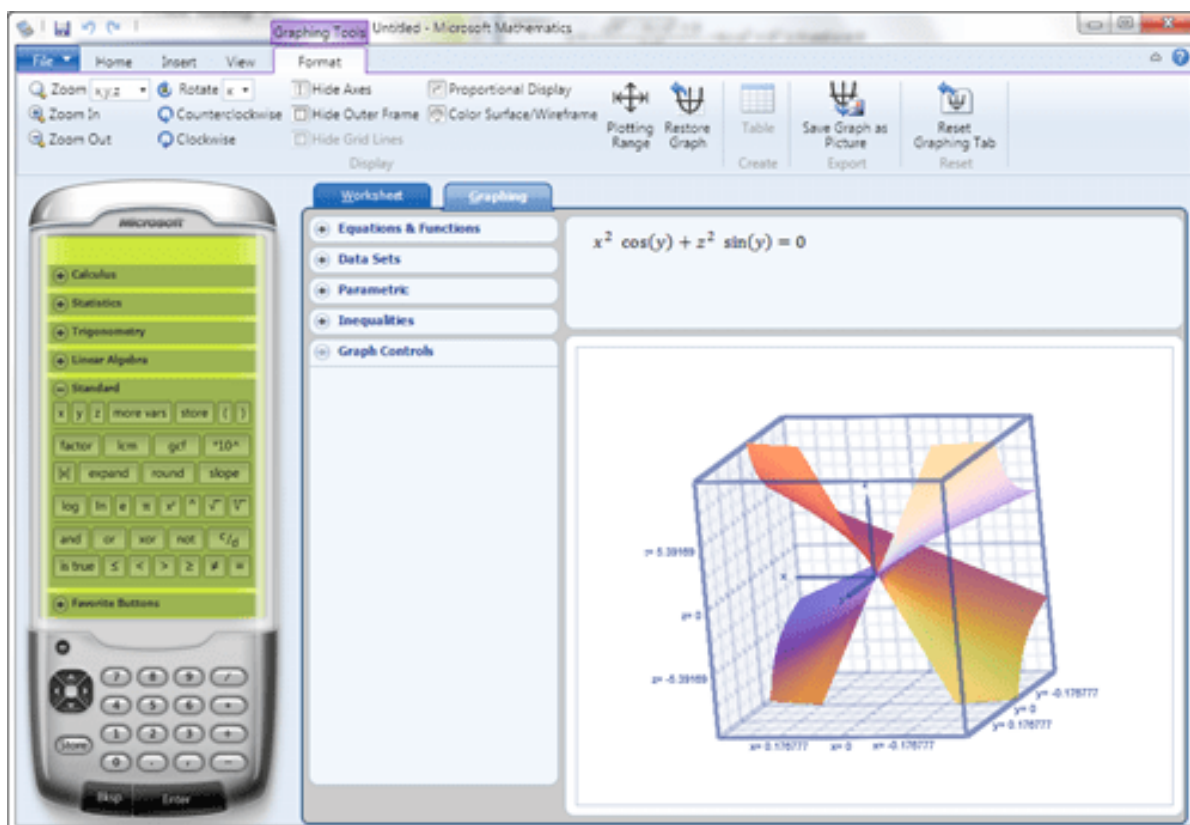
## 4.1. Mathematical software tools - examples

As already pointed out, in modern times, a number of mathematical software tools are available, which are very easily downloaded via the Internet and installed on users' personal computers. They not only serve to solve various mathematical, arithmetic and geometric problems and problems, but also help to master mathematical knowledge and skills.

### 4.1.1. Microsoft Mathematics

This is a free Microsoft software tool that offers help to students in solving complex math problems. It is used for this purpose by many other participants, and makes a special contribution to solving geometric problems. Specifically, it is used to draw 2D and 3D diagrams. In addition, it is intended to solve complex equations, explaining in detail step by step (MeraBheja, 2017). An overview of the same follows (Figure 1).

Microsoft Mathematics provides a set of math tools that help users quickly and easily solve math problems and gain reference knowledge. It allows them to better understand the basic concepts of algebra, trigonometry, physics, chemistry and calculus. Provides an extensive collection of mathematical symbols and structures to represent clearly formed mathematical expressions.



**Figure 1. Microsoft Mathematics software tool**

Source: MeraBheya (2017) Top 20 Best Free Math software you can use. Available at: <https://merabheja.com/best-free-math-softwares/>

This Microsoft tool includes a graphing calculator that has a full-featured design, which is designed to work like a handheld calculator. Additional mathematical tools help calculate standard mathematical

functions such as roots and logarithms, calculate trigonometric functions, derivatives and integrals, operations with matrices, and many other mathematical problems.

### 4.1.2. Math Editor

Math Editor is a specialized mathematical software tool for creating mathematical equations of various types. It allows you to spontaneously create, save and edit equations, download equations as image files to be used as office documents or web links, and allows you to copy any part of the equation and paste it into documents. Although it helps create mathematical equations on a computer, it is compatible with various formats such as JPEG, GIF, PNG, TIFF, and BMP (Math Chia, 2019). An overview of the Math Editor follows below (Figure 2).

It is a multifunctional mathematical software tool, which is popular among students, but also professors. Since it offers the possibility of converting equations into concrete diagrams, as well as full or partial insertion into documents, it is often used in both the public and business sectors, when preparing studies, documents and similar acts.

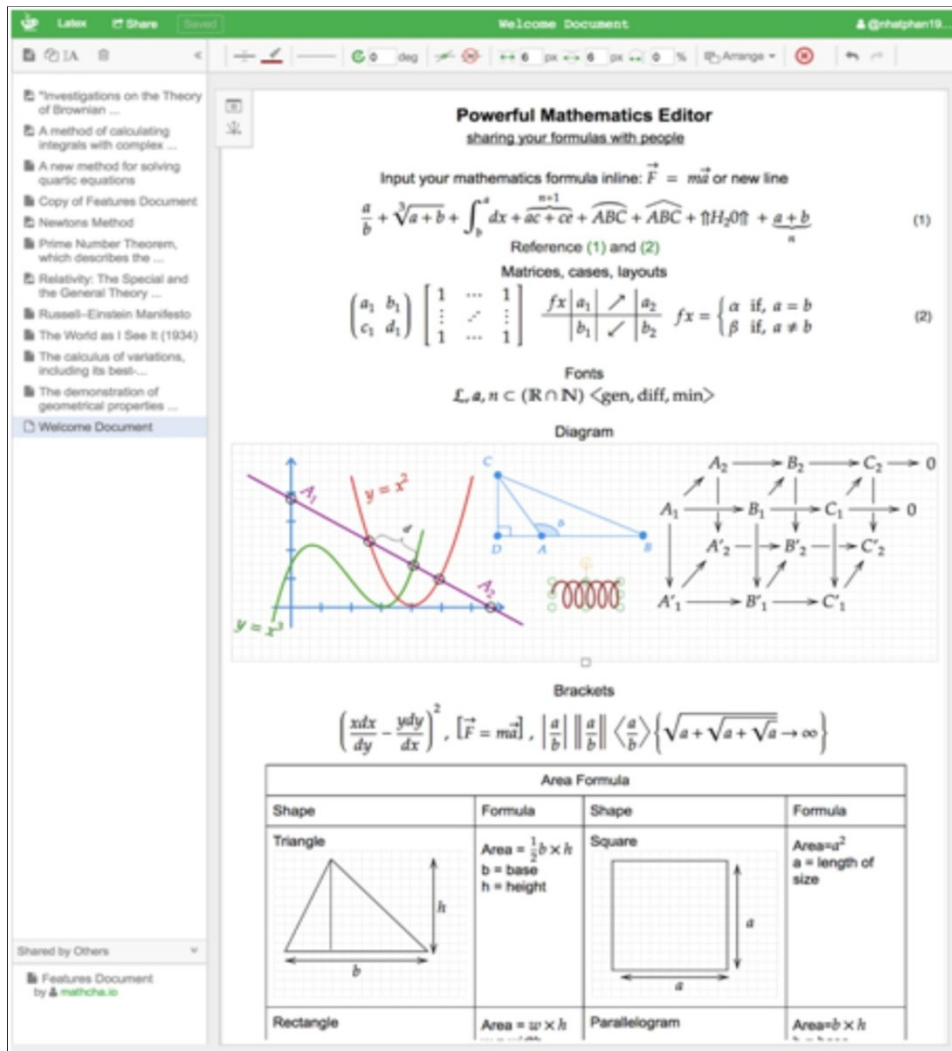


Figure 2. Math Editor

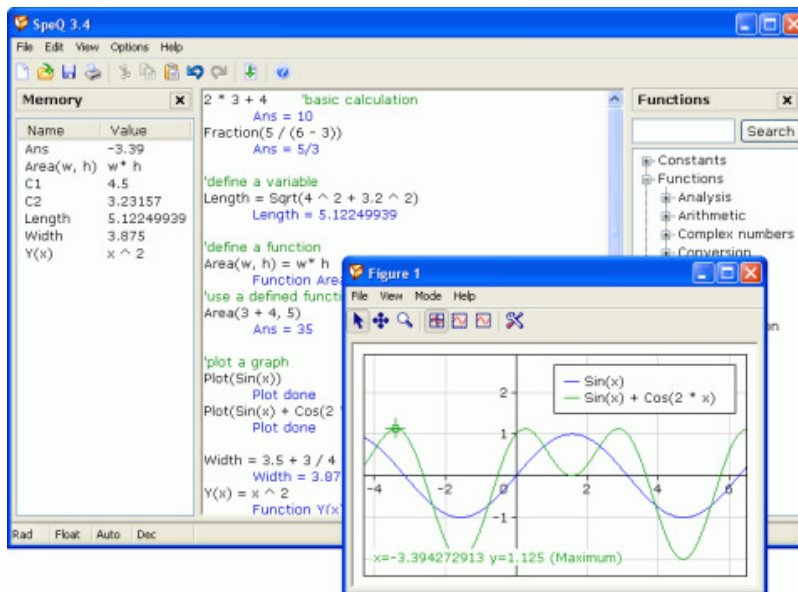
Source: MathChia (2019) Math Editor. Available at: <https://www.mathcha.io/>

### 4.1.3. SpeQ Mathematics

This mathematical software tool is intended for solving differentiated mathematical problems and tasks. It comes with built-in support for a wide range of constants, mathematical functions and variables. While it helps define custom variables and functions, it also helps solve trigonometry problems. It comes with a simple interface that contains a spreadsheet and on which it is possible to add, edit and perform calculations (SpeQ Mathematics, 2019).

It is basically a small but extensive math program with a simple, intuitive interface. All calculations are entered in the sheet. SpeQ supports all common functions, constants and units of measure. Furthermore, it is possible to define custom variables and functions and draw graphs of its functions. His account follows (Figure 3). The features of this software tool are numerous. They can be considered systematically as follows (SpeQ Mathematics, 2019):

- expression calculations with arithmetic, conditional, logical and essential operators;
- about 100 built-in mathematical, physical and conditional constants;
- about 60 built-in functions for analysis, arithmetic, complex numbers, probability, numerical systems, statistics, trigonometry;



**Figure 3. SpeQ Math**

Source: SpeQ Mathematics (2019) Home. Available at: <https://www.speqmath.com/>

- the ability to define custom variables and functions for use in calculations;
- the ability to draw fully customizable charts;
- extensive unit support;
- decimal, binary, hexadecimal and octal number system;
- settlements with lists;
- complex number calculations.

SpeQ is written as an advanced tool, but very easy to use. It is simple and fast and works on any Windows computer, also with a USB key or floppy disk. SpeQ is useful for simple, short calculations as well as for creating worksheets with extensive variable definitions, functions, and complex calculations. SpeQ is usable for engineers, students and pupils or simply as a replacement for a Windows calculator.



#### 4.1.4. Sketchpad

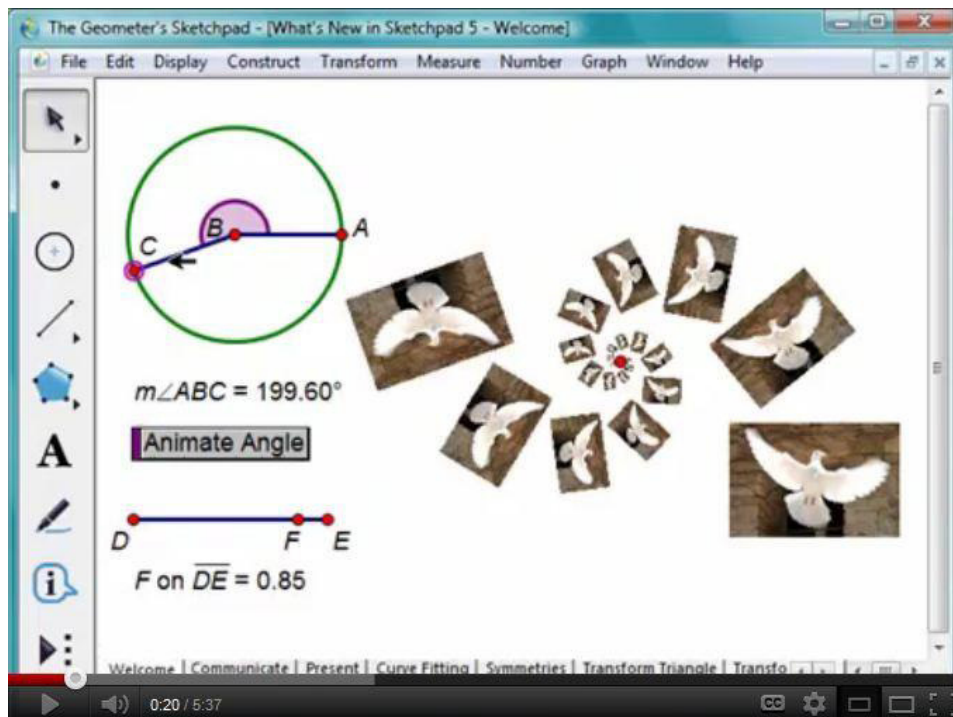
Sketchpad is a mathematical software program used by teachers, pupils and students to explore and visualize mathematical concepts and construct mathematical and scientific models. It is very popular more precisely in its education system, so it is the subject of interest of this paper. This mathematical software tool is created as a product of a kind of project called Hr. Sketchpad.CARNet.

The contribution of the above was manifested, and still is today, in the context of providing opportunities for the participants in the use of quality and pedagogically appropriate standardized educational technology. The objectives of the project in question can be systematized in the following way (The Geometer's Sketchpad, 2019):

- to acquaint students with the modern approach in learning mathematics;
- educate students and train them to research problems and generate working hypotheses and, after shaping, prove and apply them;
- teach young people to write down and present their "discoveries" in the classical sense in the form of an article, and in the form of lectures / presentations (using all modern computer tools).

In the context of the functionality and purpose of this software tool, in addition to the above, it should be noted that its development began in 1991 in the world. At that time, the Key Curriculum Press (KCP) coined and protected the term Dynamic Geometry. The name describes the ability of this tool to explain and apply mathematical relationships in geometric construction when users move construction components using a mouse. Dynamic Geometry allows students to see and understand mathematical relationships (The Geometer's Sketchpad, 2019).

In principle, it can be described as an open mathematical tool, which is widely available, and serves to adopt and research the topics of elementary geometry and numerical concepts. The Java component of Sketchpad allows the program to create HTML Dynamic Geometry pages that can be further viewed and modified. Only Internet access is required for this purpose. An overview of this tool follows below (Figure 4).



**Figure 4. Sketchpad**

Source: *The Geometer's Sketchpad* (2019) Key Curriculum. Available at: <https://www.keycurriculum.com/node/583.html>



Like most other mathematical programming tools, Sketchpad is also often used in other areas of science. It refers, for example, to the creation of visual models in physics, optics, chemistry, biology and mechanical engineering, and is often applicable in fine arts and technical programs.

#### 4.1.5. GeoGebra

GeoGebra is another popular mathematical software tool in Croatia. This interactive digital tool serves to master the knowledge and skills in the field of geometry and algebra, as the name suggests. As a software solution of this nature, it stands out for its ease of use and suitability for multiple types of devices.

GeoGebra is particularly popular with teachers and professors, and its popularity is partly based on the fact that it comes as open source software and is free for non-commercial use. This tool supports two ways of scripting, which enables communication with objects within the module, at a higher level. It supports the ability to use multiple commands, and is often used for the purpose of programming interactive exercises and games.

Accordingly, it is primarily used in primary and secondary education and teaching, but it is also applicable at other levels of education. GeoGebra connects the areas of interactive geometry, algebra, spreadsheets, statistics, analysis and graphing. Although it is mainly used for mathematical purposes, it is present and applicable in other sciences as well (Figure 5).

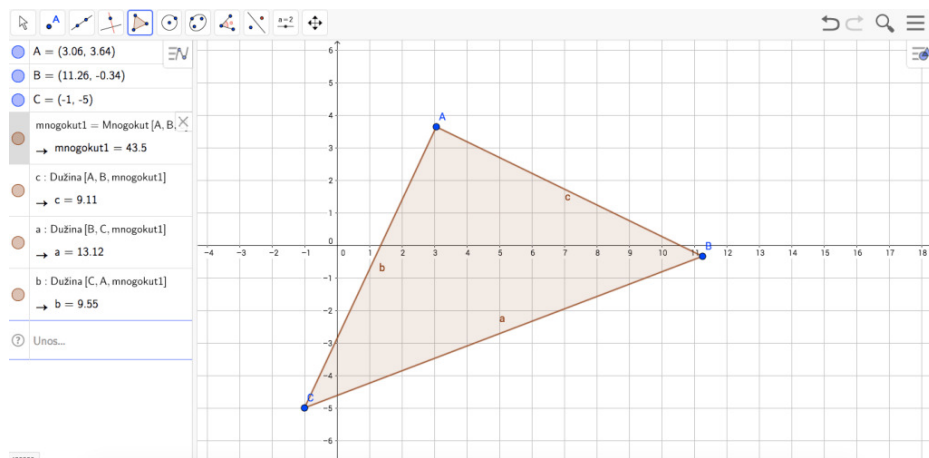


Figure 5. GeoGebra

Source: Negulic, T. (2017) *GeoGebra - interactive mathematics*. Available at: <http://e-laboratorij.carnet.hr/geogebra-interaktivna-matematika/>

This mathematical software tool is supported on several types of devices and operating systems, as already pointed out. This allows its wider use, and can be used as a network application or standalone.

Devices that support its use are (Negulic, 2017):

- tablets that use the Windows operating system download the application on the Windows Store platform, those that use the Android operating systems in Google Play and those that use iOS operating systems on the Apple Store platform;
- smartphones, depending on GeoGebra applications (basic application or additional applications), Android and iOS devices are supported;
- computers for Windows, Linux and MacOS operating systems as an installer or directly through a network browser."

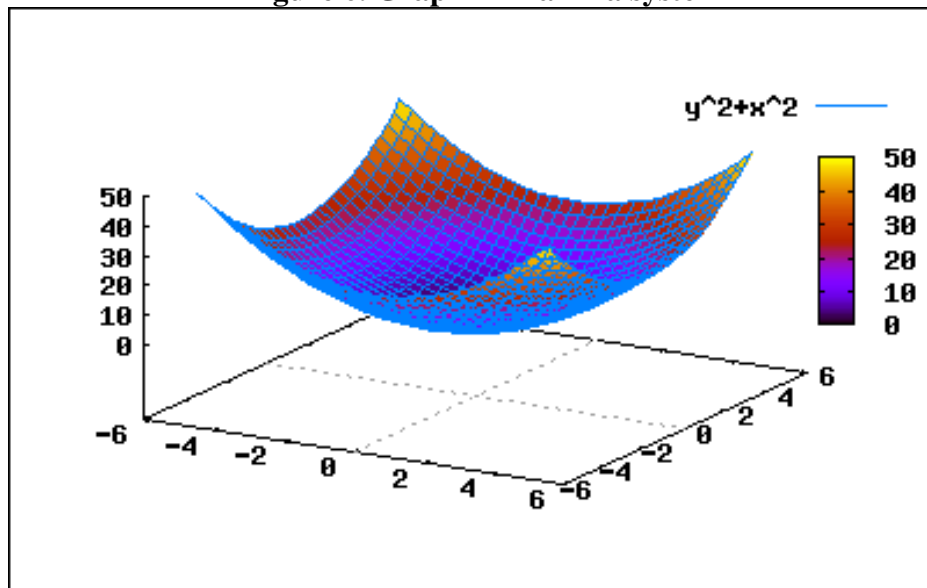
To enable the use of this tool, as in other examples, it is primarily necessary to perform the installation, which is free. It is carried out via the Internet, on one of the listed devices. In doing so, it is important to open a user account in order to log in to this system.

Once GeoGebra is installed, its ease of use begins. In the user interface, in the right corner, clicking on the "New" icon opens a menu in which to run this tool, enter material or create a document. It is possible to use text, multimedia content and pdf files.

#### 4.1.6. Maxima

This mathematical software tool appears as a modified form of the former tool, of the same purpose, called Macsyme, and developed as early as the 1960s. Maxima is the only system based on the former intention. Maxima began operations in 1998, and has been regularly updated since then to fix bugs and improve code and documentation. An overview of the chart created in this system follows (Figure 6).

**Figure 6. Graph in Maxima system**



**Figure 6. Graph in Maxima system**

Source: Math Blog Team (2007) 3 awesome free Math programs. Available at: <https://mathblog.com/3-awesome-free-math-programs/>

It is basically an algebraic system that can perform symbolic manipulation to solve common problems. It covers an extremely wide range of functionalities. The tool itself is presented to users as a manipulation system, and serves symbolic and numerical expressions including differentiation, integration, Taylor series, Laplace transforms, ordinary differential equations, systems of linear equations, polynomials and sets, lists, vectors, matrices and tensors. Maxima gives high precision numerical results using accurate fractions, arbitrary precision numbers and variable numbers of precision floating points. The tool can draw functions and data in two and three dimensions (SW Math, 2019).

#### 4.1.7. Matlab

Matlab is a complex mathematical software tool, which is used to solve various mathematical problems. In addition, it is intended for various calculations and simulations related to signal processing, control and system identification (Petkovic, 2005).

The first version of this program was created in the late 1970s, and was initially intended for matrix theory, linear algebra, and numerical analysis. In the early 80's it switched to the C programming language and new features were added to it. Modification of the same has led to its wider application in practice, and since 1984 it has been available as a commercial product of MathWorks (Petkovic, 2005).

Today it represents an interactive system and programming language for general technical and scientific calculations. In addition to the basic system, there are numerous software packages that further expand it, and thus it covers various areas of engineering (Petkovic, 2005):

- signal and image processing;
- 2D and 3D graphics;
- automatic steering;
- system identification;
- statistical processing;
- time and frequency domain analysis;
- symbolic mathematics and many others.

It should be noted in particular that all data in MATLAB are treated as matrices whose dimensions do not need to be stored as separate variables. In doing so, all data are written in double float format, which contributes to a large dynamic range and accuracy in various applications. An example of this software tool, or one of the solutions, follows below (Figure 7).

This example shows basic techniques and functions for working with matrices in the MATLAB® language.

First, let's create a simple vector with 9 elements called a.

```
a = [1 2 3 4 6 4 3 4 5]
```

```
a = 1×9
```

```
1 2 3 4 6 4 3 4 5
```

Now let's add 2 to each element of our vector, a, and store the result in a new vector.

Notice how MATLAB requires no special handling of vector or matrix math.

```
b = a + 2
```

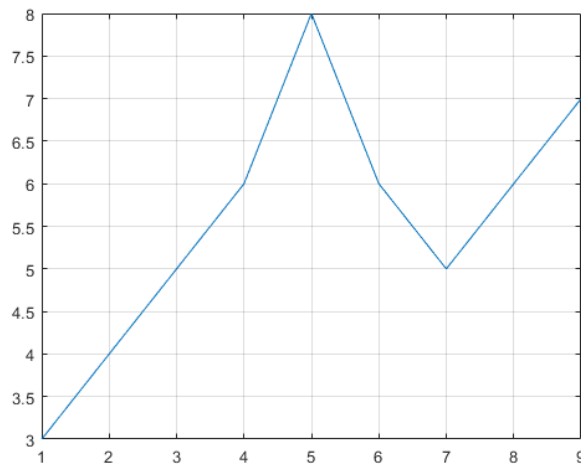
```
b = 1×9
```

```
3 4 5 6 8 6 5 6 7
```

Creating graphs in MATLAB is as easy as one command. Let's plot the result of our vector addition with grid lines.

```
plot(b)
```

```
grid on
```



**Figure 7. Matlab**

Source: Math works (2019) MATLAB. Available at:  
<https://nl.mathworks.com/help/matlab/math/basic-matrix-operations.html>

It can be seen that the problem task was set first, ie the vector with nine elements. After that, a second vector was created, to which nine were also defined elements, with the addition of 2. Based on these conditions, with the presented commands this tool provides a solution in the form of a graphical representation. It should be noted how it is possible to generate and different forms of graphs (columns, key points, etc.).

The basic operations covered by this tool are (Petkovic, 2005):

- basic arithmetic operations - addition, subtraction, multiplication and division; operations by elements - a dot (.) is placed before the mathematical operator;
- relational operators - smaller, larger, equal;
- special matrices;
- Matrix operations.

It should be noted that this is an advanced interactive tool, which is used by many today stakeholders. In doing so, it goes beyond educational functions and purpose, and is especially popular in engineering and business problems.

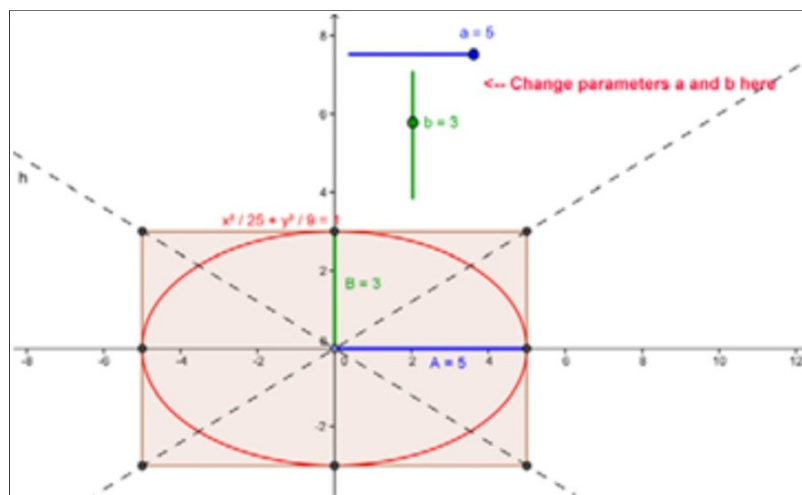
**5. INTEGRATING EDUCATIONAL MATERIALS IN TEACHING**

The results of the diagnostic testing led me to continue with everything while performing students, I pay attention to the low level of prior knowledge. It was necessary, yes, before that at all I try to process some mathematical concept by analogy, to take a step back and virtually i process the base of the analogy first, because, one cannot expect an analogy to work when the base is an analogy vague, blurry.

For example, when processing the contents of a square surface, before moving on to the study of ellipsoids, I worked out the ellipse in detail. For this purpose, I used applets created with free open source software, GeoGebra intended for learning dynamic mathematical concepts relating to two-dimensional mathematics. That way, the students had a chance to see as the shape of the ellipse changes.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

by changing the values of parameters *a* and *b*.

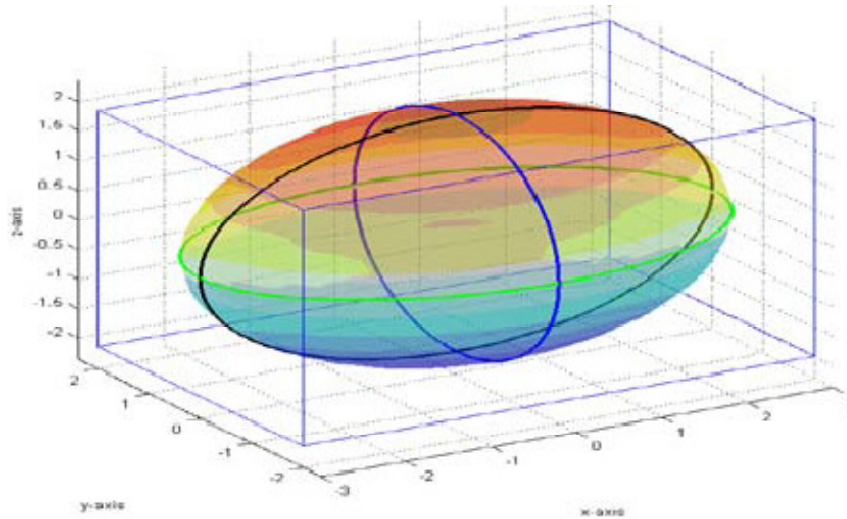


**Figure 8. GeoGebra applet: changes in the shape of an ellipse caused by changing parameter values** Only after that did i explain by analogy how the shape of an ellipsoid changes.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

by changing the values of parameters  $a$ ,  $b$  and  $c$ .

For that purpose, i used visualization tools made in Matlab.



**Figure 9. Changes in the shape of ellipsoids caused by changing parameter values**

In this way, interactivity between students and multimedia tools as educational materials was achieved. In each situation where this was possible, the dynamic nature of a particular problem or mathematical situation was visualized by appropriate animation.

## 6. CONCLUSION

Information and communication technologies are technologies that computers use for collection, processing, storage, protection and transmission of information such as portable computers, tablets, smart mobile devices, interactive whiteboards ... Today's educational the system is subject to change and will be implemented from this school year (2018/2019) Experimental program "School for Life". This program wants to be tested advantages and disadvantages of new curricula for certain subjects.

The experimental program is also conducted in the teaching of mathematics. With numerous didactic teaching aids and tools, the experimental program is recommended and use of digital technology in teaching, ie use of ICT. For teaching mathematics it is recommended to use various applications and software tools such as Matific, Wordwall, Kahoot, Quizizz ... These programs make learning easier content in mathematics and are interesting to students. Numerous studies have been conducted in schools have shown how ICT technology positively affects learning teaching content in mathematics. Research conducted on teachers showed how teachers want to use ICT in their educational process as well that they would like to be further educated for such a form of teaching. I think every teacher would should develop their competencies and abilities, and thus competencies in natural sciences and technology. By completing college obligations and obtaining diplomas, the teacher's education should not stop. The task of the teacher is to educates throughout life and uses the various forms and methods of teaching that are in accordance with the needs and requirements of the society in which we live.

Learning mathematics at a higher education level is a very complex process. In order to answer the questions of how mathematical thinking is acquired, and what thought processes a person goes through creating his system of mathematical knowledge, in the future we should turn to mathematical cognition - an area that will provide answers to questions that may be the basis of new directions in mathematics methodology. Following modern trends in mathematics methodology and innovations in mathematics teaching, it is very important that more research is conducted to examine the impact of ICT on knowledge related to more specific areas of mathematics at all educational levels, as well as student experiences during learning mathematics with innovative approaches. modern mathematics education.

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**LINKS**

<https://www.azvo.hr/hr/pojmovnik/78-kompetencije>

<https://hrcak.srce.hr/74954>

The links marked individual images in the paper

<https://hrcak.srce.hr/61536>

<https://mis.element.hr/fajli/824/46-03.pdf>



## PROPERTIES OF WEYL TENSOR AND COTTON TENSOR IN A RIEMANNIAN MANIFOLD

**Monika Sati and K.C. Petwal**

Department of Mathematics, H.N.B. Garhwal University (A Central University),  
S.R.T Campus Badshahithaul, TehriGarhwal, Uttarakhand, India  
E-mail : monikasati123@gmail.com

### ABSTRACT :

*In the present paper, we study some algebraic and geometric properties of Weyl tensor and Cotton tensor in a Riemannian manifold. Here we find that the Weyl tensor and Cotton tensor yields the traceless properties in a Riemannian curvature tensor. Also by contraction method, we show that the Riemannian tensor and Cotton tensor vanishes identically in a Riemannian manifold. The relative properties of Weyl tensor and Cotton tensor are also investigated.*

*Keywords: Weyl tensor, Cotton tensor, Riemannian curvature tensor, Kulkarni-Nomizu product, Manifold.*

*MSC(2010): 53B20, 53B20, 53A45*

### 1. INTRODUCTION

J. A. Wheeler [9] named Weyl curvature tensor which is a measure of the curvature of space-time. Hermann Klaus Hugo Weyl (1955), one of the German Mathematicians of the 19th century, published technical and some general work on space, time, matter, philosophy, logic symmetry and, visualized general relativity with the laws of electromagnetism. In 1918, Hermann Weyl [7] first appeared the Weyl tensor is of enormous consequence in theoretical physics and the general theory of relativity. The Weyl tensor is similar to Riemannian curvature tensor and behaves like a force that a body feels when moving on a geodesic. This force is called tidal force which changes the shape of the body and like Riemann curvature tensor, it does not pass on the information that how the volume of the body changes. In three dimensional Weyl space, Kumar et al [8] discuss the evolution of tidal tensor as relativistic equation and defined a new kind of tensorial quantity called “Newtonian tidal tensor” which is trace free part of gravitational field tensor, where comma denotes derivative concerning for to Eulerian coordinate system. The Weyl tensor is a totally trace-free component of the decomposition of the Riemannian curvature tensor and it satisfies the first Bianchi identity.

E. Cotton [6] in 1899 first appeared the Cotton tensor is similar and closely define in conformal geometry and Weyl tensor by its properties and plays important role in the Hamiltonian formulation of general relativity [12]. The Cotton tensor is a third-order tensor in three- dimensional Riemannian manifolds, which serves as on barrier to local conformal flatness manifold. Cotton tensor possesses properties that

are very similar to those of the Weyl tensor; in particular, it is conformally invariant. The Weyl tensor and Cotton tensor plays an important role in modern mathematics and theoretical physics.

**1.1 Preliminaries**

Let  $M$  be an  $n$ -dimensional Riemannian manifold with the fundamental positive definite metric,  $ds^2 = g_{ji}(x)dx^j dx^i$ , covered by any system of coordinates  $x^k$ , where  $i, j, k$  run over the range  $1, 2, 3, \dots, n$ . If  $g_{ji}$  are constant then  $M$  is called locally Euclidean manifold if and only if the curvature tensor  $F_{kij}^h = 0$ .

The curvature tensor appears also in the Ricci identities [11]:

$$\nabla_h \nabla_j v^h - \nabla_j \nabla_h v^h = F_{hji}^h v^i \tag{1.1}$$

$$\nabla_h \nabla_j W_i - \nabla_j \nabla_h W_i = -F_{hji}^h W_h \tag{1.2}$$

$$\nabla_l \nabla_k T_{ji}^h - \nabla_k \nabla_l T_{ji}^h = F_{lka}^h T_{ji}^a - F_{lkj}^a T_{ai}^h - F_{lki}^a T_{ja}^h \tag{1.3}$$

Also the Riemannian Curvature tensors are:

$$F_{ij} = -F_{ji}, \quad i, j = 1, 2, 3, \dots, n \tag{1.4}$$

$$F_{ijkh} = -F_{ijhk} \tag{1.5}$$

$$F_{ijk} + F_{ikj} + F_{kij} = 0 \tag{1.6}$$

Here the curvature tensor

$$F_{ij} = F_{aij}^a = g^{ab} F_{aijb} = g^{ab} F_{iabj} \tag{1.7}$$

is called the Ricci tensor. It is symmetric tensor i.e.  $F_{ij} = F_{ji}$  and the scalar  $F = g^{ij} F_{ij}$  is called the curvature scalar.

Here equation (1.3) is the first Bianchi identity or algebraically Bianchi identity of Riemannian curvature tensor. The second Bianchi identity or differential Bianchi identity takes the form:

$$\nabla_l F_{kji}^h + \nabla_k F_{jli}^h + \nabla_j F_{lki}^h = 0, \tag{1.8}$$

Where  $\nabla_i, \nabla_j, \nabla_k$  are covariant derivative of Riemannian curvature tensor.

From this we find by contraction

$$\nabla_a F_{kji}^a = \nabla_h F_{ji} - \nabla_j F_{ki} \tag{1.9}$$

And from this by contraction with  $g^{ji}$

$$\nabla_a F_k^a = \frac{1}{2} \nabla_h F \tag{1.10}$$



Further, in the Riemannian Curvature tensor, the second-order Ricci tensor  $R_j^h$  in terms of contracted Bianchi identity of Ricci scalar-tensor  $\mathcal{R}$  is

$$\nabla_h R_j^h = \frac{1}{2} \nabla_j \mathcal{R} \quad (1.11)$$

In Riemannian Curvature tensor, Kulkarni-Nomizu studied and described the product of two rank tensors. In virtue of this if S and Q are symmetric two rank tensors then we described the product

$$\begin{aligned} (S \odot Q)(W, X, Y, Z) &= S(W, Y)Q(X, Z) + S(X, Z)Q(W, Y) \\ &\quad - S(W, Z)Q(X, Y) - S(X, Y)Q(W, Z) \\ &= \begin{vmatrix} S(W, Y) & S(W, Z) \\ Q(X, Y) & Q(X, Z) \end{vmatrix} + \begin{vmatrix} Q(W, Y) & Q(W, Z) \\ S(X, Y) & S(X, Z) \end{vmatrix} \end{aligned} \quad (1.12)$$

where W, X, Y, Z are the tangent vectors and  $|\cdot|$  is the matrix determinant. In terms of local coordinate system the above equation we can be written in the following form

$$(S \odot Q)_{hijk} = S_{jh}Q_{ik} + S_{ik}Q_{jh} - S_{ik}Q_{kh} - S_{kh}Q_{ik} \quad (1.13)$$

The Weyl Tensor (or Weyl curvature tensor) and Cotton tensor (or Cotton curvature tensor) have the similar geometrical properties. In general relativity, the Weyl tensor governs the propagation of gravitational waves also is the part of curvature that exist in free space and solution of vacuum Einstein equation. Weyl tensor is similar to Riemannian tensor of the type (0, 4). In  $n$ -dimensional manifold the (0, 4) type Weyl tensor represented by Petersen [20]

$$L = \mathcal{R} - \frac{1}{n-2} \left( Ric - \frac{s}{n} g \right) \odot g - \frac{s}{2n(n-1)} g \odot g, \quad (1.14)$$

where  $g$  is metric,  $\mathcal{R}$  is Riemann tensor,  $Ric$  is the Ricci-tensor,  $s$  is the scalar curvature and  $h \odot k$  denote the Kulkarni-Nomizu product of the symmetric scalar tensor. The component (extended) form of this tensor is

$$\begin{aligned} L_{ijkl} &= R_{ijkl} + \frac{1}{n-2} (R_{il}g_{jk} - R_{ik}g_{jl} + R_{jk}g_{il} - R_{jl}g_{ik}) \\ &\quad + \frac{\mathcal{R}}{(n-1)(n-2)} (g_{ik}g_{jl} - g_{il}g_{jk}). \end{aligned} \quad (1.15)$$

The Weyl tensor can also be expressed using the Schouten tensor, which is a trace adjusted multiple of the Ricci tensor

$$P = \frac{1}{n-2} \left( Ric - \frac{s}{2(n-1)} g \right). \quad (1.16)$$

Then  $L = \mathcal{R} - P \odot g$  or in extended form

$$L_{ijkl} = R_{ijkl} - \frac{2}{n-2} (g_{i[k}R_{l]j} - g_{j[k}R_{l]i}) + \frac{2}{(n-1)(n-2)} \mathcal{R}g_{i[k}R_{l]j}, \tag{1.17}$$

where  $R_{ijkl}$ ,  $R_{ij}$ ,  $\mathcal{R}$  are the Riemann, Ricci, Ricci scalar tensor respectively and square brackets around indices refers to the anti-symmetric part. The above equation is also equivalently

$$L_{ij}^{kl} = R_{ij}^{kl} - 4 S [{}^k_i \delta_j^l], \tag{1.18}$$

where  $S$  denote the Schouten tensor.

The Weyl tensor is invariant under conformal changes so it is called the conformal tensor which has the same symmetries as the Riemann tensor as the following:

$$L_{jkl}^i = L_{jkl}^i; L_{ijkl} = -L_{jikl} = -L_{ijlk}; \tag{1.19}$$

$$L_{ijkl} + L_{ilkj} + L_{iljk} = 0 \text{ and } L_{jik}^i = 0.$$

Weyl tensor shows in second Bianchi identity that

$$\nabla_i L_{jkl}^i = 2(n-3) \nabla_{[k} S_{l]j}, \tag{1.20}$$

where  $S$  is the Schouten-tensor. Here the valance tensor  $\nabla_i L_{jkl}^i$  of the type  $(0, 3)$  is called Cotton tensor. The Cotton tensor is denoted by  $\tilde{C}_{jk}^i$  and its extended form is

$$\tilde{C}_{jk}^i = g^{im} \tilde{C}_{mjk} = \nabla_k \varphi_j^i - \nabla_j \varphi_k^i + \frac{1}{2(n-1)} (\delta_k^i \nabla_j \mathcal{R} - \delta_j^i \nabla_k \mathcal{R}). \tag{1.21}$$

## 2. WEYL TENSOR AND COTTON TENSOR IN A RIEMANNIAN MANIFOLD

**Theorem 2.1** *In the Riemannian manifold the Riemannian curvature tensor  $F_{ijk}^h$  yields the traceless Weyl tensor  $L_{ijk}^h$ .*

**Proof:** The weyl tensor in terms of Riemannian curvature tensor is characterized by

$$L_{ijk}^h = F_{ijk}^h + \frac{1}{(n-2)} (\varphi_{ij} \delta_k^h - \varphi_{ij} \delta_k^h + g_{ij} g^{nh} \varphi_{nk} - g_{ik} g^{nh} \varphi_{nj}) + \frac{\mathcal{R}}{(n-1)(n-2)} (g_{ik} \delta_j^h - g_{ij} \delta_k^h). \tag{2.1}$$

If we contract the equation (2.1) with respect to indices  $h$  and  $j$ , we have

$$L_{ihk}^h = F_{ihk}^h + \frac{1}{(n-2)} (\varphi_{ih} \delta_k^h - \varphi_{ik} \delta_h^h + g_{ih} g^{nh} \varphi_{nk} - g_{ik} g^{nh} \varphi_{nh}) + \frac{\mathcal{R}}{(n-1)(n-2)} (g_{ik} \delta_h^h - g_{ih} \delta_k^h). \tag{2.2}$$

Using (1.6) and definition of kronecker delta the equation (2.2) reduces to the form

$$\begin{aligned} L_{ihk}^h &= \varphi_{ik} + \frac{1}{(n-2)} (\varphi_{ik} - n\varphi_{ik} + \varphi_{ik} - g_{ik} \mathcal{R}) + \frac{\mathcal{R}}{(n-1)(n-2)} (ng_{ik} - g_{ik}) \\ &= \varphi_{ik} + \frac{1}{(n-2)} \left[ (2\varphi_{ik} - n\varphi_{ik} - g_{ik} \mathcal{R}) + \frac{\mathcal{R}}{(n-1)} (n-1)g_{ik} \right] \\ &= \varphi_{ik} + \frac{1}{(n-2)} [-\varphi_{ik}(n-2) - g_{ik} \mathcal{R} + g_{ik} \mathcal{R}] \\ &= \varphi_{ik} - \frac{1}{(n-2)} \cdot (n-2)\varphi_{ik} \\ &= \varphi_{ik} - \varphi_{ik} = 0. \end{aligned} \tag{2.3}$$

Similarly, after contraction of the indices  $h$  and  $k$ , and noting that  $L_{ijk}^h = L_{ikj}^h$ , we get

$$L_{ihj}^h = \varphi_{ij} + \frac{1}{(n-2)} \left[ -\varphi_{ij}(n-2) - g_{ij}\mathcal{R} + \frac{\mathcal{R}}{(n-1)}g_{ij}(n-1) \right] = 0 \quad (2.4)$$

The rest indices  $h$  and  $i$  are non-trivial trace, however under this procedure the weyl tensor is identically zero due to its anti-symmetry. Hence the Riemannian curvature tensor  $F_{ijk}^h$  yields traceless Weyl tensor  $L_{ijk}^h$ .

**Theorem 2.2** *The weyl tensor  $n \geq 4$  satisfies the first Bianchi identity of Riemannian Curvature tensor i.e.*

$$L_{hijk} + L_{hkij} + L_{hjki} = 0 \text{ or } L_{h[ijk]} = 0. \quad (2.5)$$

**Proof:** For the Kulkarni-Nomizu product of two symmetric tensors  $n \geq 2$ , it is sufficient to prove that the Weyl tensor  $n \geq 4$  satisfies the first Bianchi identity of Riemannian Curvature tensor by the direct computation  $(\forall i, j, k, h)$

Let us define

$$L_{hijk} = (S \odot Q)_{hijk},$$

$$L_{hkij} = (S \odot Q)_{hkij} \text{ And } L_{hjki} = (S \odot Q)_{hjki}.$$

Adding these three equations and using the equation (1.5), we get

$$\begin{aligned} (S \odot Q)_{hijk} + (S \odot Q)_{hkij} + (S \odot Q)_{hjki} \\ = S_{hk}Q_{ij} + S_{ij}Q_{hk} - S_{hj}Q_{ik} - S_{ik}Q_{hj} + S_{hj}Q_{ki} + S_{ki}Q_{hj} \\ - S_{hi}Q_{kj} - S_{kj}Q_{hi} + S_{hi}Q_{jk} + S_{jk}Q_{hi} - S_{hk}Q_{ji} - S_{ji}Q_{hk}. \end{aligned} \quad (2.6)$$

The right hand terms of this equation are cancelled as  $S_{hk} = S_{kh}$  and  $Q_{ij} = Q_{ji}$  etc., therefore

$$(S \odot Q)_{hijk} + (S \odot Q)_{hkij} + (S \odot Q)_{hjki} = 0 \quad (2.7)$$

$$\text{Or } L_{hijk} + L_{hkij} + L_{hjki} = 0.$$

Hence the weyl tensor  $n \geq 4$  satisfies the first Bianchi identity of Riemannian Curvature tensor.

**Theorem 2.3** *Let  $(M, g)$  be an  $n$ -dimensional Riemannian Manifold with metric  $g$ , then by contraction the Riemann tensor and Weyl tensor vanishes identically.*

**Proof:** In three dimensional Riemannian manifold, consider the Riemann tensor  $F_{ijk}^h$  as

$$F_{ijk}^h = \varphi_{ij}\delta_k^h - \varphi_{ij}\delta_k^h + g_{ij}g^{nh}\varphi_{nk} - g_{ik}g^{nh}\varphi_{nj} + \frac{\mathcal{R}}{2}(g_{ik}\delta_j^h - g_{ij}\delta_k^h). \quad (2.8)$$

Multiplying throughout by  $g^{im}$  and using  $g^{im}F_{ijk}^h = F_{jk}^{hm}$ , then equation (2.8) reduces to

$$F_{jk}^{hm} = \varphi_j^m\delta_k^h - \varphi_k^m\delta_j^h + \delta_j^m g^{nh}\varphi_{nk} - \delta_k^m g^{nh}\varphi_{nj} + \frac{\mathcal{R}}{2}(\delta_k^m\delta_j^h - \delta_j^m\delta_k^h) \quad (2.9)$$

Now we arise some cases by assuming that at least two indices of Riemann tensor are equal, therefore:

(a) If  $m = h$ , then equation (2.9) implies

$$\begin{aligned} F_{jk}^{hh} &= \varphi_j^h\delta_k^h - \varphi_k^h\delta_j^h + \delta_j^h g^{nh}\varphi_{nk} - \delta_k^h g^{nh}\varphi_{nj} + \frac{\mathcal{R}}{2}(\delta_k^h\delta_j^h - \delta_j^h\delta_k^h) \\ &= \varphi_j^h\delta_k^h - \varphi_k^h\delta_j^h + \delta_j^h\varphi_k^h - \delta_k^h\varphi_j^h + \frac{\mathcal{R}}{2}(\delta_k^h\delta_j^h - \delta_j^h\delta_k^h) \end{aligned}$$

Using the property of skew-symmetric (anti symmetric), we have

$$F_{jk}^{hh} = -F_{kj}^{hh} = 0.$$

(b) If  $j = h$ , then equation (2.9) implies

$$\begin{aligned} F_{hk}^{hm} &= \varphi_h^m \delta_k^h - \varphi_k^m \delta_h^h + \delta_h^m g^{nh} \varphi_{nk} - \delta_k^m g^{nh} \varphi_{nh} + \frac{\mathcal{R}}{2} (\delta_k^m \delta_n^h - \delta_h^m \delta_k^h). \\ &= \varphi_h^m \delta_k^h - 3\varphi_k^m + \delta_h^m \varphi_k^h - \delta_k^m \mathcal{R} + \frac{\mathcal{R}}{2} (3\delta_k^m - \delta_k^m). \\ &= \varphi_k^m - 3\varphi_k^m + \varphi_k^m - \mathcal{R}\delta_k^m + \frac{\mathcal{R}}{2} (2\delta_k^m). \\ &= -\varphi_k^m. \end{aligned}$$

Since  $F_{hk}^{hm} = -g^{mn} F_{nhk}^h = -g^{mn} \varphi_{nk} = -\varphi_k^m$ , therefore the equality holds.

(c) If  $k = h$ , then equation (2.9) implies

$$\begin{aligned} F_{jh}^{hm} &= \varphi_j^m \delta_h^h - \varphi_h^m \delta_j^h + \delta_j^m g^{nh} \varphi_{nh} - \delta_h^m g^{nh} \varphi_{nj} + \frac{\mathcal{R}}{2} (\delta_h^m \delta_j^h - \delta_j^m \delta_h^h). \\ &= 3\varphi_j^m - \varphi_j^m + \delta_j^m \mathcal{R} - \delta_h^m \varphi_j^h + \frac{\mathcal{R}}{2} (\delta_j^m - 3\delta_j^m). \\ &= 3\varphi_j^m - \varphi_j^m - \varphi_j^m + \delta_j^m \mathcal{R} + \frac{\mathcal{R}}{2} (-2\delta_j^m). \\ &= \varphi_j^m. \end{aligned}$$

Since  $F_{jh}^{hm} = g^{mn} F_{njh}^h = g^{mn} \varphi_{nj} = \varphi_j^m$ , therefore the equality holds.

(d) If  $j = m$ , then equation (2.9) implies

$$\begin{aligned} F_{mk}^{hm} &= \varphi_m^m \delta_k^h - \varphi_k^m \delta_m^h + \delta_m^m g^{nh} \varphi_{nk} - \delta_k^m g^{nh} \varphi_{nm} + \frac{\mathcal{R}}{2} (\delta_k^m \delta_m^h - \delta_m^m \delta_k^h). \\ &= \mathcal{R}\delta_k^h - \varphi_k^h + 3\varphi_k^h - \varphi_k^h + \frac{\mathcal{R}}{2} (\delta_k^h - 3\delta_k^h). \\ &= \varphi_k^h. \end{aligned}$$

Since  $F_{mk}^{hm} = g^{nh} F_{nmk}^m = g^{nh} \varphi_{nk} = \varphi_k^h$ , therefore the equality holds

(e) If  $k = m$ , then equation (2.9) implies

$$\begin{aligned} F_{jm}^{hm} &= \varphi_j^m \delta_m^h - \varphi_m^m \delta_j^h + \delta_j^m g^{nh} \varphi_{nm} - \delta_m^m g^{nh} \varphi_{nj} + \frac{\mathcal{R}}{2} (\delta_m^m \delta_j^h - \delta_j^m \delta_m^h). \\ &= \varphi_j^h - \mathcal{R}\delta_j^h + \varphi_j^h - 3\varphi_j^h + \frac{\mathcal{R}}{2} (3\delta_j^h - \delta_j^h). \\ &= -\varphi_j^h. \end{aligned}$$

Since  $F_{jm}^{hm} = -g^{nh} F_{njm}^m = -g^{nh} \varphi_{nj} = -\varphi_j^h$ , therefore the equality holds.

(f) If  $j = k$ , then equation (2.9) implies

$$\begin{aligned} F_{jk}^{hm} &= \varphi_k^m \delta_k^h - \varphi_k^m \delta_k^h + \delta_k^m g^{nh} \varphi_{nk} - \delta_k^m g^{nh} \varphi_{nk} + \frac{\mathcal{R}}{2} (\delta_k^m \delta_k^h - \delta_k^m \delta_k^h). \\ &= \varphi_k^m \delta_k^h - \varphi_k^m \delta_k^h + \varphi_k^m \delta_k^h - \varphi_k^m \delta_k^h. \\ &= 0. \end{aligned}$$

Since  $F_{kk}^{hm} = -F_{kk}^{hm} = 0$ , therefore the equality holds.

Again, consider the three indices out of four are equal and using the above cases together with the anti-symmetric properties, we have

Case I: If  $h = m = j$  then  $F_{jk}^{jj} = 0$

Case II: If  $h = m = k$  then  $F_{jk}^{kk} = 0$

Case III: If  $h = k = j$  then  $F_{jj}^{im} = 0$

Case IV: If  $m = k = j$  then  $F_{jj}^{im} = 0$

Hence by contraction method, Riemann tensor vanishes. In the similar way we also show that the Weyl tensor vanishes identically.

**Theorem 2.4** *In  $n -$  dimensional Riemannian Curvature tensor, the (1, 2)-type Cotton tensor  $\tilde{C}$  is traceless.*

**Proof:** Contract the equation (1.21) with respect to indices  $i$  and  $j$ , we have

$$\begin{aligned} \tilde{C}_{ik}^i &= g^{im} \tilde{C}_{mik} \\ &= \nabla_k \varphi_i^i - \nabla_i \varphi_k^i + \frac{1}{2(n-1)} (\delta_k^i \nabla_i \mathcal{R} - \delta_k^i \nabla_i \mathcal{R}). \\ &= \nabla_k \mathcal{R} - \frac{1}{2} \nabla_k \mathcal{R} + \frac{1}{2(n-1)} (\nabla_k \mathcal{R} - n \nabla_k \mathcal{R}). \because \varphi_i^i = \mathcal{R} \text{ and } \nabla_i \varphi_k^i = \frac{1}{2} \nabla_k \mathcal{R} \\ &= \frac{1}{2} \nabla_k \mathcal{R} - \frac{(n-1)}{2(n-1)} \nabla_k \mathcal{R}. \\ &= \frac{1}{2} \nabla_k \mathcal{R} - \frac{1}{2} \nabla_k \mathcal{R} = 0 \end{aligned} \quad (2.10)$$

Similarly, contract indices  $k$  and  $i$ , we have

$$\begin{aligned} \tilde{C}_{jk}^k &= g^{ik} \tilde{C}_{ijk} = \nabla_k \varphi_j^k - \nabla_j \varphi_k^k + \frac{1}{2(n-1)} (\delta_k^k \nabla_j \mathcal{R} - \delta_j^k \nabla_k \mathcal{R}). \\ &= \frac{1}{2} \nabla_j \mathcal{R} - \mathcal{R} \nabla_j + \frac{1}{2(n-1)} (n \nabla_j \mathcal{R} - \nabla_j \mathcal{R}). \\ &= \frac{1}{2} \nabla_j \mathcal{R} - \mathcal{R} \nabla_j + \frac{(n-1)}{2(n-1)} \nabla_j \mathcal{R}. \\ &= -\frac{1}{2} \nabla_j \mathcal{R} + \frac{1}{2} \nabla_j \mathcal{R} = 0. \end{aligned} \quad (2.11)$$

Hence, in Riemannian curvature tensor the (1,2)-type Cotton tensor  $\tilde{C}$  is traceless.

### 3. Relative Properties of Weyl Tensor and Cotton tensor

**Theorem 3.1** *By using second Bianchi identity of Riemann Curvature tensor, the Weyl tensor  $L_{ijk}^l$  and Cotton tensor  $\tilde{C}_{ijk}$  represent the following relation:*

$$\begin{aligned} &\nabla_h L_{ijk}^l + \nabla_j L_{ikh}^l + \nabla_k L_{ihj}^l \\ &= \frac{1}{n-2} (\delta_h^l \tilde{C}_{ijk} + \delta_j^l \tilde{C}_{ikh} + \delta_k^l \tilde{C}_{ihj} + g_{ik} \tilde{C}_{jh}^l + g_{ih} \tilde{C}_{kj}^l + g_{ij} \tilde{C}_{hk}^l), \\ \text{where } \tilde{C}_{ijk} &= \nabla_k \varphi_{ij} - \nabla_j \varphi_{ik} + \frac{1}{2(n-1)} (g_{ik} \nabla_j \mathcal{R} - g_{ij} \nabla_k \mathcal{R}). \end{aligned} \quad (3.1)$$

**Proof:** Connect the Levi-Civita connections in the components of Weyl tensor as:

$$\begin{aligned} \nabla_h L_{ijk}^l &= \nabla_h F_{ijk}^l + \frac{1}{n-2} (\delta_j^l \nabla_h \varphi_{ik} - \delta_k^l \nabla_h \varphi_{ij} + g_{ik} \nabla_h \varphi_j^l - g_{ij} \nabla_h \varphi_k^l) \\ &\quad + \frac{\mathcal{R}}{(n-1)(n-2)} (g_{ij} \delta_k^l - g_{ik} \delta_j^l) \nabla_h. \end{aligned} \quad (3.2)$$

$$\begin{aligned} \nabla_j L_{ikh}^l &= \nabla_j F_{ikh}^l + \frac{1}{n-2} (\delta_k^l \nabla_j \varphi_{ih} - \delta_h^l \nabla_j \varphi_{ik} + g_{ih} \nabla_j \varphi_k^l - g_{ik} \nabla_j \varphi_h^l) \\ &\quad + \frac{\mathcal{R}}{(n-1)(n-2)} (g_{ik} \delta_h^l - g_{ih} \delta_k^l) \nabla_j. \end{aligned} \tag{3.3}$$

$$\begin{aligned} \nabla_k L_{ihj}^l &= \nabla_k F_{ihj}^l + \frac{1}{n-2} (\delta_h^l \nabla_k \varphi_{ij} - \delta_j^l \nabla_k \varphi_{ih} + g_{ih} \nabla_k \varphi_j^l - g_{ij} \nabla_k \varphi_h^l) \\ &\quad + \frac{\mathcal{R}}{(n-1)(n-2)} (g_{ih} \delta_j^l - g_{ij} \delta_h^l) \nabla_k. \end{aligned} \tag{3.4}$$

Adding (3.2), (3.3) and (3.4), we have

$$\begin{aligned} \nabla_h L_{ijk}^l + \nabla_j L_{ikh}^l + \nabla_k L_{ihj}^l &= \nabla_h F_{ijk}^l + \frac{1}{n-2} (\delta_j^l \nabla_h \varphi_{ik} - \delta_k^l \nabla_h \varphi_{ij} + g_{ik} \nabla_h \varphi_j^l - g_{ij} \nabla_h \varphi_k^l) \\ &\quad + \frac{\mathcal{R}}{(n-1)(n-2)} (g_{ij} \delta_k^l - g_{ik} \delta_j^l) \nabla_h + \nabla_j F_{ikh}^l \\ &\quad + \frac{1}{n-2} (\delta_k^l \nabla_j \varphi_{ih} - \delta_h^l \nabla_j \varphi_{ik} + g_{ih} \nabla_j \varphi_k^l - g_{ik} \nabla_j \varphi_h^l) \\ &\quad + \frac{\mathcal{R}}{(n-1)(n-2)} (g_{ik} \delta_h^l - g_{ih} \delta_k^l) \nabla_j + \nabla_k F_{ihj}^l \\ &\quad + \frac{1}{n-2} (\delta_h^l \nabla_k \varphi_{ij} - \delta_j^l \nabla_k \varphi_{ih} + g_{ih} \nabla_k \varphi_j^l - g_{ij} \nabla_k \varphi_h^l) \\ &\quad + \frac{\mathcal{R}}{(n-1)(n-2)} (g_{ih} \delta_j^l - g_{ij} \delta_h^l) \nabla_k. \end{aligned}$$

Or  $\nabla_h L_{ijk}^l + \nabla_j L_{ikh}^l + \nabla_k L_{ihj}^l = \nabla_h F_{ijk}^l + \nabla_j F_{ikh}^l + \nabla_k F_{ihj}^l$

$$\begin{aligned} &\quad + \frac{1}{n-2} [(\delta_j^l \nabla_h \varphi_{ik} - \delta_k^l \nabla_h \varphi_{ij} + g_{ik} \nabla_h \varphi_j^l - g_{ij} \nabla_h \varphi_k^l + \delta_k^l \nabla_j \varphi_{ih} \\ &\quad - \delta_h^l \nabla_j \varphi_{ik} + g_{ih} \nabla_j \varphi_k^l - g_{ik} \nabla_j \varphi_h^l + \delta_h^l \nabla_k \varphi_{ij} - \delta_j^l \nabla_k \varphi_{ih} + g_{ih} \nabla_k \varphi_j^l - g_{ij} \nabla_k \varphi_h^l) \\ &\quad + \frac{1}{n-1} (\delta_k^l g_{ij} \nabla_h \mathcal{R} - \delta_j^l g_{ik} \nabla_h \mathcal{R} + \delta_h^l g_{ik} \nabla_j \mathcal{R} - \delta_k^l g_{ih} \nabla_j \mathcal{R} + \delta_j^l g_{ih} \nabla_k \mathcal{R} - \delta_h^l g_{ij} \nabla_k \mathcal{R})] \end{aligned}$$

Using equation (1.8) i.e.  $\nabla_h F_{ijk}^l + \nabla_j F_{ikh}^l + \nabla_k F_{ihj}^l = 0$ , we get

$$\begin{aligned} \nabla_h L_{ijk}^l + \nabla_j L_{ikh}^l + \nabla_k L_{ihj}^l &= \frac{1}{n-2} [\delta_j^l (\nabla_h \varphi_{ik} - \nabla_k \varphi_{ih}) + \delta_k^l (\nabla_j \varphi_{ih} - \nabla_h \varphi_{ij}) + \delta_h^l (\nabla_k \varphi_{ij} - \nabla_j \varphi_{ik})] \\ &\quad + \frac{1}{n-2} [g_{ik} (\nabla_h \varphi_j^l - \nabla_j \varphi_h^l) + g_{ij} (\nabla_k \varphi_h^l - \nabla_h \varphi_k^l) + g_{ih} (\nabla_j \varphi_k^l - \nabla_k \varphi_j^l)] \\ &\quad + \frac{1}{(n-1)(n-2)} (g_{ij} \delta_k^l \nabla_h \mathcal{R} - g_{ik} \delta_j^l \nabla_h \mathcal{R} + g_{ik} \delta_h^l \nabla_j \mathcal{R} - g_{ih} \delta_k^l \nabla_j \mathcal{R} + g_{ih} \delta_j^l \nabla_k \mathcal{R} - g_{ij} \delta_h^l \nabla_k \mathcal{R}). \end{aligned}$$

Or

$$\begin{aligned} &\nabla_h L_{ijk}^l + \nabla_j L_{ikh}^l + \nabla_k L_{ihj}^l \\ &= \frac{1}{n-2} (\delta_h^l \tilde{C}_{ijk} + \delta_j^l \tilde{C}_{ikh} + \delta_k^l \tilde{C}_{ihj} + g_{ik} \tilde{C}_{jh}^l + g_{ih} \tilde{C}_{kj}^l + g_{ij} \tilde{C}_{hk}^l) \\ &\quad + \frac{1}{(n-1)(n-2)} \underbrace{(g_{ij} \tilde{C}_{kh}^l - g_{ik} \tilde{C}_{jh}^l + g_{ik} \tilde{C}_{hj}^l - g_{ih} \tilde{C}_k^l + g_{ih} \tilde{C}_{jk}^l - g_{ij} \tilde{C}_{hk}^l)}_0. \end{aligned} \tag{3.5}$$

Here we put  $(\nabla_h \varphi_{ik} - \nabla_k \varphi_{ih}) = \tilde{C}_{ikh}$ ;  $(\nabla_j \varphi_{ih} - \nabla_h \varphi_{ij}) = \tilde{C}_{ihj}$  and  $(\nabla_k \varphi_{ij} - \nabla_j \varphi_{ik}) = \tilde{C}_{ijk}$ .

Also, we put  $(\nabla_h \varphi_j^l - \nabla_j \varphi_h^l)$ ,  $(\nabla_k \varphi_h^l - \nabla_h \varphi_k^l)$  and  $(\nabla_j \varphi_k^l - \nabla_k \varphi_j^l)$  to  $\tilde{C}_{jh}^l$ ,  $\tilde{C}_{hk}^l$  and  $\tilde{C}_{kj}^l$  respectively, then

second part of the equation is zero.

Hence equation (3.5) reduces to the form

$$\nabla_h L_{ijk}^l + \nabla_j L_{ikh}^l + \nabla_k L_{ihj}^l = \frac{1}{n-2} (\delta_h^l \tilde{C}_{ijk} + \delta_j^l \tilde{C}_{ikh} + \delta_k^l \tilde{C}_{ihj} + g_{ik} \tilde{C}_{jh}^l + g_{ih} \tilde{C}_{kj}^l + g_{ij} \tilde{C}_{hk}^l),$$

Where 
$$\tilde{C}_{ijk} = \nabla_k \varphi_{ij} - \nabla_j \varphi_{ik} + \frac{1}{2(n-1)} (g_{ik} \nabla_j \mathcal{R} - g_{ij} \nabla_k \mathcal{R}).$$

**Theorem 3.2** *Let  $(M, g)$  be an  $n$ -dimensional Riemannian manifold with dimension  $n > 3$  then the relation between the co-ordinate expressions of the Weyl tensor and Cotton tensor is*

$$\nabla_h L_{ijk}^h = \frac{n-3}{n-2} \tilde{C}_{ijk}. \quad (3.6)$$

**Proof:** Contract the equation (3.1) with respect to indices  $l$  and  $h$ , we have

$$\begin{aligned} \nabla_h L_{ijk}^h + \nabla_j L_{ikh}^h + \nabla_k L_{ihj}^h &= \frac{1}{n-2} (\delta_h^h \tilde{C}_{ijk} + \delta_j^h \tilde{C}_{ikh} + \delta_k^h \tilde{C}_{ihj} + g_{ik} \tilde{C}_{jh}^h + g_{ih} \tilde{C}_{kj}^h + g_{ij} \tilde{C}_{hk}^h). \\ \text{Or } \nabla_h L_{ijk}^h + \nabla_j L_{ikh}^h + \nabla_k L_{ihj}^h &= \frac{1}{n-2} (n \tilde{C}_{ijk} + \tilde{C}_{ikj} + \tilde{C}_{ikj} + g_{ik} \tilde{C}_{jh}^h + g_{ih} \tilde{C}_{kj}^h + g_{ij} \tilde{C}_{hk}^h). \end{aligned} \quad (3.7)$$

Now from the traceless properties of Weyl tensor

$$\nabla_j L_{ikh}^h = 0; \nabla_k L_{ihj}^h = 0$$

and the traceless properties of Cotton tensor

$$g_{ih} \tilde{C}_{kj}^h = 0; g_{ij} \tilde{C}_{hk}^h = 0; g_{ih} \tilde{C}_{kj}^h = \tilde{C}_{ikj},$$

The equation (3.7) reduces to the form

$$\begin{aligned} \nabla_h L_{ijk}^h &= \frac{1}{n-2} (n \tilde{C}_{ijk} - \tilde{C}_{ijk} - \tilde{C}_{ijk} - \tilde{C}_{ijk}) \because \tilde{C}_{ijk} = -\tilde{C}_{ijk} \\ &= \frac{1}{n-2} (n \tilde{C}_{ijk} - 3 \tilde{C}_{ijk}) \end{aligned}$$

Or

$$\nabla_h L_{ijk}^h = \frac{n-3}{n-2} \tilde{C}_{ijk}$$

## CONCLUSION

The properties of Weyl tensor and Cotton tensor in a Riemannian manifold is divided in three sections. The first section is an introductory one in which we introduce some preliminaries related to the article. In section two, we investigated some theorems related to the traceless properties of Weyl tensor and Cotton tensor and show how the Weyl tensor satisfied the first Bianchi identity. In consequence of third section by using second Bianchi identity, the relation as well as coordinate expression between Weyl tensor and Cotton tensor has been investigated.

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## WEKA CLASSIFICATION AND PREDICTION OF COVID-19 PANDEMIC AFFECTED DISTRICTS OF TAMILNADU - A CASE STUDY

Manimannan G\* & R. Lakshmi Priya\*\*

\*Department of Mathematics, TMG College of Arts and Science, Chennai

\*\*Department of Statistics, Dr. Ambedkar Government Arts, College, Vyasarpadi, Chennai

E-mail : manimannang@gmail.com, priyagayu2006@gmail.com

### ABSTRACT :

*This study attempts to identify COVID-19 pandemic affected districts of Tamilnadu using different data mining techniques and visualize with the help of Geographical Information System (GIS). The secondary data sources were collected from Health and Family Welfare Department, Tamilnadu. In this research paper Diagnosed cases, Deaths, Recovered cases, Active Cases, Population and Total case per 1,000, 000 population were used as parameters. The main objective of this paper is to classify the districts of Tamilnadu based on the above parameters and database. Weka (Waikato Environment for Knowledge Analysis) is a very powerful data mining tool in the area of Data Science. Initially, the given databases are subject to normalize using Apriori algorithm and subsequently applying simple k-means algorithm to identify the groups and are named as three zones, Green (Low Spread Districts), Orange (Moderate Spread Districts) and Red (High Spread Districts). Finally, to cross validate the results of three clusters using various data mining model. In addition, the summary statistics, confusion matrix, visualization of k-mean cluster and classification of various states based on GIS has been included. The correct classification results achieved hundred percent of data mining techniques. The different data mining techniques are explained in the part of methodology.*

*Keywords:; Normalization, k-means, Naive Baye's Classifier, Multi Layer Perceptron (MLP) and Random Forest.*

### 1. INTRODUCTION

In December 2019, originated the virus Covid-19 from wuhan, China and slowly spread all over the world. Many peoples lost their lives because of this virus. This virus was intially identified with some symptoms from the affected person, like throat infection, continuous fever, cough, etc. Recently so many people lost their lives without any symptoms. In super power countries all were in struggle to lead their regular life and lost so many peoples. In our country, the first case was confirmed from Kerala on 21<sup>st</sup> January 2020, returned from China. Majority of the cases are from international pasengers who returned to their own country and slowly spreaded all over the nation. Now we are in first place of corona virus affected country in Asia. The government is taking so many safety measurements, asking the people to wear face mask, maintain social distacnce, lockdown the entire country in the first, second, third and fourth phases. In the mean time so many people migrated within the county itself. In international and national airports passengers are screened by thermal scanners. The entire nation was divided into three zones, namely Red Zone, Orange Zone and Green Zone. The place with high number of active cases are

marked as red Zone, the places in which the number is less is marked as orange zone, and Green zone is the area where there are no case of COVID-19.

In Tamilnadu, the virus spread and affected capital city and neighbouring districts of Kancheepuram, Chengalpet and Thiruvallur. In capital city many people live from other districts and states, but now they moved to their native places. They are tested in the district entrance and put them isolated in a place. If positive, government makes them to move to government hospital, else permitting to their own place and advising them for seven days of self quarantine as per the government rules. In Tamilnadu mortality rate is very less when compared with other states and also with other countries. Tamil nadu Government tests more samples and identifying positive cases every day, particularly the capital city is isolated from other districts. In forthcoming days, from 19<sup>th</sup> to 30<sup>th</sup> June government announced intense lockdown for the capital city and its three neighbouring districts namely, Chengalpet, Kancheepuram and Thiruvallur districts.. The government cancelled the Tenth exam and announced all pass and marks will be awarded based on their previous exam and attendance. In the mean time many higher education institutions organizing webinar, E-quiz, online conferences, Online Faculty Development programmes etc. Most of the people have lost their profession and striving hard to lead their life. The government offices are working with 33% of workers in a regular shift basis. Majority of the private sectors and IT companies are trying hard to serve the employees to work from home option.

## **2. REVIEW OF LITERATURE:**

In the previous study of our research papers, an attempt is made to predict the future using time series model and classify the Indian states using k-means algorithm for Covid-19 cases [1,2]. The first positive case of this corona virus was found in 2019 (COVID-19) in Wuhan city of China by the World Health Organization (WHO) [3]. Many statistical hypotheses attribute the origin of this virus from seafood and bat. COVID-19 spreads internationally in most of the countries and was defined pandemic by WHO in March 2020 [4] and COVID-19 affected 186 countries and territories around the world with more than 280,000 confirmed cases and most of the people lost their lives [5]. India is one of the top ten countries affected by this virus [6]. COVID-19 spread is very exponential in the world. The main objectives of this research paper is to identify Data mining classifier and prediction models which can help in health resources management and planning for prevention purposes.

## **3. DATABASE**

In this research paper Diagnosed cases, Deaths, Recovered cases, Active Cases, Population and Total case per 1,000, 000 population of 37 districts were used as parameters. The main objective of this paper is to classify the districts of Tamilnadu based on the above parameters and database. Weka (Waikato Environment for Knowledge Analysis) is a very powerful data mining tool in the area of Data Science. The secondary data is one of the information resources that contain useful parameters to predict and classify epidemics [7]. The WEKA data mining algorithms and techniques are well-known tools for predictive and classification model development and data analysis.

## 4. METHODOLOGY:

The data mining classifiers are theoretically described below:

### 4.1 Naive Baye's Classifier

Probability is the science of data mining. The Naive Baye's classifier is a simple probability based algorithm, it uses the Baye's Theorem, but assumes that instances are independent of each other, which is rather a unrealistic assumption in the practical world. The naive bayes algorithm offer fast model building and scoring both binary and multiclass situations for relatively high volumes of data [8]. The algorithm makes predictions using Baye's Theorem, which incorporate evidence or prior knowledge in its prediction [Hand et.al. 2001]. The statement of the Baye's Theorem is,

$$P(C/A) = \frac{P(A/C)P(C)}{P(A)} \propto L(A/C)P(C)$$

Where  $P$  stands for probability of the variable within parentheses, and  $L(A/C)$  is referred to as the likelihood of  $A$  given  $A/C$ . The conventional names of each term are:

*Step 1:*  $P(C)$  is the *prior probability* or marginal probability of  $C$ . It is 'prior' in the sense that it has not yet accounted for the information available in  $A$ .

*Step 2:*  $P(C/A)$  is the *conditional probability* of  $C$ , given  $A$ . It is also called the posterior probability because it has already incorporated the outcome of event  $A$ .

*Step 3:*  $P(A/C)$  is the conditional probability of  $A$  given  $C$ .

*Step 4:*  $P(A)$  is the prior marginal probability of  $A$ , which is normally the evidence, with the terminology, the theorem may be put in words as

$$\text{Posterior} = \frac{\text{Likelihood} * \text{Prior}}{\text{Normalising Constant}}$$

The ratio  $P(A/C)/P(A)$  is sometimes called the standardised likelihood.

To understand the development of the Naive Baye's algorithm, let us start with basics that a classifier is conditional upon the attributes  $A_1, A_2, \dots, A_n$ . Then  $P(C/A_1, A_2, \dots, A_n)$  where  $n$  is the number of attributes, with Baye's Theorem.

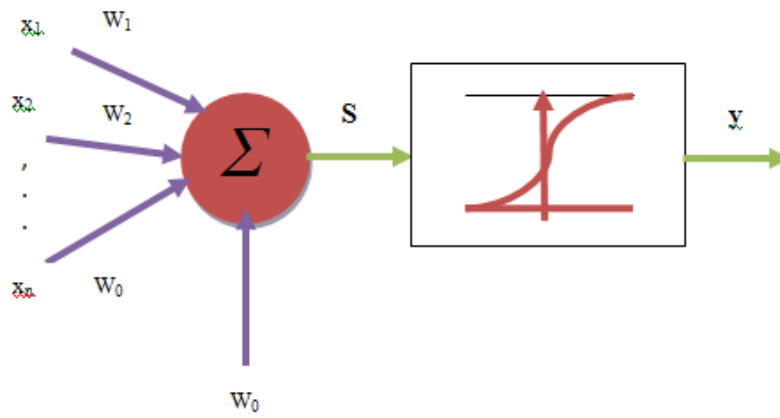
$$P(C/A_1, A_2, \dots, A_n) = \frac{P(A_1, A_2, \dots, A_n/C)P(C)}{P(A_1, A_2, \dots, A_n)}$$

Ignoring the normalising constant and assuming that the attributes are independent, we get

$$P(C/A_1, A_2, \dots, A_n) = P(A_1/C) * P(A_2/C) * \dots * P(A_n/C).$$

### 4.2 Multi Layer Perceptron (MLP)

The theoretical aspects of artificial neural network model proposed by McCulloch and Pitts in the 40's and were generalized later in many ways. The most popular approach is:



**Figure 1. Artificial Neural Neuron**

The above neurons compute the weighted sum of  $n$  inputs, add threshold value and then applies an activation function to the result in order to compute the output.

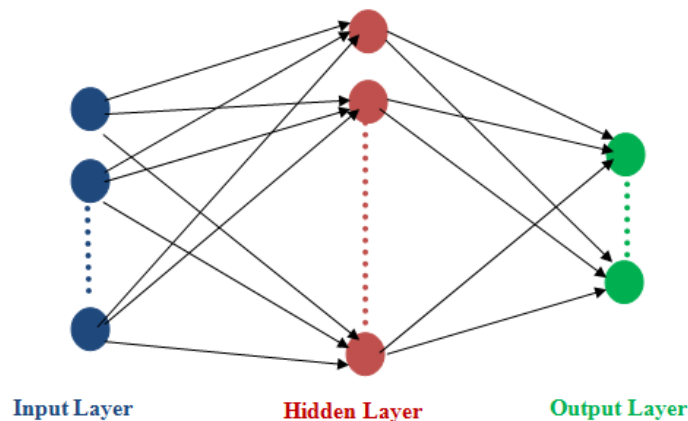
$$S = \sum_{i=1}^n x_i w_i + w_o \quad y = f(S)$$

Most of the researchers have used activation function that is sigmoid function which is defined as ,

$$sigmoid(x) = \frac{1}{1 + e^{-x}}$$

The nonlinearity of the function is essential for the power of neural networks model and also the function scales, the output [0-1] range. The previous described perceptron can classify only linear separable input vectors. This was proven since 1969 by Minsky and Papert and has reduced the interest of researchers for neural networks. To solve the problem of multilayer perceptron that had to be used but it was not known how to update the weights of hidden (intermediate) layers. The updating rule for the weights (briefly described below) was discovered only in late 80's and was the basis of the explosion of neural networks field.[9]

Most familiar architecture for neural networks is the Multilayer Perceptron (MLP) where each neuron is connected to all neurons from the previous layer. The only exception is the first layer whose units repeat only their inputs. The figure below exemplifies the most common approach with one single hidden layer.



**Figure 2. Artificial Neural Network with three Layers**

In case of supervised learning, desired output for each input vector is achieved. Therefore, the representation error  $E$  that appears can also be computed based on Euclidian distance between obtained output and desired output vectors. The learning rule falls in the category of “error-correction rules”. The most general rule to update a weight  $w$  from any layer is:  $\Delta w = \eta \frac{\partial E}{\partial w} w$ , where  $E$  is the error (as a function of  $w$ ) and  $\eta$  is the learning rate.

Each layer is made up of units. The inputs are fed simultaneously into the input layer and then its output are weighted and fed to a hidden layer. The outputs of the hidden layer units can be input to another hidden layer, and so on. The weighted outputs of the last hidden layer are output to units making up the output layer, which are the network’s prediction for given tuples.

### 4.3 Random Forest Classifier

Random forest classifier is a supervised learning method of algorithm which is used for both classification as well as regression. Random Forest classifier is mainly used for classification purpose. A forest is made up lots of trees and more trees mean more stout forest. In addition, random forest classifier algorithm creates decision trees on data samples and then gets the forecast from each of them and finally selects the most prominent solution by means of determination.

#### 4.3.1 Random Forest Algorithm

Random Forest algorithm is performed with the help of following steps:

**Step 1:** First, start with the selection of random samples from a given dataset (COVID-19).

**Step 2:** Next, this algorithm will construct a decision tree for every sample. Then it will get the prediction result from every decision tree.

**Step 3:** In this step, voting will be performed for every predicted result.

**Step 4:** At last, select the most voted prediction result as the final forecast result. The following diagram illustrates Random forest classification.

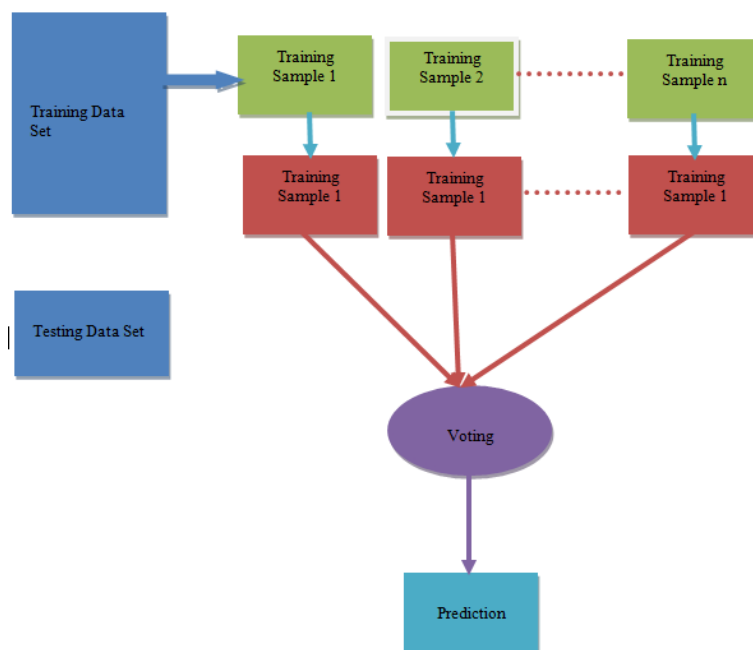


Figure 3. Random Forest Classification

## 5. RESULT AND DISCUSSION

The various algorithms and results of COVID-19 affected districts of Tamilnadu Dataset were discussed below:

### 5.1 Proposed Algorithm for three data mining Classifiers:

In this research paper, initially the COVID-19 database is normalized and gives as a test data for all the three methods.

*Step 1:* Go to WEKA explorer window and open existing file “COVID\_19. CSV” from file path.

*Step 2:* Go to filter menu and choose normalization method.

*Step 3:* Apply normalization for the given data and get basic Statistics and visualization bar chart of all the parameters of COVID-19 affected districts.

*Step 4:* The above data is split into various clusters using k-means clustering method, the initial seeds of  $n = 2,3,4,\dots,n$ . finally the researcher identifies three meaningful clusters; subsequently the results are to be cross validate using three data mining techniques and visualize the data of COVID-19 dataset.(Figure 4)

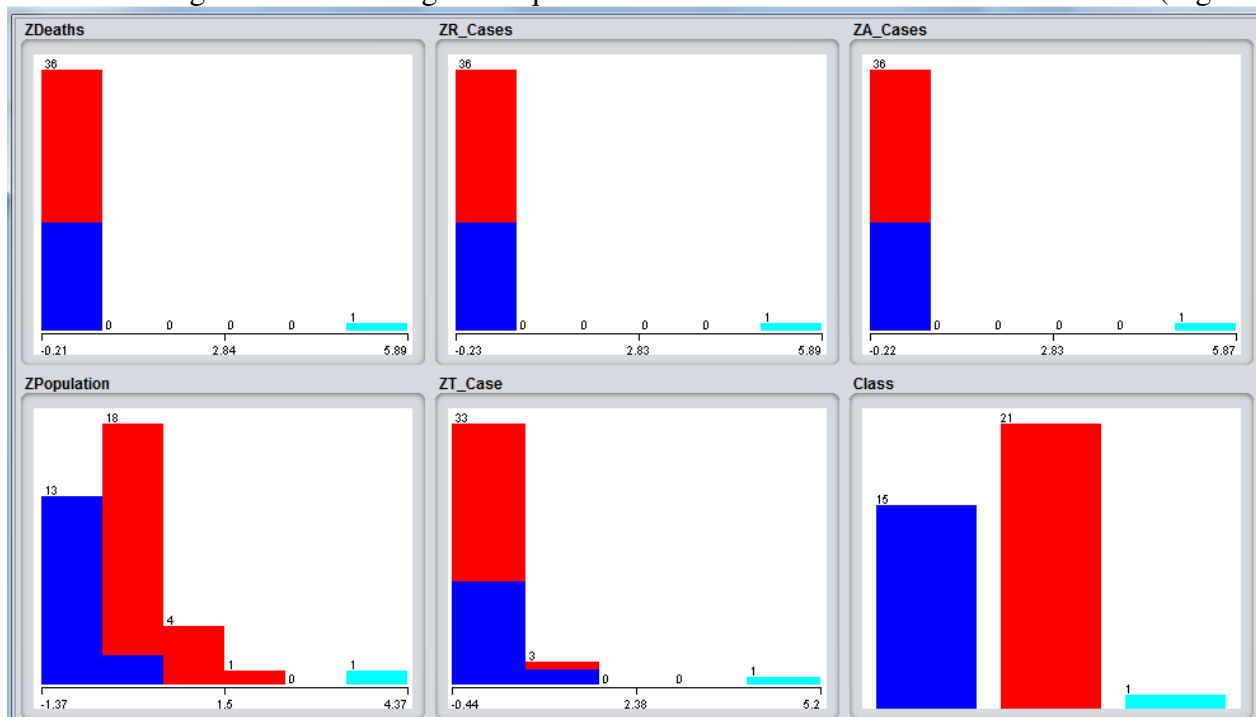


Figure 4. Visualization of COVID-19 Parameters and Districts

#### 5.1.1 Naive Baye;s Classifier

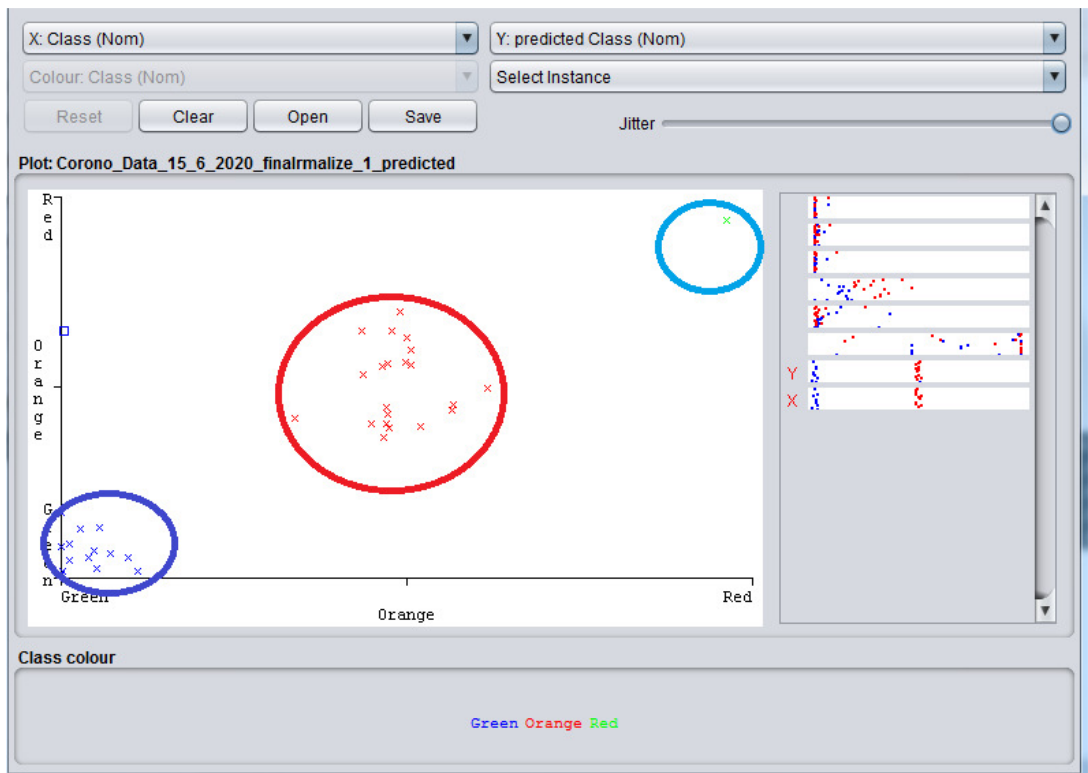
*Step 1:* Open “COVID\_19.CSV” data file from the specific path file and choose the Naive Baye’s classifier as listed in Weka under the “Baye’s” sub directory.

*Step 2:* After Selecting Naive Baye’s classifier, keep the default settings of WEKA environment and the classifier parameters. Next, click on Start button to execute the program and to achieve the results in output window.

The performance of Naive Baye’s algorithm for COVID\_19 data is 100% correctly classified. The Classification Model, Confusion matrix, Model Build Time, Classification Percentage and visualization of predicted districts are in Table 1, 2, 3 and Figure 5.

Attribute	Total Instances	Classification Model			Model Build Time	Confusion Matrix			Classification Percentage
		Green	Orange	Red		Green	Orange	Red	
Naive Baye's	37	15	21	1	0	14	0	0	100
	37	15	21	1	0	0	21	0	100
	37	15	21	1	0	0	0	1	100
MLP	37	15	21	1	0	14	0	0	100
	37	15	21	1	0	0	21	0	100
	37	15	21	1	0	0	0	1	100
Random Forest	37	15	21	1	0	14	0	0	100
	37	15	21	1	0	0	21	0	100
	37	15	21	1	0	0	0	1	100

**Table 1. Classification Model, Confusion Matrix, Model Build Time, Classification Percentage of districts**



**Figure 5. Naive Baye's Predicted Classification of Districts of Tamilnadu**

**5.1.2 Multilayer Perceptron Classifier (MLP)**

The MLP algorithm presented here has the following sequence of steps:

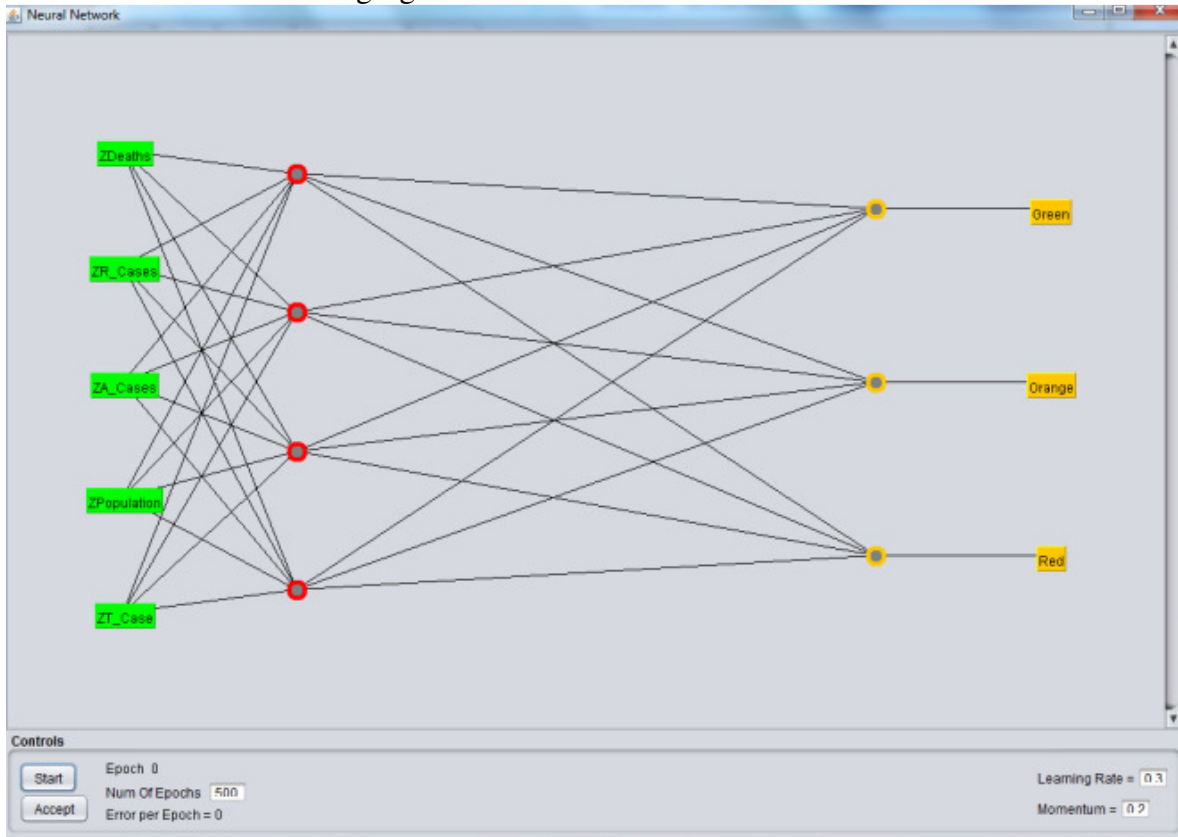
**Step 1:** Open “COVID\_19.CSV” in data file description available in WEKA data directory with the data retrieval information presented in Weka Explorer window.

**Step 2:** Next to do classification using neural networks, click on the Classify tab on the Weka window and select “function” folder and choose multilayer perceptron.

**Step 3:** After selecting and clicking on the MLP classifier, the Weka window with default parameters will appear. The learning rate (1) is to be 0.3 and momentum (M) is to be 0.2. The parameter default settings can be changed, if necessary.

**Step 4:** Then build a MLP network diagram. Go to MLP properties and change the GUI to true. Click OK button. Next move to control menus. Click ‘start’ and ‘accept’ button to get the MLP results and diagram from the given data (Figure 6, 7 and Table 1, 2, 3).

If the GUI is activated for an input vector having 6 attributes and 4 neurons on the hidden layer then network will look like the following figure.



**Figure 6. GUI Representation on four neurons MLP Model for COVID-19 parameters**

**Table 2. Summary Statistics of COVID-19 Data of Tamilnadu**

Models	Correctly Classified Instances	Incorrectly Classified Instances	Kappa Statistic	Mean Absolute Error	Root Mean Squared Error	Relative Absolute Error	Root Relative Squared Error
Naive Baye’s	37	0	1	0.0539	0.1165	15.434	28.1628
MLP	37	0	1	0.0539	0.1165	15.434	28.1628
Logistic	37	0	1	0.0000	0.0000	0.0003	0.0010
Random Forest	37	0	1	0.0000	0.0000	0.0211	0.0010



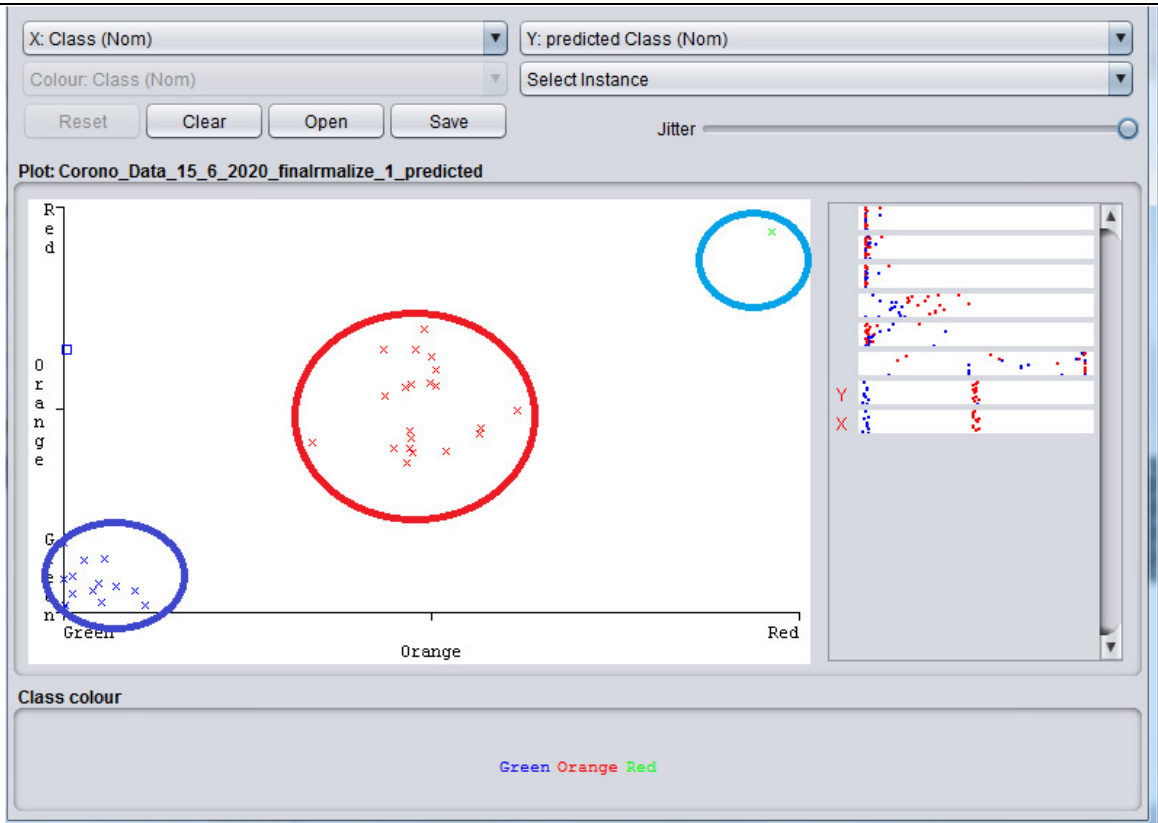


Figure 7. MLP Predicted Classification of Districts of Tamilnadu

**5.1.3 Random Forest Classifier (RFC)**

The Random Forest Classifier (RFC) algorithm presented here has the sequence of steps:

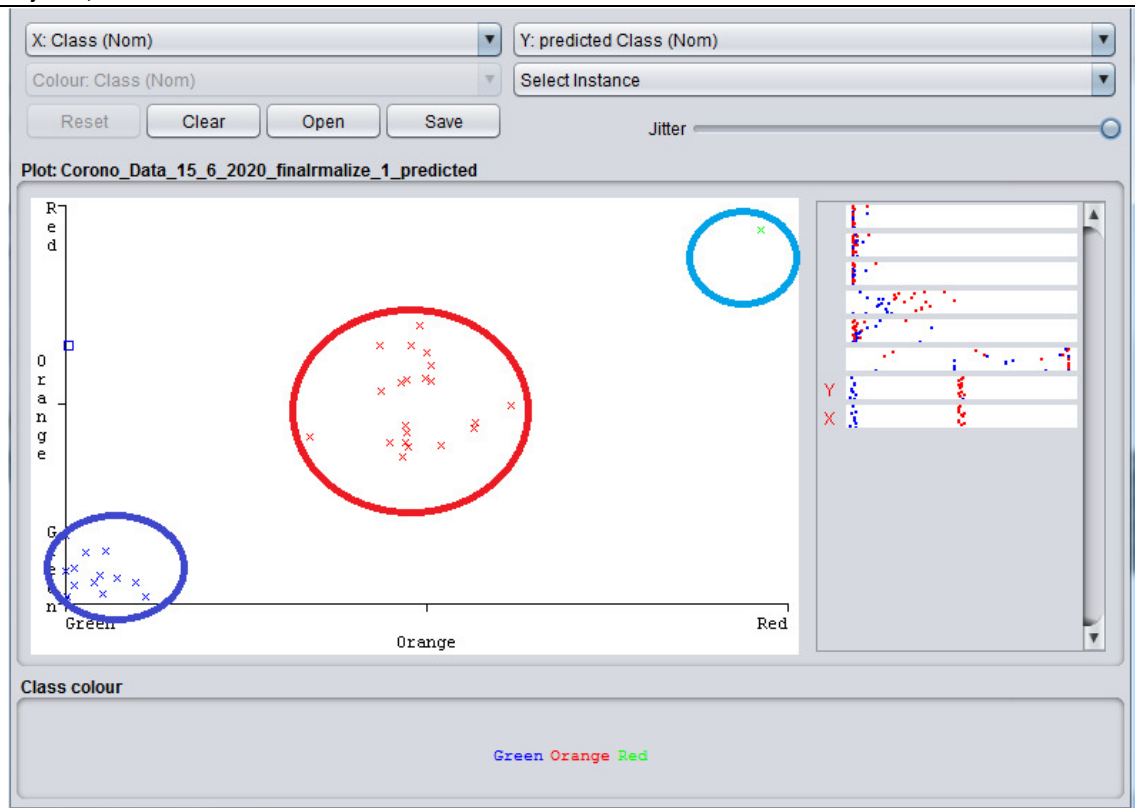
**Step 1:** Open “COVID\_19.CSV” file description that is available in WEKA data directory with the data retrieval information presented in Weka Explorer window.

**Step 2:** Next to do classification using RFC, click on the “Trees tab” on the Weka window and select “Random forest” model.

**Step 4:** Then build a RFC classifier by using the data set. Then click Start button to execute the RFC algorithm. The output window of Weka shows the results of summary statistics, confusion matrix and classification model etc. for given database (Figure 8 and Table 1, 2, 3).

**Table 3. Detailed accuracy by class using COVID-19 Data of Tamilnadu**

Model	TP Rate	FP Rate	Precision	Recall	F-Measure	MCC	ROC Area	PRC Area	Class
Naive Baye’s	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Green
	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Orange
	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Red
MLP	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Green
	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Orange
	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Red
Logistic	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Green
	1.000	0.000	1.000	1.000	1.000	1.000	0.931	0.831	Orange
	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Red
Random Forest	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Green
	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Orange
	1.000	0.000	1.000	1.000	1.000	1.000	1.000	1.000	Red



**Figure 8. RFC Predicted Classification of Districts of Tamilnadu**

The summary Statistics of all the three methods produced more or less the same results. Every Statistical tools have formula like, Kappa Statistics, Mean Absolute Error (MAE), Root Mean Squared Error (RMSE), Relative Absolute Error (RAE), Root Relative Squared Error (RASE), True Positive Rate (TP Rate), False Positive Rate (FP Rate), Precision, Recall, F-measure. Mathews Correlation Coefficient (MCC) is used in machine learning as a measure of the quality of binary classifications. It takes into account true and false positives and negatives and is generally regarded as a balanced measure which can be used even if the classes are of very different sizes. Receiver Operating Characteristic (ROC), Precision and Recall (PRC) methods have been used. All the thirty seven districts are classified and predicted based on COVID-19 parameters using WEKA data mining. The results have achieved three classification zones and are labeled as:

Green (Low Spread Areas), fifteen districts: Ariyalur, Dharmapuri, Kallakurichi, Kanyakumari, Karur, Nilgiris, Perambalur, Ramanathapuram, Ranipet, Sivagangai, Tenkasi, Theni, Tirupattur, Tiruvarur and Villuppuram.

Orange (Moderate Spread Areas) twenty one districts: Chengalpet, Coimbatore, Cuddalore, Dindigul, Erode, Kancheepuram, Krishnagiri, Madurai, Nagapattinam, Namakkal, Pudukkottai, Salem, Thanjavur, Thoothukudi, Tiruchirappalli, Tirunelveli, Tiruppur, Tiruvallur, Tiruvannamalai, Vellore and Virudhunagar.

Red (High Spread Area) Only one, Chennai, the capital city of Tamilnadu. These three Zones are represented in the Geographical Information System (GIS) Map (Figure 10).



All the thirty seven districts are classified and predicted based on COVID-19 parameters using WEKA data mining. Finally, three classifiers were identified with help of statistical measures, classification results, confusion matrix, etc., and are labelled as:

Green (Low Spread Areas) fifteen districts: Ariyalur, Dharmapuri, Kallakurichi, Kanyakumari, Karur, Nilgiris, Perambalur, Ramanathapuram, Ranipet, Sivagangai, Tenkasi, Theni, Tirupattur, Tiruvarur and Villuppuram.

Orange (Moderate Spread Areas) twenty one districts : Chengalpet, Coimbatore, Cuddalore, Dindigul, Erode, Kancheepuram, Krishnagiri, Madurai, Nagapattinam, Namakkal, Pudukkottai, Salem, Thanjavur, Thoothukudi, Tiruchirappalli, Tirunelveli, Tiruppur, Tiruvallur, Tiruvannamalai, Vellore and Virudhunagar.

Red (High Spread Area) Only one, Chennai, the capital city of Tamilnadu. These three Zones are represented in the Geographical Information System (GIS). In addition, the capital city is more affected with COVID-19, one among the largest capital city in India. The current scenario is most of the people who work in Chennai, now migrates to their native place and thereby spreading to their home towns also. Instead, they can follow the regular instructions given by the Government of Tamilnadu and stay safe by wearing face masks, maintaining social distance and regular hand wash. Public can also boost their immune system by taking kabasura kudineer, Nilavembu Kudineer and Arsenica Album 30 which is distributed to them.

**Acknowledgement :** Public must be very thankful to all Police Personnels, Doctors, Nurses, Health workers, Government officials and volunteers for their complete dedication and commitment during this pandemic period. Public has to cooperate with the Government of Tamilnadu to eradicate COVID-19 completely from our state.

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## INCLUSIVE DEVELOPMENT THROUGH MICRO, SMALL AND MEDIUM ENTERPRISES

**Dr. Abhishek Kumar Jha**

L.N. Mithila University, Darbhanga (Bihar) India  
E-mail : touch2ashish@gmail.com

### ABSTRACT

*Small enterprises in India have been flourishing despite a challenging environment. The Government is taking steps to strengthen the sector as it not only contributes significantly to employment generation but also contributes to the nation's economy. The government has launched many programmes to reorient the sector for scaling up the economic activity in the sector. For rejuvenating this sector some financial and technical initiatives have also been undertaken. These include re-implementation of Public Procurement Policy, MUDRA Yojana, Make in India, Startup India, and Skill India. These initiatives are aimed at empowering the sector so that it grows strongly and its share in GDP increases. The implementation of GST Bill in 2017 is a landmark step towards eradicating indirect taxes, creating more transparency in tax process and also in contributing to business growth in unconventional sectors as well.*

*Small enterprises can play a big role in making India self-reliant by an innovative approach and power to use cheap and local resources. They act as low-cost alternatives to the corporate, relying solely on their expertise in their niche, unique manufacturing process and ability to bring about innovation. Provided with the right support at the right time, small enterprises in India can not only spearhead growth of the Indian industry, but also help cut its high dependence on costly exports.*

**Key Words:** *Indian MSME Sector, Make in India, Startup India, Sustainable Development Goals*

### INTRODUCTION

The MSME are credited with generating the highest rates of employment growth and account for a major share of industrial production and exports. As far as India's growth opportunity is concerned, MSME can be the backbone for the existing and future high growth business. The Indian MSME sector provides maximum opportunities for both self-employment and wage-employment outside the agricultural sector and contributes in building an inclusive and sustainable society in innumerable ways through creation of non-farm livelihood at low cost, balanced regional development, gender and social balance, environmentally sustainable development etc.

The MSME sector in India continues to demonstrate remarkable resilience in the face of trialing global and domestic economic circumstances. The sector has sustained an annual growth rate of over 10% for the past few years. With its agility and dynamism, the sector has shown admirable innovativeness and adaptability to survive economic shocks, even of the gravest nature.

The small-scale industries sector plays a vital role in the growth of the country. It contributes almost 40% of the gross industrial value added in the Indian economy. It has been estimated that a million Rs. of

investment in fixed assets in the small scale sector produces 4.62 million worth of goods or services with an approximate value addition of ten percentage points. When the performance of this sector is viewed against the growth in the manufacturing and the industry sector as a whole, it instills confidence in the resilience of the small-scale sector.

SSI Sector in India creates largest employment opportunities for the Indian populace, next only to Agriculture. It has been estimated that 100,000 rupees of investment in fixed assets in the small scale sector generates employment for four persons.

### **DYNAMICS OF MSMEs**

Cutting across all sectors of production and services, MSME sector is truly a strategic asset for the economy of the country. On the one hand, we have the village and rural industries including the khadi industry, which are primarily located in the rural landscape and provide an important ingredient in the local eco-system. On the other hand, in extreme contrast and the opposite side of spectrum are the micro, small and medium enterprises which are producing an extremely wide variety of goods which are exported as well as have the reach out to the domestic consumers.

The sector has a wide range of stakeholders including the regulators, facilitators and the beneficiaries. The various stakeholders include

- MSME (both existing and prospective).
- Large enterprises including multinationals (as producer of goods and services).
- State/Union territories.
- Central Ministries/Departments.
- Banks/Financial Institutions.
- Entrepreneurship and skill Development Institutes, both in the public and in the private.
- Research and Development Institutions.
- Educational Institutions.
- Organisations under the administrative control of the Ministry.

In India in the post-liberalized era, the entrepreneurs in the MSME sector who have been able to survive have realized the value of collaboration, strategic alliance, outsourcing, and flexible production system and have a global mindset. Increasing the efficiency of the MSME sector depends on strengthening the following three pillars:

- Innovation
- Technology
- Finance

Entrepreneurs preparing for the global market need to focus on the following:

- Global mindset
- Lean Manufacturing
- Mergers and Acquisition
- Complementing large industries
- Alternate Finance
- Prepare for non-tariff barriers
- Explore options of Co-operation, Fair trade, IPR and FDI

The winning formula for enhancing competitiveness combines

- Elimination of waste
- Enhancement of Technology
- Use of Quality and productivity tools
- Total Employee Involvement
- Flow of information to all enterprise stakeholders
- Use of ICT

### **ROLE OF MSME IN MAKE IN INDIA**

MSMEs can play a critical role to strengthen the ‘Make in India’ initiative by addressing the issues of job creation, increasing the manufacturing share in GDP and export promotion. Already, the sector contributes to around 45 per cent of the total manufacturing output and around 40 per cent of the country’s exports.

There has been a sustained contribution of the MSME both in terms of output and employment, in spite of the slowdown in the economy as a whole and the manufacturing sector in particular. However, MSMEs remained a cause of concern due to the major problems faced by the sector, in particular, having limited access to capital and credit facility, lack of technology know-how, limited access, awareness and interaction with the global market, inadequate infrastructure like roads, power and water supply, multiplicity of labour laws, lack of skilled man-power for manufacturing and marketing of the products.

The major schemes run by the Government through National Manufacturing Competitiveness Programme, Credit Guarantee Scheme, Cluster Development, and the announcement of Micro Units Development Refinance Agency (MUDRA) Bank with a corpus of 20,000 crores is a welcome step to strengthen this sector’s contribution in ‘Make in India’. The Apprentice Protsahan Yojana would boost the manufacturing sector of MSME. An innovative initiative is the launch of low cost and high quality MSME products on-line delivery system via [www.msmeshopping.com](http://www.msmeshopping.com) facilitated by National Small Industries Corporation (NSIC).

### **SUSTAINABLE DEVELOPMENT GOALS AND MSMEs**

The critical contribution of MSMEs to broader social economic objectives including job creation makes them a key priority area for achieving the Sustainable Development Goals (SDGs). Job creation through MSMEs will often directly benefit the poor and vulnerable, particularly women and youth, thereby directly reducing poverty, increasing income and positively impacting on household investments in education and health over time. MSME development has the potential for wide reaching impacts on the SDGs globally, including SDG 1 (end poverty) SDG 2 (zero hunger), SDG 3 (good health and well-being), SDG 5 (gender equality), SDG 8 (promote inclusive and sustainable economic growth, employment and decent work), and SDG 9 (improve sustainable industrialization and fostering innovation) (Liu, 2017).

### **WAY AHEAD**

China's image as a low-cost location for manufacturing is rapidly changing due to labour shortages and increases in wages and in response, China is shifting its specialization from basic to relatively more sophisticated manufacturing. With the increasing wage costs, Chinese firms in the labour intensive industries are increasingly under pressure and have started looking for other low cost locations such as

Vietnam and Indonesia. An important question in this context, is: Can India become the next workshop of the world?

Recognizing the importance of a strong manufacturing sector for employment generation, the Prime Minister has recently launched “Make in India” campaign, with an aim to boost India’s manufacturing sector. It is important to situate these new initiatives in the context of growing global production networks in manufacturing industries. Greater integration of domestic industries with global production networks must form an essential part of the “Make in India” initiative. What is important is the creation of an environment that allows entrepreneurs to freely search, and identify opportunities in the vertically integrated global supply chains of various industries. Based on imported parts and components, India has a huge potential to emerge as a major hub for final assembly in several industries. However, it is important to resist the temptation of extending tariff protection for final goods assembly as it will have the detrimental effect of breeding inefficiencies. A level playing field should be created for different types of business entities domestic, foreign and joint ventures. The domestic market for goods should be as contestable as is the export market for competing suppliers from around the world.

A flexible labour market, with appropriate social safety nets, is a crucial necessary condition for the growth of labour Intensive manufacturing in India. The recent amendments in labour laws in states such as Rajasthan and efforts to improve the ease of doing business are all moves in the right direction.

## CONCLUSION

In India, almost 31 million MSMEs have been a dominant force in creating ample opportunities for the inclusive socioeconomic growth and also acting as a nurturing seedbed for entrepreneurial talent of perspective global players. Over the past five decades, the Indian Government had initiated various progressive credit policies and practices, new schemes to support the MSME sector the promotion, development and enhancing competitiveness of these enterprises, preference in Government procurements to products and services of the micro and small enterprises, m effective mechanisms for mitigating the problems of delayed payments to micro and small enterprises and simplification of the process of closure business by all three categories of enterprises.

However, there is still a lack of awareness in access to finance and marketing, adoption innovative technologies are the major challenges I the MSMEs to sustain in the present markets. With the increasing global competition and uncertain due to global meltdown, there is a need for the MSMEs to incorporate the latest technology in. their production process as well as in their financing marketing and management functions, to cut do the cost, gain efficiency and consistency. This will help them to become successful and contribute t the Indian economy in the long run.

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## THE EPIDEMIC : SCENARIO & LEARNING

**Dr. Sapna Nagar**

Assistant Professor, Department of Mathematics, Mihir Bhoj (PG) College, Dadri Gautam Buddha Nagar U.P. -203207  
E-mail : sapna.nagar3@gmail.com

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### ABSTRACT

*In this Paper we present the study of Epidemic like the corona virus (COVID-19) outbreak have shaped politics, crushed revolutions, and entrenched racial and economic discrimination. Epidemics have also altered the societies they have spread through, affecting personal relationships, the work of artists and intellectuals, and the man-made and natural environment of recent scenario in Epidemics. This is a category of disease that seems to hold up the mirror to human beings as to who we really are. That is to say, they obviously have everything to do with our relationship to our mortality, to death, to our lives. They also reflect our relationships with the environment-the built environment that we create and the natural environment that responds. They show the moral relationships that we have toward each other as people, and we're seeing that to day.*

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### INTRODUCTION

The coronavirus pandemic began as a simple outbreak in December 2019 in Wuhan, China. However, it quickly propagated to other countries and became a primary global threat. It seems that most countries were not prepared for this pandemic. As a consequence, hospitals were over-crowded with patients and death rates due to the disease skyrocketed. In particular, as of the time of this writing (20th May 2020), the coronavirus outcome resulted in over 4.2 million cases and over 280 thousand deaths worldwide as a cause of the induced disease, COVID-19.

In order to reduce the impact of the disease spread, most governments implemented social distancing restrictions such as closure of schools, airports, borders, restaurants and shopping malls [1]. In the most severe cases there were even lockdowns - all citizens were prohibited from leaving their homes. This subsequently led to a major economic downturn: stock markets plummeted, international trade slowed down, businesses went bankrupt and people were left unemployed. While in some countries the implemented restrictions had a significant impact on reducing the expected shock from the coronavirus, the extent of the disease spread in the population greatly varied from one economy to another.

A multitude of social and economic criteria have been attributed as potential determinants for the observed variety in the coronavirus outcome. Some experts say that the hardest hit countries also had an aging population [2,3] or an underdeveloped healthcare system [4,5]. Others emphasize the role of the natural environment [6,7]. In addition, while the developments in most of the countries follow certain common patterns, several countries are notably outliers, both in the number of officially documented cases and in number of diseased people due to the disease. Having in mind the ongoing debate, a comprehensive empirical study of the critical socio-economic determinants of the coronavirus epidemic country level outcome would not only provide a glimpse on their potential impact, but would also offer a guidance for future policies that aim at preventing the emergence of epidemics.

## **EPIDEMIC SCENARIO**

Past studies have shown that famine mortality occurred in two phases. First, in the less lethal stage, the fundamental causes of death were cholera, dysentery, and smallpox, all of which were spread by migration, unsanitary conditions, close contact between people in relief camps, and consumption of decayed old food. Then, in the far more lethal phase, the main cause of death was malaria. This happened, somewhat paradoxically, in the second year of famine-when rain had fallen, food cost had declined, supplies had increased, and food nutrition among the working poor had risen gradually. However, the development of epidemic malaria was aided by torrential rain, which favored the breeding of the anopheles mosquito, the vector responsible for transmitting the malariaparasites.

As far as India is concerned, there have been only two major, significant pandemics throughout history. While cholera had been predominant throughout the 19th century with increasing death tolls every year, the influenza pandemic came later on in the early 20th century [8,9]. The influenza pandemic was short but devastating and after a long time, quite recently, came yet flu pandemic by the H1N1 strain [10]. Though, it is almost impossible to analyze all epidemics and Pandemics throughout Indian history, effort has been made to include most of the significant ones.

## **HISTORICAL VIEW OF PANDEMICS**

### **The 19th Century:**

#### **Ist Cholera Pandemic (1817):**

This is considered to be the first major epidemic of the 19th century in British- colonized India and was described as probably the most terrifying of all [11]. The first case was reported on 23rd August 1817 by a civil surgeon of Jessore [12].The overall estimation of the mortality is not available as the data collection in India began much later, probably in the late 1860s. As for the geography, it is important to note that the year 1817 had brought a very heavy rainfall leading to flooding which could have been the cause for such a rapid spread [13]. While the Europeans living in India then and the elite were not seriously affected, the slum dwellers and people in rural poverty were hit the worst [14]. This was probably due to the differences in living conditions, personal hygiene and practices.

#### **IInd Cholera Pandemic (1829):**

The second outbreak started around 1826 from Bengal and spread through the rivers to various parts of northern India. After affecting the United Provinces (UP), its impact was huge on areas around Punjab and Delhi but most significant is its pandemic spread to countries like China [15]. Cholera spread far and wide, all along the trade routes from China affecting several cities and villages alike. In each place, it lasted for a few weeks and killed hundreds of people everyday .

#### **IIIrd Cholera Pandemic (1852):**

This third cholera pandemic started around 1852 and lasted till the late 1860s. It is significant in history because of its spread to countries that were until then not affected. Though India was not its major area of impact, in the later phase of pandemic, small spurts of cases were noted in Bengal. It spread to several other countries like Persia, Arabia and then to Russia [16]. This was due to the worldwide spread of El Tor serotype of Vibrio which was initially endemic to India[17].

**IVth Cholera Pandemic (1863):**

This began around 1863. While some suggest that the major cholera epidemic in 1865 was brought to Mecca by the Haj pilgrims from India, others disagree stating that it was just a recrudescence [18]. However, it is agreed that it was from Mecca that the infection spread to several countries [19,20]. The Kumbh Mela at Hardwar in April 1867 has been considered to be responsible for the epidemic spread of cholera in northern India. The Madras Presidency in 1877 was the worst hit and the cholera epidemic was responsible for about 10% of the annual mortality then[21].

**Vth Cholera Pandemic (1881):**

The fifth cholera pandemic was considerably less fatal as compared to the previous four. It was during this pandemic (1881-1869) that Robert Koch proved that cholera was transmitted through the fecal-oral route, after studying the outbreaks in Calcutta and Egypt [22]. It spread to United Provinces and Punjab after which it spread to other countries like Afghanistan, Persia, parts of Russia and then to Europe[23].

**BOMBAY PLAGUE EPIDEMIC (1896):**

This plague began in September 1896 in colonial Bombay creating a lot of social and political frenzy. The rapid growth of commerce in Bombay led to an increase in population and thereby overcrowding. The anti-plague campaign was started to battle this epidemic and it was based on the belief that the focus of the infections was from the slums. The plague killed thousands and many people were forced out of the city[24].

**VIth Cholera Pandemic (1899):**

The sixth cholera pandemic began around 1899 and major outbreaks were noted in Bombay, Calcutta, and Madras [8]. While the infection throughout the 20th century was caused by O1 serotype of *Vibrio cholera* and confined mostly through the Asian subcontinent, the sixth cholera pandemic brought about surprising challenges. This cholera infection was caused by an unknown, non-O1 serotype of *V. cholera* and spread to many distant countries including the United States [25]. The sixth cholera pandemic lasted for about 25 years(1899-1973)[26].

**THE 20TH CENTURY****Influenza Pandemic (1918):**

This is also known as the Spanish Flu of 1918-19. This has been known to have caused around 20- 50 million deaths worldwide and is considered most devastating [27].

This was caused by the H1N1 strain of Influenza and was severe. The first episode of the disease began in early 1918 and later in autumn, it began to spread all around the world, India considered to be the foci [28]. The second wave of the attack began in Bombay in 1918 and spread to other parts of northern India and Sri Lanka from where it spread worldwide [29]. Improvement in the virulence and velocity of the virus strain and the monsoon bringing humidity are considered to be the key factors in increasing the severity and spread[28].

**Polio Epidemic (1970-1990):**

India was the worst affected by polio among the developing countries until the late 1990s after which the EPI was initiated [30]. The incidence of polio in India was very high in both urban and rural states and the most affected was the state of Uttar Pradesh [31]. Its worst sequel was reported to be post-polio paralysis

and in the district of Vellore, about 6/1000 preschool children were affected [32]. It was in 1964 in Bombay and 1965 in Vellore that the oral polio vaccine was introduced [33]. India had a choice between Salk's IPV and Sabin's OPV. Even after the introduction of the OPV in EPI there was no improvement to be noted for 10 years [33]. But with improvement in surveillance, the desired results were achieved and India was declared polio-free status in January 2011 and emphasis has been laid on maintaining the guard to prevent resurgence[30].

#### **Small Pox Epidemic (1974):**

It is known as one of the worst small pox epidemics of the 20th century. India contributed to about 85% of this epidemic worldwide. This epidemic broke out in three different villages of West Bengal, Bihar and Odissa but it was impossible to establish a connection between the men hence it was treated as three different epidemics. The disease was introduced into different areas by different sources. While over 15,000 people died in this epidemic, thousands of the survived but most of them but most ended up with disfigurement and blindness [34, 35]. Small pox was eradicated in by the WHO small pox eradication program. It was the first disease to be combated globally and was declared eradicated by WHO in 1980 [36].

#### **SURAT PLAGUE EPIDEMIC (1994):**

Plague cases in Surat were first reported in Sept 1994 and which it spread to other cities in India. Fewer than 1,200 people were found positive and it lasted for less than two weeks but it is considered important due to its high fatality and created worldwide repercussions. It is said to have been initially difficult for doctors to diagnose it but when they did, all necessary precautions are taken to contain its spread [37].

THE 21st Century

#### **Plague of Northern India (2002):**

The Plague of Northern India broke out in Shimla district of Himachal Pradesh in February 2002. It was a small and less serious epidemic. Also, as soon as the plague was detected, immediate measures were taken like fumigation, evacuation, and chemoprophylaxis that lead to further control of the epidemic[38].

#### **Dengue Epidemic (2003):**

In 2003 during September, there occurred an outbreak of DF/DHF in Delhi. It reached its peak around October-November and lasted until early December. The mortality rate was around 3%.It became a major outbreak in India in spite of the widespread preventive measures taken to control DF [39].

#### **SARS Epidemic (2003):**

SARS (severe acute respiratory syndrome), is considered as the first serious infectious disease out break of the twenty-first century. It initially started in the Guandong province of China in 2003 and spread quickly to about 30 countries across Asia, Americas and Europe and accounted for a total of 8,439 cases and 812 deaths, within 7 to 8 months [40,41].

#### **MENINGOCOCCAL MENINGITIS EPIDEMIC (2005):**

In early 2005, a sudden surge had been noted in meningococemia and meningococcal meningitis cases in India. Cases were reported from Delhi and the surrounding states of Uttar Pradesh and Maharashtra. Around 430 cases of meningococcal meningitis were reported as of June 2005 [42]. Case management,early detection through surveillance was aimed at prevention of spread [43].

**Chikungunya Outbreak (2006):**

Around 3.4 million cases of Chikungunya were reported in Ahmedabad 2006 with 2,944 deaths estimated. The mortality rate in 2006 epidemic was substantially increased when compared with that in the previous four years [44]. In December, there occurred another epidemic in South India where the states of Andhra Pradesh, Karnataka and Tamil Nadu were affected. The volatile nature of this epidemic was attributed to the herd immunity to the then isolated genotype [45]. Major efforts were taken for mosquito control and several awareness campaigns were initiated by the television and print media[46].

**Dengue Outbreak (2006):**

The outbreak began in early September of 2006 and the first case was reported from Delhi. By the end of September, it began to spread to other states like Rajasthan, Kerala, Gujarat, Chandigarh and Uttar Pradesh [47]. The ministry of health set up a control room to monitor the outbreak and provide technical assistance that led to the efficient management of the disease[48].

**GUJARAT JAUNDICE EPIDEMIC (2009):**

Modasa town in Gujarat witnessed the outbreak of hepatitis B in 2009[49] This is of significance because almost all outbreaks of viral hepatitis in India were considered to be due to hepatitis E which is feco-orally transmitted [50]. It was a long-lasting epidemic and control was achieved by mass public awareness and healthactions.

**H1N1 Flu Pandemic (2009):**

The H1N1 Flu pandemic began in May 2009 and spread globally by July 2009. By August 2010, it was declared pandemic and around 18,500 deaths were reported from all around the world [51, 52]. Three strains of influenza viruses were circulating then of which the Inf A (H1N1) and Inf A (H3N2) viruses were largely replaced by the pdm H1N1 strain [53,54].

**Odisha Jaundice Epidemic (2014):**

The outbreak began in November 2014 in Kantalbai, a remote village in Odissa. This led to a district level investigation and it was confirmed to be jaundice caused by the Hepatitis E virus [55]. This 2014 Odisha Jaundice epidemic was one of the many outbreaks in Odisha and the most common cause being HEV [56]. This is transmitted enterically and has affected several people, especially of the low socioeconomic category. Surveillance for clean water and sanitation was proposed as the control measure[57].

**INDIAN SWINE FLU OUTBREAK (2015):**

It refers to the outbreak of the 2009 H1N1 flu pandemic in India which was still present as of March 2015. This outbreak in 2015 is considered as a resurgence of the infection and the most plausible reasons are considered to be low temperature, decreasing hostimmunity and failure of vaccination campaign after 2010[58]. According to the NCDC data in India, Rajasthan, Maharastra, and Gujarat were the worst affected states in India during this pandemic [59].

**NIPAH OUTBREAK (2018):**

This Nipah virus outbreak began in May 2018 in Kozhikode District, Kerala. Previously this virus was noted in the late 1990s in Singapore and Malaysia. The natural host for this disease is the fruit bat and

transmission is from direct person to person contact [60]. The Spread of awareness about this infection, isolation of the infected and post-outbreak surveillance led to the control of this outbreak [61].

### **CORONA VIRUS (2019) / COVID-19 :**

The coronavirus which is known as COVID-19 worldwide pandemic began as a simple outbreak in December 2019 in Wuhan, China. However, it quickly propagated to other countries and became a primary global threat. It seems that most countries were not prepared for this pandemic. As a consequence, hospitals were overcrowded with patients and death rates due to the disease skyrocketed. In particular, as of the time of this writing (20th May 2020), the coronavirus outcome resulted in over 5.2 million cases and over 380 thousand deaths worldwide as a cause of the induced disease, COVID-19.

### **EPIDEMIC LEARNING:**

Among a plethora of messages on COVID-19 was a request to God to reboot 2020 as it has a virus. But the reality is that nations across the globe will have to reboot their thoughts and creativity to tackle the deadly spread of the pandemic. One consolation is that a pandemic does not anchor itself for eternity; it will die. But before that, the 'Corona Effect' would have changed many aspects of our lives ranging from healthcare to social norms and leave behind a deep economic scar that would take years to heal.

COVID-19 has already sounded a warning bell - that despite our scientific claims and achievements, we are deeply unprepared to handle a pandemic. It shaped politics, crushed revolutions, and entrenched racial and economic discrimination. Some areas are given below:

#### **1. Health Care**

The first learning in this current pandemic is to build a robust healthcare infrastructure for better economy and overall a better country which takes care of all its citizens. How do we insulate people from pandemics that come without a working manual on possible vaccines or drugs? It comes with a DIY (do it yourself) kit with absolutely no tools. War like preparation required on healthcare front: Future healthcare systems should be war- prepared to battle pandemics whenever they breakout. We have near-complete knowledge on tackling major killers like heart ailments, cancer, lifestyle diseases, AIDS etc., but not a virus. We also need to critically look at our healthcare ecosystem, especially the critical care segment. Hence, when a pandemic strikes, there should be an action plan that can be implemented in the shortest time frame - one that can balloon our healthcare infrastructure by removing bottlenecks and creating more critical care units and isolation centres without affecting non-pandemic critical care patients. This can be done by involving the private sector - both in the healthcare segment and hospitality sector which can become isolation units.

#### **2. Early Warning System**

We need to set 'early warning systems' in place in the society. Unlike in the past, viruses in the 21st century not only are 'hyperactive', but also leave their tales of devastation with 'viral speed'. This is because of globalisation. Again, this is not new. In his seminal book *The Columbian Exchange*, AB Alfred W Crosby Jr writes: "When the isolation of the New World was broken, when Columbus brought the two halves of the planet together, the American Indian met for the first time his most hideous enemy; not the white man nor his black servant, but the invisible killers which these men brought in their blood and breath." [62] Within a span of just 70 odd years, 80-100 million natives perished because of the disease brought by Europeans from across the seas: smallpox, influenza, diphtheria. Today, it will not

take 70 odd years, but just a few months to leave thousands dead due to an unknown virus. Hence, the second lesson is that we cannot stop globalisation with countries interconnected by air, but when an unknown virus breaks its barriers and jumps to humans, there should be warning bells early enough. In the case of COVID-19, that did not happen. Had there been an early warning system in place and had nations isolated themselves, the virus could have been contained in small pockets.

### **3. Social Distancing**

Social distancing must become a strict norm of our regular habits for the future living. When a new virus comes knocking, there should be a self-imposed social distancing to break the chain. Even in the absence of a virus, it is better to observe social distancing as a part of life. Intimacy may be good and may reflect part of one's culture, but distancing, without offending the other, is the best.

### **4. Hygiene & Cleanliness**

World has to evolve a mechanism to maintain hygiene and cleanliness. It should become a norm for 100 per cent humanity. For this, the urge should come from within. Parts of cities and towns are buried to the nose with garbage and filth. This has to end, so too our unbridled drive to pollute air and water in the name of economic progress and gains.

### **5. Look After The Patient Previously**

A Guideline for at-risk people-elderly and patients with comorbidity- should be in place and in practice: Whenever there are signs of a pandemic, World should be able to isolate the elderly, patients with comorbidity and children who are either underweight or malnourished. For this, what is of paramount importance is a robust health surveillance system at the state level and a constantly updated registry of population on a national scale. Unless the vulnerable sections are pre-identified through strong data collection, states and nations would not only invite trouble but also leave the doors wide open for the virus to enter.

### **6. Reseach & Development**

Need to promote original R&D and strengthen our Drug & Vaccination Development Programme further: Coronavirus has had a happy flip side too. It has shown the world the strength of our pharma segment. Global exchange of life-saving drugs should be set between the major countries for protect the world. There is a need to further strengthen our Drug & Vaccine Development Programme by speed-tracking vaccine and drug development when a pandemic strikes.

### **7. Speedy Testing Labs**

Push our indigenous diagnostics manufacturing to make it sufficient for world Needs: COVID-19 has exposed the world to unpreparedness in having rapid testing kits and PPEs. Along with developing frontline vaccines, the lesson that COVID-19 has taught is the need to have millions of testing kits. This is because, when a pandemic strikes, the only way to check community spread is testing.

### **8. Economics & Politics**

Every Government needs to make appropriate provisions for supporting the BoP population in case of such calamities: A pandemic always hard-knocks and downs economies world over. COVID-19 is no exception. 2020 could be the worst year for the global economy & Political relation in nearly a century.

The global economy is expected to contract by 3-5 per cent this year because of economic damage from the coronavirus pandemic - the steepest downturn since the Great Depression of the 1930s, according to the International Monetary Fund. But in future, Every Country should have a strong mechanism to take care of the people at the base of the pyramid who actually oil and run the wheels of the nation's economy. Only they can put the economy back on track and make the political relationsound.

## CONCLUSION:

Now it is clear that improving action on the impact of globalization on infectious disease on an priori basis is a highly cost-effective policy intervention. It is increasingly recognized that the long-term sustainability of globalization requires greater attention to the social (including health) costs of current forms of globalization. What needs to be understood more fully is that resources committed to infectious diseases prevention, treatment and control, in a globalizing world, is a worthwhile investment. As demonstrated by the outbreak of plague in Surat, India, cholera in Peru, and SARS in Asia and elsewhere, the economic costs of infectious disease outbreaks are potentially magnified in the context of globalization. As well as the costs of prevention, treatment and control of the disease, countries risk losing substantial revenues from reduced trade, investment and tourism. Furthermore, economic costs can be imposed indiscriminately. In the case of SARS, countries throughout SE Asia, regardless of the number of cases reported, suffered significant loss of earnings. Given the integrated nature of the global economy, including capital flows, population mobility, and trade in goods and services, there is shared interest in investing greater resources in strengthening global capacity surrounding infectious diseases. Finally What we need to do what it takes to reassure people that the society does care and that their minimum well-being should be secure. The bottom line is that the world has to have a robust action plan when a pandemic strikes - a plan that is creative, disciplined and, above all, sensitive.

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## CENTRIFUGE SYSTEM IN THERMAL POWER PLANT CONSIDERING IMPACT OF VARIOUS FAULTS ON PERFORMANCE : A REVIEW

Anil Kumari\*, Pooja Bhatia\*\*

\*Assistant Prof (Math) G.C.Dujana & Research Scholar, Deptt. Of Mathematics, BMU Rohtak

\*\*Associate professor, Deptt. Of Mathematics, B.M. University, Asthal Bohar, Rohtak (Haryana)

E-mail : kumarianil2012@gmail.com\*, budhirajapooja92@gmail.com\*\*

### ABSTRACT

*An overview of research study on Reliability and Availability analysis made by Pooja Bhatia and Rajiv Kumar [5],[6],[8],[9] from 2011 to 2013 is presented in this review paper. Their work is on availability and profit analysis of reliability model, impact of ignored faults on reliability and availability and profit evaluation of the stochastic model on a centrifuge system working in a thermal power plant. Their papers deal with a single unit centrifuge system that undergoes periodic rest wherein faults are classified as major, minor and ignored faults. The occurrence of a minor faults leads to degradation of the system whereas occurrence of a major faults leads to failure of the system. Various measures of the system effectiveness are obtained using Markov processes and regenerative point technique. The analysis of the system is carried out on the basis of the graphical studies and conclusions are drawn regarding the reliability, availability, profit and cost of the system. The objective is to highlight and compare different aspect of their research study, explore application and other related manufacturing plant and identify potential research directions from reviewed study.*

*Keywords: Availability, centrifuge system, Markov process, MTSF profit, regenerative point technique.*

### INTRODUCTION

A large number of researchers in the field of reliability modelling analyzed several systems considering different types of failure, repairs/replacements policies, inspections, etc. Goel et al. (1986) obtained the reliability analysis of a system with preventive maintenance. Gupta and Goel (1989) discussed a two-unit priority standby system with administrative delay in repair. Tuteja and Taneja (1991, 93) investigated reliability and profit analysis of one/two-unit standby systems considering two identical repairman, minor repair, partial failure and random inspection. Gupta et al. (1997) dealt with the analysis of a system with three non-identical-units super-priority, priority and ordinary, with arbitrary distributions. Taneja and Nanda (2003) analyzed a two-unit cold standby system with resume and repeat repair policies. Gupta et al. (2008) investigated a single unit system with one helping unit that save the system from the external effects. Veena and Bhatia (2020) analyzed the performance of a paper machine considering its minor/major faults and power failure.

Thus, by the time the realm of reliability applications become as wider as to cover various industrial processes where purposeful and decision making is involved.

Many scientists [1-4] are trying to improve the performance of plants and industries in India. In many practical situations, for example in thermal power plant for oil purification, milk plant for making butter,

laboratories, blood fractionation, wine clarification, etc. centrifuge system are used and act as the main systems or sub systems. The reliability and availability of the centrifuge system has very significant role in these situations and that's why Kumar and Bhatia analyzed them. While collecting the primary data from Thermal Power Plant, (Haryana), they observed that a minor fault lead to degradation of the system whereas a major fault lead to complete failure of the system.

## DISCUSSION

In paper [4], the reliability and availability analyses of a centrifuge system considering major, minor and ignored faults has been carried out. But centrifuge systems were not studied considering periodic rest period, ignored faults and stoppage on occurrence of emergency so far. The paper [5] deals with a single unit centrifuge system considering major, minor and ignored faults in which a minor fault degrades a system whereas a major fault leads to the complete failure of the system. The system undergoes periodic rest and in this rest period the repairman inspects the system whether the fault is repairable or not and accordingly carry out repair or replacement. But sometimes system has to be stopped forcibly during functioning time. These things make their research more realistic. To analyze the reliability and availability of the system Kumar and Bhatia[5]used Markov processes, regenerative point technique and graphical analyses were drawn. Faults in the centrifuge system lesser the values of the availability and the reliability of the system. It was concluded that delay in repairs of ignored faults reduces the availability.

In paper [6], Kumar and Bhatia studied the impact of ignored faults on reliability and availability of a centrifuge system that undergoes periodic rest. Some faults such as vibration, abnormal sound, etc. are ignored /delayed for repair during the operation of the system until the system goes to rest or to complete failure. These faults sometimes may lead to complete failure of the system. Keeping this in view, the paper [6] deals with a single unit centrifuge system considering the above motioned fault.

In this paper they concluded that higher the rate of occurrence of ignored faults in the centrifuge system lesser the reliability of the system. Also, it had been observed that the availability ( $A_0$ ) of the system decreased with the increase in the above motioned faults and also decreased with increase in delay time.

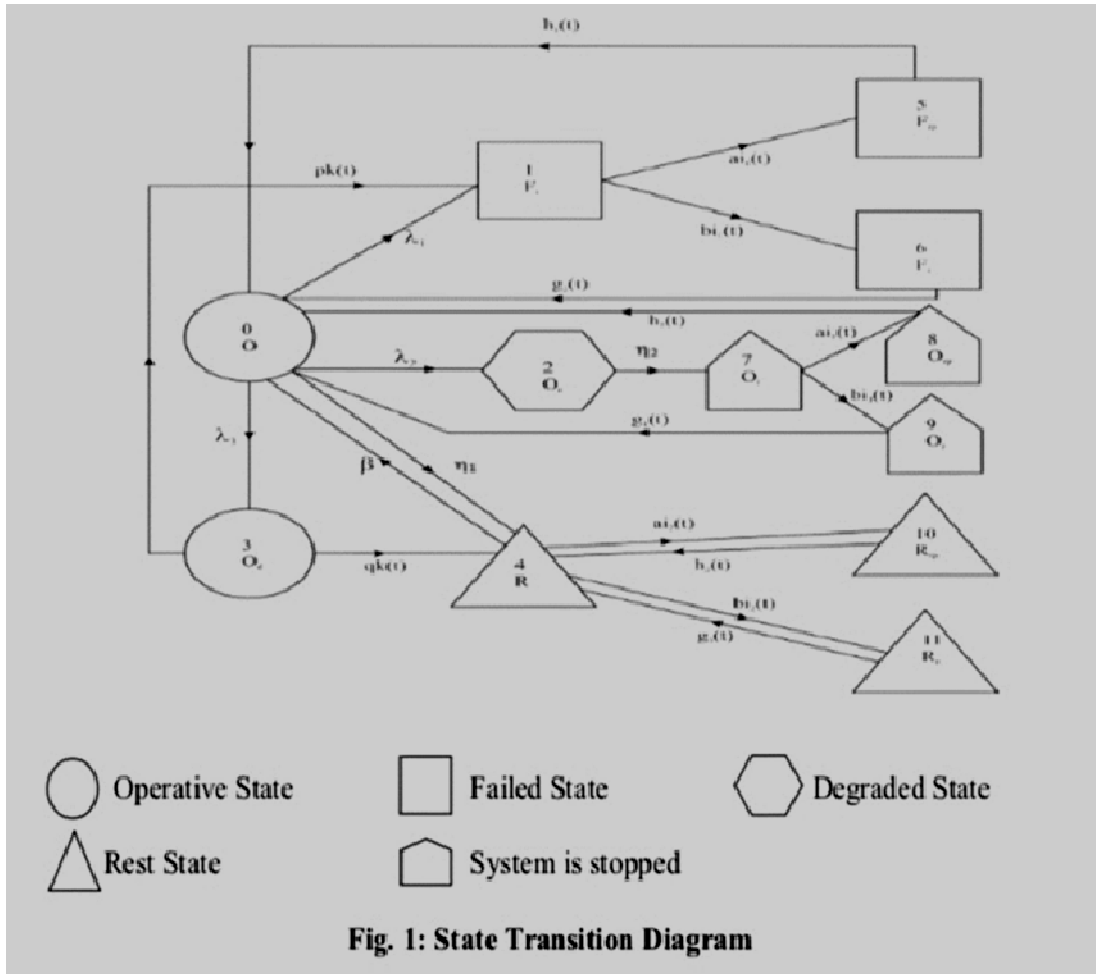
In 2013, Kumar and Bhatia [8] published a new paper on single unit centrifuge system. In this paper they analyzed that in practice some faults in the system were generally neglected or delayed for repair during operation of the system until the system went to rest or complete failure and that were taken as neglected faults. The system sometimes had to be halted on occurrence of minor faults that aspect also had been considered. In this paper various measures of system effectiveness were obtained and conclusions regarding reliability, expected uptime and cost of the system were drawn on the basis of the graphical studies. They found that MTSF deceases with increase in the values of rates of occurrence of major and neglected faults.

Kumar and Bhatia [9] formulated a stochastic model for a single unit centrifuge system considering major, minor and neglected faults. Using Markov processes, regenerative point and graphical presentation various conclusion were drawn regarding the reliability and profit of the system. It is clear from the graph in that profit decreases with the increase in the rate due to occurrence of minor faults and delay in repair of neglected faults respectively when other parameters remain fixed.

Thus, in paper [5], they have taken only minor and major faults. In paper [6] they added neglected faults. In paper [8], they included rest stage along with all the above said faults. Then in paper [9], they

included halt stage (forcibly stopped). Last but not least, in **Master Model** they have taken all real -life situations collectively. In this way they have tried to show a real- life situation of centrifuge system’s modeling and tried to discuss how to improve reliability.

The Master Model of their research is shown in fig. 1. This model has taken all above said situations. The epochs of entry in to state 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 are regenerative point and thus all the states are regenerative states. It is assumed that faults are self -announcing, single repairman reaches to the system in negligible time, switching is perfect and instantaneous, system is as good as new after each repair /replacement. All the faults and failure are repaired/replaced during rest time but in an emergency, system can be stopped.



Using the probabilistic arguments and the theory of regenerative processes, the availability  $A_i(t)$ , the probability that the system is up at instant  $t$ , is obtained. By recursive relations and Laplace transformation the availability is obtained given by

$$A_0 = \lim_{s \rightarrow 0} (s A_0^*(s)) = \frac{N_i}{D_i}$$

Various graphs drawn for the MTSF and the steady – state availability ( $A_0$ ) for the different values of the rates of occurrence of major, minor and ignored ( $\lambda_1, \lambda_2, \lambda_3$ ), repair rates ( $\beta_1, \beta_2, \beta_3$ ), replacement rates ( $\gamma_1, \gamma_2, \gamma_3$ ), inspection rates ( $\alpha_1, \alpha_2, \alpha_3$ ), rates of the system rest, stoppage and delay for repair ( $\eta_1, \eta_2, \delta$ ) respectively.

Fig. 2 gives the graph between MTSF ( $T_0$ ) and different values of failure rate due to major and minor faults ( $\lambda_1, \lambda_2$ ). The graph reveals that the MTSF decreases with increase in the values of the failure rates due to major and minor faults.

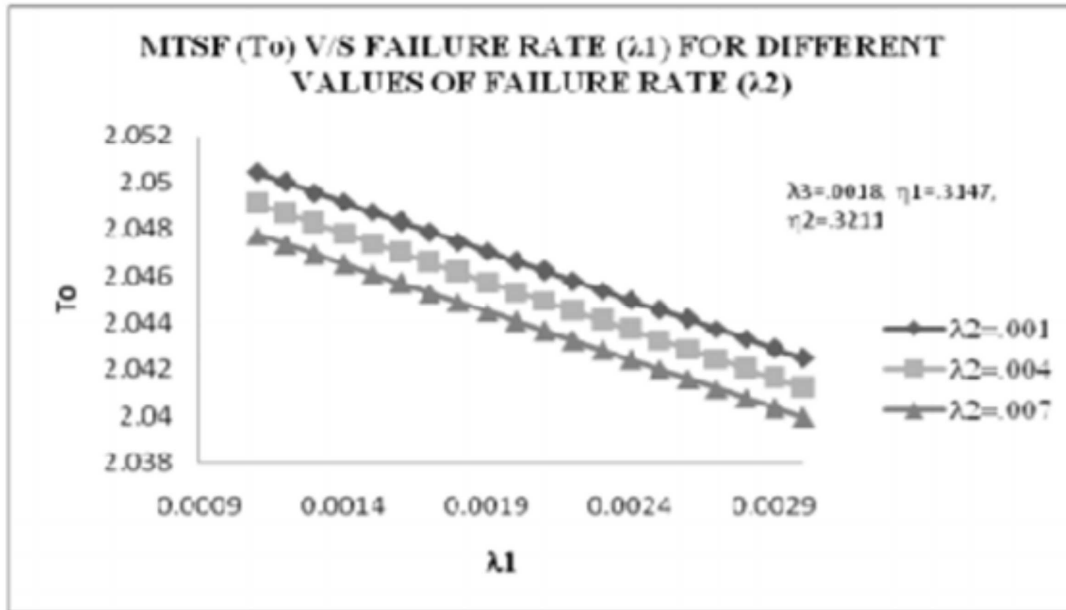


Fig. 2

Curves in Fig.3 shows that the increase in the rate of ignored faults ( $\lambda_3$ ), for small values of rate of delay in repair of these faults ( $\delta$ ), the availability increases whereas for higher values of  $\delta$ , the availability decreases.

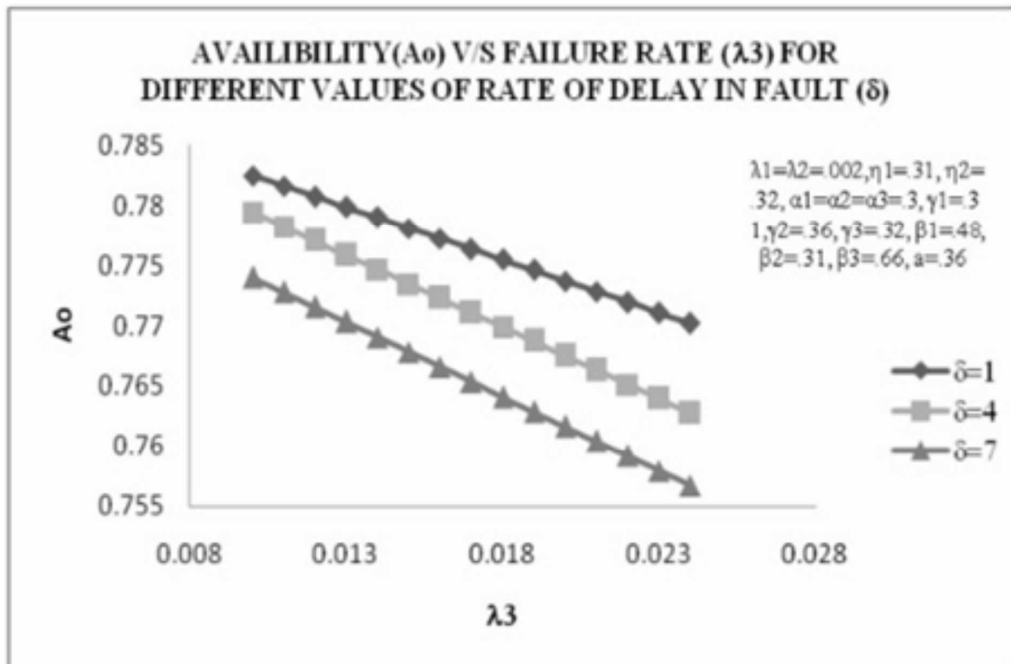


Fig.3

Thus, it was concluded that higher the values of the rates of occurrence of major, minor and ignored faults in the centrifuge system lesser the values of the availability and the reliability of the system. It was also concluded that delay in repairs of ignored faults reduces the availability.

The curve in the figure 4 shows the behavior of the profit with respect to rate of occurrence of minor faults (2) of the system for different values of rate of delay in repair of neglected faults ( $\delta$ ).

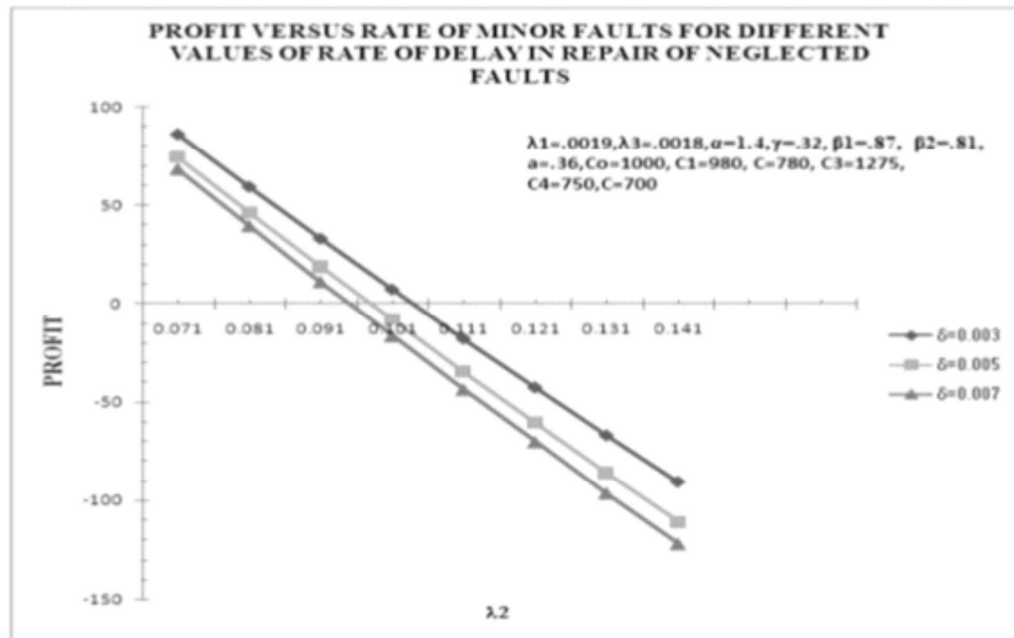


Fig.4

## CONCLUSION

In the literature of reliability, analysis of centrifuge system considering periodic rest, neglected faults and stoppage on occurrence of minor faults has not been reported so far. In rest period, the repairman can check the faults and can repair or replacement of faulty component. It saves the working time. Also, sometimes on the occurrence of minor faults system has to be stopped forcibly. Some faults such as vibration, abnormal sound i.e. neglected faults have not been analyzed. But these situations occur in real life and very important for the reliability of a machine. Kumar and Bhatia bridge this gap. That's why their work is very near the real situations. It's a strong reason to review their work. It is the importance of their work that many authors and research scholars referred their work in their research work. We conclude that mean time to the system failure, availability and the profit per unit time of centrifuge system decreases with the increase in the values of the rate of minor, major as well as neglected faults. Further it can also be concluded that major and neglected faults put more effect on profit and availability than minor faults. Therefore, more attention should be paid for removal of major as well as neglected faults (because in long run they converted to major faults).

## FUTURE SCOPE

Although, Shakeel and Vinod Kumar have done further work on two -unit centrifuge system (by taking Rajeev Kumar and Pooja Bhatia's reference). One can do further research on centrifuge multiple

system by taking redundancy. One can develop more practical useful models on considering Weibull distribution parameter or applying different methodology/approaches such as Regenerative graphical point technique (RGPT) etc. the research can also be extended by taking two or more repairman and then a comparative study can be made.

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## FINITE DEFORMATION PROBLEMS : UNIFORM EXTENSION

**Dr. Rajkala**

Assoc Prof, Mathematics, Dr B.R.Ambedkar Govt College, Kaithal, Haryana  
 E-mail : rajkalamaths@gmail.com

### ABSTRACT :

*In this paper, we apply fixed cartesian axes to define the unstrained body and the identity strain of deformation in terms of relative displacement of particles in the body that excludes rigid body motion.*

**Keywords:** strain, deformation, coordinates, rigid, uniform, tensors

### INTRODUCTION

Here, we study the theory of elasticity which describes the mechanical behaviors of elastic bodies under the action of applied loads. The main objective of the paper is to study the elastic response of a material using different approaches.

### FINITE DEFORMATION PROBLEMS

We take a fixed cartesian axes  $(x,y,z)$  to define the unstrained body  $B_0$  and identify the  $\theta_i$  coordinate with  $(x,y,z)$  so that

$$\theta_1 = x_1 = x, \theta_2 = x_2 = y, \theta_3 = x_3 = z \quad \dots(1.1.1)$$

We now consider a strain in which a unit cube of  $B_0$  whose sides are parallel to the axes is deformed into a cube of dimensions  $\lambda_1, \lambda_2, \lambda_3$  parallel to the  $x, y, z$  axes respectively. The coordinates of the strained body  $B$  may be referred to a fixed cartesian set of axes  $y_i$  which coincide with  $x, y, z$  so that

$$Y_1 = \lambda_1 x, Y_2 = \lambda_2 y, Y_3 = \lambda_3 z \quad \dots(1.1.2)$$

Using covariant metrics tensors the components of metric tensors  $g^{ik}, g_{ik}, G^{ik}, G_{ik}$  can be obtained as

$$g^{ik} = g_{ik} = Q_{ik}, g=1 \quad \dots(1.1.3)$$

And

$$G_{ik} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \quad G^{ik} = \begin{bmatrix} \lambda_1^{-2} & 0 & 0 \\ 0 & \lambda_2^{-2} & 0 \\ 0 & 0 & \lambda_3^{-2} \end{bmatrix},$$

$$G = \lambda_1^2 \lambda_2^2 \lambda_3^2 \quad \dots \quad \dots(1.1.4)$$

Using (1.1.3), (1.1.4) the strain invariants

$$\begin{aligned}
 I_1 &= g^{11}G_{11} + g^{22}G_{22} + g^{33}G_{33}, \\
 I_1 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\
 I_2 &= \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 + \lambda_1^2 \lambda_2^2, \\
 I_3 &= \lambda_1^2 \lambda_2^2 \lambda_3^2. \quad \dots(1.1.5)
 \end{aligned}$$

Using scalar invariants we get

$$\begin{aligned}
 B^{11} &= I_1 g^{11} - [g^{11}g^{11}G_{11} + g^{11}g^{12}G_{12} + g^{11}g^{13}G_{13} \\
 &\quad + g^{12}g^{11}G_{21} + g^{12}g^{12}G_{22} + g^{12}g^{13}G_{23} + g^{13}g^{11}G_{31} \\
 &\quad + g^{13}g^{12}G_{32} + g^{13}g^{13}G_{33}], \\
 B^{11} &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_1^2 = \lambda_2^2 + \lambda_3^2, \\
 B^{22} &= I_1 g^{22} - [g^{21}g^{21}G_{11} + g^{21}g^{22}G_{12} + g^{21}g^{23}G_{13} + g^{22}g^{21}G_{21} \\
 &\quad + g^{22}g^{22}G_{22} + g^{22}g^{23}G_{23} + g^{23}g^{21}G_{31} + g^{23}g^{22}G_{32} + g^{23}g^{23}G_{33}] \\
 B^{22} &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_2^2 = \lambda_1^2 + \lambda_3^2. \\
 B^{33} &= I_1 g^{33} - g^{33}g^{33}G_{33}, = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_3^2 = \lambda_1^2 + \lambda_2^2. \\
 B^{12} &= I_1 g^{12} - [g^{11}g^{21}G_{11} + g^{11}g^{22}G_{12} + g^{11}g^{23}G_{13} + g^{12}g^{21}G_{21} \\
 &\quad + g^{12}g^{22}G_{22} + g^{12}g^{23}G_{23} + g^{13}g^{21}G_{31} + g^{13}g^{22}G_{32} + g^{13}g^{23}G_{33}], \\
 B^{12} &= 0 \\
 B^{23} &= 0 \\
 B^{31} &= 0
 \end{aligned}$$

$$B^{ik} = \begin{bmatrix} \lambda_2^2 + \lambda_3^2 & 0 & 0 \\ 0 & \lambda_1^2 + \lambda_3^2 & 0 \\ 0 & 0 & \lambda_1^2 + \lambda_2^2 \end{bmatrix} \quad \dots(1.1.6)$$

And stress components Using (1,1,3), (1,1,6) and (1,1,4) from are given by

$$\begin{aligned}
 \tau^{11} &= \bar{\Phi} g^{11} + \bar{\Psi} B^{11} + b g^{11} \\
 \tau^{11} &= \bar{\Phi} + \bar{\Psi} (\lambda_2^2 + \lambda_3^2) + b \lambda_1^{-2}, \\
 \tau^{22} &= \bar{\Phi} + \bar{\Psi} (\lambda_1^2 + \lambda_3^2) + b \lambda_2^{-2}, \\
 \tau^{33} &= \bar{\Phi} + \bar{\Psi} (\lambda_2^2 + \lambda_1^2) + b \lambda_3^{-2} \quad \dots(1.1.6)(a) \\
 \tau^{12} &= 0 \\
 \tau^{13} &= 0 \\
 \tau^{23} &= 0
 \end{aligned}$$

And the stress equation of equilibrium are satisfied when body forces are absent using (1,1,4) and (1,1,6) (a) the physical stress can be written as

$$\begin{aligned}
 \sigma_{11} &= \frac{\sqrt{g_{11}}}{\sqrt{G^{11}}} \tau^{11} = \frac{\sqrt{\lambda_1^2}}{\sqrt{\lambda_1^2}} \tau^{11} = \lambda_1^2 \tau^{11}, \\
 \sigma_{11} &= \lambda_1^2 \bar{\Phi} + \lambda_1^2 (\lambda_2^2 + \lambda_3^2) \bar{\Psi} + b \\
 \sigma_{22} &= \lambda_2^2 \tau^{22} = \lambda_2^2 \bar{\Phi} + \lambda_2^2 (\lambda_3^2 + \lambda_1^2) \bar{\Psi} + b \\
 \sigma_{33} &= \lambda_3^2 \tau^{33} = \lambda_3^2 \bar{\Phi} + \lambda_3^2 (\lambda_1^2 + \lambda_2^2) \bar{\Psi} + b \\
 \sigma_{12} &= \sigma_{23} = \sigma_{31} = 0 \quad \dots(1.1.7)
 \end{aligned}$$

Using (1,1,5) the above can be written as

$$\begin{aligned}
 \sigma_{11} &= \lambda_1^2 \bar{\Phi} - \lambda_2^2 \lambda_3^2 \bar{\Psi} + b' \\
 \sigma_{22} &= \lambda_2^2 \bar{\Phi} - \lambda_3^2 \lambda_1^2 \bar{\Psi} + b' \\
 \sigma_{33} &= \lambda_3^2 \bar{\Phi} - \lambda_1^2 \lambda_2^2 \bar{\Psi} + b' \quad \dots(1.1.8)
 \end{aligned}$$

$$\text{where } b' = b + \bar{\Psi} I_2 \quad \dots(1.1.9)$$

When the body is incompressible the stresses (1,1,8) can be written as

$$\sigma_{11} = \lambda_1^2 \Phi - \Psi/\lambda_1^2 + b', \quad \dots(1.1.10)$$

$$\text{since } \lambda_1^2 \lambda_2^2 \lambda_3^2 = 1 \quad \dots(1.1.11)$$

For the particular case of simple extensions under a force parallel to the x-axis,  $\lambda_2 = \lambda_3$ ,  $\sigma_{22} = \sigma_{33} = 0$ . Hence for a compressible body (1,1,8) may be written as

$$b' = -\lambda_2^2 \Phi + \lambda_1^2 \lambda_2^2 \Psi, \quad \dots(1.1.12)$$

$$\text{and } \sigma_{11} = (\lambda_1^2 - \lambda_2^2) \Phi + \Psi \lambda_2^2 \quad \dots(1.1.13)$$

For an incompressible body

$$\lambda_2^2 = 1/\lambda_1 \quad \text{and therefore}$$

$$\sigma_{11} = (\lambda_1^2 - \frac{1}{\lambda_1}) (\Phi + \Psi/\lambda_1) \quad \dots(1.1.14)$$

## CONCLUSION

This paper deals with the solution of some problems of finite deformation, for bodies which are homogeneous and isotropic as from Green and Zerna (12,1954). First of all coordinates of a specified point are written under formed and deformed state of body. Writing the metric tensor on both states, we calculate value of strain invariants. Substituting these values in stress strain relations, we get stress acting on the body considered. From these stresses we get physical components of stresses and finally the forces and the couple necessary to keep the body in equilibrium are calculated.

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## APPLICATION OF INVERSE OF MATRIX IN NETWORK SECURITY

**Arun Pratap Singh**

Department of General Studies, Jubail Industrial College, Jubail, Saudi Arabia  
E-mail : arungeit@yahoo.co.in

### ABSTRACT

*Computer data often travels from one computer to another, leaving the safety of its protected physical surroundings. Once the data is out of hand, people with bad intention could modify or forge your data, either for amusement or for their own benefit. In this paper, we have presented the application of matrices in network security.*

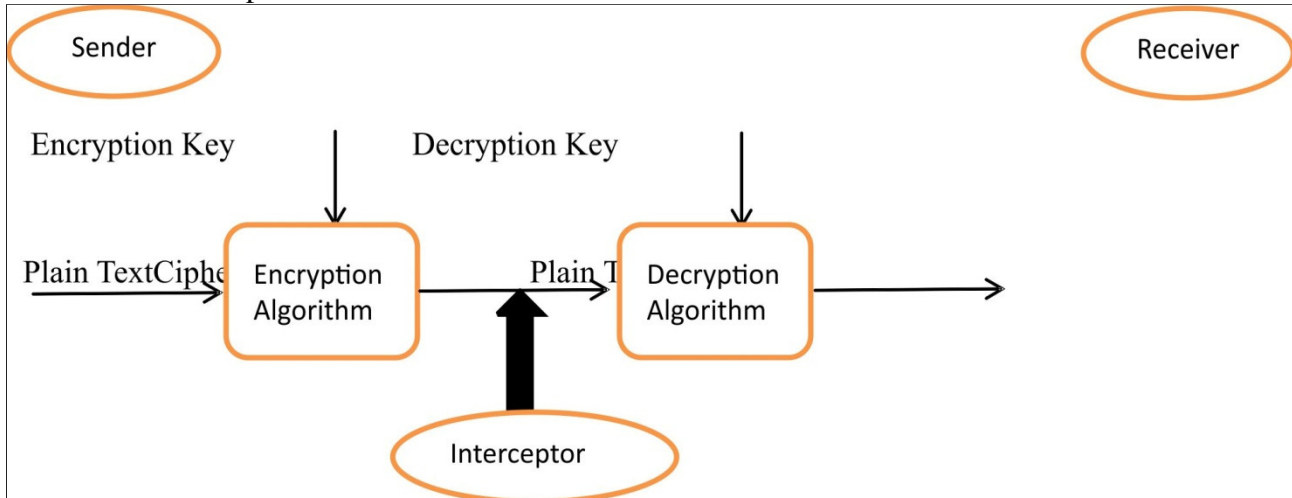
*Keywords: Cryptology, Encryption, Decryption, Encoder, Decoder, Key matrix.*

### 1. INTRODUCTION

Cryptography is a mathematical science of controlling access to information. The basic idea of cryptography is that the information can be encoded using encryption scheme and decoded by anyone who knows the scheme. There are lots of encryption schemes ranging from very simple to very complex. Most of them are mathematical in nature.

The art of cryptology is considered to be born along with the art of writing. As civilizations evolved human beings got organized in tribes, groups and kingdoms. This led to the emergence of ideas such as power, battles, supremacy and politics. These ideas further fueled the natural need of people to communicate secretly with selective recipient which in turn ensured the continuous evolution of cryptology as well.

The basic model is depicted as:



A sender wants to transfer some sensitive data to a receiver in such a way that any party intercepting or eavesdropping on the communication channel cannot extract the data.

## 2. APPLICATION OF INVERSE OF MATRIX

The encoder is a matrix and the decoder is its inverse. Let  $A$  be the encoding matrix,  $P$  is the message matrix and  $X$  will be encrypted matrix, then

$$AP = X$$

The size of  $A$  and  $P$  will have to be consistent and will determine the size of  $X$ . Let someone has  $X$  and knows  $A$  and want to recover  $P$ , the original message

$$P = A^{-1} X$$

**ENCRYPTION:** To encrypt a message using the Hill Cipher we must first turn our keyword into a key matrix (a  $2 \times 2$  matrix for working with digraphs, a  $3 \times 3$  matrix for working with trigraphs, etc.). We also turn the plaintext into digraphs (or trigraphs) and each of these into a column vector. We then perform matrix multiplication modulo the length of the alphabet (i.e. 26) on each vector. These vectors are then converted back into letters to produce the ciphertext.

We shall encrypt the plain text message "short example" using the keyword *hill* and a  $2 \times 2$  matrix. The first step is to turn the keyword into a matrix. If the keyword was longer than the 4 letters needed, we would only take the first 4 letters, and if it was shorter, we would fill it up with the alphabet in order.

$$\begin{bmatrix} H & I \\ L & L \end{bmatrix}$$

With the keyword in a matrix, we need to convert this into key matrix. We do this by converting each letter into a number by its position in the alphabet (starting at 0). So A=0, B=1, C=2, D=3, etc.

$$\begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix}$$

We now split the plain text into digraphs, and write these as column vectors. That is, in the first column vector we write the first plaintext letter at the top, and the second letter at the bottom. Then we move to the next column vector, where the third plaintext letter goes at the top, and the fourth at the bottom. This continues for the whole plain text fourth at the bottom. This continues for the whole plain text

$$\begin{bmatrix} s \\ h \end{bmatrix} \begin{bmatrix} o \\ r \end{bmatrix} \begin{bmatrix} t \\ e \end{bmatrix} \begin{bmatrix} x \\ a \end{bmatrix} \begin{bmatrix} m \\ p \end{bmatrix} \begin{bmatrix} l \\ e \end{bmatrix}$$

Now we must convert the plaintext column vectors in the same way that we converted the keyword into the key matrix. Each letter is replaced by its appropriate number.

$$\begin{bmatrix} 18 \\ 7 \end{bmatrix} \begin{bmatrix} 14 \\ 17 \end{bmatrix} \begin{bmatrix} 19 \\ 4 \end{bmatrix} \begin{bmatrix} 23 \\ 0 \end{bmatrix} \begin{bmatrix} 12 \\ 15 \end{bmatrix} \begin{bmatrix} 11 \\ 4 \end{bmatrix}$$

Now we perform matrix multiplication. We multiply the key matrix by each column vector in turn. We write the key matrix first, followed by column vector.

$$\begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} 18 \\ 7 \end{bmatrix}$$

To perform matrix multiplication we combine the top row of the key matrix with the column vector to get the top element of the resulting column vector. The way we combine the four numbers to get a single is that we multiply the first element of the key matrix row by the top element of the column vector and multiply the second element of the key matrix row by the bottom element of the column vector. We then add together these two answers.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

The algebraic rule for matrix multiplication

$$\text{In our case, } \begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} 18 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \times 18 + 8 \times 7 \\ 11 \times 18 + 11 \times 7 \end{bmatrix} = \begin{bmatrix} 182 \\ 275 \end{bmatrix}$$

Now reducing the resultant column vector modulo 26 (i.e. divide by 26 and take the remainder)

$$\begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} 18 \\ 7 \end{bmatrix} = \begin{bmatrix} 182 \\ 275 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \end{bmatrix} \pmod{26}$$

Now we have to convert these numbers to the letters, 0 becomes A and 15 becomes P

$$\begin{bmatrix} H & I \\ L & L \end{bmatrix} \begin{bmatrix} s \\ h \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} 18 \\ 7 \end{bmatrix} = \begin{bmatrix} 182 \\ 275 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \end{bmatrix} \pmod{26} = \begin{bmatrix} A \\ P \end{bmatrix}$$

Similarly,

$$\begin{bmatrix} H & I \\ L & L \end{bmatrix} \begin{bmatrix} o \\ r \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} 14 \\ 17 \end{bmatrix} = \begin{bmatrix} 234 \\ 341 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \pmod{26} = \begin{bmatrix} A \\ D \end{bmatrix}$$

$$\begin{bmatrix} H & I \\ L & L \end{bmatrix} \begin{bmatrix} t \\ e \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} 19 \\ 4 \end{bmatrix} = \begin{bmatrix} 165 \\ 253 \end{bmatrix} = \begin{bmatrix} 9 \\ 19 \end{bmatrix} \pmod{26} = \begin{bmatrix} J \\ T \end{bmatrix}$$

$$\begin{bmatrix} H & I \\ L & L \end{bmatrix} \begin{bmatrix} x \\ a \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} 23 \\ 0 \end{bmatrix} = \begin{bmatrix} 161 \\ 253 \end{bmatrix} = \begin{bmatrix} 5 \\ 19 \end{bmatrix} \pmod{26} = \begin{bmatrix} F \\ T \end{bmatrix}$$

$$\begin{bmatrix} H & I \\ L & L \end{bmatrix} \begin{bmatrix} m \\ p \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 204 \\ 297 \end{bmatrix} = \begin{bmatrix} 22 \\ 11 \end{bmatrix} \pmod{26} = \begin{bmatrix} W \\ L \end{bmatrix}$$

$$\begin{bmatrix} H & I \\ L & L \end{bmatrix} \begin{bmatrix} l \\ e \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix} \begin{bmatrix} 11 \\ 4 \end{bmatrix} = \begin{bmatrix} 109 \\ 165 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix} \pmod{26} = \begin{bmatrix} F \\ J \end{bmatrix}$$

This gives us final cipher text of “APADJTFTWLFJ”.

**DECRYPTION:** To decrypt a cipher text encoded using Hill cipher, we must find the inverse matrix. In general, to find the inverse of the key matrix, we perform the calculation as

$$K^{-1} = d^{-1} \times adj(K)$$

Where K is the key matrix, d is the determinant of key matrix, adj(K) is the adjugate matrix of K.

### Step 1: Multiplicative Inverse of the Determinant

The determinant is the number that relates directly to the entries of the matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Then we need to take the number modulo 26.

$$\begin{vmatrix} 7 & 8 \\ 11 & 11 \end{vmatrix} = 7 \times 11 - 8 \times 11 = -11 = 15 \pmod{26}$$

We now have to find the multiplicative inverse of the determinant working modulo 26 i.e. the number between 1 to 25 that gives an answer of 1 when we multiply it by the determinant. So in this case, we are looking for the number that we need to multiply 15 by to get an answer of 1 modulo 26.

$$d d^{-1} = 1 \pmod{26}$$

$$15 \times x = 1 \pmod{26}$$

$$15 \times 7 = 105 = 1 \pmod{26}$$

Therefore the multiplicative inverse of the determinant modulo 26 is 7.

### Step 2: Find Adjugate matrix

The adjugate matrix is a matrix of the same size as the original matrix.

$$adj \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$



Once we have these values we will need to take each of them modulo 26 (we need to add 26 to the negative values to get a number between 0 and 25)

$$\text{adj} \begin{bmatrix} 7 & 8 \\ 11 & 11 \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ -11 & 7 \end{bmatrix} = \begin{bmatrix} 11 & 18 \\ 15 & 7 \end{bmatrix}$$

### Step 3: Inverse key matrix

We now multiply the inverse determinant (7) from step 1 by each of the elements of the adjoint matrix from step 2, we take each of these answers modulo 26.

$$7 \times \begin{bmatrix} 11 & 18 \\ 15 & 7 \end{bmatrix} = \begin{bmatrix} 77 & 126 \\ 105 & 49 \end{bmatrix} = \begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \text{ mod } 26$$

Therefore, 
$$K^{-1} = \begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix}$$

Now, we have to convert the cipher text into column vectors and multiply the inverse matrix by each column vector in turn, take the results modulo 26 and convert these back into letters to get the plaintext.

$$\begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} A \\ P \end{bmatrix} = \begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} 0 \\ 15 \end{bmatrix} = \begin{bmatrix} 330 \\ 345 \end{bmatrix} = \begin{bmatrix} 18 \\ 7 \end{bmatrix} \text{ mod } 26 = \begin{bmatrix} s \\ h \end{bmatrix}$$

$$\begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} A \\ D \end{bmatrix} = \begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 66 \\ 69 \end{bmatrix} = \begin{bmatrix} 14 \\ 17 \end{bmatrix} \text{ mod } 26 = \begin{bmatrix} o \\ r \end{bmatrix}$$

$$\begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} J \\ T \end{bmatrix} = \begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} 9 \\ 19 \end{bmatrix} = \begin{bmatrix} 643 \\ 446 \end{bmatrix} = \begin{bmatrix} 19 \\ 4 \end{bmatrix} \text{ mod } 26 = \begin{bmatrix} t \\ e \end{bmatrix}$$

$$\begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} F \\ T \end{bmatrix} = \begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} 5 \\ 19 \end{bmatrix} = \begin{bmatrix} 543 \\ 442 \end{bmatrix} = \begin{bmatrix} 23 \\ 0 \end{bmatrix} \text{ mod } 26 = \begin{bmatrix} x \\ a \end{bmatrix}$$

$$\begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} W \\ L \end{bmatrix} = \begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} 22 \\ 11 \end{bmatrix} = \begin{bmatrix} 792 \\ 275 \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \end{bmatrix} \text{ mod } 26 = \begin{bmatrix} m \\ p \end{bmatrix}$$

$$\begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} F \\ J \end{bmatrix} = \begin{bmatrix} 25 & 22 \\ 1 & 23 \end{bmatrix} \begin{bmatrix} 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 330 \\ 345 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix} \text{ mod } 26 = \begin{bmatrix} l \\ e \end{bmatrix}$$

Therefore we get our original text “short example”.

**CONCLUSION:**

The application of matrices in network security has been presented. In cryptography, the matrices play an important role, specially the inverse of a matrix. We can use various methods for finding inverse of the matrices.

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## MODULAR LABELING OF SOME CLASSES OF GRAPHS

**K. Muthugurupackiam\*, S. Ramya\*\***

\*Department of Mathematics, Rajah Serfoji Govt. College, Affiliated to Bharathidasan University, Thanjavur, India,

\*\*Departments of Mathematics, Bharathidasan University Constituent College, Orathanadu, India.,

E-mail : gurupackiam@yahoo.com\*, manoramyia.8036@gmail.com\*\*

### ABSTRACT

*In this paper, Modular strength of three families of graphs are determined which are obtained by corona product of  $C_n$  with  $P_1$ ,  $C_n$  with  $P_2$  and  $C_n$  with  $P_3$ .*

*Keywords: irregular labeling, modular labeling, irregularity strength, modular strength.*

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### 1. INTRODUCTION

By a graph labeling we mean an assignment of numbers to graph elements such as vertices or edges or both, under some conditions. These conditions are normally expressed on the basis of some values (weights) of an evaluating function. Chartrand, Jacobson, Lehel, Oellermann, Ruiz and Saba[1] introduced the concept of irregular labeling as follows:

Let  $G = (V,E)$  be a connected graph of order  $n \geq 3$  together with an edge  $h$ -labeling  $\varphi: E(G) \rightarrow \{1,2,3,\dots,h\}$ . Weight of a vertex  $v$  (denoted by  $\omega(v)$ ) is the sum of the labelings of the edges which are incident with that vertex  $v$ , i.e.,  $\omega(v) = \sum_{u \in N(v)} \varphi(uv)$ .  $\varphi$  is called irregular if all vertex weights of

$G$  are distinct. The smallest positive integer  $h$  for which there exists an irregular  $h$ -labeling of  $G$  is called irregularity strength of  $G$  and is denoted by  $s(G)$ . If it does not exist, we write  $s(G) = \infty$ .

Further, they determined the lower bound of  $s(G)$  as follows:

$$s(G) \geq \max_{1 \leq i \leq \Delta} \left\{ \frac{i-1+n_i}{i} \right\} \quad 1$$

where  $n_i$  denotes the number of vertices of degree  $i$  and  $\Delta$  is the maximum degree of graph  $G$ . Stanislav Jendrol and Michal Tkac [4] studied the irregularity strength of the union of  $t$  copies of the complete graph  $K_p$ . Gyarfás [2] determined the irregularity strength of  $K_n - mK_2$ . Jeffrey H. Dinitz et al., [3] determined the irregularity strength of the  $m \times n$  grid for certain  $m$  and  $n$ . Olivier Togni [5] studied the irregularity strength of the toroidal grid. D. Amar and O. Togni [6] established that the irregularity

strength of any tree with no vertices of degree 2 is its number of pendent vertices. For a  $d$ -regular graph  $G$ ,  $d \geq 2$ , Faudree and Lehel [7] proved that  $\lceil (n+d-1)/d \rceil \leq s(G) \leq \lceil n/2 \rceil + 9$ , and they conjectured that  $s(G) \leq \lceil n/d \rceil + c$  for some constant  $c$ . Przybylo in [8] showed that  $s(G) \leq 16n/d + 6$ . Kalkowski, Karonski and Pfender [9] proved that  $s(G) \leq 6n/\delta + 6$ , where  $\delta$  is the minimum degree of graph  $G$ . Currently Majerski and Przybylo [10] showed that  $s(G) \leq (4 + o(1))n/\delta + 4$  for graphs with minimum degree  $\delta \geq \sqrt{\ln n}$ .

Motivated by the irregularity strength of graphs, M. Baca, K. Muthugurupackiam, KM. Kathiresan and S. Ramya [12] introduced and studied modular irregularity strength (or simply modular strength) of graphs. The modular irregularity strength of graph naturally arise in the study of modular version of irregularity strength. Let  $G = (V, E)$  be a graph of order  $n$  with no component of order 2. An edge labeling of  $G$ ,  $\mu: E(G) \rightarrow \{1, 2, 3, \dots, k\}$  is called modular irregular  $k$ -labeling if there exists a bijective weight function  $\sigma: V(G) \rightarrow Z_n$  defined by  $\sigma(x) = \sum \mu(xy)$  called modular weight of the vertex  $x$ , where  $Z_n$  is the group of integers modulo  $n$  and the sum is over all vertices  $y$  adjacent to  $x$ . They defined the modular irregularity strength of a graph  $G$ , denoted by  $ms(G)$ , as the minimum  $k$  for which  $G$  has a modular irregular  $k$ -labeling. If there doesn't exist a modular irregular labeling for  $G$ , they defined  $ms(G) = \infty$ . In [12], modular strength of paths, cycles, stars, triangular graph and gear graph are estimated. Further they proved the following lemmas and a lower bound for the modular strength.

**Lemma 1.1.[12]** Let  $G = (V, E)$  be a graph with no component of order 2. Every modular irregular  $k$ -labeling of  $G$  is also its irregular assignment.

**Lemma 1.2.[12]** Let  $G = (V, E)$  be a graph with no component of order  $\leq 2$  and let  $s(G) = k$ . If there exists an irregular assignment of  $G$  with edge values at most  $k$ , where weights of vertices constitute a set of consecutive integers, then  $s(G) = ms(G) = k$ .

**Theorem 1.3.[12]** Let  $G = (V, E)$  be a graph with no component of order  $\leq 2$ . Then  $s(G) \leq ms(G)$ .

**Theorem 1.4.[12]** If  $G$  is a graph of order  $n$ ,  $n \equiv 2 \pmod{4}$ , then  $G$  has no modular irregular  $k$ -labeling i.e.,  $ms(G) = \infty$ .

In [11], K. Muthugurupackiam and S. Ramya determined modular irregularity strength of some families of graphs, namely, bistar, cartesian product of  $K_2$  with cycle  $C_n$  and corona product of path  $P_n$  with  $K_1$ . Also they presented a lower bound of modular irregularity strength as follows:

**Theorem 1.5.[11]** Let  $G$  be a graph of order  $n \geq 3$ , and let  $\delta$  and  $\Delta$  be the minimum and maximum

$$\text{degree of } G \text{ respectively, then } ms(G) \geq \begin{cases} \left\lceil \frac{n + \delta + 1}{\Delta} \right\rceil, & \text{if } n \equiv 0 \pmod{4}; n \equiv 1 \pmod{4} \text{ and } \delta \text{ is even;} \\ & n \equiv 3 \pmod{4} \text{ and } \delta \text{ is odd,} \\ \left\lceil \frac{n + \delta}{\Delta} \right\rceil, & n \equiv 1 \pmod{4} \text{ and } \delta \text{ is odd;} \\ & n \equiv 3 \pmod{4} \text{ and } \delta \text{ is even.} \end{cases}$$

The main aim of this paper is to determine the precise values of modular strength for three families of graphs.

## 2. MAIN RESULTS

In this section, modular strength of corona product of cycles  $C_n$  with paths of length one, two and three are determined.

### Definition 2.1.

The corona  $G_1 \otimes G_2$  of two graphs  $G_1$  and  $G_2$  (where  $G_i$  has  $n_i$  vertices and  $m_i$  edges) is defined as the graph  $G$  obtained by taking one copy of  $G_1$  and  $n_1$  copies of  $G_2$ , and then joining by a line the  $i^{\text{th}}$  point of  $G_1$  to every point in the  $i^{\text{th}}$  copy of  $G_2$ . It follows from the definition of the corona that  $G_1 \otimes G_2$  has  $n_1(1+n_2)$  vertices and  $m_1 + n_1m_2 + n_1n_2$  edges.

The sun graph  $S_n$ ,  $n \geq 3$ , is a graph obtained by corona product of  $C_n$  with  $P_1$  i.e.,  $S_n = C_n \otimes P_1$ . Following theorem gives the modular strength of the sun graph.

**Theorem 2.2.** For  $n \geq 3$ , the modular strength of sun graph of order  $2n$  is  $n$  when  $n$  is even and  $\infty$  when  $n$  is odd. i.e.,  $ms(C_n \otimes P_1) = \begin{cases} n, & n \text{ is even,} \\ \infty, & n \text{ is odd.} \end{cases}$

Proof. Let  $V(C_n \otimes P_1) = \{u_i, v_i : 1 \leq i \leq n\}$  be the vertex set and  $E(C_n \otimes P_1) = \{u_i v_i, v_i v_{i+1} : 1 \leq i \leq n\}$  be the edge set of the sun graph  $C_n \otimes P_1$ . Define  $n$ -labeling  $\varphi : E(C_n \otimes P_1) \rightarrow \{1, 2, 3, \dots, n\}$  by

$$\varphi(u_i v_i) = i, \quad 1 \leq i \leq n,$$

$$\varphi(v_i v_{i+1}) = \frac{n}{2}, \quad 1 \leq i \leq n.$$

Induced weight function  $\sigma : V(C_n \otimes P_1) \rightarrow Z_{2n}$  is defined as follows:

$$\sigma(u_i) = i, \quad 1 \leq i \leq n.$$

$$\sigma(v_i) = n + i, \quad 1 \leq i \leq n - 1.$$

$$\sigma(v_n) = 2n \equiv 0 \pmod{2n}.$$

Hence, modular weights of the vertices of the sun graph are  $0, 1, 2, \dots, 2n - 1$ . Clearly,  $\varphi$  is a modular irregular labeling of  $C_n \otimes P_1$  and hence  $ms(C_n \otimes P_1) \leq n$ . Since, there are  $n$  pendants in  $C_n \otimes P_1$ , at least  $n$  different edge labelings are essential to obtain the irregular labeling for  $C_n \otimes P_1$ , i.e.,  $ms(C_n \otimes P_1) \geq n$ . Thus,  $ms(C_n \otimes P_1) = n$ .

By theorem 1.4,  $ms(C_n \otimes P_1) = \infty$ , when  $n$  is odd. This concludes the proof.

Next theorem shows the modular strength of the graph  $G_2$  obtained by corona product of  $C_n$  with  $P_2$  i.e.,  $G_2 = C_n \otimes P_2$ .

**Theorem 2.3.** For  $n \geq 3$ ,  $ms(C_n \otimes P_2) = \begin{cases} n+1, & n \not\equiv 2 \pmod{4} \\ \infty, & n \equiv 2 \pmod{4}. \end{cases}$

**Proof.** Let  $V(C_n \otimes P_2) = \{u_{i,1}, u_{i,2}, v_i : 1 \leq i \leq n\}$  be the vertex set and let  $E(C_n \otimes P_2) = \{u_{i,1}v_i, u_{i,2}v_i, u_{i,1}u_{i,2}, v_i v_{i+1} : 1 \leq i \leq n\}$  be the edge set of the graph  $C_n \otimes P_2$ . Define  $n+1$ -labeling  $\varphi: E(C_n \otimes P_2) \rightarrow \{1, 2, 3, \dots, n+1\}$  as follows:

If  $n \equiv 0, 3 \pmod{4}$  then,

$$\varphi(v_i u_{i,1}) = \varphi(u_{i,1} u_{i,2}) = \begin{cases} 1, & i=1, \\ 2i-2, & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

$$\varphi(v_i u_{i,2}) = \begin{cases} 2, & i=1, \\ 2i-1, & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

$$\varphi(v_{n+1-i} u_{n+1-i,1}) = \varphi(u_{n+1-i,1} u_{n+1-i,2}) = 2i+1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

$$\varphi(v_{n+1-i} u_{n+1-i,2}) = 2i+2, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

$$\varphi(v_i v_{i+1}) = \begin{cases} n+1-i, & \text{when } i \text{ is odd,} \\ n-i, & \text{when } i \text{ is even,} \end{cases} \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

$$\varphi(v_n v_1) = n-1.$$

$$\varphi(v_{n-i} v_{n+1-i}) = n-1-i, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Induced weight function  $\sigma: V(C_n \otimes P_2) \rightarrow Z_{3n}$  is defined as follows:

$$\sigma(u_{i,1}) = \begin{cases} 2, & i=1, \\ 4i-4, & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

$$\sigma(u_{i,2}) = \begin{cases} 3, & i=1, \\ 4i-3, & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

$$\sigma(u_{n+1-i,1}) = 4i+2, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

$$\sigma(u_{n+1-i,2}) = 4i+3, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

$$\sigma(v_i) = \begin{cases} 2n+2, & i=1, \\ 2n+2i-1, & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor. \end{cases}$$

$$\sigma(v_{n+1-i}) = 2n+2i+2, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 2.$$

$$\sigma\left(v_{\left\lfloor \frac{n}{2} \right\rfloor + 1}\right) = 3n+1 \equiv 1 \pmod{3n}.$$

$$\sigma\left(v_{\left\lfloor \frac{n}{2} \right\rfloor + 2}\right) = 3n \equiv 0 \pmod{3n}.$$

Under the labeling  $\varphi$ , we can obtain weights of the vertices as  $2, 3, 4, \dots, 3n+1$ . Since the order of  $C_n \otimes P_2$  is  $3n$ , the modular weights of its vertices becomes  $0, 1, 2, \dots, 3n-1$ . Clearly,  $\varphi$  is a modular irregular labeling of  $C_n \otimes P_2$  when  $n \equiv 0, 3 \pmod{4}$ , furthermore the maximum edge label under  $\varphi$  is  $n+1$  and hence  $ms(C_n \otimes P_2) \leq n+1$ . By theorem 1.3 and equation 1 we can easily obtain  $ms(C_n \otimes P_2) \geq n+1$ .

If  $n \equiv 1 \pmod{4}$  then we define labeling  $\varphi_1 : E(C_n \otimes P_2) \rightarrow \{1, 2, 3, \dots, n+1\}$  as follows:

$$\varphi_1(v_i u_{i,1}) = \begin{cases} 1, & i=1, \\ 2i-2, & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

$$\varphi_1(v_i u_{i,2}) = \varphi_1(u_{i,1} u_{i,2}) = \begin{cases} 2, & i=1, \\ 2i-1, & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

$$\varphi_1(v_{n+1-i} u_{n+1-i,1}) = 2i+1, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

$$\varphi_1(v_{n+1-i} u_{n+1-i,2}) = \varphi_1(u_{n+1-i,1} u_{n+1-i,2}) = 2i+2, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

$$\varphi_1(v_i v_{i+1}) = n+1-i, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

$$\varphi_1(v_n v_1) = n.$$

$$\varphi_1(v_{n-i} v_{n+1-i}) = \begin{cases} n-1-i, & \text{when } i \text{ is odd,} \\ n-i, & \text{when } i \text{ is even,} \end{cases} \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.$$

Induced weight function  $\sigma : V(C_n \otimes P_2) \rightarrow Z_{3n}$  is defined as follows:

$$\sigma(u_{i,1}) = \begin{cases} 3, & i=1, \\ 4i-3, & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1. \end{cases}$$

$$\sigma(u_{i,2}) = \begin{cases} 4, & i=1, \\ 4i-2, & 2 \leq i \leq \lfloor \frac{n}{2} \rfloor + 1. \end{cases}$$

$$\sigma(u_{n+1-i,1}) = 4i+3, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1.$$

$$\sigma(u_{n+1-i,2}) = 4i+4, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1.$$

$$\sigma(v_i) = \begin{cases} 2n+3, & i=1, \\ 2n+2i, & 2 \leq i \leq \lfloor \frac{n}{2} \rfloor. \end{cases}$$

$$\sigma(v_{n+1-i}) = 2n+2i+3, \quad 1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 2.$$

$$\sigma\left(v_{\lfloor \frac{n}{2} \rfloor + 1}\right) = 3n+1 \equiv 1 \pmod{3n}.$$

$$\sigma\left(v_{\lfloor \frac{n}{2} \rfloor + 2}\right) = 3n+2 \equiv 2 \pmod{3n}.$$

$$\sigma\left(v_{\lfloor \frac{n}{2} \rfloor + 3}\right) = 3n \equiv 0 \pmod{3n}.$$

Under the labeling  $\varphi_1$  vertex weights of  $C_n \otimes P_2$  are  $3,4,5,\dots,3n+2$  and its modular weights are  $0,1,2,\dots,3n-1$ . Clearly,  $\varphi_1$  is a modular irregular labeling of  $C_n \otimes P_2$  and hence  $ms(C_n \otimes P_2) \leq n+1$ . In this case the weights of vertices of  $C_n \otimes P_2$  cannot be  $2,3,4,\dots,3n+1$  furthermore,  $ms(C_n \otimes P_2) \geq n+1$ . Thus  $ms(C_n \otimes P_2) = n+1$ . By theorem 1.4,  $ms(C_n \otimes P_2) = \infty$ , when  $n \equiv 2 \pmod{4}$ .

Let  $G_3 = C_n \otimes P_3$ . Modular strength of  $G_3$  is estimated as below:

**Theorem 2.4.** For  $n \geq 3$ , modular strength of  $C_n \otimes P_3$  is  $n+1$ .

Proof. Let  $V(C_n \otimes P_3) = \{v_i, u_{i,j}, : 1 \leq i \leq n, 1 \leq j \leq 3\}$  be the vertex set and let  $E(C_n \otimes P_3) = \{u_{i,1}v_i, u_{i,2}v_i, u_{i,3}v_i, u_{i,1}u_{i,2}, u_{i,2}u_{i,3}, v_i v_{i+1} : 1 \leq i \leq n\}$  be the edge set of the graph  $C_n \otimes P_3$ .

Define  $n+1$ -labeling  $\varphi: E(C_n \otimes P_3) \rightarrow \{1,2,3,\dots,n+1\}$  as follows:

$$\varphi(u_{i,1}u_{i,2}) = i, \quad 1 \leq i \leq n.$$

$$\varphi(u_{i,2}u_{i,3}) = n, \quad 1 \leq i \leq n.$$



$$\varphi(v_i u_{i,1}) = 1, \quad 1 \leq i \leq n.$$

$$\varphi(v_i u_{i,2}) = n+1, \quad 1 \leq i \leq n.$$

$$\varphi(v_i u_{i,3}) = i+1, \quad 1 \leq i \leq n.$$

$$\varphi(v_i v_{i+1}) = n-1, \quad 1 \leq i \leq n.$$

Induced weight function  $\sigma : V(C_n \otimes P_3) \rightarrow Z_{4n}$  is defined as follows:

$$\sigma(u_{i,1}) = i+1, \quad 1 \leq i \leq n.$$

$$\sigma(u_{i,2}) = 2n+i+1, \quad 1 \leq i \leq n.$$

$$\sigma(u_{i,3}) = n+i+1, \quad 1 \leq i \leq n.$$

$$\sigma(v_i) = 3n+i+1, \quad 1 \leq i \leq n-2.$$

$$\sigma(v_n) = 4n+1 \equiv 1 \pmod{4n}.$$

$$\sigma(v_{n-1}) = 4n \equiv 0 \pmod{4n}.$$

Under the labeling  $\varphi$ , the weights of the vertices of  $C_n \otimes P_3$  are  $2, 3, 4, \dots, 4n+1$ , and their corresponding modular weights are  $0, 1, 2, \dots, 4n-1$ . Clearly,  $\varphi$  is a modular irregular labeling of  $C_n \otimes P_3$  and hence  $ms(C_n \otimes P_3) \leq n+1$ . Since, there are  $2n$  vertices of  $C_n \otimes P_3$  having degree 2,  $ms(C_n \otimes P_3) \geq n+1$ . Hence the proof.

### 3. CONCLUSION

In this paper, we have evaluated the precise value of the modular strength of some graphs, viz., corona product of cycles with  $P_1$ , corona product of cycles with  $P_2$  and corona product of cycles with  $P_3$ . As further research, we will investigate a modular strength of corona product of cycles with  $P_4$  and  $P_5$  (in progress). Moreover, we give an open problem for further research to generalize the above theorems.

**Open Problem:** What is the modular strength of  $C_n \otimes P_n$ ?

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## SOME PROPERTIES OF LOCAL FRACTIONAL MELLIN - FRACTIONAL DOUBLE LAPLACE TRANSFORM

**R.V. Kene**

Assistant Professor, Rajarshee Shahu Science College, Chandur Rly, Dist. Amravati (Maharashtra)  
E-mail : rekha.korpe@rssc.edu.in

### ABSTRACT

*In this article we introduce definition of Local Fractional, Mellin-Fractional Double Laplace Transform of real order  $\alpha$ ,  $0 < \alpha \leq 1$  Some main properties for Local Fractional Mellin-Fractional Double Laplace Transform are established. Further convolution theorem for Local Fractional Mellin-Fractional Double Laplace Transform have also been presented.*

*Keywords: local Fractional Mellin -Fractional Double laplace transform, local fractional derivatives, Fractional calculus.*

### 1 INTRODUCTION

There are various integral transforms in the literature which are used in astronomy, physics and also in engineering. The integral transforms were vastly applied to obtain the solution of differential equations; therefore there are different kinds of integral transforms like Mellin, Laplace, Fourier and so on.

In the recent years, Mellin transform is found to be very useful in signal processing as a tool to investigate scale invariance. Altes[1] had applied Mellin transform for mammalian hearing. V.D.Sharma and P.B.Deshmukh [2] has explained operational Transform Formulae for two dimensional fractional Mellin Transform. Hence number of extensions of Mellin transform was studied by different mathematicians. Mellin Transform acts as a basic tool to analyze the behavior of many important functions in mathematical physics mainly in electronics and has many applications as in quantum calculus, electromagnetic stress distribution, signal processing, optics, pattern recognition, cryptographic scheme, navigator, radar etc. The transform is also applied in solving fractional differential equations.

Fractional calculus is a generalization of the classical calculation and it has been used successfully in various fields of science and engineering. In fact, there are new opportunities in mathematics and theoretical physics appear, when order differential operator or operator becomes an integral arbitrary parameter. The fractional calculus is a powerful tool for the physical description systems that have long term memory and long term spatial interactions see Podlubny (1999), Miller and Ross (1993), Hilfer (2000), Kilbas et al.(2006).

There are different types of fractional derivatives in the current literature. One of the new fractional derivatives that was recently proposed is called Caputo-Fabrizio derivative see Caputo and Fbrizio (2015). However in our study ,Riemann-Liouville and Caputo derivatives have been used.

Local fractional calculus [3 – 5,7,8,14 – 17,19,20] is a generalization of differentiation and integration of the function defined fractal sets and it has been used successfully in various fields of science and engineering. There are many definitions of local fractional derivatives and local fractional integral (also called fractal calculus) [3 – 10,14 – 17,19 – 24]. Here we write down Gao-Yang-Kang local fractional derivative by [14 – 17,19 – 23].

**Notations:**

Through this article, we denote Local Fractional Mellin-Fractional Double Laplace transform by using the symbol  $M_{\alpha}L_{\beta}^2\{f(x, y, t)\} = F_{\alpha,\beta}^2[s, p, q]$

Here we will gather some information about fractional derivative and notation, for more specifics we refer to [10 – 12,17,23].

The purpose of this paper is to establish Local Fractional Mellin-Fractional Double Laplace Transform based on fractional calculus.

This paper is organized as follows.

The second section is devoted to definition of Local Fractional Mellin-Fractional Double Laplace transform.

In the third section some properties of Local Fractional Mellin-Fractional Double Laplace transform are established.

Convolution theorem is presented in Fourth section.

**2. LOCAL FRACTIONAL MELLIN- FRACTIONAL DOUBLE LAPLACE TRANSFORM**

**Definition:** If  $f(x, y, t)$  is a function where  $x, y, t > 0$  then Local Fractional Mellin-Fractional Double Laplace transform of  $f(x, y, t)$  is defined as

$$M_{\alpha}L_{\beta}^2\{f(x, y, t)\} = F_{\alpha,\beta}^2[s, p, q] = \frac{1}{\Gamma(1+\alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} x^{\alpha(s-1)} E_{\beta}(-(py + qt)^{\beta}) f(x, y, t) (dx)^{\alpha} (dy)^{\beta} (dt)^{\beta} - \text{-----(1)}$$

Where  $s, p, q \in \mathbb{C}$  and  $E_{\beta}(y)$  is the Mittag- Leffler function.

By using the Mittag-Leffler property then we can rewrite the formula (1) as the following

$$M_{\alpha}L_{\beta}^2\{f(x, y, t)\} = F_{\alpha,\beta}^2[s, p, q] = \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} x^{\alpha(s-1)} E_{\beta}(-(py)^{\beta}) E_{\beta}(-(qt)^{\beta}) f(x, y, t) (dx)^{\alpha} (dy)^{\beta} (dt)^{\beta}$$

### 3. SOME PROPERTIES OF LOCAL FRACTIONAL MELLIN-FRACTIONAL DOUBLE LAPLACE TRANSFORM

In this section ,various properties of Local Fractional Mellin-Fractional Double Laplace Transform are discussed and proved-

#### I Linearity Property:

Let  $f_1(x, y, t)$  and  $f_2(x, y, t)$  be functions of the variables  $x, y, t$  then

$$M_\alpha L_\beta^2 [a_1 f_1(x, y, t) + a_2 f_2(x, y, t) ] = a_1 M_\alpha L_\beta^2 + a_2 M_\alpha L_\beta^2$$

Where  $a_1$  and  $a_2$  are constants.

**Proof:** We can simply get the proof by applying the definition (1)

$$\begin{aligned} &M_\alpha L_\beta^2 [a_1 f_1(x, y, t) + a_2 f_2(x, y, t) ] \\ &= \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty \int_0^\infty \int_0^\infty x^{\alpha(s-1)} E_\beta(-py + qt)^\beta [a_1 f_1(x, y, t) + a_2 f_2(x, y, t) ] (dx)^\alpha (dy)^\beta (dt)^\beta \\ &= \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty \int_0^\infty \int_0^\infty x^{\alpha(s-1)} E_\beta(-(py + qt)^\beta) a_1 f_1(x, y, t) (dx)^\alpha (dy)^\beta (dt)^\beta \\ &\quad + \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty \int_0^\infty \int_0^\infty x^{\alpha(s-1)} E_\beta(-(py + qt)^\beta) a_2 f_2(x, y, t) (dx)^\alpha (dy)^\beta (dt)^\beta \\ &= a_1 \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty \int_0^\infty \int_0^\infty x^{\alpha(s-1)} E_\beta(-(py + qt)^\beta) f_1(x, y, t) (dx)^\alpha (dy)^\beta (dt)^\beta \\ &\quad + a_2 \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty \int_0^\infty \int_0^\infty x^{\alpha(s-1)} E_\beta(-(py + qt)^\beta) f_2(x, y, t) (dx)^\alpha (dy)^\beta (dt)^\beta \\ &= a_1 M_\alpha L_\beta^2 + a_2 M_\alpha L_\beta^2 \end{aligned}$$

#### II Scaling Property:

If  $M_\alpha L_\beta^2 \{f(x, y, t)\} = F_{\alpha,\beta}^2 [s, p, q]$  then

$$M_\alpha L_\beta^2 \{f(ax, by, ct)\} = \frac{1}{a^{\alpha s}} \frac{1}{b^\beta} \frac{1}{c^\beta} F_{\alpha,\beta}^2 \left[ s, \frac{p}{b}, \frac{q}{c} \right]$$

**Proof:**  $M_\alpha L_\beta^2 \{f(ax, by, ct)\} = \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \int_0^\infty \int_0^\infty x^{\alpha(s-1)} E_\beta(-(py + qt)^\beta) f(ax, by, ct) (dx)^\alpha (dy)^\beta (dt)^\beta$  -  
------(2)

By substituting  $ax = u, by = v, ct = w$  in (2) we obtain,

$$M_\alpha L_\beta^2 \{f(ax, by, ct)\} = \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty \int_0^\infty \int_0^\infty \left(\frac{u}{a}\right)^{\alpha(s-1)} E_\beta \left( - \left(\frac{pv}{b} + \frac{qw}{c}\right)^\beta \right) f(u, v, w) \left(\frac{du}{a}\right)^\alpha \left(\frac{dv}{b}\right)^\beta \left(\frac{dw}{c}\right)^\beta$$

$$\begin{aligned}
 &= \frac{1}{\alpha^{\alpha s}} \frac{1}{b^{\beta}} \frac{1}{c^{\beta}} \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} (u)^{\alpha(s-1)} E_{\beta} \left( - \left( \frac{pv}{b} + \frac{qw}{c} \right)^{\beta} \right) f(u, v, w) (du)^{\alpha} (dv)^{\beta} (dw)^{\beta} \\
 &= \frac{1}{\alpha^{\alpha s}} \frac{1}{b^{\beta}} \frac{1}{c^{\beta}} F_{\alpha, \beta}^2 \left[ s, \frac{p}{b}, \frac{q}{c} \right]
 \end{aligned}$$

**III Shifting Property:**

If  $M_{\alpha} L_{\beta}^2 \{f(x, y, t)\} = F_{\alpha, \beta}^2 [s, p, q]$  then  $M_{\alpha} L_{\beta}^2 \{x^{a\alpha} E_{\beta}(-by + ct)^{\beta} f(x, y, t)\} = F_{\alpha, \beta}^2 [s + a, p + b, q + c]$

**Proof:**

$$M_{\alpha} L_{\beta}^2 \{x^{a\alpha} E_{\beta}(-by + ct)^{\beta} f(x, y, t)\} = \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} x^{\alpha(s-1)} x^{a\alpha} E_{\beta}(-by + ct)^{\beta} E_{\beta}(-(py + qt)^{\beta}) f(x, y, t) (dx)^{\alpha} (dy)^{\beta} (dt)^{\beta}$$

By using the equality  $E_{\beta}(\lambda(y + t)^{\beta}) = E_{\beta}(\lambda y^{\beta}) E_{\beta}(\lambda t^{\beta})$  which implies that

$$\begin{aligned}
 &M_{\alpha} L_{\beta}^2 \{x^{a\alpha} E_{\beta}(-by + ct)^{\beta} f(x, y, t)\} \\
 &= \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} x^{\alpha(s+a-1)} E_{\beta}(-(b + p)y + (c + q)t)^{\beta} f(x, y, t) (dx)^{\alpha} (dy)^{\beta} (dt)^{\beta}
 \end{aligned}$$

Hence

$$M_{\alpha} L_{\beta}^2 \{x^{a\alpha} E_{\beta}(-by + ct)^{\beta} f(x, y, t)\} = F_{\alpha, \beta}^2 [s + a, p + b, q + c]$$

**III Multiplication by  $x^{\alpha} y^{\beta} t^{\beta}$**

Let  $M_{\alpha} L_{\beta}^2 \{f(x, y, t)\} = F_{\alpha, \beta}^2 [s, p, q]$

$$= \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} x^{\alpha(s-1)} E_{\beta}(-(py)^{\beta}) E_{\beta}(-(qt)^{\beta}) f(x, y, t) (dx)^{\alpha} (dy)^{\beta} (dt)^{\beta}$$

then

$$M_{\alpha} L_{\beta}^2 \{x^{\alpha} y^{\beta} t^{\beta} f^{(\alpha)}(x, y, t)\} = \frac{-\Gamma(1 + s\alpha)}{\Gamma(1 + (s - 1)\alpha)} \frac{\partial^{2\beta}}{\partial p^{\beta} \partial q^{\beta}} M_{\alpha} L_{\beta}^2 \{f(x, y, t)\}$$

Provided  $x^{\alpha} f(x)$  vanishes at  $x=0$  and as  $x \rightarrow \infty$

**Proof:**

$$\begin{aligned}
 &M_{\alpha} L_{\beta}^2 \{x^{\alpha} y^{\beta} t^{\beta} f^{(\alpha)}(x, y, t)\} \\
 &= \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} x^{\alpha} x^{\alpha(s-1)} y^{\beta} E_{\beta}(-(py)^{\beta}) t^{\beta} E_{\beta}(-(qt)^{\beta}) f^{\alpha}(x, y, t) (dx)^{\alpha} (dy)^{\beta} (dt)^{\beta}
 \end{aligned}$$

By using the fact  $x^{\alpha} f(x)$  vanishes at  $x=0$  and as  $x \rightarrow \infty$  and  $D_p^{\beta} (E_{\beta}(-p^{\beta} y^{\beta})) = -y^{\beta} E_{\beta}(-p^{\beta} y^{\beta})$

then

$$\begin{aligned}
 & M_{\alpha}L_{\beta}^2\{x^{\alpha}y^{\beta}t^{\beta}f^{(\alpha)}(x,y,t)\} \\
 &= \frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}x^{\alpha}x^{\alpha(s-1)}\frac{\partial^{\beta}}{\partial p^{\beta}}E_{\beta}(-(py)^{\beta})\frac{\partial^{\beta}}{\partial q^{\beta}}E_{\beta}(-(qt)^{\beta})f^{\alpha}(x,y,t)(dx)^{\alpha}(dy)^{\beta}(dt)^{\beta} \\
 &= x^{\alpha(s-1)}x^{\alpha}f(x)\Big|_0^{\infty} \\
 &\quad - \frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}(x^{\alpha}x^{\alpha(s-1)})^{\alpha}\frac{\partial^{\beta}}{\partial p^{\beta}}E_{\beta}(-(py)^{\beta})\frac{\partial^{\beta}}{\partial q^{\beta}}E_{\beta}(-(qt)^{\beta})f(x,y,t)(dx)^{\alpha}(dy)^{\beta}(dt)^{\beta} \\
 &= -\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}(x^{\alpha s})^{\alpha}\frac{\partial^{2\beta}}{\partial p^{\beta}\partial q^{\beta}}E_{\beta}(-(py)^{\beta})E_{\beta}(-(qt)^{\beta})f(x,y,t)(dx)^{\alpha}(dy)^{\beta}(dt)^{\beta} \\
 &= -\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-1)\alpha)}\frac{1}{\Gamma(1+\alpha)}\frac{\partial^{2\beta}}{\partial p^{\beta}\partial q^{\beta}}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}x^{\alpha(s-1)}E_{\beta}(-(py)^{\beta})E_{\beta}(-(qt)^{\beta})f(x,y,t)(dx)^{\alpha}(dy)^{\beta}(dt)^{\beta} \\
 &= -\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s-1)\alpha)}\frac{\partial^{2\beta}}{\partial p^{\beta}\partial q^{\beta}}M_{\alpha}L_{\beta}^2\{f(x,y,t)\}
 \end{aligned}$$

**IV Local Fractional Mellin-Fractional Double Laplace Transform of Derivatives:**

Firstly we start by Local Fractional Mellin-Fractional Double Laplace Transform of fractional derivative respect to  $x, t$  as follows

If  $M_{\alpha}L_{\beta}^2\{f(x,y,t)\} = F_{\alpha,\beta}^2[s,p,q]$  then  $M_{\alpha}L_{\beta}^2\{f^{\alpha,\beta}(x,y,t)\} = -\frac{\Gamma(1+(s-1)\alpha)}{\Gamma(1+(s-2)\alpha)}[q^{\beta}F_{\alpha,\beta}^2(s-1,p,q) - \Gamma(1+\alpha)F_{\alpha,\beta}^2(s-1,p,0)]$

**Proof:**

$$\begin{aligned}
 M_{\alpha}L_{\beta}^2\{f^{\alpha,\beta}(x,y,t)\} &= \frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}x^{\alpha(s-1)}E_{\beta}(-p^{\beta}y^{\beta})E_{\beta}(-q^{\beta}t^{\beta})\frac{\partial^{\beta}}{\partial t^{\beta}}f^{\alpha}(x,y,t)(dx)^{\alpha}(dy)^{\beta}(dt)^{\beta} \\
 &= x^{\alpha(s-1)}f(x)\Big|_0^{\infty} - \frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\int_0^{\infty}(x^{\alpha(s-1)})^{\alpha}E_{\beta}(-p^{\beta}y^{\beta})\int_0^{\infty}\left[E_{\beta}(-q^{\beta}t^{\beta})\frac{\partial^{\beta}}{\partial t^{\beta}}f(x,y,t)(dt)^{\beta}\right](dy)^{\beta}(dx)^{\alpha} \\
 &= -\frac{\Gamma(1+(s-1)\alpha)}{\Gamma(1+(s-2)\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}x^{\alpha(s-2)}\left\{\int_0^{\infty}E_{\beta}(-p^{\beta}y^{\beta})\left[\Gamma(1+\alpha)E_{\beta}(-q^{\beta}t^{\beta})f(x,y,t)\Big|_{t=0}^{\infty} - \int_0^{\infty}\left(\frac{\partial^{\beta}}{\partial t^{\beta}}E_{\beta}(-q^{\beta}t^{\beta})\right)f(x,y,t)(dt)^{\beta}\right](dy)^{\beta}\right\}(dx)^{\alpha} \\
 &= \\
 &= -\frac{\Gamma(1+(s-1)\alpha)}{\Gamma(1+(s-2)\alpha)}\left\{\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}x^{\alpha(s-2)}\left[-\Gamma(1+\alpha)\int_0^{\infty}E_{\beta}(-p^{\beta}y^{\beta})f(x,y,0)(dy)^{\beta}\right](dx)^{\alpha} + \right. \\
 &\quad \left. q^{\beta}\frac{1}{\Gamma(1+\alpha)}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}x^{\alpha(s-2)}E_{\beta}(-p^{\beta}y^{\beta})E_{\beta}(-q^{\beta}t^{\beta})f(x,y,t)(dx)^{\alpha}(dy)^{\beta}(dt)^{\beta}\right\} \\
 &= -\frac{\Gamma(1+(s-1)\alpha)}{\Gamma(1+(s-2)\alpha)}[q^{\beta}F_{\alpha,\beta}^2(s-1,p,q) - \Gamma(1+\alpha)F_{\alpha,\beta}^2(s-1,p,0)]
 \end{aligned}$$

### 4. CONVOLUTION THEOREM FOR LOCAL FRACTIONAL MELLIN-FRACTIONAL DOUBLE LAPLACE TRANSFORM

**Statement:** The convolution of order  $\alpha, \beta$  of functions  $f(x, y, t)$  and  $g(x, y, t)$  can be defined as the expression

$$[f(x, y, t) **_{\beta} g(x, y, t)] = \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} f(\xi, t - \eta, y - \theta) g\left(\frac{x}{\xi}, \eta, \theta\right) (d\xi)^{\alpha} (d\eta)^{\beta} (d\theta)^{\beta}$$

Therefore one has the equality

$$M_{\alpha} L_{\beta}^2 \{ (f **_{\beta} g)(x, y, t) \} = M_{\alpha} L_{\beta}^2 \{ f(x, y, t) \} \cdot M_{\alpha} L_{\beta}^2 \{ g(x, y, t) \} = F_{\alpha, \beta}^2(s, p, q) \cdot G_{\alpha, \beta}^2(s, p, q)$$

**Proof:** By the definition of Local Fractional Mellin- Fractional Double Laplace transform and fractional double convolution and  $\frac{x}{\xi} = \varphi$ , we obtain

$$\begin{aligned} &M_{\alpha} L_{\beta}^2 \{ (f **_{\beta} g)(x, y, t) \} \\ &= M_{\alpha} \left\{ \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} E_{\beta}(-pt)^{\beta} E_{\beta}(-qy)^{\beta} (f **_{\beta} g)(x, y, t) \frac{1}{\xi^{\alpha}} (dt)^{\beta} (dy)^{\beta} \right\} (d\xi)^{\alpha} \\ &= \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} x^{\alpha(s-1)} \left\{ \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} E_{\beta}(-pt)^{\beta} E_{\beta}(-qy)^{\beta} \left[ \int_0^t \int_0^y f(\xi, t - \eta, y - \theta) g\left(\frac{x}{\xi}, \eta, \theta\right) \frac{1}{\xi^{\alpha}} (d\xi)^{\alpha} (d\eta)^{\beta} (d\theta)^{\beta} \right] (dx)^{\alpha} (dt)^{\beta} (dy)^{\beta} \right\} \end{aligned}$$

Let  $u = t - \eta, v = y - \theta$  and taking limit from 0 to  $\infty$  gives

$$\begin{aligned} &= \frac{1}{\Gamma(1 + \alpha)} \left\{ \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} x^{\alpha(s-1)} [E_{\beta}(-p^{\beta}(u + \eta)^{\beta}) E_{\beta}(-q^{\beta}(v + \theta)^{\beta})] \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} f(\xi, u, v) g\left(\frac{x}{\xi}, \eta, \theta\right) \frac{1}{\xi^{\alpha}} (d\xi)^{\alpha} (d\eta)^{\beta} (d\theta)^{\beta} \right\} (dx)^{\alpha} (du)^{\beta} (dv)^{\beta} \\ &= \left\{ \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} E_{\beta}(-p^{\beta}u^{\beta}) E_{\beta}(-q^{\beta}v^{\beta}) f(\xi, u, v) \frac{1}{\xi^{\alpha}} (d\xi)^{\alpha} (du)^{\beta} (dv)^{\beta} \right\} \\ &\quad \left\{ \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} E_{\beta}(-p^{\beta}\eta^{\beta}) E_{\beta}(-q^{\beta}\theta^{\beta}) (\xi\varphi)^{\alpha(s-1)} g(\varphi, \eta, \theta) (d\xi\varphi)^{\alpha} (d\eta)^{\beta} (d\theta)^{\beta} \right\} \\ &= \left\{ \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} E_{\beta}(-p^{\beta}u^{\beta}) E_{\beta}(-q^{\beta}v^{\beta}) f(\xi, u, v) \frac{1}{\xi^{\alpha}} (d\xi)^{\alpha} (du)^{\beta} (dv)^{\beta} \right\} \\ &\quad \left\{ \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} (\xi)^{\alpha(s-1)} (\xi)^{\alpha} (\varphi)^{\alpha(s-1)} g(\varphi, \eta, \theta) (d\varphi)^{\alpha} (d\eta)^{\beta} (d\theta)^{\beta} \right\} \\ &= \left\{ \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \frac{1}{\xi^{\alpha}} (\xi)^{\alpha(s-1)} (\xi)^{\alpha} E_{\beta}(-p^{\beta}u^{\beta}) E_{\beta}(-q^{\beta}v^{\beta}) f(\xi, u, v) (d\xi)^{\alpha} (du)^{\beta} (dv)^{\beta} \right\} \\ &\quad \left\{ \frac{1}{\Gamma(1 + \alpha)} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} (\varphi)^{\alpha(s-1)} g(\varphi, \eta, \theta) (d\varphi)^{\alpha} (d\eta)^{\beta} (d\theta)^{\beta} \right\} \end{aligned}$$



$$\begin{aligned}
&= \left\{ \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \int_0^\infty \int_0^\infty (\xi)^{\alpha(s-1)} E_\beta(-p^\beta u^\beta) E_\beta(-q^\beta v^\beta) f(\xi, u, v) (d\xi)^\alpha (du)^\beta (dv)^\beta \right\} \\
&\left\{ \frac{1}{\Gamma(1+\alpha)} \int_0^\infty \int_0^\infty \int_0^\infty (\varphi)^{\alpha(s-1)} g(\varphi, \eta, \theta) (d\varphi)^\alpha (d\eta)^\beta (d\theta)^\beta \right\} \\
&= F_{\alpha,\beta}^2(s, p, q) \cdot G_{\alpha,\beta}^2(s, p, q)
\end{aligned}$$

## CONCLUSION

In the present work, Local Fractional Mellin-Fractional Double Laplace Transform is defined and several properties of Local Fractional Mellin-Fractional Double Laplace Transform have been discussed. Moreover convolution theorem is presented.

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## (3, 2) - JECTION OPERATOR CRITERION IN HILBERT SPACE

**Navin Kumar Singh**

V.K.S., University, Ara, Bihar (India)  
E-mail : singhnavinkumar839@gmail.com

### ABSTRACT :

*In the present paper, we introduce the general form of (3, 2)-jection operator in Hilbert space and the general form of (3, 2) - jection operator in Hilbert space when it is projection.*

*Key Words : Projection ,Inner product, Adjoint operator, Hilbert space.*

### 1. INTRODUCTION

At first some of the conventions and terminology that will be adhered to throughout this paper will be set down. They are listed only so that there can be no ambiguity when terms such as Projection, Adjoint of an operator, Inner Product, Hilbert space are used later. After establishing our rules, certain basic notions pertaining to (3, 2) - jection operator will be touched upon.

The general form of (3, 2) - jection operator is very important, with the help of which a (3, 2) - jection operator in Hilbert space can be identified and it also helps to prove whether a given operator is (3, 2) - jection operator or not.

It will also be shown that the general form of (3, 2) - jection operator on  $\mathbb{R}^2$  when it is projection.

### 2. SOME PREREQUISITE MATERIAL AND CONVENTIONS

In this section, we shall deal briefly with the following definitions :

**2.1 Inner Product :** Let  $L$  be a linear space over the field  $F$  ( $= \mathbb{R}$  or  $\mathbb{C}$ ). An inner product on  $L$  is a function  $(\dots)$  from  $L \times L$  into  $F$ , which assigns to each ordered pair of vector  $x, y$  in  $L$  a scalar  $(x, y)$  in  $F$ , satisfying the following condition.

$$(2.01) \quad (x, y) = (y, x)$$

$$(2.02) \quad (\alpha x, y) = \alpha (x, y) \text{ for all } x, y \in L \text{ and } \alpha \in F$$

$$(2.03) \quad (x+y, z) = (x, z) + (y, z) \text{ for all } x, y, z \in L.$$

$$(2.04) \quad (x, x) \geq 0 \text{ and } (x, x) = 0 \Leftrightarrow x = 0.$$

**2.2 Hilbert Space :** A complete inner product space with the norm induced by the inner product, is a Hilbert space.

**2.3 Adjoint of an operator on a Hilbert space  $H$  :** Let  $T$  be an operator on a Hilbert space  $H$ . Then there exists a unique operator  $T^*$  on  $H$  corresponding to  $T$  on  $H$  such that  $(Tx, y) = (x, T^*y)$  for all  $x, y \in H$ . Then the operator  $T^*$  is called the adjoint of  $T$ .

**2.4 (3, 2) – jection operator :** In linear algebra and functional analysis a projection is a linear operator  $E$  on a vector space such that  $E^2 = E$ . That is, whenever  $E$  is applied twice to any element of a linear space (or vector space), it gives that same result as if it were applied once.

As stated above, projection are a special case of idempotent. On the basis of above definition of projection we develop a new operator called a (3, 2) - jection operator which is generalization of projection defined as a linear operator on a linear space such that  $E^3 = E^2$ . This definition of (3, 2) - jection operator can be carried over verbaton to Hilbert space with an additional condition that  $E^* = E$ .

Hence formal definition of (3, 2) - jection operator in Hilbert space is as follows :

A (3, 2) - jection operator in a Hilbert space  $H$  is defined as an operator  $E$  on  $H$  such that  $E^3 = E^2$  and  $E^* = E$ .

### 3. MAIN RESULTS

In this section, we shall provide the general form of (3, 2) - jection operator on a Hilbert space and the general form of (3, 2) - jection operator which is a projection also, with the following theorems :

**Theorem (3.1) :** Let  $E$  be a (3, 2) - jection operator on a Hilbert space  $R^2$  then the general form of  $E$  is given by

$$E(x, y) = (ax + by, bx + dy) \text{ where } (x, y) \in R^2 \& a, b, d \in R.$$

**Proof :** Let  $z = (x, y) \in R^2$   
 $z' = (x', y') \in R^2$

For  $E$  to be a (3, 2) - jection operator, we must have

$$E = E^* \tag{3.01}$$

We have

$$\begin{aligned} (Ez, z') &= (z, E^* z') \\ &= (z, Ez') \text{ \{by (3.01)\}} \end{aligned}$$

$$\text{Thus, } (Ez, z') = (z, Ez') \tag{3.02}$$

$$\text{Let } E(x, y) = (ax + by, cx + dy) \text{ where } a, b, c, d \in R \tag{3.03}$$

Then,

$$\begin{aligned} (Ez, z') &= ((ax + by, cx + dy), (x', y')) \\ &= (ax + by)x' + (cx + dy)y' \\ &= axx' + byx' + cxy' + dyy' \end{aligned} \tag{3.04}$$

and,

$$\begin{aligned} (z, Ez') &= ((x, y), (ax' + by', cx' + dy')) \\ &= x(ax' + by') + y(cx' + dy') \\ &= axx' + bxy' + cyx' + dyy' \end{aligned} \tag{3.05}$$

Using (3.04) and (3.05) in (3.02), we get

$$\begin{aligned} axx' + byx' + cxy' + dyy' &= axx' + bxy' + cyx' + dyy' \\ byx' + cxy' &= bxy' + cyx' \end{aligned}$$

$$\Rightarrow (b - c)yx' - (b - c)xy' = 0$$

$$\Rightarrow (b - c)(yx' - xy') = 0$$

$$\Rightarrow b - c = 0 \text{ (As } z \text{ and } z' \text{ are distinct)}$$

$$\Rightarrow b = c$$

Substituting  $b$  for  $c$  in (3.03); we get

$$E(x, y) = (ax + by, bx + dy).$$

**Theorem (3.2) :** A  $(3, 2)$  - jection operator  $E$  on  $\mathbb{R}^2$  is given

$$E(x, y) = (ax + \sqrt{a-a^2}y, \sqrt{a-a^2}x + (1-a)y)$$

if  $E$  is projection.

**Proof :** We have already seen in Theorem (3.1) that a  $(3, 2)$  - jection operator  $E$  on  $\mathbb{R}^2$  is given by

$$E(x, y) = (ax + by, bx + dy) \quad \dots (3.001)$$

Now,

$$\begin{aligned} E^2(x, y) &= E(E(x, y)) \\ &= E((ax + by, bx + dy)) \\ &= (a(ax + by) + b(bx + dy), b(ax + by) + d(bx + dy)) \\ &= ((a^2 + b^2)x + b(a+d)y, b(a+d)x + (b^2 + d^2)y) \quad \dots (3.002) \end{aligned}$$

Let  $E$  be a projection.

Then  $E^2 = E$ , so by using (3.001) and (3.002), we get

$$\Rightarrow a^2 + b^2 = a, b(a+d) = b, b^2 + d^2 = d$$

**Case 3.2.0.1 :**

When  $b = 0$ ; we have

$$a^2 = a \text{ and } d^2 = d$$

$$\Rightarrow a = 0, 1 \text{ and } d = 0, 1$$

**Subcase 3.2.0.1.1 :**

When  $a = 0, b = 0, d = 0$ , we get

$$E(x, y) = (0, 0)$$

$$\Rightarrow E = 0 \text{ which is trivial case.}$$

**Subcase 3.2.0.1.2 :**

When  $a = 0, b = 0, d = 1$ , we get

$$E(x, y) = (0, y)$$

**Subcase 3.2.0.1.3 :**

When  $a = 1, b = 0, d = 0$ , we get

$$E(x, y) = (x, 0)$$

**Subcase 3.2.0.1.4 :**

When  $a = 1, b = 0, d = 1$ , we get

$$E(x, y) = (x, y)$$

**Case 3.2.0.2 :**

When  $b \neq 0$ ; we have

$$\Rightarrow b(a + d) = b$$

$$\Rightarrow a + d = 1$$

$$\begin{aligned} \Rightarrow d &= 1 - a \\ \text{or } d &= 1 - d \\ \text{Also, } a^2 + b^2 &= a \\ \Rightarrow b^2 &= a - a^2 \\ \text{And } b^2 + d^2 &= d \\ \Rightarrow b^2 &= d - d^2 \\ &= d(1 - d) \\ &= (1 - a)a \text{ \{since } d = 1 - a\} \end{aligned}$$

So,  $b \neq 0$ ,  $a \neq 0$  and  $a \neq 1$

Hence, In general, we get

$$E(x, y) = (ax + \sqrt{a-a^2}y, \sqrt{a-a^2}x + (1-a)y) \text{ which complete the proof.}$$

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## AUTOMATIC PREDICTION AND CROSS VALIDATION OF TEXTILE FABRICS DATA USING MACHINE LEARNING METHODS

**R. Lakshmi Priya\***, **Manimannan G\*\***, **M. Salomi\*\*\***

\*Assistant Professor, Department of Statistics, Dr. Ambedkar Govt. Arts College, Chennai

\*\*Assistant Professor, Department of Mathematics, TMG College of Arts and Science, Chennai

\*\*\*Assistant Professor, Department of Statistics, Madras Christian College, Chennai

E-mail: priyagayu2006@gmail.com, manimannang@gmail.com and salomim.phill@gmail.com

### ABSTRACT

*The principles of machine learning methods have been used for classification and prediction of 105 sample fabrics. The secondary database was collected from Textile Department, Chennai. Physical properties of fabrics were measured using the well known Kawabata Instrument. Fabric properties such as Tensile Energy, Tensile Resilience, Bending Properties, Shear properties, Compression Properties and Surface Properties have been measured on fabric samples and used in the present study. Initially, different methods of classification have been constructed for three known categories of fabrics made up of Polyester, Lyocell / Viscose and Treated Polyester. The classification yielded cent percent prediction and accuracy. The application of SVM, RF, kNN and Logistic Regression test score, prediction and confusion matrix validated the grouping in case of individual category of fabrics as well as combined group. The categorization of fabrics is very essential for textile industry and other fields also.*

*Keywords: Textile Fabrics, Classification, Support Vector Machine, Random Forest Classification, Logistic Regression and k-NN*

### 1. INTRODUCTION

A textile industry uses fibers, yarns and fabrics for testing their properties. Generally, for any end - use, the mechanical properties of fabrics are of special interest, and studies of how these properties are related to fabric and yarn properties could help in deciding how to produce a suitable fabric at a minimum cost when its likely range of use is known. Textile industry depends on the raw materials such as cotton, jute and silk fibers. Fibers obtained from natural resources are then sent to spinning mills for processing and yarns were produced. Yarn structure and properties are primarily influenced by fiber, spinning method and process variables. Grading of yarns is another important need in assessing textile quality for commercial purposes, since it provides a useful means of expressing yarn standards in the market. Traditionally, it is done by a comparison of measurements of various quality related parameters with their values as recommended in a standard, which is different for different countries. Yarns are used for producing fabrics. These fabrics can be classified according to weave structure, knitting method, base material used like cotton, jute, silk or synthetic material such as polyester, softness or roughness of the

material, usage of fabrics such as dress material. The categorization is very essential for textile industry. Textile technologists have introduced various methods for solving the above mentioned problems. Measurements of various properties were obtained using Kawabata instrument. Kawabata [1] and his co-worker's established the Kawabata Evaluation System for Fabrics (KES-F), which is used to measure the mechanical properties of fabrics. KES-F is shown to offer advantage over other instruments in routine measurements of the properties. Fabric is tested for tensile, bending, shearing, compression, surface properties and also for thickness and weight.

To classify a fabric based on measurements some scientific methodology is to be adopted. The relationship between yarn structure and tensile properties has been studied extensively and a number of statistical models have been developed. A number of publications of research papers show that fabric handle received wide attention from scientist and technologists from very early days. Fabric handle has become one of the most active areas of textile research during the past twenty years. Fabric 'hand' or 'handle' is defined as the quality of a fabric or yarn assessed by the reaction obtained from the sense of touch. KES-F system of assessment of hand value has been used by fiber, yarn and fabric producers all over the world. Many research workers have used multivariate statistical methods such as Cluster Analysis, Principal Component Analysis in the study of fabric hand.

In recent days, dress materials are available in various forms. The base materials are made up of cotton, jute, wool, furry and also synthetic varieties. The quality and type of fabric are determined by the properties of yarns used for producing fabrics. Textiles are now increasingly used for many purposes, from traditional uses like garments and furniture to others such as conveyor belts, technical textiles and aerospace applications. Probably there are two reasons for this wide range of utilization, the first being that textile materials covers a wide range of different physical and mechanical properties, and second is that textile technology is tending to become a science of well established rules and predictable results, so that fabrics can be designed for specific purposes.

The end-users of textiles decide the way they should be selected and constructed, suggesting that some properties should act in a certain way, while others remain of less importance. For example, outerwear fabrics need to satisfy some aesthetic, physical and thermal properties as well as being required to be durable. On the other hand, fabrics used for industrial purposes are made for strength and flexibility, and possibly for certain electrical and thermal properties aesthetics are relatively unimportant.

## **2. REVIEW OF LITERATURE**

Generally, for any end-use, the mechanical properties of fabrics are of special interest, and studies of how these properties are related to fabric and yarn properties could help in deciding how to produce a suitable fabric at a minimum cost when its likely range of use is known. The relationship between yarn structure and tensile properties has been studied extensively and a number of statistical models have been developed. Grading of yarns is another important need in assessing textile quality for commercial purposes, since it provides a useful means of expressing yarn standards in the market. Traditionally, it is



done by comparison of measurements of various quality related parameters with their values as recommended in a standard, which is different for different countries. Many Scholars developed different statistical models and they are listed in this section, Zeronian and Ryu [2] suggest statistical methods such as PCA, Variable Cluster Analysis, D-optimal method and colinearity tests for identifying the most important mechanical properties. Rong, Slater and Fei [3] have used cluster analysis method for grading of yarns in textile industry.

Yarns are woven to produce different types of fabrics. Fabrics may be of different types such as dress materials, shirting, and suiting materials etc. are made up of natural fibers or synthetic fibers such as, polyester/viscose. Each fabric will have some properties which can be measured by specified instruments. In the past, the quality of fabrics was determined by experience, that is, judgment by hand and specification of fiber types (wool, cotton, silk). With the rapidly increasing use of synthetic fibers, characterizing the fabric by experience is more and more difficult. Therefore, development of a scientific and practical method will facilitate fabric selection. The first research on measuring fabric properties can be traced back to the European airship programmes around the turn of century. Kawabata [1] and his co-workers established the Kawabata Evaluation System for Fabrics (KES-F), which is used to measure mechanical properties of fabrics. KES-F is shown to offer advantage over other instruments in the routine measurements of the properties. Fabric is tested for tensile, bending, shearing, compression, surface properties and also for thickness and weight. ANN techniques have been applied to textile industry [Ramesh, Rajamanickam and Jayaraman [4]. Pan, Pan, Yen, Zhao and Yang [5] have discussed classification of fabrics by hierarchical clustering methods. In the present study, an attempt is made to classification, AUC, CA, F1 Precision and Recall of 105 sample textile fabrics using various Machine learning methods

### 3. DATABASE

he three data sets are combined together which consists of 105 fabric samples with 16 KES-F parameters. In the present study, the combined data set as well as individual data sets is analyzed. All data sets were subjected to normality test and equality of variance test and the same ascertained. The data set measured using KES-F instrument is obtained from the Department of Textile Technology, Anna University, Chennai, India. The KES-F instrument is very costly and for testing one fabric sample it cost over thousand rupees, there by costing over a lakh of rupees for the whole data set. The following three data sets were considered for statistical analysis.

Data Set 1: comprises different types of Polyester Fabrics (Regular/Micro fibres).

Data Set 2: consists of 40 Lyocell/Viscose fabric samples with 16 KES-F parameters.

Data set 3: Chemically Treated Polyester fabric samples. Sixteen KES-F properties are measured for 38 fabric samples.

The present study deals with textile fabrics of mechanical properties and an attempt is made to classify and prediction of 105 sample textile fabrics using SVM, Random Forest and kNN. Logistic Regression is

constructed for classification and prediction of three known categories of fabrics made up of *Polyester*, *Lyocell/Viscose* and *Treated-Polyester*. The data matrix is in the form of  $n * m$ , let  $A = (a_1, a_2, a_3, \dots, a_n)$  be a set of  $n$  samples described along a set of  $m$  attributes  $X = (x_1, x_2, x_3, \dots, x_m)$ .

The present study is different from the problem of predicting mechanical properties of textile fabrics database in the sense that number of groups  $p$  is unknown in the database. The primary objective of this study is to grade the quality of fabrics on the basis of certain properties of textile data. The following KFS-F parameters are used in this study.

Parameters	Calculation method of Parameters
<b>1. Linearity of the Stress Strain Curve (LT)</b>	Linearity of the stress strain curve is defined by $\left(\frac{WT}{WL}\right) * 100$ where $WL$ is the tensile energy when the load extension curve is linear from zero strain to EMT and $WT$ is the curve obtained during recovery.
<b>2. Tensile Energy (WT):</b>	It will give an indication of tensile energy
<b>3. Tensile Resilience (RT)</b>	RT is measured by the load extension curve.
<b>4. Bending Rigidity (B):</b>	In the KES-F system, the fabric sample is bent to a maximum curvature of $0.25\text{mm}^{-1}$ , the bending rigidity is taken as the average slope of the part of the bending curve between $0.05\text{mm}^{-1}$ and $0.15\text{mm}^{-1}$ curvature.
<b>5. Hysteresis of Bending (2HB):</b>	This is coercive couple taken as a width of the bending hysteresis curve at curvature of $0.1\text{mm}^{-1}$ .
<b>6. Shear rigidity (G):</b>	This is a measure of shear rigidity of fabric which is directly related to the draping quality of the fabrics; these values show the tailoring and comfort aspect of the fabric in garment manufacture.
<b>7. Hysteresis of shear force (2HG):</b>	This is a measure of the hysteresis of the shear force on the shear hysteresis curve at 0.5 degree shear.
<b>8. Hysteresis of shear force (2HG5) :</b>	This is a measure of the hysteresis of the shear force on the shear hysteresis curve at 5 degree shear.
<b>9. Linearity of the Compression Curve (LC):</b>	It is linearity of compression curve.
<b>10. Work of Compression Curve (WC):</b>	A circular specimen of $2\text{ cm}^2$ area of a circle is compressed by two circular plates of steel with $2\text{ cm}^2$ area. The velocity of compression is $6.66\text{ micro/sec}$ and when the pressure attains $10\text{gf/cm}^2$ , the recovery process is measured by the same velocity.
<b>11. Compressional Resilience (RC):</b>	$RC = (WC^1/WC) * 100$ , Where, $WC^1 =$ Work of decompression ( $\text{J/m}^2$ ) $WC =$ Work of compression $RC = (\%)$ It shows the resilience of the material in compression. It is the ratio of work of decompression to work of compression expressed as percentage
<b>12. Coefficient of surface friction (MIU):</b>	It is a measure of the mean value of coefficient of friction between fabric surface and metallic piano model surface detector, whose surface is simulated to the finger surface.
<b>13. Mean deviation of coefficient of friction (MMD):</b>	This is a deviation of the coefficient of friction measured and has a significant effect on the handle of the fabric
<b>14. Surface Roughness (SMD):</b>	This is a measure of the mean deviation of surface roughness (mean deviation of thickness (micron)).
<b>15. Fabric Thickness (T):</b>	It is the thickness of the fabric in mm.
<b>16. Fabric Weight (W):</b>	It is the weight of the fabric in $\text{mg/cm}^2$ .

The .Main Objective of this research paper is to predict and Classify using Orange Data Mining Tools, like SVM, kNN, RF Model and Logistic Model. Also, to visualize the main categories of textile Parameter using all methods of orange Data mining tools.

## 4. METHODOLOGY

### 4.1 Proposed Algorithm

#### 4.1.1 Logistic Regression

*STEP 1:* TO OPEN THE TEXTILE DATABASE FILE AND CONNECT TO LOGISTIC REGRESSION WIDGET

*Step 2:* The logistic regression widget chooses from various machine learning method and connects to test score widget.

*Step 3:* Open logistic regression widget and select regularization type (Ridge L2 by default and widely used model).

*Step 4:* The logistic regression strength must always  $C=1$  is in the middle of the model and remaining two extreme points of left and right side of the line are labeled as weak and strong strength.

*Step 5:* Repeat the step 3 with various folds and  $C$  value, to get a better model

#### 4.1.2 Random Forest Model

*Step 1:* To open the textile database file and connect to RF Model widget.

*Step 2:* The Random Forest widget chooses from various machine learning method and will be connected to test score widget.

*Step 3:* Open Random Forest widget and select growth control model and number of trees are 10 from basic properties with number of attributes considered at each split at 5.

*Step 4:* Repeat step 3 with various size of trees and splits to get a better model. .

#### 4.1.3 Support Vector Machine

*Step 1:* To open the textile database file and connect to SVM widget.

*Step 2:* The Support Vector Machine widget chooses from various machine learning method and will be connected to test score widget.

*Step 3:* Open Support Vector Machine (SVM) type and select SVM cost type  $C = 100$ , Regression loss epsilon ( $\epsilon$ ) = 0,10 and choose the menu of optimization parameter set to 0.0010 with iteration limit 100

*Step 4:* Repeat step 2 with various cost, epsilon, optimization parameter and iteration limit, to get better model of SVM.

#### 4.1.4 k- Nearest Neighbor's Algorithm (k-NN)

*Step 1:* To open the textile database file and connect to kNN widget.

*Step 2:* The kNN widget chooses from various machine learning method and will be connected to test score widget.

*Step 3:* Open kNN type and select SVM cost type  $C = 1$ , Predicted classes and predicted probabilities set for given database to train and assign classification.

*Step 4:* Repeat step 2 with trained parameters to achieve hundred percent classifications and iterate on limit, to get better classification of kNN [8]

In this research chapter, the researcher uses orange data mining software. Orange is an open source machine learning and data visualization for learner as well as experts. Interactive data analysis work flows with a large toolbox available in this package. The software is developed with python script. Python is an interpreted high-level programming language for general-purpose programming with GUI and it was created by Guido van Rossum [6].

## 5. RESULT AND DISCUSSION

The data set is split into training and test cases randomly. The centroids of the clusters from training cases can be used to cluster test cases. The centroids of clusters formed by test data are then computed and compared with the training data. Comparable results validate the clustering that has achieved.

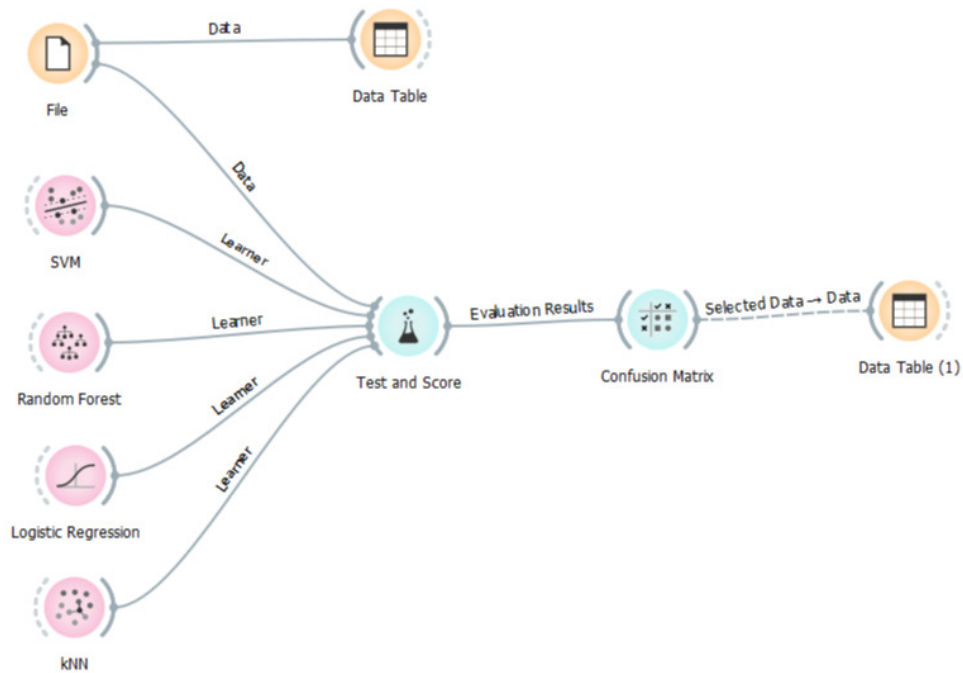
### 5.1 Classification and Prediction Workflow Model

*Step 1:* Initially, three sets of textile data given through file widget and connect through work flow with Test score widget with Logistic Regression (LR), Random Forest (RF) and Support Vector Machine (SVM) and kNN.

*Step 2:* The Test score widget assign training and testing data sets with 10 folds cross validation. The training data sets are 76 percent of the original database using random sampling method up to target class 24 percent.

*Step 3:* The data table widget connected to file widget for checking the original data.

*Step 4:* Repeat the train or test database to reach best model or repeat from step 1, change the training data and cross validation folds. Figure 1 represents the data mining work flow for various data mining Techniques, Confusion Matrix and Visualization of the database.

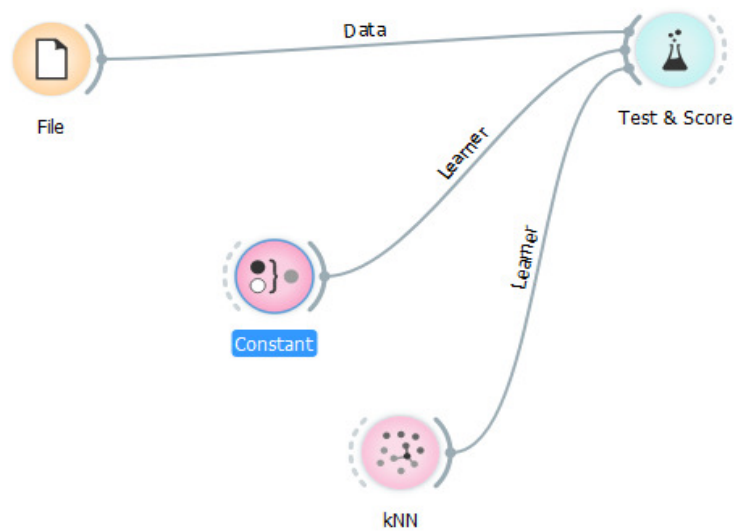


**Figure 1. Work Flow Diagram Models of classification, Cross Validation and Machine Learning Methods**

## 5.2 Model Design, Input and Outputs

### 5.2.1 k-Nearest Neighbors Algorithm (k-NN)

The given database is to predict according to the nearest training instances. The **kNN** widget uses kNN algorithm that searches for k closest training database in feature space and uses their average as prediction (Figure 2).



**Figure 2. Work Flow Diagram Models of k- Nearest Neighbors algorithm and Constant Method using Test Score**

The second workflow is a regression task. The workflow below shows how to use the *Learner* output. For the purpose of this database, the researcher used textile *fabrics* dataset. Input the kNN prediction model into prediction and observe the predicted values (Figure 3 and Table 1).

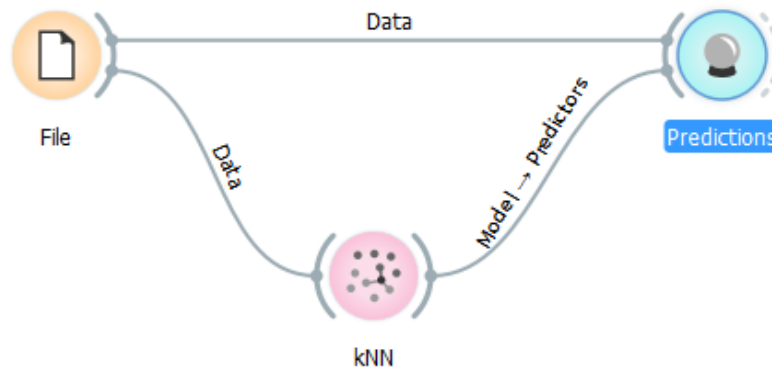


Figure 3. Work Flow Diagram Models of k- Nearest Neighbors Algorithm using Test Score

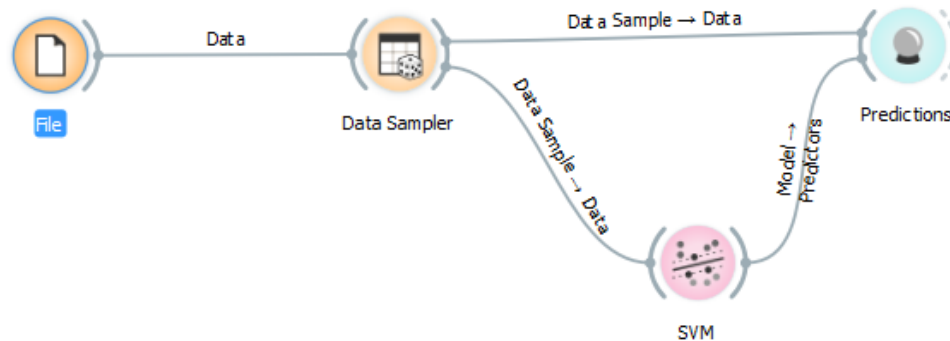
	kNN	WC	RC	T	
1	1.00 : 0.00 : 0.00 - C1	0.0340	71.30	0.390	6
2	1.00 : 0.00 : 0.00 - C1	0.0320	76.59	0.380	7
3	1.00 : 0.00 : 0.00 - C1	0.0290	73.23	0.370	7
4	0.60 : 0.00 : 0.40 - C1	0.0420	65.04	0.400	4
5	0.80 : 0.00 : 0.20 - C1	0.0390	65.33	0.390	4
6	0.80 : 0.00 : 0.20 - C1	0.0300	76.32	0.290	4
7	0.80 : 0.00 : 0.20 - C1	0.0170	90.14	0.240	6
8	0.80 : 0.00 : 0.20 - C1	0.0210	90.14	0.280	6
9	1.00 : 0.00 : 0.00 - C1	0.0180	91.96	0.260	7
10	0.80 : 0.00 : 0.20 - C1	0.0460	61.12	0.490	1
11	1.00 : 0.00 : 0.00 - C1	0.0480	58.53	0.500	1
12	1.00 : 0.00 : 0.00 - C1	0.0520	62.57	0.540	1
13	1.00 : 0.00 : 0.00 - C1	0.0880	62.93	0.580	1
14	1.00 : 0.00 : 0.00 - C1	0.1040	64.83	0.620	1
15	1.00 : 0.00 : 0.00 - C1	0.0750	70.52	0.540	1
16	1.00 : 0.00 : 0.00 - C1	0.0980	58.29	0.580	1
17	1.00 : 0.00 : 0.00 - C1	0.1010	59.74	0.620	1
18	1.00 : 0.00 : 0.00 - C1	0.0920	56.93	0.600	1
19	0.60 : 0.00 : 0.40 - C1	0.0940	63.51	0.570	1

Table 1. Classification Tables for Textile Data using k- Nearest Neighbors Algorithm

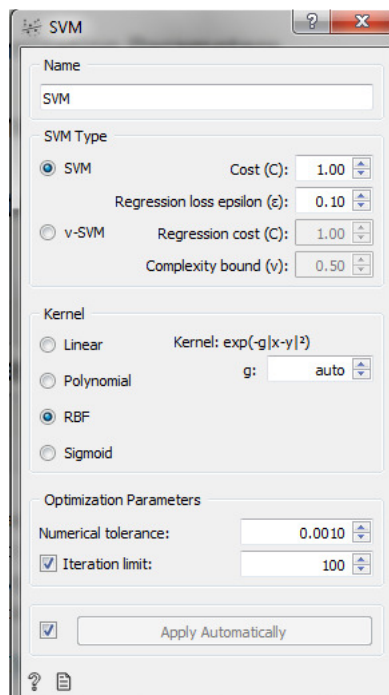
### 5.2.2 Support Vector Machine

Support Vector Machine tool inputs to higher-dimensional feature spaces. Support Vector Machine (SVM) is a machine learning technique that separates attribute space with a hyperplane, thus maximizing the margin between instances of different classes or class values. The technique often yields supreme predictive performance results. Orange data mining software pushed in a popular implementation of SVM from the LIBSVM package. This widget is its graphical user interface.

Regression analysis tasks, SVM perform linear regression in a high dimension database feature space using  $\epsilon$ -insensitive loss. SVM evaluation accuracy depends on a good setting of  $C$ ,  $\epsilon$  and kernel parameters. The SVM widget outputs class predictions based on SVM Regression. The SVM widget works for both classification and regression analysis (Figure 4 and Table 2)

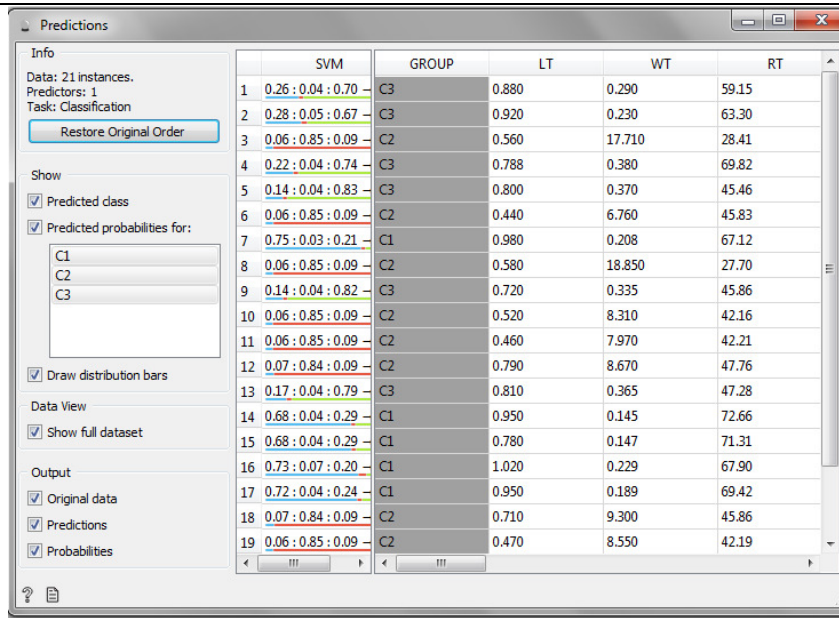


**Figure 4. Work Flow Diagram Models of Support Vector Machine using Predict Score**



**Table 2. Sample SVM train tables for textile data using SVM**

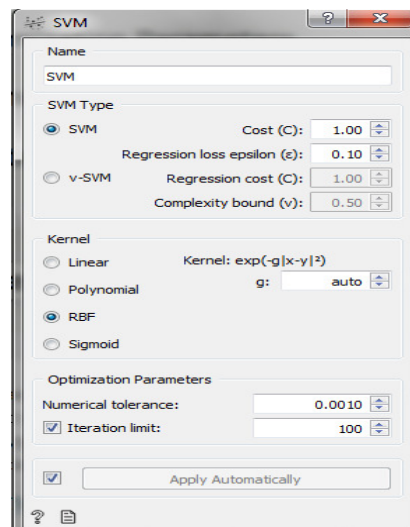
In the first (regression) example, *textile* dataset was used and split data into two data subsets (*Data Sample* and *Remaining Data*) with Data Sampler. The sample was sent to SVM which produced a *Model*, which was then used in Predictions to predict values in *Remaining Data*. A similar schema can be used if data is already in two separate files; in this case, two File widgets would be used instead of File - Data Sampler combination. (Table 3)



	SVM	GROUP	LT	WT	RT
1	0.26 : 0.04 : 0.70	C3	0.880	0.290	59.15
2	0.28 : 0.05 : 0.67	C3	0.920	0.230	63.30
3	0.06 : 0.85 : 0.09	C2	0.560	17.710	28.41
4	0.22 : 0.04 : 0.74	C3	0.788	0.380	69.82
5	0.14 : 0.04 : 0.83	C3	0.800	0.370	45.46
6	0.06 : 0.85 : 0.09	C2	0.440	6.760	45.83
7	0.75 : 0.03 : 0.21	C1	0.980	0.208	67.12
8	0.06 : 0.85 : 0.09	C2	0.580	18.850	27.70
9	0.14 : 0.04 : 0.82	C3	0.720	0.335	45.86
10	0.06 : 0.85 : 0.09	C2	0.520	8.310	42.16
11	0.06 : 0.85 : 0.09	C2	0.460	7.970	42.21
12	0.07 : 0.84 : 0.09	C2	0.790	8.670	47.76
13	0.17 : 0.04 : 0.79	C3	0.810	0.365	47.28
14	0.68 : 0.04 : 0.29	C1	0.950	0.145	72.66
15	0.68 : 0.04 : 0.29	C1	0.780	0.147	71.31
16	0.73 : 0.07 : 0.20	C1	1.020	0.229	67.90
17	0.72 : 0.04 : 0.24	C1	0.950	0.189	69.42
18	0.07 : 0.84 : 0.09	C2	0.710	9.300	45.86
19	0.06 : 0.85 : 0.09	C2	0.470	8.550	42.19

**Table 3. Sample SVM predictions for textile data using SVM**

The second shows that how to use SVM in combination with Scatter Plot and the following workflow trains a SVM model on *textile fabrics* data and outputs support vectors, which are those data instances that were used as support vectors in learning phase. One can observe these data instances in scatter plot visualization. Note that for the workflow to work correctly, you must set the links between widgets as demonstrated in the screenshot below (Table 4).



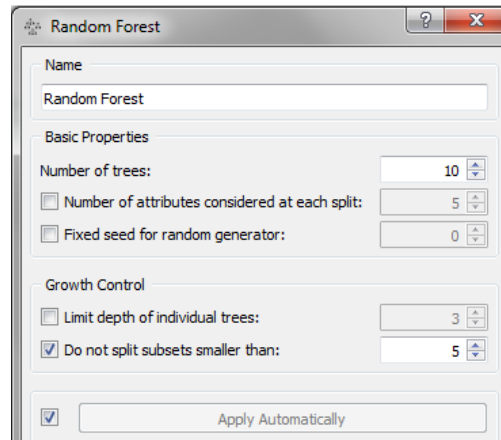
**Table 4. Trained SVM train tables for textile data using SVM**

### 5.2.3 Random Forest

Random forests are a combination of tree predictors such that each tree depends on the values of a random vector sampled independently and with the same distribution for all trees in the forest. Random forest is a group learning method used for classification, regression and prediction, etc. This method is first



proposed by Tin Kam Ho and further developed by Leo Breiman [7] and Adele Cutler. Random Forest model constructs a set of decision trees. Each tree is developed from a textile fabrics sample from the training data. When developing individual trees, an arbitrary subset of attributes is drawn, from which best attribute for the split is selected. Final model is based on the majority textile fabrics from individually developed trees in the forest. Random Forest works for both classification and regression analysis. (Table 5)



**Table 5. Trained Random Forest Model for textile data**

### 5.2.4 Logistic Regression

The logistic regression classification algorithm is set the value of LASSO (L1) or ridge (L2) regularization. For classification tasks, textile fabrics dataset were used and connect it to Predictions. Then connect file to Random **Forest** and Tree and connect them further to Predictions. Finally, observe the predictions for the two models in Table 6.

	Random Forest	Tree	LC	WC
1	1.00 : 0.00 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.570	0.0340
2	1.00 : 0.00 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.560	0.0310
3	0.97 : 0.03 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.510	0.0290
4	1.00 : 0.00 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.590	0.0400
5	1.00 : 0.00 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.580	0.0420
6	1.00 : 0.00 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.530	0.0270
7	1.00 : 0.00 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.520	0.0250
8	0.87 : 0.10 : 0.03 - C1	1.00 : 0.00 : 0.00 - C1	0.570	0.0310
9	0.80 : 0.00 : 0.20 - C1	1.00 : 0.00 : 0.00 - C1	0.640	0.0190
10	0.97 : 0.00 : 0.03 - C1	1.00 : 0.00 : 0.00 - C1	0.680	0.0190
11	0.97 : 0.00 : 0.03 - C1	1.00 : 0.00 : 0.00 - C1	0.720	0.0190
12	0.90 : 0.00 : 0.10 - C1	1.00 : 0.00 : 0.00 - C1	0.640	0.0150
13	0.92 : 0.00 : 0.08 - C1	1.00 : 0.00 : 0.00 - C1	0.720	0.0170
14	0.82 : 0.15 : 0.03 - C1	1.00 : 0.00 : 0.00 - C1	0.609	0.0520
15	0.89 : 0.00 : 0.11 - C1	1.00 : 0.00 : 0.00 - C1	0.641	0.0510
16	1.00 : 0.00 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.600	0.0340
17	1.00 : 0.00 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.530	0.0320
18	1.00 : 0.00 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.560	0.0290
19	1.00 : 0.00 : 0.00 - C1	1.00 : 0.00 : 0.00 - C1	0.600	0.0420

**Table 6. Random Forest Mode Prediction for Textile Data**

## 6. RESULT AND DISCUSSION

In general, the test and scoring methods are AUC, CA (Classification Accuracy), F1, Precision and Recall measures are displayed in output window. The definition of each measure is given below.

**6.1 Area Under the Curve (AUC)**

The Area under the curve is a performance matrix for a binary classification in data mining. By comparing the Receiver Operating Characteristic Curve with Area Under the Curve (AUC), it captures the extent to which the curve is up in Northwest corner. The AUC score less than 0.5 is not a better random estimate. The measure of AUC with 0.9 would be a very good model, but a score of 0.999 would be the best model to be true and indicate correct model (Table 7).

**6.2 Classification Accuracy (CA)**

The classification accuracy with 1 is the model which is the best and 0 is the model as worst (Table 7), the following formula is used to calculate the CA measure based on Type I and Type II errors of statistics,

$$CA = \frac{\text{true positive} + \text{true negative}}{\text{true positive} + \text{true negative} + \text{false positive} + \text{false negative}}$$

$$= \frac{tp + tn}{tp + tn + fp + fn}$$

**6.3 F1 Score**

In statistical binary classification the  $F_1$  score, is alternatively named as ( $F$ -score or  $F$ -measure) a measure of a test's precision. The  $F_1$  score considers both precision  $p$  and recall  $r$  of the test to compute the score:  $p$  is the number of correct positive results divided by the number of all positive results returned by the classifier.  $r$  is the number of correct positive results divided by the number of all relevant samples (all samples that has been identified as positive).  $F_1$  Score is harmonic mean of precision and Recall, where a  $F_1$  score reaches its best model at 1 (that is perfect Precision and Recall score) and worst model at 0 (Table 7). The general formula for  $F_1$  score is the harmonic average of  $p$  and  $r$

$$F_1 = 2 * \frac{1}{\frac{1}{r} + \frac{1}{p}}$$

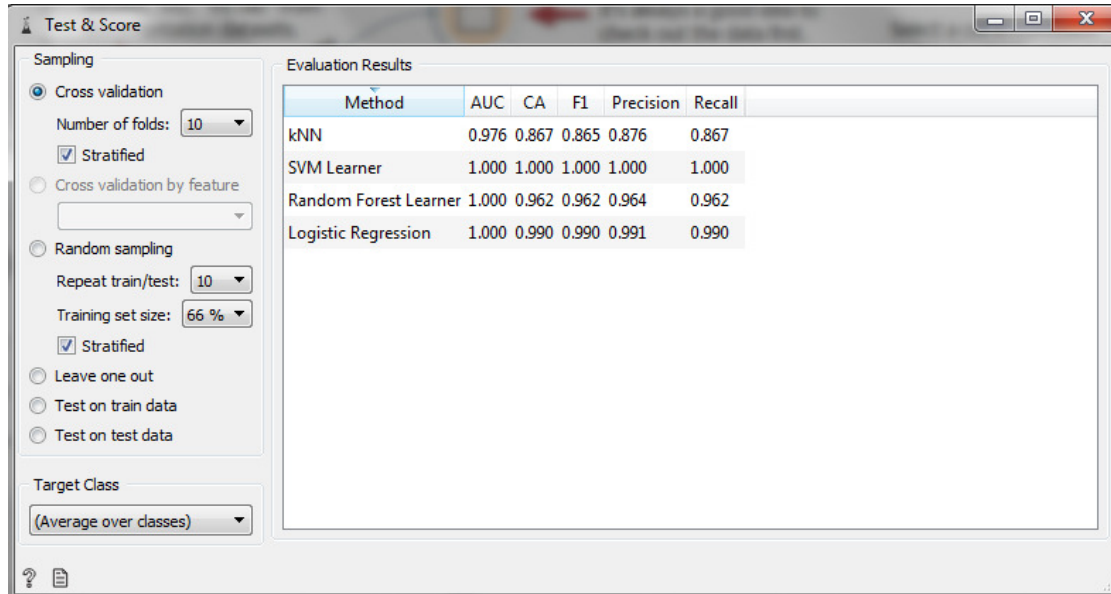
$$= 2 * \frac{p * r}{p + r}$$

**6.4 Precision**

The precision measure is the ratio of  $P = \frac{\text{true positive}}{\text{true positive} + \text{false positive}} = \frac{tp}{tp + fp}$  where  $tp$  is the number of true positives and  $fp$  is the number of false positives. The precision is naturally the ability of classifier not to label a sample as positive that is negative. The precision value 1 is the best model and 0 is the worst model (Table 7).

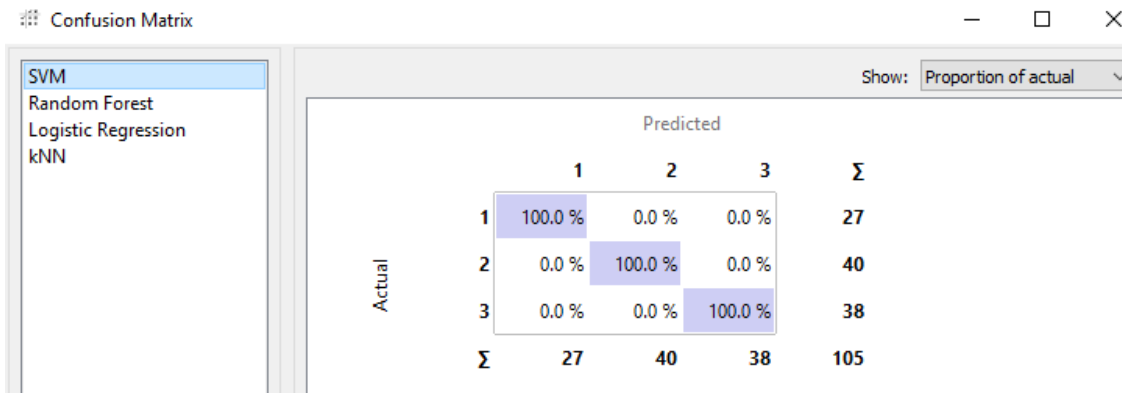
## 6.5 Recall of Test Score

Recall measure is the ratio of  $R = \frac{\text{true positive}}{\text{true positive} + \text{false negative}} = \frac{tp}{tp + fn}$  where  $tp$  is the number of true positives and  $fn$  is the number of false negatives [8]. The precision is naturally the ability of classifier not to label a sample as positive which is negative. The precision value is 1 (Table 7).



Method	AUC	CA	F1	Precision	Recall
kNN	0.976	0.867	0.865	0.876	0.867
SVM Learner	1.000	1.000	1.000	1.000	1.000
Random Forest Learner	1.000	0.962	0.962	0.964	0.962
Logistic Regression	1.000	0.990	0.990	0.991	0.990

**Table 7. Test and Score of Orange Data Mining Methods**



		Predicted			Σ
		1	2	3	
Actual	1	100.0 %	0.0 %	0.0 %	27
	2	0.0 %	100.0 %	0.0 %	40
	3	0.0 %	0.0 %	100.0 %	38
	Σ	27	40	38	105

**Table 8. Sample Classification and Prediction of Textile Data**

The above table exhibits clearly the test and their scores of various data mining techniques. All the three methods of AUC, CA, F1, Precision and Recall values are closer to unity. The results achieved best model and cross validation of textile database. The classification and confusion matrix of four models were classified and predicted accuracy to cent percent (Table 8) due to data variation. Based on this classification and prediction accuracy, textile database were classified and labeled as High, Moderate, and Low categories.

## 7. CONCLUSION

This research attempts to identify a meaningful classification, prediction and model evaluation of textile fabrics database and make possible the result in three set of data sets. The secondary database was collected from Department of Textile Technology from Anna University, Chennai for one year duration. The application of machine learning method of kNN, Logistic Regression, Support Vector Machine and Random Forest as data mining tools to explore the classification model and prediction of the present dataset of the study. Four classification models were applied first and extracted the best model. Area under the Curve (AUC), Classification Accuracy (CA), F1 Score, Precision and Recall are all closer to unity. All the above measures show that three major categories of textile data base are well predicted and classified. Finally, machine learning methods achieved best model test and score and they are labeled as High (H), Moderate (M) and Low (L). The results of the present study indicate that the machine learning Data Mining Tools can be used as a feasible tool for analysis of large set of textile database.

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## AN INVENTORY COST MODEL WITH WEIBULL DISTRIBUTED DETERIORATION, SHORTAGES AND PARTIALLY BACKLOGGING USING TRAPEZODIAL FUZZY NUMBERS

**Harish Kumar Yadav\*, T.P.Singh\*\*, Vinod Kumar\*\*\***

\*Research Scholar, Baba. Mast Nath. University, Asthal Bohar, Rohtak

\*\*Visiting Professor, Department of Mathematics, MM University Mullana

\*\*\*Registrar & Professor, Department of Mathematics, BM University, Asthal Bohar, Rohtak

E-mail: hrkyadav12@gmail.com\*, tpsingh78@yahoo.com\*\*, kakoriavinod@rediffmail.com

### ABSTRACT

*Inventory control plays important role in manufacturing and supply chain management system. In this paper, we propose a fuzzy inventory model having time varying demand which starts and end at zero with shortage period along with Weibull deterioration rate and partial back logging. The demand rate is to be considered as quadratic function of time. As Inventory modeling fuzzy set theory is the closest path towards reality. The Optimal Inventory Cost has been explored for the developed model after defuzzification of cost function. Centroid method and Graded Mean Representation Method are used for defuzzification of cost function. A numerical illustration has been proposed to validate the model. The sensitivity analysis has been carried out through numerical examples.*

**Keywords :** *Fuzzy Inventory model, Centroid method., Graded Mean Representation Method, defuzzification Process.*

### 1. INTRODUCTION

It is well known fact that in inventory management, deterioration of items is a realistic phenomenon. The deterioration of goods creates many problems such as loss of profit, shortages, loss of goodwill in organization for the buyers. Also due to cut throat competition, customer's behavior, priority level change or arriving high tech better product in the market the sales for the product may decline. These all situations generate the backlogging or partial backlogging factors to be considered for both customer's and supplier's point of view as well. Rosenberg (1979), Wu and Yao(2010), Kazemi et al. (2010) , Harish & Singh T.P.( 2019) studied some EOQ models with partial backorders .

It has been observed that in inventory control problems most of the parameters are uncertain in nature or depends upon the situation and requirement. The fuzzy logic given by Zadeh in 1965 provides a justified solution in such situation. Yager (1981) applied the fuzzy set theory to the inventory problems through fuzzy decision making model. Urgeletti (1983), Chang Wang (1996), developed the inventory models in fuzzy sense minimizing the total cost function. Yao and Lee (1999), Wang and Chen (2001) have derived

the models with or without backorder under Fuzzy parameters. Kao and Hsu (2002), Rawat (2011), Ritha W.etal. ( 2011, 2013), Dutta. D and kumar Pawan (2013) taking major parameters as fuzzy in nature determined the total inventory cost. Rajeswari. N, etal (2015) Rajalakshmi, R.M, & Michael.G.(2017) and Singh T.P and Yadav H.K.( 2019) derived Fuzzy Inventory Model for Constantly Deteriorating Items and time varying holding cost..

Recently Harish &. Singh T.P. etal. (2020 ) explored an inventory model for Weibull rate of deteriorating items, shortages and quadratic demand rate with partial backlogging with triangular fuzzy parameters to find optimal value of production cycle time, minimizing the stock level & total average cost over a time horizon ,the cycle time being constant. The present study is further an extension of the said model to study it with trapezoidal fuzzy parameters. The uncertainty is described by trapezoidal fuzzy numbers.

Following an introductory part rest of the paper is organized as follows. In section 2, some definitions and properties about fuzzy sets related to this work are given In. section 3 notations and assumptions are described in brief for developing the model. Section 4 presents the mathematical formulations of the model and solution based on Harish etal. (2020) study.. In section 5, the fuzzification process on the model has been carried out while in section 6 the defuzzification process is being made using well known technique. Section 7 provides numerical example to illustrate the results of the proposed model while in section 8 sensitive analysis of the optimal solution has been carried out. Finally, conclusions are given in last section.

**2. PRELIMINARIES**

**Definition 2.1 Fuzzy Set and Membership Function ( Characteristic Function ) :** Let X be the collection of objects then a ‘ Fuzzy Set ’in X is defined as a set of ordered pairs  $\hat{A} = \{ ( x, \mu_A(x) : x \in X \}$ , Here  $\mu_A(x)$  is called **Characteristic Function. The value of  $\mu_A(x)$  is permitted to have any value belongs to [0,1], which denotes the degree of membership of an element in a given set and mathematically we express**

$$\mu_A(x) = \begin{cases} 1, & \text{if and only if } x \in \hat{A} \\ 0, & \text{if and only if } x \notin \hat{A} \end{cases}$$

**Definition 2.2  $\alpha$ - Cut Set:** The  $\alpha$ -Cut or  $\alpha$  level set of a Fuzzy set  $\hat{A}$  is a crisp set defined by

$$\hat{A}_\alpha = \{ x : \mu_A(x) \geq \alpha \} \text{ and Strong } \alpha\text{-cut } \hat{A}_\alpha = \{ x : \mu_A(x) > \alpha \}$$

**Definition 2.2 Trapezoidal Membership Function:** A Triangular membership function is specified by three parameters ( a,b,c,d) as follows:

$$\mu_A(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{otherwise} \end{cases}$$

The parameters ( a,b,c,d) with  $a \leq b \leq c \leq d$  determined the x coordinate of trapezoidal membership function.

**Definition 2.3**

If  $\hat{A} = (a,b,c,d)$  is a trapezoidal fuzzy number then the graded mean integration representation of  $\hat{A}$  is defined as

$$P(\hat{A}) = \frac{a+2b+2c+d}{6}$$

**Definition 2.5**

The Centroid for a trapezoidal fuzzy number  $\hat{A} = (a, b, c, d)$  is defined as

$$P(\hat{A}) = \frac{a+b+c+d}{4}$$

**Definition 2.6 Convex Fuzzy Set :**

Let  $\hat{A}$  be a fuzzy set defined on a universal set X then  $\hat{A}$  is called convex fuzzy set if

$$\hat{A}(\lambda x_1 + (1-\lambda)x_2) \geq \text{Min of } [\hat{A}(x_1), \hat{A}(x_2)] \text{ for all } x_1, x_2 \in X \text{ and } \lambda \in [0,1].$$

**3. ASSUMPTIONS AND NOTATIONS**

The model is developed under the following assumptions;

- 1) Replenishment size is constant and the replenishment rate is infinite.
- 2) Lead time is zero.
- 3) T is the length of each production cycle
- 4)  $\hat{t}$  is the fuzzy cycle length.
- 5) A be the Ordering Cost.
- 6)  $C_1$  is the inventory holding cost per unit per unit time;
- 7)  $C_2$  is Purchase cost per unit;
- 8)  $C_3$  is the cost of each deteriorated unit;
- 9)  $C_4$  is Backordered cost per unit;
- 10)  $C_5$  is Lost sales cost per unit;
- 11)  $C(t_1)$  is the total inventory cost ;
- 12)  $\bar{G}(\hat{t})$  is fuzzy total cost by Graded mean representation method

13)  $\zeta(t)$  is fuzzy total cost by Centriod method.

14) The deterioration rate function  $\theta(t)$  represents the on-hand inventory deteriorates per unit time and Moreover in the present study the function assumed in the form

$$\theta(t) = \alpha\beta t^{\beta-1}; 0 < \alpha < 1, \beta > 0, t > 0.$$

When  $\beta = 1$ ,  $\theta(t)$  becomes a constant which is a case of exponential decay. When  $\beta < 1$ , the rate of deterioration is decreasing with  $t$  and when  $\beta > 1$ , the rate of deterioration is increasing with  $t$ .

15) The demand rate starts from zero and ends at zero during the inventory period. It is assumed of the form  $D(t) = at(T-t)$  where  $T$  is the cycle period.

16) During stock out period, the backlogging rate is variable and is dependent on the length of the waiting time for next replenishment. So that the backlogging rate for negative inventory is,

$$B(t) = \frac{1}{1+\gamma(T-t)}$$

$\gamma$  is backlogging parameter and  $t - T$  is waiting time and  $t_1 < t < T$ .

#### 4. MATHEMATICAL MODEL AND ANALYSIS

Assuming an amount  $S$  ( $S > 0$ ) as an initial inventory, we find inventory level gradually diminishes due to reasons of market demand and deterioration of the items and ultimately falls to zero at time  $T$ . Let  $I(t)$  be the on hand inventory at any time  $t$ . Clear, the differential equations which on hand inventory  $I(t)$  must satisfy the following :

$$\frac{dI(t)}{dt} + \theta I(t) = -D(t) \quad 0 \leq t \leq t_1 \quad \dots(1)$$

$$\frac{dI(t)}{dt} = -\frac{D(t)}{1+\gamma(T-t)}, \quad t_1 \leq t \leq T \quad \dots(2)$$

Solution of differential Equations (1) and (2) on the basis of study made by Yadav H.K., Singh T.P. etal (June 2020) can be written in (3) and (4) as:

$$I(t) = -a \left[ \frac{Tt^2}{2} - \frac{(\alpha\beta)Tt^{\beta+2}}{2(\beta+2)} - \frac{t^3}{3} + \frac{(\alpha\beta)t^{\beta+3}}{3(\beta+3)} - \frac{Tt_1^2}{2} - \frac{\alpha T t_1^{\beta+2}}{\beta+2} + \frac{t_1^3}{3} + \frac{\alpha t_1^{\beta+3}}{\beta+3} \right] + a\alpha \left[ \frac{\alpha T t^{2\beta+2}}{\beta+2} - \frac{\alpha t^{2\beta+3}}{\beta+3} - \frac{T t^\beta t_1^2}{2} - \frac{\alpha T t^\beta t_1^{\beta+2}}{\beta+2} + \frac{t^\beta t_1^3}{3} + \frac{\alpha t^\beta t_1^{\beta+3}}{\beta+3} \right] \quad \dots(3)$$

$$I(t) = \frac{a(t_1^2 - t^2)}{2\gamma} + \frac{a(t - t_1)}{\gamma^2} - \frac{a(1+\gamma T)(t_1 - t)}{\gamma^2} \quad t_1 \leq t \leq T \quad \dots(4)$$

Hence total amount of deteriorated units (D)=I(0)–stock loss due to demand =  $S - \int_0^{t_1} at(T - t) dt$



$$\begin{aligned}
 &= a \left( \frac{Tt_1^2}{2} + \frac{\alpha Tt_1^{\beta+2}}{\beta+2} - \frac{t_1^3}{3} - \frac{\alpha t_1^{\beta+3}}{\beta+3} \right) - a \left( \frac{Tt_1^2}{2} - \frac{t_1^3}{3} \right) \\
 &= a \left( \frac{\alpha Tt_1^{\beta+2}}{\beta+2} - \frac{\alpha t_1^{\beta+3}}{\beta+3} \right) \dots\dots\dots(5)
 \end{aligned}$$

Total Inventory held ( I<sub>1</sub> ) = C<sub>1</sub> ∫<sub>0</sub><sup>t<sub>1</sub></sup> (I(t) dt

$$I_1 = C_1 \left\{ -a \left[ -\frac{\alpha(\beta^3+4\beta^2+\beta)t_1^{\beta+4}}{3(\beta+3)(\beta+1)} + \frac{\alpha(\beta^3+3\beta^2+\beta)Tt_1^{\beta+3}}{2(\beta+2)(\beta+1)} + \frac{t_1^4}{4} - \frac{Tt_1^3}{3} \right] + a\alpha \left[ -\frac{\alpha Tt_1^{2\beta+3}}{(\beta+1)(2\beta+3)} + \frac{\alpha t_1^{2\beta+4}}{(\beta+1)(2\beta+4)} \right] \right\} \dots\dots\dots(6)$$

Cost of deteriorated items = C<sub>3</sub> × total amount of deteriorated units

$$= C_3 a \left( \frac{\alpha Tt_1^{\beta+2}}{\beta+2} - \frac{\alpha t_1^{\beta+3}}{\beta+3} \right) \dots\dots\dots(7)$$

Backordered cost per cycle = C<sub>4</sub> ∫<sub>t<sub>1</sub></sub><sup>T</sup> -I(t) dt

$$\begin{aligned}
 &= -C_4 \int_{t_1}^T \frac{a(t_1^2-t^2)}{2\gamma} + \frac{a(t-t_1)}{\gamma^2} - \frac{a(1+\gamma T)(t_1-t)}{\gamma^2} dt \\
 &= C_4 \left\{ \frac{a(T^3-t_1^3)}{6\gamma} - \frac{a(T-t_1)t_1^2}{2\gamma} + \frac{a(T-t_1)^2}{2\gamma^2} + \frac{a(1+\gamma T)(t_1-T)^2}{2\gamma^2} \right\} \dots\dots\dots(8)
 \end{aligned}$$

Lost sales per cycle = C<sub>5</sub> ∫<sub>t<sub>1</sub></sub><sup>T</sup> (1 - 1/(1+γ(T-t))) at(T-t) dt

$$= C_5 \left\{ \frac{a(T^3-Tt_1^2)}{2} - \frac{a(T^3-t_1^3)}{3} - \frac{a(T^2-t_1^2)}{2\gamma} + \frac{a(T-t_1)}{\gamma^2} + \frac{a(1+\gamma T)}{\gamma^2} (T-t_1) \right\} \dots\dots\dots(9)$$

Purchase cost per cycle = C<sub>2</sub> a ( Tt<sub>1</sub><sup>2</sup>/2 + αTt<sub>1</sub><sup>β+2</sup>/β+2 - t<sub>1</sub><sup>3</sup>/3 - αt<sub>1</sub><sup>β+3</sup>/β+3 ) .....(10)

Average total cost per unit time C(t<sub>1</sub>) = 1/T [ Total cost per unit time ] = 1/T [ Ordering Cost+ Total Inventory held +Cost of deterioration items+ Backordered cost per cycle + Lost sales per cycle + Purchase cost per cycle ]

$$\begin{aligned}
 &= \frac{1}{T} \left\{ A + C_1 \left\{ -a \left[ -\frac{\alpha(\beta^3+4\beta^2+\beta)t_1^{\beta+4}}{3(\beta+3)(\beta+1)} + \frac{\alpha(\beta^3+3\beta^2+\beta)Tt_1^{\beta+3}}{2(\beta+2)(\beta+1)} + \frac{t_1^4}{4} - \frac{Tt_1^3}{3} \right] + a\alpha \left[ -\frac{\alpha Tt_1^{2\beta+3}}{(\beta+1)(2\beta+3)} + \frac{\alpha t_1^{2\beta+4}}{(\beta+1)(2\beta+4)} \right] \right\} + C_3 a \left( \frac{\alpha Tt_1^{\beta+2}}{\beta+2} - \frac{\alpha t_1^{\beta+3}}{\beta+3} \right) + C_4 \left\{ \frac{a(T^3-t_1^3)}{6\gamma} - \frac{a(T-t_1)t_1^2}{2\gamma} + \frac{a(T-t_1)^2}{2\gamma^2} + \frac{a(1+\gamma T)(t_1-T)^2}{2\gamma^2} \right\} + C_5 \left\{ \frac{a(T^3-Tt_1^2)}{2} - \frac{a(T^3-t_1^3)}{3} - \frac{a(T^2-t_1^2)}{2\gamma} + \frac{a(2+\gamma T)}{\gamma^2} (T-t_1) \right\} + C_2 a \left( \frac{Tt_1^2}{2} + \frac{\alpha Tt_1^{\beta+2}}{\beta+2} - \frac{t_1^3}{3} - \frac{\alpha t_1^{\beta+3}}{\beta+3} \right) \right\} \dots\dots\dots(11)
 \end{aligned}$$

For minimum C(T) ,the necessary condition is

$\frac{dC(t_1)}{dt_1} = 0$  After solving we get an equation of odd degree whose last term is negative, then there exists

a solution  $t_1^* \in (0, T)$  also satisfy the condition  $\frac{d^2C}{dt_1^2} > 0$  at  $t_1 = t_1^*$

$\therefore C(t_1)$  is minimum at  $t_1 = t_1^*$

So optimum value of  $t_1$  is  $t_1^*$

### Section 5

#### 6. FUZZIFICATION OF THE MODEL

Let us describe the cycle time as triangular fuzzy parameter,  $\dot{t} = (t_1 - \mathcal{E}, t_1 - \varphi, t_1 + \lambda, t_1 + \mu)$ .

From equation (10) the total cost function with fuzzy cycle time is

$$C(\dot{t}) = \frac{1}{T} \{ A + C_1 \left\{ -a \left[ -\frac{\alpha(\beta^3 + 4\beta^2 + \beta)t_i^{\beta+4}}{3(\beta+3)(\beta+1)} + \frac{\alpha(\beta^3 + 3\beta^2 + \beta)Tt_i^{\beta+3}}{2(\beta+2)(\beta+1)} + \frac{t_i^4}{4} - \frac{Tt_i^3}{3} \right] + a\alpha \left[ -\frac{\alpha Tt_i^{2\beta+3}}{(\beta+1)(2\beta+3)} + \frac{\alpha t_i^{2\beta+4}}{(\beta+1)(2\beta+4)} \right] \right\} + C_3 a \left( \frac{\alpha Tt_i^{\beta+2}}{\beta+2} - \frac{\alpha t_i^{\beta+3}}{\beta+3} \right) + C_4 \left\{ \frac{a(T^3 - t_i^3)}{6\gamma} - \frac{a(T - t_i)t_i^2}{2\gamma} + \frac{a(T - t_i)^2}{2\gamma^2} + \frac{a(1 + \gamma T)(t_i - T)^2}{2\gamma^2} \right\} + C_5 \left\{ \frac{a(T^3 - Tt_i^2)}{2} - \frac{a(T^3 - t_i^3)}{3} - \frac{a(T^2 - t_i^2)}{2\gamma} + \frac{a(2 + \gamma T)(T - t_i)}{\gamma^2} \right\} + C_2 a \left( \frac{Tt_i^2}{2} + \frac{\alpha Tt_i^{\beta+2}}{\beta+2} - \frac{t_i^3}{3} - \frac{\alpha t_i^{\beta+3}}{\beta+3} \right) \} \text{ where } i = 1, 2, 3, 4 \dots\dots(12)$$

As  $C(\dot{t}) = (A_1, A_2, A_3, A_4)$  therefore

$$A_1 = \frac{1}{T} \left\{ A + C_1 \left\{ -a \left[ -\frac{\alpha(\beta^3 + 4\beta^2 + \beta)(t_1 - \mathcal{E})^{\beta+4}}{3(\beta+3)(\beta+1)} + \frac{\alpha(\beta^3 + 3\beta^2 + \beta)T(t_1 - \mathcal{E})^{\beta+3}}{2(\beta+2)(\beta+1)} + \frac{(t_1 - \mathcal{E})^4}{4} - \frac{T(t_1 - \mathcal{E})^3}{3} \right] + a\alpha \left[ -\frac{\alpha T(t_1 - \mathcal{E})^{2\beta+3}}{(\beta+1)(2\beta+3)} + \frac{\alpha(t_1 - \mathcal{E})^{2\beta+4}}{(\beta+1)(2\beta+4)} \right] \right\} + C_3 a \left( \frac{\alpha T(t_1 - \mathcal{E})^{\beta+2}}{\beta+2} - \frac{\alpha(t_1 - \mathcal{E})^{\beta+3}}{\beta+3} \right) + C_4 \left\{ \frac{a(T^3 - (t_1 - \mathcal{E})^3)}{6\gamma} - \frac{a(T - (t_1 - \mathcal{E}))(t_1 - \mathcal{E})^2}{2\gamma} + \frac{a(T - (t_1 - \mathcal{E}))^2}{2\gamma^2} + \frac{a(1 + \gamma T)((t_1 - \mathcal{E}) - T)^2}{2\gamma^2} \right\} + C_5 \left\{ \frac{a(T^3 - T(t_1 - \mathcal{E})^2)}{2} - \frac{a(T^3 - (t_1 - \mathcal{E})^3)}{3} - \frac{a(T^2 - (t_1 - \mathcal{E})^2)}{2\gamma} + \frac{a(2 + \gamma T)(T - (t_1 - \mathcal{E}))}{\gamma^2} \right\} + C_2 a \left( \frac{T(t_1 - \mathcal{E})^2}{2} + \frac{\alpha T(t_1 - \mathcal{E})^{\beta+2}}{\beta+2} - \frac{(t_1 - \mathcal{E})^3}{3} - \frac{\alpha(t_1 - \mathcal{E})^{\beta+3}}{\beta+3} \right) \} \dots\dots(13)$$

$$A_2 = \frac{1}{T} \left\{ A + C_1 \left\{ -a \left[ -\frac{\alpha(\beta^3 + 4\beta^2 + \beta)(t_1 - \varphi)^{\beta+4}}{3(\beta+3)(\beta+1)} + \frac{\alpha(\beta^3 + 3\beta^2 + \beta)T(t_1 - \varphi)^{\beta+3}}{2(\beta+2)(\beta+1)} + \frac{(t_1 - \varphi)^4}{4} - \frac{T(t_1 - \varphi)^3}{3} \right] + a\alpha \left[ -\frac{\alpha T(t_1 - \varphi)^{2\beta+3}}{(\beta+1)(2\beta+3)} + \frac{\alpha(t_1 - \varphi)^{2\beta+4}}{(\beta+1)(2\beta+4)} \right] \right\} + C_3 a \left( \frac{\alpha T(t_1 - \varphi)^{\beta+2}}{\beta+2} - \frac{\alpha(t_1 - \varphi)^{\beta+3}}{\beta+3} \right) + C_4 \left\{ \frac{a(T^3 - (t_1 - \varphi)^3)}{6\gamma} - \frac{a(T - (t_1 - \varphi))(t_1 - \varphi)^2}{2\gamma} + \frac{a(T - (t_1 - \varphi))^2}{2\gamma^2} + \frac{a(1 + \gamma T)((t_1 - \varphi) - T)^2}{2\gamma^2} \right\} + C_5 \left\{ \frac{a(T^3 - T(t_1 - \varphi)^2)}{2} - \frac{a(T^3 - (t_1 - \varphi)^3)}{3} - \frac{a(T^2 - (t_1 - \varphi)^2)}{2\gamma} + \frac{a(2 + \gamma T)(T - (t_1 - \varphi))}{\gamma^2} \right\} + C_2 a \left( \frac{T(t_1 - \varphi)^2}{2} + \frac{\alpha T(t_1 - \varphi)^{\beta+2}}{\beta+2} - \frac{(t_1 - \varphi)^3}{3} - \frac{\alpha(t_1 - \varphi)^{\beta+3}}{\beta+3} \right) \} \dots\dots(14)$$

$$\begin{aligned}
 A_3 = & \frac{1}{T} \{ A+ C_1 \{ - a \left[ -\frac{\alpha(\beta^3+4\beta^2+\beta)(t_1+\lambda)^{\beta+4}}{3(\beta+3)(\beta+1)} + \frac{\alpha(\beta^3+3\beta^2+\beta)T(t_1+\lambda)^{\beta+3}}{2(\beta+2)(\beta+1)} + \frac{(t_1+\lambda)^4}{4} - \frac{T(t_1+\lambda)^3}{3} \right] + \\
 & \alpha \left[ -\frac{\alpha T(t_1+\lambda)^{2\beta+3}}{(\beta+1)(2\beta+3)} + \frac{\alpha(t_1+\lambda)^{2\beta+4}}{(\beta+1)(2\beta+4)} \right] \} + C_3 \left( \frac{\alpha T(t_1+\lambda)^{\beta+2}}{\beta+2} - \frac{\alpha(t_1+\lambda)^{\beta+3}}{\beta+3} \right) + C_4 \left\{ \frac{a(T^3-(t_1+\lambda)^3)}{6\gamma} - \right. \\
 & \left. \frac{a(T-(t_1+\lambda))(t_1+\mathcal{E})^2}{2\gamma} + \frac{a(T-(t_1+\lambda))^2}{2\gamma^2} + \frac{a(1+\gamma T)((t_1+\lambda)-T)^2}{2\gamma^2} \right\} + C_5 \left\{ \frac{a(T^3-T(t_1+\lambda)^2)}{2} - \frac{a(T^3-(t_1+\lambda)^3)}{3} - \right. \\
 & \left. \frac{a(T^2-(t_1+\lambda)^2)}{2\gamma} + \frac{a(2+\gamma T)}{\gamma^2} (T - (t_1 + \lambda)) \right\} + C_2 \left( \frac{T(t_1+\lambda)^2}{2} + \frac{\alpha T(t_1+\lambda)^{\beta+2}}{\beta+2} - \frac{(t_1+\lambda)^3}{3} - \frac{\alpha(t_1+\lambda)^{\beta+3}}{\beta+3} \right) \} \\
 & \dots\dots(15)
 \end{aligned}$$

$$\begin{aligned}
 A_4 = & \frac{1}{T} \{ A+ C_1 \{ - a \left[ -\frac{\alpha(\beta^3+4\beta^2+\beta)(t_1+\mu)^{\beta+4}}{3(\beta+3)(\beta+1)} + \frac{\alpha(\beta^3+3\beta^2+\beta)T(t_1+\mu)^{\beta+3}}{2(\beta+2)(\beta+1)} + \frac{(t_1+\mu)^4}{4} - \frac{T(t_1+\mu)^3}{3} \right] + \\
 & \alpha \left[ -\frac{\alpha T(t_1+\mu)^{2\beta+3}}{(\beta+1)(2\beta+3)} + \frac{\alpha(t_1+\mu)^{2\beta+4}}{(\beta+1)(2\beta+4)} \right] \} + C_3 \left( \frac{\alpha T(t_1+\mu)^{\beta+2}}{\beta+2} - \frac{\alpha(t_1+\mu)^{\beta+3}}{\beta+3} \right) + C_4 \left\{ \frac{a(T^3-(t_1+\mu)^3)}{6\gamma} - \right. \\
 & \left. \frac{a(T-(t_1+\mu))(t_1+\mu)^2}{2\gamma} + \frac{a(T-(t_1+\mu))^2}{2\gamma^2} + \frac{a(1+\gamma T)((t_1+\mu)-T)^2}{2\gamma^2} \right\} + C_5 \left\{ \frac{a(T^3-T(t_1+\mu)^2)}{2} - \frac{a(T^3-(t_1+\mu)^3)}{3} - \right. \\
 & \left. \frac{a(T^2-(t_1+\mu)^2)}{2\gamma} + \frac{a(2+\gamma T)}{\gamma^2} (T - (t_1 + \mu)) \right\} + C_2 \left( \frac{T(t_1+\mu)^2}{2} + \frac{\alpha T(t_1+\mu)^{\beta+2}}{\beta+2} - \frac{(t_1+\mu)^3}{3} - \frac{\alpha(t_1+\mu)^{\beta+3}}{\beta+3} \right) \} \dots\dots\dots(16)
 \end{aligned}$$

7. DEFUZZIFICATION:

7.A. Chen’s Defuzzification Method

$$\bar{G}(t_1) = \frac{1}{6} [ A_1+2A_2+2A_3+A_4]$$

$$\bar{G}(t_1) =$$

$$\begin{aligned}
 & \frac{1}{T} \{ A + C_1 \{ \\
 & - a \left[ -\frac{\alpha(\beta^3+4\beta^2+\beta)(2(t_1-\varphi)^{\beta+4}+(t_1-\mathcal{E})^{\beta+4}+(t_1+\mu)^{\beta+4}+2(t_1+\lambda)^{\beta+4})}{18(\beta+3)(\beta+1)} + \right. \\
 & \left. \frac{\alpha(\beta^3+3\beta^2+\beta)T(2(t_1-\varphi)^{\beta+3}+(t_1-\mathcal{E})^{\beta+3}+(t_1+\mu)^{\beta+3}+2(t_1+\lambda)^{\beta+3})}{12(\beta+2)(\beta+1)} + \frac{(2(t_1-\varphi)^4+(t_1-\mathcal{E})^4+(t_1+\mu)^4+2(t_1+\lambda)^4)}{24} - \right. \\
 & \left. \frac{T(2(t_1-\varphi)^3+(t_1-\mathcal{E})^3+(t_1+\mu)^3+2(t_1+\lambda)^3)}{18} \right] + \alpha \left[ -\frac{\alpha T(2(t_1-\varphi)^{2\beta+3}+(t_1-\mathcal{E})^{2\beta+3}+(t_1+\mu)^{2\beta+3}+2(t_1+\lambda)^{2\beta+3})}{6(\beta+1)(2\beta+3)} + \right. \\
 & \left. \frac{\alpha(2(t_1-\varphi)^{2\beta+4}+(t_1-\mathcal{E})^{2\beta+4}+(t_1+\mu)^{2\beta+4}+2(t_1+\lambda)^{2\beta+4})}{6(\beta+1)(2\beta+4)} \right] \} + C_3 \left( \frac{\alpha T(2(t_1-\varphi)^{\beta+2}+(t_1-\mathcal{E})^{\beta+2}+(t_1+\mu)^{\beta+2}+2(t_1+\lambda)^{\beta+2})}{6(\beta+2)} - \right. \\
 & \left. \frac{\alpha(2(t_1-\varphi)^{\beta+3}+(t_1-\mathcal{E})^{\beta+3}+(t_1+\mu)^{\beta+3}+2(t_1+\lambda)^{\beta+3})}{36(\beta+3)} \right) + C_4 \left\{ \frac{a(T^3-(2(t_1-\varphi)^3+(t_1-\mathcal{E})^3+(t_1+\mu)^3+2(t_1+\lambda)^3))}{36\gamma} - \right. \\
 & \left. \frac{a(T-6t_1-2\lambda-\mu+\mathcal{E}+2\varphi))(2(t_1-\varphi)^2+(t_1-\mathcal{E})^2+(t_1+\mu)^2+2(t_1+\lambda)^2)}{12\gamma} + \frac{a(2+\gamma T)[(2(t_1-\varphi)^2+(t_1-\mathcal{E})^2+(t_1+\mu)^2+2(t_1+\lambda)^2)}{12\gamma^2} \right\} + \\
 & C_5 \left\{ \frac{a(T^3-T((2(t_1-\varphi)^2+(t_1-\mathcal{E})^2+(t_1+\mu)^2+2(t_1+\lambda)^2))}{12} - \frac{a(T^3-(2(t_1-\varphi)^3+(t_1-\mathcal{E})^3+(t_1+\mu)^3+2(t_1+\lambda)^3))}{18} - \right. \\
 & \left. \frac{a(T^2-((2(t_1-\varphi)^2+(t_1-\mathcal{E})^2+(t_1+\mu)^2+2(t_1+\lambda)^2))}{12\gamma} + \frac{a(2+\gamma T)}{6\gamma^2} (T - 6t_1 - 2\lambda - \mu + \mathcal{E} + 2\varphi) \right\}
 \end{aligned}$$

$$+C_2a\left\{\frac{T((2(t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+2(t_1+\lambda)^2))}{12} + \frac{\alpha T(2(t_1-\varphi)^{\beta+2}+(t_1-\varepsilon)^{\beta+2}+(t_1+\mu)^{\beta+2}+2(t_1+\lambda)^{\beta+2})}{6(\beta+2)} - \frac{(2(t_1-\varphi)^3+(t_1-\varepsilon)^3+(t_1+\mu)^3+2(t_1+\lambda)^3)}{18} - \frac{\alpha(2(t_1-\varphi)^{\beta+3}+(t_1-\varepsilon)^{\beta+3}+(t_1+\mu)^{\beta+3}+2(t_1+\lambda)^{\beta+3})}{6(\beta+3)}\right\} \dots\dots(17)$$

$$\begin{aligned} \frac{d\bar{G}(t_1)}{dt_1} &= \frac{1}{T} \{C_1 \left\{ -a \left[ -\frac{\alpha(\beta^3+4\beta^2+\beta)(\beta+4)(2(t_1-\varphi)^{\beta+3}+(t_1-\varepsilon)^{\beta+3}+(t_1+\mu)^{\beta+3}+2(t_1+\lambda)^{\beta+3})}{18(\beta+3)(\beta+1)} + \frac{\alpha(\beta+3)(\beta^3+3\beta^2+\beta)T(2(t_1-\varphi)^{\beta+2}+(t_1-\varepsilon)^{\beta+2}+(t_1+\mu)^{\beta+2}+2(t_1+\lambda)^{\beta+2})}{12(\beta+2)(\beta+1)} + \frac{(2(t_1-\varphi)^3+(t_1-\varepsilon)^3+(t_1+\mu)^3+2(t_1+\lambda)^3)}{6} - \frac{T(2(t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+2(t_1+\lambda)^2)}{6} \right] + \alpha a \left[ -\frac{\alpha T(2(t_1-\varphi)^{2\beta+2}+(t_1-\varepsilon)^{2\beta+2}+(t_1+\mu)^{2\beta+2}+2(t_1+\lambda)^{2\beta+2})}{6(\beta+1)} + \frac{\alpha(2(t_1-\varphi)^{2\beta+3}+(t_1-\varepsilon)^{2\beta+3}+(t_1+\mu)^{2\beta+3}+2(t_1+\lambda)^{2\beta+3})}{6(\beta+1)} \right] \right\} + C_3 \dots\dots a( \\ &\frac{\alpha T(2(t_1-\varphi)^{\beta+1}+(t_1-\varepsilon)^{\beta+1}+(t_1+\mu)^{\beta+1}+2(t_1+\lambda)^{\beta+1})}{6} - \frac{\alpha(2(t_1-\varphi)^{\beta+2}+(t_1-\varepsilon)^{\beta+2}+(t_1+\mu)^{\beta+2}+2(t_1+\lambda)^{\beta+2})}{36} \} + C_4 \{ \\ &\frac{5a(2(t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+2(t_1+\lambda)^2)}{12\gamma} - \frac{a(T-6t_1-2\lambda-\mu+\varepsilon+2\varphi)(6t_1+\mu-\varphi+\lambda-\varepsilon)}{6\gamma} + \frac{a(2+\gamma T)(6t_1+\mu-2\varphi+2\lambda-\varepsilon-6T)}{6\gamma^2} \} \\ &+ C_5 \left\{ \frac{(-aT(6t_1+\mu-\varphi+\lambda-\varepsilon))}{6} + \frac{a(2(t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+2(t_1+\lambda)^2)}{6} + \frac{a(6t_1+\mu-\varphi+\lambda-\varepsilon)}{6\gamma} - \frac{a(2+\gamma T)}{\gamma^2} \right\} + C_2sa( \\ &\frac{T(6t_1+\mu-\varphi+\lambda-\varepsilon)}{12} + \frac{\alpha T(2(t_1-\varphi)^{\beta+1}+(t_1-\varepsilon)^{\beta+1}+(t_1+\mu)^{\beta+1}+2(t_1+\lambda)^{\beta+1})}{6} - \frac{(2(t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+2(t_1+\lambda)^2)}{6} - \\ &\frac{\alpha(2(t_1-\varphi)^{\beta+2}+(t_1-\varepsilon)^{\beta+2}+(t_1+\mu)^{\beta+2}+2(t_1+\lambda)^{\beta+2})}{6} \} \dots\dots(18) \end{aligned}$$

To minimize total cost function per unit time  $\bar{G}(t_1)$ , the optimal value of  $t_1$  called  $\bar{t}$  can be obtained by solving the following equation:

$\frac{d\bar{G}(t_1)}{dt_1} = 0$  Further, for the total cost function  $\bar{G}(t_1)$  to is convex, the following conditions must be satisfied  $\frac{d^2\bar{G}(t_1)}{dt_1^2} > 0$ .

The second derivatives of the total cost function  $\bar{G}(t_1)$  are complicated and it is very difficult to prove the convexity mathematically. Hence, the convexity of total cost function can be established graphically.

**7.B. Using Centroid method, we get the defuzzified value of C(t<sub>1</sub>)**

$$\begin{aligned} \hat{C}(t_1) &= \frac{1}{4} [ A_1+A_2+A_3+A_4] \\ \hat{C}(t_1) &= \frac{1}{T} \{ A + C_1 \{ \\ &- a \left[ -\frac{\alpha(\beta^3+4\beta^2+\beta)((t_1-\varphi)^{\beta+4}+(t_1-\varepsilon)^{\beta+4}+(t_1+\mu)^{\beta+4}+(t_1+\lambda)^{\beta+4})}{12(\beta+3)(\beta+1)} + \right. \\ &\left. \frac{\alpha(\beta^3+3\beta^2+\beta)T((t_1-\varphi)^{\beta+3}+(t_1-\varepsilon)^{\beta+3}+(t_1+\mu)^{\beta+3}+(t_1+\lambda)^{\beta+3})}{8(\beta+2)(\beta+1)} + \frac{((t_1-\varphi)^4+(t_1-\varepsilon)^4+(t_1+\mu)^4+(t_1+\lambda)^4)}{16} - \right. \end{aligned}$$

$$\begin{aligned} & \left. \frac{T((t_1-\varphi)^3+(t_1-\varepsilon)^3+(t_1+\mu)^3+(t_1+\lambda)^3)}{8} \right] + a\alpha \left[ - \frac{\alpha T((t_1-\varphi)^{2\beta+3}+(t_1-\varepsilon)^{2\beta+3}+(t_1+\mu)^{2\beta+3}+(t_1+\lambda)^{2\beta+3})}{4(\beta+1)(2\beta+3)} + \right. \\ & \left. \frac{\alpha((t_1-\varphi)^{2\beta+4}+(t_1-\varepsilon)^{2\beta+4}+(t_1+\mu)^{2\beta+4}+(t_1+\lambda)^{2\beta+4})}{4(\beta+1)(2\beta+4)} \right] + C_3 \ a \left( \frac{\alpha T((t_1-\varphi)^{\beta+2}+(t_1-\varepsilon)^{\beta+2}+(t_1+\mu)^{\beta+2}+(t_1+\lambda)^{\beta+2})}{4(\beta+2)} - \right. \\ & \left. \frac{\alpha((t_1-\varphi)^{\beta+3}+(t_1-\varepsilon)^{\beta+3}+(t_1+\mu)^{\beta+3}+(t_1+\lambda)^{\beta+3})}{24(\beta+3)} \right) + C_4 \{ \\ & \frac{a(T^3-((t_1-\varphi)^3+(t_1-\varepsilon)^3+(t_1+\mu)^3+(t_1+\lambda)^3))}{24\gamma} - \frac{a(T-4t_1-\lambda-\mu+\varepsilon+\varphi)((t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+(t_1+\lambda)^2)}{12\gamma} + \\ & \frac{a(2+\gamma T)[((t_1-\varphi-T)^2+(t_1-\varepsilon-T)^2+(t_1+\mu-T)^2+(t_1+\lambda-T)^2)}{8\gamma^2} \} + C_5 \left\{ \frac{a(T^3-T((t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+(t_1+\lambda)^2))}{8} - \right. \\ & \frac{a(T^3-((t_1-\varphi)^3+(t_1-\varepsilon)^3+(t_1+\mu)^3+(t_1+\lambda)^3))}{12} - \frac{a(T^2-((t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+(t_1+\lambda)^2))}{8\gamma} + \frac{a(2+\gamma T)}{4\gamma^2} (T - 4t_1 - \\ & \lambda - \mu + \varepsilon + \varphi) \} \\ & + C_2 a \left\{ \frac{T((t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+(t_1+\lambda)^2)}{8} + \frac{\alpha T((t_1-\varphi)^{\beta+2}+(t_1-\varepsilon)^{\beta+2}+(t_1+\mu)^{\beta+2}+(t_1+\lambda)^{\beta+2})}{4(\beta+2)} - \right. \\ & \left. \frac{((t_1-\varphi)^3+(t_1-\varepsilon)^3+(t_1+\mu)^3+(t_1+\lambda)^3)}{8} - \frac{\alpha((t_1-\varphi)^{\beta+3}+(t_1-\varepsilon)^{\beta+3}+(t_1+\mu)^{\beta+3}+(t_1+\lambda)^{\beta+3})}{4(\beta+3)} \right\}. \quad \dots(19) \end{aligned}$$

$$\begin{aligned} \frac{d\zeta(t_1)}{dt_1} &= \frac{1}{T} \{ C_1 \left\{ - a \left[ - \frac{\alpha(\beta^3+4\beta^2+\beta)(\beta+4)((t_1-\varphi)^{\beta+3}+(t_1-\varepsilon)^{\beta+3}+(t_1+\mu)^{\beta+3}+(t_1+\lambda)^{\beta+3})}{12(\beta+3)(\beta+1)} + \right. \right. \\ & \frac{\alpha(\beta+3)(\beta^3+3\beta^2+\beta)T((t_1-\varphi)^{\beta+2}+(t_1-\varepsilon)^{\beta+2}+(t_1+\mu)^{\beta+2}+(t_1+\lambda)^{\beta+2})}{8(\beta+2)(\beta+1)} + \frac{((t_1-\varphi)^3+(t_1-\varepsilon)^3+(t_1+\mu)^3+(t_1+\lambda)^3)}{4} - \\ & \left. \frac{T((t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+(t_1+\lambda)^2)}{4} \right] + a\alpha \left[ - \frac{\alpha T((t_1-\varphi)^{2\beta+2}+(t_1-\varepsilon)^{2\beta+2}+(t_1+\mu)^{2\beta+2}+(t_1+\lambda)^{2\beta+2})}{4(\beta+1)} + \right. \\ & \left. \frac{\alpha((t_1-\varphi)^{2\beta+3}+(t_1-\varepsilon)^{2\beta+3}+(t_1+\mu)^{2\beta+3}+(t_1+\lambda)^{2\beta+3})}{4(\beta+1)} \right] \} + C_3 \ a \left( \frac{\alpha T((t_1-\varphi)^{\beta+1}+(t_1-\varepsilon)^{\beta+1}+(t_1+\mu)^{\beta+1}+(t_1+\lambda)^{\beta+1})}{4} - \right. \\ & \left. \frac{\alpha((t_1-\varphi)^{\beta+2}+(t_1-\varepsilon)^{\beta+2}+(t_1+\mu)^{\beta+2}+(t_1+\lambda)^{\beta+2})}{24} \right) + C_4 \{ \\ & \frac{5a((t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+(t_1+\lambda)^2)}{24\gamma} - \frac{a(T-4t_1-\lambda-\mu+\varepsilon+\varphi)(4t_1+\mu-\varphi+\lambda-\varepsilon)}{6\gamma} + \frac{a(2+\gamma T)((4t_1+\mu-\varphi+\lambda-\varepsilon)-4T)}{8\gamma^2} \} + \\ & C_5 \left\{ \frac{(-aT(4t_1+\mu-\varphi+\lambda-\varepsilon))}{4} + \frac{a((t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+(t_1+\lambda)^2)}{4} + \frac{a(4t_1+\mu-\varphi+\lambda-\varepsilon)}{4\gamma} - \frac{a(2+\gamma T)}{\gamma^2} \right\} + C_2 \ a \left( \right. \\ & \left. \frac{T(4t_1+\mu-\varphi+\lambda-\varepsilon)}{8} + \frac{\alpha T((t_1-\varphi)^{\beta+1}+(t_1-\varepsilon)^{\beta+1}+(t_1+\mu)^{\beta+1}+(t_1+\lambda)^{\beta+1})}{4} - \frac{((t_1-\varphi)^2+(t_1-\varepsilon)^2+(t_1+\mu)^2+(t_1+\lambda)^2)}{4} - \right. \\ & \left. \frac{\alpha((t_1-\varphi)^{\beta+2}+(t_1-\varepsilon)^{\beta+2}+(t_1+\mu)^{\beta+2}+(t_1+\lambda)^{\beta+2})}{4} \right] \} \quad \dots\dots(20) \end{aligned}$$

To minimize total cost function per unit time  $\zeta(t_1)$ , the optimal value of  $t_1$  called  $t_1^*$  can be obtained by solving the following equation:

$$\frac{d\zeta(t_1)}{dt_1} = 0$$

Further, for the total cost function  $\tilde{G}(t_1)$  to is convex, the following conditions must be satisfied  $\frac{d^2 \tilde{G}(t_1)}{dt_1^2} > 0$ . The second derivatives of the total cost function  $\tilde{G}(t_1)$  are complicated and it is very difficult to prove the convexity mathematically. Hence, the convexity of total cost function can be established graphically.

**Section 6**

**8. NUMERICAL ILLUSTRATION**

To illustrate the model we consider following numerical values of the parameters.

$a = 1000, \alpha = 0.01, \beta = 2, C_1 = 5, C_2 = 40, C_3 = 5, C_4 = .8, C_5 = 0.5, \gamma = 1.1, \phi = .02, \theta = .09, \lambda = 0.04, \mu = 0.09$

we obtain for the crisp model Total Cost = 7293.715 and cycle time  $t = 0.9619$  years and for the fuzzy model  $C_1 = (3, 4, 7, 7.64), C_2 = (35, 37, 43, 46.32)$  by Centroid method total cost = 7276.84 and cycle time = 0.9569 years., by Grad mean representation method total cost = 7272.51 and cycle time = 0.953567 years.

**8. SENSITIVITY ANALYSIS**

Change in Parameter	CRISP MODEL		CENTRIOD METHOD		GRAD MEAN REPRESENTATION METHOD	
	$t_1$	$C(t_1)$	$\dot{t}$	$\dot{C}(\dot{t})$	$\ddot{t}$	$\ddot{G}(\ddot{t})$
0.01	0.9619	7293.715	0.9569	7276.84	0.953567	7272.51
0.02	0.96	7306.627	0.9475	7282.30	0.944167	7281.49
0.03	0.959	7326.33	0.954	7305.85	0.950667	7301.37
0.04	0.9582	7352.371	0.9532	7330.07	0.949867	7325.62
0.05	0.9573	7384.553	0.9523	7360.44	0.948967	7356.01
0.06	0.9565	7423.008	0.9515	7397.07	0.948167	7392.68
0.07	0.9568	7468.329	0.9518	7440.48	0.948467	7436.32
0.08	0.9549	7518.548	0.9499	7488.98	0.946567	7484.68
0.09	0.9541	7575.636	0.9491	7544.26	0.945767	7540.01
0.1	0.9534	7638.996	0.9484	7605.81	0.945067	7601.62

**9. CONCLUSION**

A fuzzy Inventory model of deteriorating products in supply chain process has been developed in which demand rate is quadratic function of time while the deterioration follows Weibull Deterioration. Cycle time, Inventory holding cost and purchasing cost has been fuzzified with the help of trapezoidal fuzzy parameters and defuzzified with the help of Centroid method and Graded Mean Representation Method. The case of shortage of items with partially backlogging has been discussed. A comparative study of the total inventory cost, obtained by different models have been done in order to measure the

sensitivity analysis. Numerical example shows that the Graded mean representation method gives minimum cost as compared to crisp model and Centroid method.

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## SOLUTION OF MULTI-OBJECTIVE LINEAR PROGRAMMING PROBLEM USING MODIFIED FOURIER ELIMINATION TECHNIQUE & AHA SIMPLEX ALGORITHM

**S. Jain\*, A. Mangal\*\***

\*Director- Research (Additional Charge), M. D. S. University, Ajmer & Department of Mathematics,  
S. P. C. Government College Ajmer, India.

\*\*Department of Mathematics, Engineering College Ajmer, Affiliated to Bikaner Technical University, Bikaner, India  
E-mail: drjainsanjay@gmail.com\*, dradarshmangal1@gmail.com\*\*

### ABSTRACT

*In this paper, an attempt has been made to solve Multi-objective Linear Programming Problem (MOLPP) by using various techniques. The individual objective function involved in the MOLPP is solved by AHA simplex algorithm. After that, with the help of values of objective function obtained by AHA simplex algorithm; the objective function of original MOLPP is transformed into an objective function of a single LPP by using various techniques viz., Chandra Sen's technique, averaging techniques and new averaging techniques and then the solution can be recovered by applying Modified Fourier elimination technique. The method is quite useful because the calculations involved are simple in nature and takes least time. The method has been illustrated by a numerical example.*

*Keywords-Multi-objective Linear Programming Problem, AHA Simplex Algorithm, Modified Fourier Elimination Technique, Objective Function, Chandra Sen's technique, Averaging technique, New Averaging technique, AHA Simplex Algorithm.*

### 1. INTRODUCTION

Multi-objective Linear Programming Problem is concerned with the linear programming problems of maximizing or minimizing the linear objective function having more than one objective; along with subject to a set of linear inequality constraints. Here, the aim is to reduce the computing time of the optimization process of the proposed problem in which objective function treated as constraints in nature. Such constraints are encountered in transportation, flow and network models.

Sen, C. [3] developed a new approach for MOLPP. He suggested an approach to construct the multi-objective function under the limitation that the optimum value of individual problem is greater than zero. Sulaiman, N. A. and Sadiq, G. W. [12] discussed the solution of linear multi-objective programming problems by using mean and median value. Sulaiman, N.A. and Hamad-Ameen, A-Q. [13] developed an optimal transformation technique to solve MOLPP. Hamad-Ameen A.O. [14] discussed an adaptive arithmetic average technique to solve MOLPP. Sulaiman, N.A. and Mustafa, R.B. [21] used harmonic mean technique to solve multi-objective linear programming problems. Nahar, S. and Alim, M.A. [22] proposed a new statistical averaging method to solve MOLPP. Different elimination techniques for linear programming problems have been studied earlier by Kohler [2], Williams [6], Kannappan et al [8] and Sharma et al [10]. Jain et al [11&15] proposed Modified Fourier elimination as well as Gauss elimination

techniques for linear fractional programming problems. Further, Jain et al [16 & 17] presented extended Gauss as well as extended Modified Fourier elimination technique for integer solution of linear fractional programming problem. Jain, S. [18] presented the modelling of Gauss elimination technique for multi-objective linear programming problems; in which, he considered all the linear objective functions as constraints to solve the multi-objective linear programming problem. Jain, S.[19] proposed modelling of Fourier elimination technique for multi-objective fractional programming problem. Jain, S.[20] developed the modelling of Gauss elimination technique for multi-objective fractional programming problem. Here, we have taken a multi-objective linear programming problem and applied the AHA simplex algorithm to solve linear programming problem for multi-objective functions. After that, the single LPP has been constructed by using various techniques (Chandra Sen, averaging techniques and new averaging techniques) from the MOLPP under consideration and solved it by Modified Fourier elimination technique. We solved the LPP thus formed by all these techniques one by one by Modified Fourier elimination technique and the result thus obtained are compared with other existing techniques available in the literature and found the same answer. The optimal answer found by this technique is the same as the result obtained from either simplex method or by graphical method (in case of only two decision variables) or by AHA simplex algorithm. The AHA simplex algorithm had been introduced to solve Linear Programming Problem (LPP) by Ansari, A.H [23]. Jain, S. and Mangal, A. [24] commented on a new statistical averaging method to solve multi-objective linear programming problems.

This paper is organized as follows: Modified Fourier elimination technique for inequalities and AHA simplex algorithm are being described in the section 2. The mathematical formulation of the problem under consideration is given in the section 3. Different seven techniques viz. Chandra Sen’s Technique, averaging techniques (arithmetic, geometric and harmonic) and new averaging techniques (new arithmetic, new geometric and new harmonic) are being briefly described in the section 4. In the section 5, the solution of each LPP by Modified Fourier elimination technique is recovered which has been obtained by applying various techniques to MOLPP. Section 6 & 7 provide results and discussions as well as conclusion of this research work.

**2. METHODOLOGY**

**2.1 Modified Fourier Elimination Technique for Inequalities**

Here, we are applying Modified Fourier elimination technique for a system of inequalities of the same nature i.e., either less than or equal to ( $\leq$ ) or greater than or equal to ( $\geq$ ). In this technique, the variables are eliminated by combining the inequalities in such a way that the inequalities and variables reduce one by one in every iteration i.e., one variable and one inequality reduce in one iteration so at last there remains only one variable with one inequality remains. This last inequality gives value of last variable in bounded form and finally taking the value of last variable either maximum or minimum according to the objective function of LPP reduced from MOLPP. Finally, we get value of other variables by back substitution of value of the last variable.

For the sake of clarity and simplicity, we consider the system of n variables and m inequalities:

$$\begin{aligned}
 &a_{11}x_1 + \dots + a_{1n}x_n \leq b_1 \\
 &a_{21}x_1 + \dots + a_{2n}x_n \leq b_2 \\
 &\dots
 \end{aligned}$$

.....  
 $a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m$

Now, construct the following pairwise disjoint sets to determine the variable to be eliminated.

$l_j^+ = \{ i : A_{i1} > 0 \}; l_j^- = \{ i : A_{i1} < 0 \}$  and  $l_j^0 = \{ i : A_{i1} = 0 \}$  where  $i = 1, 2, \dots, m$ , and  $j = 1, 2, \dots, n$ .

If at any stage  $l_j^+$  or  $l_j^-$  becomes empty for a variable, then the given LPP has an unbounded solution otherwise calculate Minimum  $\{ |l_j^+| \times |l_j^-| \}$ ; where  $1 \leq j \leq n$ . Now, we have to select the index  $j$  corresponding to the minimum value obtained from the above relation. After choosing the variable corresponding to the index  $j$ , apply the Modified Fourier elimination technique to eliminate that variable. Now after first iteration, we get  $(n-1)$  variables and  $(m-1)$  inequalities. Repeating this process or after  $(n-1)$  iterations we have only one variable remains. Finally, one can get the value of other variables by back substitution of the value of last variable. It is possible that some redundant constraints may present in the system.

**2.2 AHA Simplex Algorithm**

**Step 1:** We have to construct as many separate linear programming problems as the number of objectives involved in the multi-objective linear programming problem under consideration.

**Step 2:** To apply AHA simplex algorithm, firstly we have to write each linear programming problem in the form given below:

Maximize  $Z_i \leq c_i^T x + \alpha_i$  Or  
 Maximize  $\begin{bmatrix} 1 & -c^T \end{bmatrix} \begin{bmatrix} Z_i \\ x \end{bmatrix} \leq 0$   
 Subject to  $\begin{bmatrix} 0 & A \end{bmatrix} \begin{bmatrix} Z_i \\ x \end{bmatrix} \leq b$   
 and  $x \geq 0$

**Step 3:** The objective function of every linear program must be in maximization form. If it is not so, then we have to convert it into maximize form by using the standard result i.e., Maximize  $Z_i = -$  Minimize  $(Z_i)$

**Step 4:** The requirement vector components i.e.,  $b_i$  must be positive always. If any one of  $b_i$  is not positive, then we have to make it positive by multiplying the corresponding inequality by -1.

**Step 5:** Now, we have to check the sign convention of all the decision variables involved in the objective inequality.

- (i) The solution will be an optimal one if all  $-c_j \geq 0$  ; where  $Z = 0$  and  $x_j = 0, \forall j$
- (ii) There is a requirement to improve the solution if at least one  $-c_j < 0$ , then go to the next step.

**Step 6:** Select the most negative value involved among all  $-c_j$ . Let it be for  $j = k$ .

- (i) If all  $a_{ik}$  are negative for all  $i$  then there exists an unbounded solution of the given linear programming problem.
- (ii) If there exists at least one  $a_{ik}$  is positive then the vector  $x_k$  corresponding to it enters the basis. The column in which  $x_k$  lies is known as pivotal column.
- (iii) Select the minimum ratio amongst the ratio  $\{ \frac{b_i}{a_{ik}} ; \forall a_{ik} > 0, \forall i \}$ . Let minimum ratio occurs for  $\{ \frac{b_s}{a_{sk}} ; \forall a_{sk} > 0 \}$ , then the row in which  $a_{sk}$  lies is said to be pivotal row; which is left blank in initial iteration. The intersection of pivotal row and pivotal column is known as pivotal element i.e.  $a_{sk}$ .

**Step 7:** To construct the next table for iteration, proceed similarly as the process adopted for ordinary simplex algorithm.

**Step 8:** Go to step 5. Repeat the procedure until either an optimal solution is achieved.

**3. PROBLEM FORMULATION**

Here we consider the Multi-objective linear programming problem as:

$Z = \{ Z_1, Z_2, \dots, Z_s \}$  where the objective is to find the

Maximum value of  $Z_1, Z_2, \dots, Z_r$  &

Minimum value of  $Z_{r+1}, \dots, Z_s$ .

subject to  $Ax \leq b$

and  $x \geq 0$

where  $A$  is a matrix of order  $m \times n$  and

$$S = \{x \in R^n | Ax \leq b, x \geq 0 \text{ and } b \in R^m (x \geq 0)\} \text{ is a feasible set and } s \geq 2.$$

Also,  $Z_i = c_i^T x + \alpha_i$ ;  $c_i^T \in R^n, \alpha_i \in R \forall i = 1, 2, 3, \dots, s$ .

**4. TECHNIQUES TO CONVERT MOLPP INTO A SINGLE LPP**

Firstly, we have to solve each LPP by simplex method or by graphical method (if no. of decision variables are two only) or by any other method and find the value of objective function.

Let us suppose

Max.  $Z_1 = \tau_1$   
 Max.  $Z_2 = \tau_2$   
 .....  
 Max.  $Z_r = \tau_r$   
 Min.  $Z_{r+1} = \tau_{r+1}$   
 .....  
 Min.  $Z_s = \tau_s$

**4.1 Chandra Sen’s Technique**

Now, as per Chandra Sen’s technique, we can convert MOLPP into a single LPP as follows:

$$\text{Max. } Z = \sum_1^r \frac{Z_i}{|\tau_i|} - \sum_{r+1}^s \frac{Z_i}{|\tau_i|} = \alpha x_1 + \beta x_2(\text{let}) ; \text{ where } \tau_i \text{ is non-zero for } i = 1, 2, \dots, s$$

subject to the constraints given in the problem under consideration. Here  $\tau_i$  may be positive or negative and  $\tau_i$  denotes the value of objective function of  $i^{\text{th}}$  objective.

**4.2 Arithmetic Averaging Technique**

$$\text{Max } Z = \sum_1^r \frac{Z_i}{A.M.(AA_i)} - \sum_{r+1}^s \frac{Z_i}{A.M.(AL_i)} = \alpha x_1 + \beta x_2(\text{let})$$

Where  $AA_i = |\tau_i|$  for  $i = 1, 2, \dots, r$ ;  $AL_i = |\tau_i|$  for  $i = r + 1, \dots, s$   
 A.M. here refers for Arithmetic Mean.

**4.3 Geometric Averaging Technique**

$$\text{Max } Z = \sum_1^r \frac{Z_i}{G.M.(AA_i)} - \sum_{r+1}^s \frac{Z_i}{G.M.(AL_i)} = \alpha x_1 + \beta x_2(\text{let})$$

Where  $AA_i = |\tau_i|$  for  $i = 1, 2, \dots, r$ ;  $AL_i = |\tau_i|$  for  $i = r + 1, \dots, s$   
 G.M. here refers for Geometric Mean.

**4.4 Harmonic Averaging Technique**

$$\text{Max } Z = \sum_1^r \frac{Z_i}{H.M.(AA_i)} - \sum_{r+1}^s \frac{Z_i}{H.M.(AL_i)} = \alpha x_1 + \beta x_2(\text{let})$$

Where  $AA_i = |\tau_i|$  for  $i = 1, 2, \dots, r$ ;  $AL_i = |\tau_i|$  for  $i = r + 1, \dots, s$   
 H.M. here refers for Harmonic Mean.

#### 4.5 New Averaging Techniques

Let  $m_1 = \text{Minimum } \{AA_i\}$  where  $AA_i = |\tau_i|$  and  $\tau_i$  is the maximum value of  $Z_i$  for  $i = 1, 2, \dots, r$ . Also let  $m_2 = \text{Minimum } \{AL_i\}$  where  $AL_i = |\tau_i|$  and  $\tau_i$  is the minimum value of  $Z_i$  for  $i = r + 1, \dots, s$ .

##### 4.5.1 New Arithmetic Averaging Technique :

Arithmetic average will be  $m = \frac{m_1 + m_2}{2}$

Hence, objective function becomes  $\text{Max. } Z = \frac{\sum_1^r Z_i - \sum_{r+1}^s Z_i}{m} = \alpha x_1 + \beta x_2 \text{ (let)}$

##### 4.5.2 New Geometric Averaging Technique :

Geometric average will be  $m = \sqrt{m_1 m_2}$

Hence, objective function becomes  $\text{Max. } Z = \frac{\sum_1^r Z_i - \sum_{r+1}^s Z_i}{m} = \alpha x_1 + \beta x_2 \text{ (let)}$

##### 4.5.3 New Harmonic Averaging Technique :

Harmonic average will be  $m = \frac{2}{\frac{1}{m_1} + \frac{1}{m_2}}$

Hence, objective function becomes  $\text{Max. } Z = \frac{\sum_1^r Z_i - \sum_{r+1}^s Z_i}{m} = \alpha x_1 + \beta x_2 \text{ (let)}$

After that, by taking objective function as constraint and all constraints of the same sign of inequality, reduced form of MOLPP for Modified Fourier elimination technique is as follows:

$$\text{Max } Z, \quad Z - \alpha x_1 - \beta x_2 \leq 0$$

$$\text{Such that, } Ax \leq b \quad \text{and} \quad -x \leq 0$$

Now, we have to combine the inequalities in such a way that the inequalities and the variables are reduced one by one in each iteration. If an absurd inequality  $0 \leq d$  is found at any stage, where  $d$  is a negative number then the given MOLPP has infeasible solution. Otherwise one can get the feasible solution.

## 5. NUMERICAL EXAMPLE

Solve the given multi-objective linear programming problem: (Samsun, N. et al (2017))

$$\begin{aligned} &\text{Maximize} && Z_1 = x_1 + 2x_2 \\ &\text{Maximize} && Z_2 = x_1 + 0x_2 \\ &\text{Minimize} && Z_3 = -2x_1 - 3x_2 \\ &\text{Minimize} && Z_4 = 0x_1 - x_2 \\ &\text{Subject to} && 6x_1 + 8x_2 \leq 48 \\ &&& x_1 + x_2 \geq 3 \\ &&& x_1 + 0x_2 \leq 4 \\ &&& 0x_1 + x_2 \leq 3 \\ &\text{and} && x_1, x_2 \geq 0 \end{aligned}$$

First of all, we have to solve each and every linear programming problem along with the given constraints to find the optimal value of each objective of MOLPP by any of the methods available in the literature. Here, we have applied AHA Simplex Algorithm to find the optimal value of each objective function. For the first linear program, the procedure is as under:

$$\begin{array}{ll}
 \text{(I) Max. } Z_1 = x_1 + 2x_2 & \text{Max. } Z_1 = x_1 + 2x_2 + 0x_3 \\
 \text{s.t. } 6x_1 + 8x_2 \leq 48 & \text{S t. } 6x_1 + 8x_2 + 0x_3 \leq 48 \\
 \quad x_1 + x_2 \geq 3 & \quad x_1 + x_2 - x_3 \leq 3 \\
 \quad x_1 + 0x_2 \leq 4 & \quad x_1 + 0x_2 + 0x_3 \leq 4 \\
 \quad 0x_1 + x_2 \leq 3 & \quad 0x_1 + x_2 + 0x_3 \leq 3 \\
 \text{and } x_1, x_2 \geq 0 & \text{and } x_1, x_2, x_3 \geq 0
 \end{array}$$



Now, we apply AHA simplex algorithm to find the optimal solution.

Table 1

$x_1$	$x_2$	$x_3$		$b_i$
-1	-2	0	$\leq$	0
6	8	0	$\leq$	48
1	1	-1	$\leq$	3
1	0	0	$\leq$	4
0	1	0	$\leq$	3

The most negative coefficient in the first row is -2 which corresponds to the variable  $x_2$ . Hence the entering variable is  $x_2$ . Here minimum positive ratio is 3 which is coming at two positions. Select arbitrarily the fourth row. Therefore variable  $x_2$  enters into the fourth row. Pivot element is 1.

Table 2

$x_1$	$x_2$	$x_3$		$b_i$
-1	0	0	$\leq$	6
6	0	0	$\leq$	24
1	0	-1	$\leq$	0
1	0	0	$\leq$	4
0	1	0	$\leq$	3

The most negative coefficient in the first row is -1 which corresponds to the variable  $x_1$ . Hence the entering variable is  $x_1$ . Here minimum positive ratio is 0, therefore variable  $x_1$  enters into the second row. Pivot element is 1.

Table 3

$x_1$	$x_2$	$x_3$		$b_i$
0	0	-1	$\leq$	6
0	0	6	$\leq$	24
1	0	-1	$\leq$	0
0	0	1	$\leq$	4
0	1	0	$\leq$	3

The most negative coefficient in the first row is -1 which corresponds to the variable  $x_3$ . Hence the entering variable is  $x_3$ . Here minimum positive ratio is 4, which is occurring from the two positions. Hence we have to select it arbitrarily. Let variable  $x_3$  enters into the first row. Pivot element is 6.

Table 4

$x_1$	$x_2$	$x_3$		$b_i$
0	0	0	$\leq$	10
0	0	1	$\leq$	4
1	0	0	$\leq$	4
0	0	0	$\leq$	0
0	1	0	$\leq$	3

Now, it can be observed that all the coefficients of  $x_j$  in the objective inequality is either zero or positive. Therefore, this is an optimal solution. The optimal solution occurs at  $x_1 = 4, x_2 = 3$  and  $x_3 = 4$  with Maximum  $Z_1 = (4) + 2(3) = 10$ .

Similarly, we can find the values of other objective functions.

For the second objective  $x_1 = 4, x_2 = 3$  with Maximum  $Z_2 = 4$

For the third objective  $x_1 = 4, x_2 = 3$  with Minimum  $Z_3 = (-2)(4) - (3)(3) = -17$

For the fourth objective  $x_1 = 4, x_2 = 3$  with Minimum  $Z_4 = -3$

The knowledge of these values are mandatory before applying the conversion techniques. That's why we have calculated.

Now each linear programming problem thus obtained by the techniques of Chandra Sen, Averaging techniques and New averaging techniques are being solved by the method of Modified Fourier elimination technique.

**Chandra Sen's Technique (CST)**

As per Chandra Sen's technique, the reduced linear programming is as follows:

$$\text{Max. } Z = \left(\frac{x_1 + 2x_2}{10}\right) + \frac{x_1}{4} + \left(\frac{2x_1 + 3x_2}{17}\right) + \frac{x_2}{3} = 0.4676 x_1 + 0.7098 x_2$$

$$\text{Subject to } 6x_1 + 8x_2 \leq 48$$

$$x_1 + x_2 \geq 3$$

$$x_1 + 0x_2 \leq 4$$

$$0x_1 + x_2 \leq 3$$

$$\text{and } x_1, x_2 \geq 0$$

Making standard form by treating objective function as constraint and all inequalities of same sign for Modified Fourier Elimination Technique, we have

Max. Z

$$Z - 0.4676 x_1 - 0.7098 x_2 \leq 0 \dots\dots\dots(1)$$

$$6x_1 + 8x_2 \leq 48 \dots\dots\dots(2)$$

$$-x_1 - x_2 \leq -3 \dots\dots\dots(3)$$

$$x_1 + 0x_2 \leq 4 \dots\dots\dots(4)$$

$$0x_1 + x_2 \leq 3 \dots\dots\dots(5)$$

$$-x_1 \leq 0 \dots\dots\dots(6)$$

$$-x_2 \leq 0 \dots\dots\dots(7)$$

For the first stage of Modified Fourier elimination technique, we have to construct  $l_j^+, l_j^-$  and  $l_j^0$ .

$$l_1^+ = \{ 2, 4 \} = \text{two constraints,}$$

$$l_1^- = \{ 1, 3, 6 \} = \text{three constraints}$$

$$\text{and } l_1^0 = \{ 5, 7 \} = \text{two constraints}$$

$$l_2^+ = \{ 2, 5 \} = \text{two constraints,}$$

$$l_2^- = \{ 1, 3, 7 \} = \text{three constraints}$$

$$\text{and } l_2^0 = \{ 4, 6 \} = \text{two constraints}$$

Then the Minimum  $\{ |l_j^+| \times |l_j^-| \}$ ; where  $1 \leq j \leq n = \text{Min.} \{ |l_1^+| \times |l_1^-|, |l_2^+| \times |l_2^-| \}$

$= \{ 2 \times 3, 2 \times 3 \} = \{ 6, 6 \} = 6$  which is coming from both the decision variables. Hence we can select arbitrarily. Select  $j = 2$  i.e., variable  $x_2$  for elimination, we have

Max. Z

$$.518 x_1 + 8 Z \leq 34.0704$$

$$-.4676 x_1 + Z \leq 2.1294$$

$$-x_1 \leq 12, \quad x_1 \leq 8, \quad x_1 \leq 4, \quad -x_1 \leq 0$$

Now, after eliminating  $x_2$ , the only left variable is  $x_1$ . Eliminate  $x_1$  by the above procedure, we have

$$Z \leq 5.0358, \quad Z \leq 4.935,$$

$$Z \leq 3.9998, \quad Z \leq 4.2588$$

We get different bounded values for Z. Out of these  $Z = 3.9998$  is the only value that satisfies all the inequalities altogether. Hence  $Z = 3.9998$ . Now, putting  $Z = 3.9998$  into the inequalities involved in second stage, we get different bounded values for variable  $x_2$ .

Out of these,  $x_2 = 3$  is the only value that satisfies all the inequalities altogether. Hence  $x_2 = 3$ .

Now putting  $Z = 3.9998$  and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ .

Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

<b>Arithmetic average technique (AAT)</b>	
Objective Function	$\text{Max } Z = \frac{(2x_1+2x_2)}{7} - \frac{(-2x_1-4x_2)}{10}$ $= 0.4857 x_1 + 0.6857 x_2$
Constraints	$6x_1 + 8x_2 \leq 48, \quad x_1 + x_2 \geq 3, \quad x_1 + 0x_2 \leq 4,$ $0x_1 + x_2 \leq 3, \quad \text{and } x_1, x_2 \geq 0$
After first stage of elimination	Max Z $.2286x_1 + 8Z \leq 32.9136, \quad -.4857x_1 + Z \leq 2.0571,$ $-x_1 \leq 12, \quad x_1 \leq 8, \quad x_1 \leq 4, \quad -x_1 \leq 0$
After second stage of elimination	$Z \leq 3.9999, \quad Z \leq 5.9427,$ $Z \leq 4.4571, \quad Z \leq 4.1142$
We get different bounded values for Z. Out of these $Z = 3.9999$ is the only value that satisfies all the inequalities altogether. Hence $Z = 3.9999$ . Now, putting $Z = 3.9999$ into the inequalities involved in second stage, we get different bounded values for variable $x_2$ . Out of these, $x_2 = 3$ is the only value that satisfies all the inequalities	



altogether. Hence  $x_2 = 3$ . Now putting  $Z = 3.9999$  and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

**Geometric average technique (GAT)**

Objective Function	$\text{Max } Z = \frac{(2x_1+2x_2)}{6.324} - \frac{(-2x_1-4x_2)}{7.1414}$ $= 0.5963 x_1 + 0.8763 x_2$
Constraints	$6x_1 + 8x_2 \leq 48, \quad x_1 + x_2 \geq 3, \quad x_1 + 0x_2 \leq 4$ $0x_1 + x_2 \leq 3, \quad \text{and } x_1, x_2 \geq 0$
After first stage of elimination	$\text{Max } Z$ $.4874x_1 + 8Z \leq 42.0624, \quad -.5963x_1 + Z \leq 2.6289,$ $-x_1 \leq 12, \quad x_1 \leq 8, \quad x_1 \leq 4, \quad -x_1 \leq 0$
After second stage of elimination	$Z \leq 5.0141, \quad Z \leq 7.3993$ $Z \leq 5.9889, \quad Z \leq 5.2578$

We get different bounded values for Z. Out of these  $Z = 5.0141$  is the only value that satisfies all the inequalities altogether. Hence  $Z = 5.0141$ . Now, putting  $Z = 5.0141$  into the inequalities involved in second stage, we get different bounded values for variable  $x_2$ . Out of these,  $x_2 = 3$  is the only value that satisfies all the inequalities altogether. Hence  $x_2 = 3$ . Now putting  $Z = 5.0141$  and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

**Harmonic average technique (HAT)**

Objective Function	$\text{Max } Z = \frac{(2x_1+2x_2)}{5.7143} - \frac{(-2x_1-4x_2)}{5.1}$ $= 0.7421 x_1 + 1.13421 x_2$
Constraints	$6x_1 + 8x_2 \leq 48, \quad x_1 + x_2 \geq 3, \quad x_1 + 0x_2 \leq 4$ $0x_1 + x_2 \leq 3, \quad \text{and } x_1, x_2 \geq 0$
After first stage of elimination	$\text{Max } Z$ $.86846x_1 + 8Z \leq 54.44208, \quad -.7421x_1 + Z \leq 3.40263$ $-x_1 \leq 12, \quad x_1 \leq 8, \quad x_1 \leq 4, \quad -x_1 \leq 0$
After second stage of elimination	$Z \leq 6.37103, \quad Z \leq 9.33943$ $Z \leq 8.10795, \quad Z \leq 6.80526$

We get different bounded values for Z. Out of these  $Z = 6.37103$  is the only value that satisfies all the inequalities altogether. Hence  $Z = 6.37103$ . Now, putting  $Z = 6.37103$  into the inequalities involved in second stage, we get different bounded values for variable  $x_2$ . Out of these,  $x_2 = 3$  is the only value that satisfies all the inequalities altogether. Hence  $x_2 = 3$ . Now putting  $Z = 6.37103$  and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

**New Arithmetic average technique (NAAT)**

Objective Function	$\text{Max } Z = \frac{(2x_1+2x_2)}{3.5} - \frac{(-2x_1-4x_2)}{3.5}$ $= 1.1428 x_1 + 1.7143 x_2$
Constraints	$6x_1 + 8x_2 \leq 48, \quad x_1 + x_2 \geq 3, \quad x_1 + 0x_2 \leq 4,$ $0x_1 + x_2 \leq 3, \quad \text{and } x_1, x_2 \geq 0$
After first stage of elimination	$\text{Max } Z$ $1.1434x_1 + 8Z \leq 82.2864, \quad -1.1428x_1 + Z \leq 5.1429$ $-x_1 \leq 12, \quad x_1 \leq 8, \quad x_1 \leq 4, \quad -x_1 \leq 0$
After second stage of elimination	$Z \leq 9.7141, \quad Z \leq 14.2853$ $Z \leq 12.0009, \quad Z \leq 10.2858$

We get different bounded values for Z. Out of these  $Z = 9.7141$  is the only value that satisfies all the inequalities altogether. Hence  $Z = 9.7141$ . Now, putting  $Z = 9.7141$  into the inequalities involved in second stage, we get different bounded values for variable  $x_2$ . Out of these,  $x_2 = 3$  is the only value that satisfies all the inequalities altogether. Hence  $x_2 = 3$ . Now putting  $Z = 9.7141$  and  $x_2 = 3$  into the inequalities involved in first stage, we get different bounded values for the variable  $x_1$ . Out of these,  $x_1 = 4$  is the only value that satisfies all the inequalities altogether. Hence  $x_1 = 4$ .

**New Geometric average technique (NGAT)**

Objective Function	$\text{Max } Z = \frac{(2x_1+2x_2)}{3.4641} - \frac{(-2x_1-4x_2)}{3.4641}$
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	$= 1.1547 x_1 + 1.73205 x_2$
Constraints	$6x_1 + 8 x_2 \leq 48$ , $x_1 + x_2 \geq 3$ , $x_1 + 0 x_2 \leq 4$ , $0x_1 + x_2 \leq 3$ , and $x_1, x_2 \geq 0$
After first stage of elimination	Max Z $1.1547x_1 + 8 Z \leq 83.1384$ , $-1.1547 x_1 + Z \leq 5.19615$ , $-x_1 \leq 12$ , $x_1 \leq 8$ , $x_1 \leq 4$ , $-x_1 \leq 0$
After second stage of elimination	$Z \leq 9.81495$ , $Z \leq 14.43375$ $Z \leq 12.12435$ , $Z \leq 10.3923$
We get different bounded values for Z. Out of these Z = 9.81495 is the only value that satisfies all the inequalities altogether. Hence Z = 9.81495. Now, putting Z = 9.81495 into the inequalities involved in second stage, we get different bounded values for variable $x_2$ . Out of these, $x_2 = 3$ is the only value that satisfies all the inequalities altogether. Hence $x_2 = 3$ . Now putting Z =9.81495 and $x_2 = 3$ into the inequalities involved in first stage, we get different bounded values for the variable $x_1$ . Out of these, $x_1 = 4$ is the only value that satisfies all the inequalities altogether. Hence $x_1 = 4$ .	
<b>New Harmonic average technique (NHAT)</b>	
Objective Function	Max Z = $\frac{(2x_1+2x_2)}{3.4285} - \frac{(-2x_1-4 x_2)}{3.4285}$ $= 1.1666 x_1 + 1.7500x_2$
Constraints	$6x_1 + 8 x_2 \leq 48$ , $x_1 + x_2 \geq 3$ , $x_1 + 0 x_2 \leq 4$ , $0x_1 + x_2 \leq 3$ , and $x_1, x_2 \geq 0$
After first stage of elimination	Max Z $1.1672 x_1 + 8 Z \leq 84$ , $-1.1666 x_1 + Z \leq 5.25$ , $-x_1 \leq 12$ , $x_1 \leq 8$ , $x_1 \leq 4$ , $-x_1 \leq 0$
After second stage of elimination	$Z \leq 9.9164$ , $Z \leq 14.5828$ $Z \leq 12.2508$ , $Z \leq 10.5$
We get different bounded values for Z. Out of these Z = 9.9164 is the only value that satisfies all the inequalities altogether. Hence Z = 9.9164. Now, putting Z = 9.9164 into the inequalities involved in second stage, we get different bounded values for variable $x_2$ . Out of these, $x_2 = 3$ is the only value that satisfies all the inequalities altogether. Hence $x_2 = 3$ . Now putting Z =9.9164 and $x_2 = 3$ into the inequalities involved in first stage, we get different bounded values for the variable $x_1$ . Out of these, $x_1 = 4$ is the only value that satisfies all the inequalities altogether. Hence $x_1 = 4$ .	

**6. RESULTS AND DISCUSSIONS**

The value of each objective function as well as value of an optimal point which is recovered from each linear programming problem obtained by MOLPP after applying Modified Fourier elimination technique is tabulated below:

Chandra Sen's technique	Arithmetic average technique	Geometric average technique	Harmonic average technique	New arithmetic average technique	New geometric average technique	New harmonic average technique
$x_1 = 4$ , $x_2 = 3$ and Z = 3.9998	$x_1 = 4$ , $x_2 = 3$ and Z = 3.9999	$x_1 = 4$ , $x_2 = 3$ and Z = 5.0141	$x_1 = 4$ , $x_2 = 3$ and Z = 6.37103	$x_1 = 4$ , $x_2 = 3$ and Z = 9.7141	$x_1 = 4$ , $x_2 = 3$ and Z = 9.81495	$x_1 = 4$ , $x_2 = 3$ and Z = 9.9164

It is obvious to say that these value are similar to the values obtained from traditional simplex method or by Graphical method (in case of having only two variables) or by any other method available in the literature. Also it is very much clear that if we use new harmonic average technique to solve MOLPP, then one can get the best optimal solution.

## 6. CONCLUSION

In this research paper, we used the AHA simplex algorithm and Modified Fourier elimination technique to solve MOLPP for the very first time. Different researchers used to solve traditional simplex method to solve MOLPP while we tried to apply different algorithms and techniques. It is observed that harmonic average technique possess dominance property than the techniques of Chandra Sen, Arithmetic averaging and Geometric averaging techniques. When we apply new statistical averaging techniques, the value of the objective function gets optimize as compared to statistical averaging techniques. New harmonic averaging technique provides best optimize value of the objective function as compared to new arithmetic and new geometric averaging techniques.

## CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

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Authors of original research should present an accurate account of the work performed and the results, followed by an objective discussion of the significance of the work. The manuscript should contain sufficient detail and references to permit others to replicate the work. Review articles should be accurate, objective and comprehensive, while editorial 'opinion' or perspective pieces should be clearly identified as such. Fraudulent or knowingly inaccurate statements constitute unethical behavior and are unacceptable.

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