

CONTENTS

RESOLVING NUMBER OF EDGE CYCLE GRAPHS J. Paulraj Joseph, N. Shunmugapriya	01-16	A STUDY ON RE-MANUFACTURING OF USED PRODUCTS IN A VENDOR-BUYER SUPPLY CHAIN W. Ritha and J. C. Eveline	133-138
MELLIN AND LAPLACE TRANSFORMS INVOLVING THE PRODUCT OF STRUVE'S FUNCTION AND I-FUNCTION OF TWO VARIABLES B. Satyanarayana, B.V. Purnima & Y. Pragathi Kumar	17-24	MATHEMATICAL PRINCIPLES IN VEDAS AND PURANAS AND ITS APPLICATIONS A. K. Sah	139-146
ANALYSIS OF HOT STANDBY DATABASE SYSTEM WITH STANDBY UNIT UNDER CONSTANT OBSERVATION Amit Manocha, Sukhvir Singh & Anil Taneja	25-32	STUDY ON FUZZY γ^* - SEMI INTERIOR AND FUZZY γ^* -SEMICLOSURE M. Vijaya, M. Rajesh	147-150
A COMMON FIXED POINT THEOREM USING COMPATIBILITY OF TYPE (A-1) AND WEAKLY COMPATIBILITY V. Nagaraju, Bathini Raju	33-38	WEAKLY COMPATIBLE MAPS AND COMMON COUPLED FIXED POINT THEOREMS IN GENERALIZED FUZZY METRIC SPACE Seema Mehra	151-158
ON LOWER SEPERATION AXIOMS VIA IDEALS Nitakshi Goyal, Navpreet Singh Noorie	39-44	HYPER-ASYMPTOTIC CURVES ON A KAEHLERIAN HYPER-SURFACE U. S. Negi	159-162
RELIABILITY AND PROFIT ANALYSIS OF A SYSTEM WITH INSTRUCTION, REPLACEMENT AND TWO OF THE THREE TYPES OF REPAIR POLICY Rashmi Gupta	45-58	COMPARATIVE ANALYSIS OF TWO RELIABILITY MODELS WITH REGARD TO PROVISION OF ASSISTANT REPAIRMAN AND DISCUSSION TIME Anil Kumar Taneja	163-172
WEAK ISOMORPHISM ON BI-POLAR TOTAL FUZZY GRAPH M. Vijaya, V. Mekala	59-66	ON A NEW SUMMATION FORMULA FOR THE H-FUNCTION Vivek Rohira, Shantha Kumari. K. & Arjun K. Rathie	173-178
APPLICATIONS OF STABLE SET PROBLEMS IN HYPERGRAPH A. Ramesh Kumar, G. Kavitha	67-76	CHARACTERISTICS OF RIEMANNIAN MANIFOLDS AND PSEUDO-RIEMANNIAN METRIC K.C. Petwal and Kulveer Singh Rana	179-190
A SURVEY OF E – SUPER VERTEX MAGIC LABELINGNESS AND V – SUPER VERTEX MAGIC LABELINGNESS Seema Mehra and Neelam Kumari	77-82	COMPARATIVE STUDY OF EFFECTS OF RAMP ANGLE ON THE NON-LINEAR PASSIVE DYNAMIC WALKING OF KNEED AND KNEE-LESS BIPED ROBOT Mahesh A. Yeolekar	191-198
Q-HYPERCONVEXITY IN T_0 -ULTRA-QUASI-METRIC SPACES AND EXISTENCE OF FIXED POINT THEOREMS Qazi Aftab Kabir, Ramakant Bhardwaj, Rizwana Jamal & Masroor Mohammad	83-88	MATHEMATICAL MODEL OF POPULATION DYNAMICS AND GROWTH IN INDIA Dr. Shankar Lal	199-208
COMPUTATION OF AUGMENTED ZAGREB INDEX AND THEIR POLYNOMIAL OF CERTAIN CLASS OF WINDMILL GRAPHS B. Chalubaraju and S. A. Diwakar	89-94	COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS SATISFYING IMPLICIT RELATION IN MODULAR METRIC SPACES Prerna Pathak, Aklesh Pariya, V. H. Badshah & Nirmala Gupta	209-216
POWER GRAPH OF SOME FINITE GROUPS Z_n AND C_n OF PRIME ORDER Ajay Kumar, Manju Pruthi	95-98	EVOLUTIONARY COMPUTATIONAL ALGORITHM AND ANN SUPERVISED CLASSIFIER FOR MICRO-ARRAY GENE EXPRESSION DATA Manaswini Pradhan	217-230
APPROXIMATION OF THE CUBIC FUNCTIONAL EQUATION IN RANDOM NORMED SPACES : DIRECT AND FIXED POINT METHOD Nawneet Hooda, Shalini Tomar	99-114		
CENTRAL TENDENCY OF ANNUAL EXTREMUM OF AMBIENT AIR TEMPERATURE AT DHUBRI Rinamani Sarmah Bordoloi, Dhritikesh Chakrabarty	115-124		
C-COMPLEX MATRIX COMPLETION PROBLEM M. S. Ponmudi & A. Ramesh Kumar	125-132		



Dr. T.P. Singh

Professor in Maths & O.R.
Chief Editor

The Journal : 10 Years of Journey

About 10 years back, a group of enthusiastic professors, mathematicians, economists and researchers from different institutes of higher learning gathered together under the registered trust/society 'Aryans Research and Educational Trust' and decided to promote research activities in interdisciplinary approach through a research journal. The title of journal was named as '**Aryabhatta Journal of Mathematics & Informatics**' (AJMI) in honor of ancient famous mathematical scientist. '**Aryabhatta**' (born in 5th century) who for the first time established the healthy tradition of scientific research in India discarding the traditional way of thinking. The responsibility for registration, title approval, getting ISSN No. and to act as Editor in chief was assigned to me. On completing all the formalities, the first issue of the journal was published in year 2009. Since then and till today the journal is publishing regularly well in time. It gives us a great pleasure to put forward before the scholars and academicians that **The Journal** from its start is making an effort to produce good quality articles. The credit goes to its editorial and reviewer team which review sincerely and furnish valuable suggestions to improve its quality. We are proud to mention that **AJMI** is among 50 Indian journals (Rank 10) based on citation per year from foreign countries (table 3.5 on page 30, a report based on Indian Citation Index 2016 under supervision of Sh. Prakash Chand, Scientist NISCAIR-CSIR and Head ICI). The Journal has 1.583 citation per paper and its impact factor is continuously increasing. The Journal has been indexed by many National and International agencies as **Copernicus, Indian Citation index(ICI), cite factor Google scholars, CNKI scholar, EBSCO Discover etc.** The Journal is in **approved list of UGC** for research advancement. AJMI discourages any type of plagiarism in the paper and motivate the authors to have self plagiarism not exceed 20% while sending the paper, if at any time it is found we remove such paper from our website.

Aryabhatta Journal of Mathematics & Informatics mainly covers area of mathematical and statistical sciences, Operational Research, data based management, economical issues and information sciences. Mathematics being an interdisciplinary approach, is the core of computer simulation, physical sciences, quantitative analysis of management and economic issues, above all it is a key of key technologies of our times while information sciences the fastest growing segment due to industrial liberation, changes business environment, globalization and the trends in world economic scenario has posed an increasing challenges for the organization to be competitive and productive. The statistics reveal that today no organization or individual is without communicating or information device. Information Technology is becoming the dominant force in our culture and will continue to transform the key and the world we live and work. Information is an asset which is currently as important as capital or work. The Journal aims to focus on all such issues in mathematical, technical and business domain using the available set of knowledge.

We are of the opinion, it is good that life should be on going search, the journey is more important than the destination.

DR. T.P. SINGH



RESOLVING NUMBER OF EDGE CYCLE GRAPHS

J. Paulraj Joseph*, **N. Shunmugapriya****

*&** Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu, India
 E-mail: prof.jpaulraj@gmail.com, nshunmugapriya2013@gmail.com

ABSTRACT :

Let $G = (V, E)$ be a simple connected graph. An ordered subset W of V is said to be a resolving set of G if every vertex is uniquely determined by its vector of distances to the vertices in W . The minimum cardinality of a resolving set is called the resolving number of G and is denoted by $r(G)$. In this paper, we introduce the edge cycle graph $G(C_k)$ of a graph G and find the exact value of the resolving number for some edge cycle graphs. We also find lower and upper bounds and characterize the extremal graphs.

AMS Subject Classification: Primary 05C12, Secondary 05C35.

Keywords: resolving number, edge cycle graph.

1. INTRODUCTION

Let $G = (V, E)$ be a finite, simple, connected and undirected graph. The *degree* of a vertex v in a graph G is the number of edges incident with v and it is denoted by $d(v)$. The maximum degree in a graph G is denoted by $\Delta(G)$ and the minimum degree is denoted by $\delta(G)$. If $\delta(G) = \Delta(G)$, then the vertices of G have the same degree and G is called *regular*. If $\deg(v) = r$ for every v of G , where $1 \leq r \leq n - 1$, then G is r -regular or regular of degree r . The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest u - v path in G . The maximum value of distance between vertices of G is called its *diameter*.

A graph H is called a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let P_n denote any *path* on n vertices, C_n denote any *cycle* on n vertices and K_n denote any *complete graph* on n vertices. For a cut vertex v of a connected graph G , suppose that the disconnected graph $G \setminus \{v\}$ has k components G_1, G_2, \dots, G_k ($k \geq 2$). The induced sub-graphs $B_i = G[V(G_i) \cup \{v\}]$ are connected and referred to as the *branches* of G at v . The *complement* G^c of a graph G is that graph whose vertex set is $V(G)$ and such that for each pair u, v of vertices of G , uv is an edge of G^c if and only if uv is not an edge of G . The *join* $G + H$ consists of $G \cup H$ and all edges joining a vertex of G and a vertex of H . For an integer $s \geq 2$, $sK_2 + K_1$ is called the *friendship graph* and is denoted by F_s . A *clique* in a graph G is a complete sub-graph of G . The order of the largest clique in a graph G is its *clique number* and is denoted by $\omega(G)$. A *proper vertex coloring* of a graph G is an assignment of colors to the vertices of G , one color to each vertex, so that adjacent vertices are colored differently. A graph G is *k-colorable* if there exists a coloring of G from a set of k colors. The minimum positive integer k for which G is k -colorable is the *chromatic number* of G and is denoted by $\chi(G)$.

2. RESOLVING NUMBER OF A GRAPH

If $W = \{w_1, w_2, \dots, w_k\} \subseteq V(G)$ is an ordered set, then the ordered k -tuple $(d(v, w_1), d(v, w_2), \dots, d(v, w_k))$ is called the representation of v with respect to W and it is denoted by $r(v|W)$. Since the representation for each $w_i \in W$ contains exactly one 0 in the i^{th} position, all the vertices of W have distinct representations. W is called a resolving set for G if all the vertices of $V \setminus W$ also have distinct representations. The minimum cardinality of a resolving set is called the resolving number of G and it is denoted by $r(G)$.

In 1975, Slater [10] introduced these ideas and used *locating set* for what we have called *resolving set*. He referred to the cardinality of a minimum resolving set in G as its *location number*. In 1976, Harary and Melter [4] discovered these concepts independently as well but used the term metric dimension rather than location number. In 2003, Ping Zhang and Varaporn Saenpholphat [8, 9] studied *connected resolving number* and in 2015, we introduced and studied *total resolving number*. In this paper, we use the term *resolving number* to maintain uniformity in the current literature.

We use the following results to prove new results in the subsequent sections.

Theorem 2.1. [2] A connected graph G of order $n \geq 2$ has resolving number 1 if and only if $G \cong P_n$.

Theorem 2.2. [2] A connected graph of order $n \geq 2$ has resolving number $n - 1$ if and only if $G \cong K_n$.

Observation 2.3. Let G be a graph of order $n \geq 2$. Then $1 \leq r(G) \leq n - 1$.

Theorem 2.4. [6] Let G be a graph with resolving number 2 and let $\{w_1, w_2\} \subseteq V(G)$ be a resolving set in G .

Then the following are true:

- There is a unique shortest path P between w_1 and w_2 .
- The degrees of w_1 and w_2 are at most 3.
- Every other vertex on P has degree at most 5.

Proposition 2.5. [5] Let v be a cut vertex in a graph G . Then each resolving set for G is disjoint from at most one component of $G \setminus \{v\}$. Moreover, if W is a resolving set for G which is not disjoint from at least two components of $G \setminus \{v\}$, then $W \setminus \{v\}$ is a resolving set for G .

Definition 2.6. A block of G containing exactly one cut vertex of G is called an end block of G .

Lemma 2.7. Let G be a 1-connected graph with $\delta(G) \geq 2$. Then every resolving set contains at least one non cut vertex of each end block.

Proof. Let B_1, B_2, \dots, B_b be the end blocks of G and v_i be the cut vertex of B_i in G . Let W be a resolving set of G . Then we claim that $|W \cap (V(B_i) \setminus \{v_i\})| \geq 1$ for all $1 \leq i \leq b$. Suppose $W \cap (V(B_i) \setminus \{v_i\}) = \emptyset$ for some i . Without loss of generality, let $W \cap (V(B_1) \setminus \{v_1\}) = \emptyset$ in G . Since $\delta(G) \geq 2$, there exist two distinct vertices $x_1, x_2 \in V(B_1)$ such that x_1 and x_2 are adjacent to v_1 . Then $d(x_1, w) = d(x_1, v_1) + d(v_1, w) = 1 + d(v_1, w) = d(x_2, v_1) + d(v_1, w) = d(x_2, w)$ for all $w \in W$, which is a contradiction. Therefore $W \cap (V(B_i) \setminus \{v_i\}) \neq \emptyset$ for all $1 \leq i \leq b$.

Corollary 2.8. If G contains b end blocks, then $r(G) \geq b$.

In this paper, we introduce edge cycle graph of a graph and investigate the resolving number of such graphs.

3. EDGE CYCLE GRAPHS

In this section, we define edge cycle graph of a graph G and give some properties of $G(C_k)$.

Definition 3.1. An edge cycle graph of a graph G is the graph $G(C_k)$ formed from one copy of G and $|E(G)|$ copies of P_k , where the ends of the i^{th} edge are identified with the ends of i^{th} copy of P_k . A graph G and its edge cycle graph $G(C_k)$ are shown in Fig 3.1.

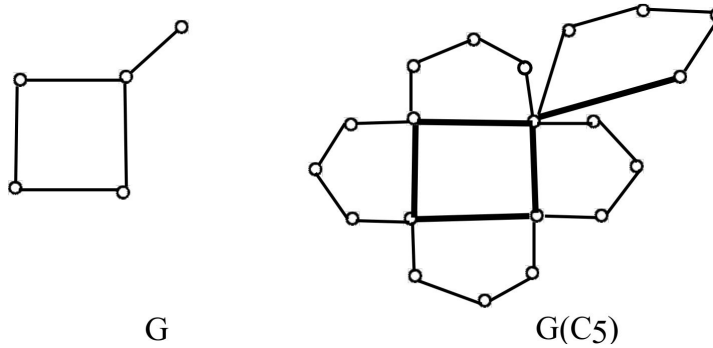


Fig 3.1 A graph G and its edge cycle graph

Some Properties of $G(C_k)$.

1. If G is a graph of order n and size m , then $|V(G(C_k))| = n + m(k - 2)$ and $|E(G(C_k))| = mk$.
2. The degree of each vertex of $V(G)$ in $G(C_k)$ is twice the degree of the vertex in G .
3. $\Delta(G(C_k)) = 2\Delta(G)$, since maximum degree in G corresponds to maximum degree in $G(C_k)$.
4. $\delta(G(C_k)) = 2$, since minimum degree corresponds to new vertices in $G(C_k)$.
5. $G(C_k)$ is connected if and only if G is connected.
6. $G(C_k)$ is Eulerian, since every vertex of $G(C_k)$ has even degree.

$$7. \text{Diam}(G(C_k)) = \begin{cases} \text{Diam}(G) + 2\left\lfloor \frac{k}{2} \right\rfloor - 1 & \text{if } k \text{ is odd} \\ \text{Diam}(G) + k - 2 & \text{if } k \text{ is even.} \end{cases}$$

8. If G is r -regular graph, then $G(C_k)$ is $(2r, 2)$ -bi-regular graph.
9. If G is a graph of order at least three, then the complement of $G(C_k)$ is connected.
10. The clique number of G and $G(C_k)$ are equal.
11. If k is even, then G is bipartite if and only if $G(C_k)$ is bipartite. If k is odd, then $G(C_k)$ is not bipartite.
12. If G is bipartite and k is even, then $\chi(G(C_k)) = \chi(G) = 2$. If G is bipartite and k is odd, then $\chi(G(C_k)) = \chi(G) + 1 = 3$. If G is not bipartite, then $\chi(G(C_k)) = \chi(G)$.
13. For any graph G , v is a cut vertex of G if and only if v is a cut vertex in $G(C_k)$.

A minimum resolving set of $G(C_3)$ with distinct representation is illustrated in Fig 3.2.

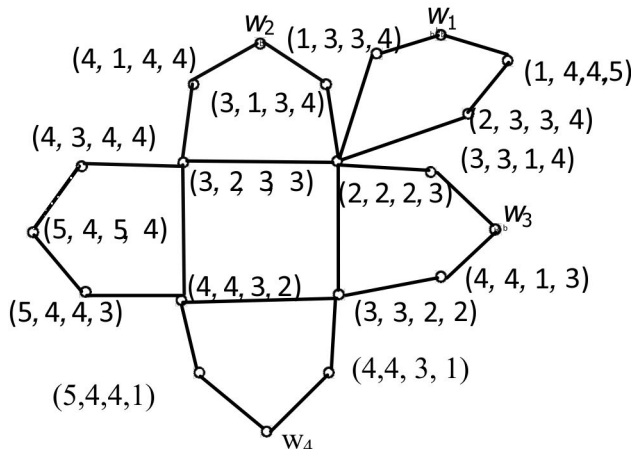


Fig 3.2 Minimum resolving set of $G(C_3)$

4. THE GRAPHS $G(C_3)$

By Property 1, $|V(G(C_3))| = n + m$ and $|E(G(C_3))| = 3m$. In this section, we determine the exact values when G is a cycle or tree. By Observation 2.3, $r(G(C_3)) \leq n + m - 1$. But we reduce the upper bound and characterize the extremal graphs.

Observation 4.1. For $n = 3$, $r(C_n(C_3)) = 2$.

Theorem 4.2. For $n \geq 4$, $r(C_n(C_3)) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd.} \end{cases}$

Proof. Let $V(C_n) = \{v_1, v_2, \dots, v_n\}$, $E(C_n) = \{v_1v_2, v_2v_3, \dots, v_nv_1\}$ and u_1, u_2, \dots, u_n be the new vertices in $C_n(C_3)$ corresponding to the edges $v_1v_2, v_2v_3, \dots, v_nv_1$. Then $V(C_n(C_3)) = V \cup U$, where $V = V(C_n)$, $U = \{u_1, u_2, \dots, u_n\}$ and $E(C_n(C_3)) = E(C_n) \cup \{u_iv_i, u_iv_{i+1} / 1 \leq i \leq n-1\} \cup \{u_nv_n, u_nv_1\}$. We consider the following two cases.

Case 1: n is even.

By Theorem 2.1, $r(C_n(C_3)) \geq 2$. Next, we claim that $r(C_n(C_3)) \leq 2$. Let $W = \{u_1, u_{\frac{n}{2}}\}$. Let x, y be two distinct vertices of $V(C_n(C_3)) \setminus W$. If $d(x, u_1) \neq d(y, u_1)$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, u_1) = d(y, u_1)$. Let $X = \{v_i / 2 \leq i \leq \frac{n}{2}\} \cup \{u_j / 2 \leq j \leq \frac{n}{2} - 1\}$. If $xy \in E(C_n(C_3))$, then $x \in U$ and $y \in V$ or $x, y \in V$. If $x \in U$ and $y \in V$, then $d(x, u_{\frac{n}{2}}) = d(y, u_{\frac{n}{2}}) + 1$. It follows that $r(x|W) \neq r(y|W)$. If $x, y \in V$, then $x = v_{\frac{n}{2}+1}$ and $y = v_{\frac{n}{2}+2}$ or $x = v_1$ and $y = v_2$. If $x = v_{\frac{n}{2}+1}$ and $y = v_{\frac{n}{2}+2}$, then $d(x, u_{\frac{n}{2}}) = 1$ and $d(y, u_{\frac{n}{2}}) = 2$. If $x = v_1$ and $y = v_2$, then y lies on x - $u_{\frac{n}{2}}$ path. It follows that $r(x|W) \neq r(y|W)$. If $xy \notin E(C_n(C_3))$, then clearly, $x \in X$ or $y \in X$ but not both. Without loss of generality, let $x \in X$. Since $d(u_1, u_{\frac{n}{2}}) = \frac{n}{2}$, $d(x, u_{\frac{n}{2}}) < d(y, u_{\frac{n}{2}})$. It follows that $r(x|W) \neq r(y|W)$. Therefore W is a resolving set of $C_n(C_3)$ and hence $r(C_n(C_3)) \leq 2$. Thus $r(C_n(C_3)) = 2$.

Case 2: n is odd.

First we claim that $r(C_n(C_3)) \geq 3$. Suppose that $r(C_n(C_3)) = 2$. Let $W = \{w_1, w_2\}$ be a resolving set of $C_n(C_3)$. By Theorem 2.4, $d(w_1) \leq 3$ and $d(w_2) \leq 3$. It follows that $w_1, w_2 \in U$. Since $\text{diam}(C_n(C_3)) = \lfloor \frac{n}{2} \rfloor + 1$, $d(w_1, w_2) = r \leq \lfloor \frac{n}{2} \rfloor + 1$. If $d(w_1, w_2) \leq \lfloor \frac{n}{2} \rfloor$, then without loss of generality, let $w_1 = u_1$ and $w_2 \in \{u_1, u_2, \dots, u_{\lfloor \frac{n}{2} \rfloor}\}$. But $d(u_n, w_1) = d(v_n, w_1) = 2$ and $d(u_n, w_2) = d(v_n, w_2) = r + 1$. It follows that $r(u_n|W) = r(v_n|W) = (2, r + 1)$, which is a contradiction. If $d(w_1, w_2) = \lfloor \frac{n}{2} \rfloor + 1$, then without loss of generality, let $w_1 = u_1$ and $w_2 = u_{\lfloor \frac{n}{2} \rfloor}$. But $d(u_2, w_1) = d(v_n, w_1) = 2$ and $d(u_2, w_2) = d(v_n, w_2) = r - 1$. It follows that $r(u_2|W) = r(v_n|W) = (2, r - 1)$, which is a contradiction. Thus $r(C_n(C_3)) \geq 3$. Next, we claim that $r(C_n(C_3)) \leq 3$. If $n = 5$, then we can easily verify $r(C_5(C_3)) \leq 3$. So we may assume that $n \geq 7$. Let $W = \{u_1, u_{\lfloor \frac{n}{2} \rfloor}, u_{n-2}\}$. If either $d(x, u_1) \neq d(y, u_1)$ or $d(x, u_{\lfloor \frac{n}{2} \rfloor}) \neq d(y, u_{\lfloor \frac{n}{2} \rfloor})$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, u_1) = d(y, u_1)$ and $d(x, u_{\lfloor \frac{n}{2} \rfloor}) = d(y, u_{\lfloor \frac{n}{2} \rfloor})$. Then $x = u_{\lfloor \frac{n}{2} \rfloor + 1}$ and $y = v_{\lfloor \frac{n}{2} \rfloor + 2}$ or $x = u_n$ and $y = v_n$. But $d(x, u_{n-2}) = d(y, u_{n-2}) + 1$. It follows that $r(x|W) \neq r(y|W)$. Therefore W is a resolving set of $C_n(C_3)$ and hence $r(C_n(C_3)) \leq 3$. Thus $r(C_n(C_3)) = 3$.

Theorem 4.3. Let T be a tree of order $n \geq 3$ and p denote the number of pendant vertices of T . Then $r(T(C_3)) = p$.

Proof. Let $V(T) = \{v_1, v_2, \dots, v_n\}$ where v_1, v_2, \dots, v_p are the pendant vertices of T . Let $E(T) = \{e_1, e_2, \dots, e_{n-1}\}$ where e_1, e_2, \dots, e_p are the pendant edges of T . Let C_i be the edge cycle of e_i . Let v_{ij} be the new vertex corresponds to the edge $v_i v_j$ in $T(C_3)$. Let $W = \{v_1, v_2, \dots, v_p\}$. We claim that W is a resolving set of $T(C_3)$. Let x, y be two distinct vertices of $V(T(C_3)) \setminus W$. We consider the following three cases.

Case 1: $x, y \in V(C_i)$ for some $1 \leq i \leq p$.

Without loss of generality, let $x, y \in V(C_1)$. Then x or y is a cut vertex of $T(C_3)$. Without loss of generality, let x be a cut vertex. Then $d(y, v_i) = d(x, v_i) + 1$ for all $2 \leq i \leq p$. It follows that $r(x|W) \neq r(y|W)$.

Case 2: $x \in V(C_i), y \in V(C_j)$ for some $1 \leq i \leq p$ and $1 \leq j \leq n - 1, i \neq j$.

Without loss of generality, let $x \in V(C_1)$ and $y \notin V(C_1)$. Since $d(x, v_1) = 1, d(y, v_1) > 1$. It follows that $r(x|W) \neq r(y|W)$.

Case 3: $x, y \notin V(C_i)$ for all $1 \leq i \leq p$.

Then we consider the following two subcases.

Subcase 1: $x, y \in V(C_i)$ for some $p + 1 \leq i \leq n - 1$.

Then x or y is a cut vertex. If x is a cut vertex in $T(C_3)$, then x lies on the path between y and a vertex v_i of W for some $1 \leq i \leq p$. Let v_1 be such a vertex. Then $d(x, v_1) < d(y, v_1)$. It follows that $r(x|W) \neq r(y|W)$.

Subcase 2: $x \in V(C_i), y \in V(C_j)$ for some $p + 1 \leq i, j \leq n - 1, i \neq j$.

If x or y is a cut vertex, say x , then x lies on the path between y and a vertex v_i of W for some $1 \leq i \leq p$. Let v_1 be such a vertex. Then clearly, $d(x, v_1) < d(y, v_1)$. If x and y are non cut vertices in $T(C_3)$, then there exist two distinct cut vertices x_1 and x_2 such that x_1 is adjacent to x and x_2 is adjacent to y . Clearly, x_1 lies on the path between x and a vertex v_i of W for some $1 \leq i \leq p$, say v_1 and simultaneously x_1 lies on the path between y and v_1 . Therefore $d(x, v_1) = d(x, x_1) + d(x_1, v_1) = 1 + d(x_1, v_1) < d(y, x_1) + d(x_1, v_1) = d(y, v_1)$. It follows that $r(x|W) \neq r(y|W)$. Thus W is a resolving set of $T(C_3)$ and hence $r(T(C_3)) \leq p$. Since $T(C_3)$ contains p end blocks, by Corollary 2.8, $r(T(C_3)) \geq p$. Hence $r(T(C_3)) = p$.

Theorem 4.4. Let G be a graph of order $n \geq 3$. Then $2 \leq r(G(C_3)) \leq n - 1$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$. Let v_{ij} be the new vertex in $G(C_3)$ corresponding to the edge $v_i v_j$. If $G \cong P_n$, then by Theorem 4.3, $r(P_n(C_3)) = 2$. If $G \cong C_n$, then by Observation 4.1 and Theorem 4.2, $r(C_n(C_3)) \leq 3$. Now we assume that $\Delta(G) \geq 3$. Let $d(v_n) = r, r \geq 3$. Let $W = \{v_1, v_2, \dots, v_{n-1}\}$. For $1 \leq i, j \leq n - 1$, i^{th} and j^{th} coordinates of the representation of v_{ij} are 1 and that of any other vertex is not 1. The first r coordinates of the representation of v_n are 1 and that of any other vertex are not 1. For $1 \leq i \leq r$, only i^{th} coordinate of the representation of v_{in} is 1 and that of any other vertex is not 1. Thus each vertex of $V(G(C_3)) \setminus W$ have distinct representations and hence W is a resolving set of $G(C_3)$. Thus $r(G(C_3)) \leq n - 1$. By Theorem 2.1, $r(G(C_3)) \geq 2$.

Theorem 4.5. Let G be a 1-connected graph of order $n \geq 3$. Then $r(G(C_3)) = 2$ if and only if $G \cong P_n$.

Proof. Let $r(G(C_3)) = 2$ and $W = \{w_1, w_2\}$ be a resolving set of $G(C_3)$. We claim that degree of each cut vertex of G is two in G . Suppose not. Let v be a cut vertex and $d(v) \geq 3$ in G . By Property 2, $d(v) \geq 6$ in $G(C_3)$. By Theorem 2.4, v does not lie on P and there exists a unique shortest path P between w_1 and w_2 . By Property 13, v is also a cut vertex in $G(C_3)$. Then there exist at least two branches at v in $G(C_3)$, say B_1 and B_2 . By the above property of the path P , it lies in either B_1 or B_2 , say B_1 and hence $W \cap V(B_2) = \emptyset$, which is a contradiction to Lemma 2.7. Thus degree of each cut vertex of G is two and hence G is a path. The proof of the converse part follows from Theorem 4.3.

Notation 4.6. The graph obtained by identifying the centre vertex of F_s and the centre vertex of $K_{1,t}$ is denoted by $F_s * K_{1,t}$. Clearly it is a 1-connected graph of order $2s + t + 1$.

Theorem 4.7. Let G be a 1-connected graph of order $n \geq 3$. Then $r(G(C_3)) = n - 1$ if and only if $G \cong K_{1,n-1}$ or $F_{s,s} \geq 2$ or $F_s * K_{1,t}, s, t \geq 1$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and v_1 be a cut vertex in G . Let v_{ij} be the new vertex in $G(C_3)$ corresponding to the edge $v_i v_j$.

Assume that $r(G(C_3)) = n - 1$. Let $C_1, C_2, \dots, C_r, r \geq 2$ be the components of $G \setminus \{v_1\}$. Then there are r branches at v . Let B_1, B_2, \dots, B_r be such branches. Then we claim that each B_i is either C_3 or K_2 for all $1 \leq i \leq r$. Suppose B_i is neither C_3 nor K_2 for some $1 \leq i \leq r$. Let B_1 be such a branch. Let $v_2 \in V(B_1)$ be adjacent to v_1 in G . Let $W = V(G) \setminus \{v_1, v_2\}$.

Let x, y be two distinct vertices of $V(G(C_3)) \setminus W$. If $x = v_1, y = v_2$, then $d(y, v) = d(y, x) + d(x, v) = 1 + d(x, v) > d(x, v)$ for all $v \in V(G(C_3)) \setminus V(B_1(C_3))$ in $G(C_3)$. It follows that $r(x|W) \neq r(y|W)$. If $x = v_1, y \neq v_2$, then $y \in V(B_1(C_3))$ or $y \in V(G(C_3)) \setminus V(B_1(C_3))$. If $y \in V(B_1(C_3))$, then $d(x, v) = 1$ for some $v \in W \cap V(B_2(C_3))$ and $d(y, v) > 1$ for all $v \in V(B_2(C_3))$ in $G(C_3)$. It follows that $r(x|W) \neq r(y|W)$. If $y \in V(G(C_3)) \setminus V(B_1(C_3))$, then $d(y, v_3) = d(y, x) + d(x, v_3) < d(x, v_3), v_3 \in V(B_1) \setminus \{v_1, v_2\}$ in $G(C_3)$. It follows that $r(x|W) \neq r(y|W)$. If $x \neq v_1$ and $y \neq v_2$, then $x, y \in \{v_{ij} / 1 \leq i, j \leq n, i \neq j\}$. If $i, j \in \{3, 4, \dots, n\}$, then clearly $r(x|W) \neq r(y|W)$. So we may assume that $x = v_{1i}$ and $y = v_{2j}$. If $i \neq j$, then clearly, $r(x|W) \neq r(y|W)$. If $i = j$, then without loss of generality, let $i = j = 3$. Since $d(x, v_3) = d(y, v_3) = 1, d(y, v) = (y, v_1) + d(v_1, v) = 2 + d(v_1, v) > 1 + d(v_1, v) = d(x, v_1) + d(v_1, v) = d(x, v)$ for all $v \in V(G(C_3)) \setminus V(B_1(C_3))$. It follows that $r(x|W) \neq r(y|W)$. Thus $r(G(C_3)) \leq n - 2$, which is a contradiction. Thus each branch at v is either C_3 or K_2 in G and hence $G \cong K_{1,n-1}$ or $F_{s,s} \geq 2$ or $F_s * K_{1,t}, s, t \geq 1$.

Conversely, let $G \cong K_{1,n-1}$ or $F_{s,s} \geq 2$ or $F_s * K_{1,t}, s, t \geq 1$. If $G \cong K_{1,n-1}$, then by Theorems 4.3 and 4.4, $r(G(C_3)) = n - 1$. In view of Theorem 4.4, it is enough, if we prove that $n - 1$ is the lower bound for $F_s(C_3), s \geq 2$ and $(F_s * K_{1,t})(C_3), s, t \geq 1$. First we consider $F_s(C_3), s \geq 2$. Let B_1, B_2, \dots, B_s be the blocks of $F_s(C_3)$ and let W be its resolving set. By Lemma 2.7, $|W \cap (V(B_i) \setminus \{v_1\})| \geq 1$ for all $1 \leq i \leq s$. By Proposition 2.5, $v_1 \notin W$. Now

we claim that $|W \cap (V(B_i) \setminus \{v_1\})| \geq 2$ for all $1 \leq i \leq s$. Suppose that $|W \cap (V(B_i) \setminus \{v_1\})| \leq 1$ for some $1 \leq i \leq s$. Let $|W \cap (V(B_1) \setminus \{v_1\})| \leq 1$. Since $|W \cap (V(B_1) \setminus \{v_1\})| \geq 1$, $|W \cap (V(B_1) \setminus \{v_1\})| = 1$. Let $V(B_1) = \{v_1, v_2, v_3, v_{12}, v_{23}, v_{13}\}$. If $v_2 \in W$, then $r(v_3|W) = r(v_{12}|W)$. If $v_{12} \in W$, then $r(v_3|W) = r(v_{13}|W)$. If $v_{23} \in W$, then $r(v_2|W) = r(v_3|W)$. Thus $|W \cap (V(B_i) \setminus \{v_1\})| \geq 2$ for all $1 \leq i \leq s$ and hence $r(F_s(C_3)) \geq 2s = n - 1$. If $G \cong F_s * K_{1,t}$, $t \geq 1$, then we can similarly prove that $r(G(C_3)) \geq n - 1$. Hence $r(G(C_3)) = n - 1$.

5. THE GRAPHS $G(C_k)$, $k \geq 4$

In this section, we prove that a minimum resolving set of $G(C_k)$ contains no vertex of $V(G)$.

Definition 5.1. A set of edges in a graph is independent if no two edges in the set are adjacent. The edge independence number $\beta_1(G)$ of a graph G is the maximum cardinality taken over all maximal independent sets in G .

Lemma 5.2. Let v be a vertex of degree r in G and e_1, e_2, \dots, e_r be edges incident with v and C_i be the edge cycle of e_i , $1 \leq i \leq r$. Then every resolving set of $G(C_k)$ contains at least one vertex of degree 2 from C_i for all $1 \leq i \leq r$ with at most one exception.

Proof. Let $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$, $1 \leq i \leq r$ and $v_{i1} = v$, $v_{i1}v_{ik} = e_i$. Let W be any resolving set of $G(C_k)$ and $A_i = \{v_{i2}, v_{i3}, \dots, v_{i(k-1)}\}$. Then we claim that $W \cap V(A_i) \neq \emptyset$ for all $1 \leq i \leq r$ with at most one exception. Suppose not. Then without loss of generality let $W \cap V(A_1) = W \cap V(A_2) = \emptyset$. Then $d(v_{12}, u) = d(v_{12}, v) + d(v, u) = 1 + d(v, u) = d(v_{22}, v) + d(v, u) = d(v_{22}, u)$ for all $u \in V(G(C_k)) \setminus (A_1 \cup A_2)$. Since $W \subset V(G(C_k)) \setminus (A_1 \cup A_2)$, $d(v_{12}, w) = d(v_{22}, w)$ for all $w \in W$. Thus $r(v_{12}|W) = r(v_{22}|W)$, which is a contradiction. Hence $W \cap V(A_i) \neq \emptyset$ for all $1 \leq i \leq r$ with at most one exception.

Lemma 5.3. Let e be an edge of degree s and e_1, e_2, \dots, e_{s-2} be the edges adjacent to e in G . If any resolving set W of $G(C_k)$ does not contain any internal vertex of the edge cycle of e , then W contains at least one internal vertex from each edge cycle of e_i , $1 \leq i \leq s - 2$.

Proof. Let C_i be the edge cycle of e_i and $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$, $e_i = v_{i1}v_{ik}$. Let $V(A_i) = \{v_{i2}, v_{i3}, \dots, v_{i(k-1)}\}$. We claim that $W \cap V(A_i) \neq \emptyset$ for all $1 \leq i \leq s - 2$. Suppose $W \cap V(A_i) = \emptyset$ for some $1 \leq i \leq s - 2$. Without loss of generality, let $W \cap V(A_1) = \emptyset$. Let C be the edge cycle of e . Then W does not contain any vertex of degree 2 from C_1 and C , which is a contradiction to Lemma 5.2. Hence $W \cap V(A_i) \neq \emptyset$ for all $1 \leq i \leq s - 2$.

Theorem 5.4. Let $E_1 = \{e_1, e_2, \dots, e_t\}$ be a subset of edges of G and W be a resolving set of $G(C_k)$. If W does not contain any internal vertex of edge cycle of e_i for all $1 \leq i \leq t$, then E_1 is independent.

Proof. The proof follows from Lemmas 5.2 and 5.3.

Theorem 5.5. Let G be a graph of order $n \geq 5$ and $\delta(G) \geq 2$. If W is a minimum resolving set of $G(C_k)$, then $W \cap V(G) = \emptyset$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ and $E(G) = \{e_1, e_2, \dots, e_m\}$. Let C_i be the edge cycle of e_i and $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$, $e_i = v_{i1}v_{ik}$. Let $V(A_i) = V(C_i) \setminus \{v_{i1}, v_{ik}\}$. Let $W_1 = W \setminus V(G)$. We claim that W_1 is a resolving set of $G(C_k)$. Let x and y be two distinct vertices of $V(G(C_k)) \setminus W_1$. We consider the following two cases.

Case 1: $x, y \in V(C_i)$ for some i . Without loss of generality, let $x, y \in V(C_1)$. Let $e_1 = v_1v_2$ and $v_1 = v_{11}, v_2 = v_{1k}$. Since $\delta(G) \geq 2$, $d(v_1) \geq 2$ and $d(v_2) \geq 2$. Therefore two distinct edges e_2 and e_3 such that e_2 is incident with v_1 and e_3 is incident with v_2 . We consider the following two sub-cases :

Subcase 1: $W \cap V(A_1) = \emptyset$.

By Lemma 5.3, $W \cap V(A_2) \neq \emptyset$ and $W \cap V(A_3) \neq \emptyset$. Let $a \in W \cap V(A_2)$ and $b \in W \cap V(A_3)$. If $d(x, a) \neq d(y, a)$, then $r(x|W_1) \neq r(y|W_1)$. So we may assume that $d(x, a) = d(y, a)$. Then x or y lies on $v_1-v_{1\lfloor \frac{k}{2} \rfloor}$ path. Without loss of generality, let x lie on $v_1-v_{1\lfloor \frac{k}{2} \rfloor}$ path. Therefore

$$d(x, b) = \begin{cases} d(y, b) + 1 & \text{if } x = v_{1\lfloor \frac{k}{2} \rfloor}, y = v_{1\lfloor \frac{k}{2} \rfloor + 1} \text{ and } k \text{ is odd} \\ d(y, b) + 2 & \text{otherwise.} \end{cases}$$

It follows that $r(x|W_1) \neq r(y|W_1)$.

Subcase 2: $W \cap V(A_1) \neq \emptyset$.

Let $a \in W \cap V(A_1)$. If $d(x, a) \neq d(y, a)$, then $r(x|W_1) \neq r(y|W_1)$. So we may assume that $d(x, a) = d(y, a)$. If $a = v_{1\lfloor \frac{k}{2} \rfloor}$, then x or y lies on $a-v_{11}$ path. Without loss of generality, let x lie on $a-v_{11}$ path. If either $W \cap V(A_2) \neq \emptyset$ or $W \cap V(A_3) \neq \emptyset$, without loss of generality, let $W \cap V(A_2) \neq \emptyset$. Let $b \in W \cap V(A_2)$. Let $v_1 = v_{21}, v_2 = v_{31}$ and $b = v_{2\lfloor \frac{k}{2} \rfloor}$. Then

$$d(y, b) = \begin{cases} d(x, b) + 1 & \text{if } k \text{ is odd} \\ d(x, b) + 2 & \text{if } k \text{ is even.} \end{cases}$$

It follows that $r(x|W_1) \neq r(y|W_1)$. So we may assume that $W \cap V(A_2) = \emptyset$ or $W \cap V(A_3) = \emptyset$ and $d(v_1) = d(v_2) = 2$. By Theorem 5.4, e_2 and e_3 are independent edges in G . Since $\delta(G) \geq 2$, $d(v_{2k}) \geq 2$. Let e_4 be incident with v_{2k} . Let $v_{2k} = v_{41}, v_{2k} = v_3$ and $v_{3k} = v_4$. Since $n \geq 5$, a vertex of $V(G) \setminus \{v_1, v_2, v_3, v_4\}$ is adjacent to v_3 or v_4 . Without loss of generality, let v_5 be adjacent to v_3 . Let

$e_4 = v_3v_5$. Since $W \cap V(A_2) = \emptyset$, by Lemma 5.3, $W \cap V(A_4) \neq \emptyset$. Let $v_3 = v_{41}$ and $b = v_{2\lfloor \frac{k}{2} \rfloor}$. Then

$$d(y, b) = \begin{cases} d(x, b) + 1 & \text{if } k \text{ is odd} \\ d(x, b) + 2 & \text{if } k \text{ is even.} \end{cases}$$

It follows that $r(x|W_1) \neq r(y|W_1)$.

Case 2: $x \in V(C_i)$ and $y \in V(C_j)$, $i \neq j$.

Without loss of generality, let $x \in V(C_1), y \in V(C_2)$. If either $W \cap V(A_1) \neq \emptyset$ or $W \cap V(A_2) \neq \emptyset$, without loss of generality, let $W \cap V(A_1) \neq \emptyset$. Let $a \in W \cap V(A_1)$ and $a = v_{1\lfloor \frac{k}{2} \rfloor}$. If $d(x, v_{1\lfloor \frac{k}{2} \rfloor}) \neq d(y, v_{1\lfloor \frac{k}{2} \rfloor})$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_{1\lfloor \frac{k}{2} \rfloor}) = d(y, v_{1\lfloor \frac{k}{2} \rfloor})$. Clearly, e_1 is incident with v_{11} or v_{1k} , say v_{11} in G . Let $v_{11} = v_{21}$. Then k is even, $x = v_{1k}$ and $y = v_{22}$ or $y = v_{2k}$. Assume that $y = v_{22}$. If $W \cap V(A_2) \neq \emptyset$, then let $v_{\frac{2k}{2}} \in W \cap V(A_2)$. But $d(x, v_{\frac{2k}{2}}) = d(y, v_{\frac{2k}{2}}) + 2$. So we may assume that $W \cap V(A_2) = \emptyset$. Since $\delta(G) \geq 2$, $d(v_{1k}) \geq 2$ and $d(v_{2k}) \geq 2$ in G . Let e_3 be incident with v_{1k} and $v_{31} = v_{1k}$. If $W \cap V(A_3) \neq \emptyset$, then let $v_{\frac{3k}{2}} \in W \cap V(A_3)$. But $d(y, v_{\frac{3k}{2}}) = d(x) + 2$. So we may assume that $W \cap V(A_3) = \emptyset$. Since $\delta(G) \geq 2$, $d(v_{3k}) \geq 2$ in G . Let e_4 be

incident with v_{3k} . Let $v_{41} = v_{3k}$. If $v_{4k} = v_{11}$, then $d(y, v_{\frac{4k}{2}}) = d(x, v_{\frac{4k}{2}}) + 1$. If $v_{4k} \neq v_{11}$, then $d(y, v_{\frac{4k}{2}}) = d(x, v_{\frac{4k}{2}}) + 2$. If $y = v_{2k}$, then we can similarly prove that $r(x|W) \neq r(y|W)$.

So we may assume that $W \cap V(A_1) = \emptyset$ and $W \cap V(A_2) = \emptyset$. By Lemma 5.3, e_1 and e_2 are independent edges in G . Let $e_1 = v_1v_2$, $e_2 = v_3v_4$. Since $d(v_1) \geq 2$, $d(v_2) \geq 2$, $d(v_3) \geq 2$ and $d(v_4) \geq 2$, let e_3 be incident with v_1 , e_4 be incident with v_2 , e_5 be incident with v_3 and e_6 be incident with v_4 . Since $W \cap V(A_1) = \emptyset$, $W \cap V(A_2) = \emptyset$ and by Lemma 5.3, $W \cap V(A_3) \neq \emptyset$, $W \cap V(A_4) \neq \emptyset$, $W \cap V(A_5) \neq \emptyset$ and $W \cap V(A_6) \neq \emptyset$. Let $a \in W \cap V(A_3)$, $b \in W \cap V(A_4)$, $c \in W \cap V(A_5)$ and $d \in W \cap V(A_6)$. If either $e_3 \neq e_5$ or $e_4 \neq e_6$, then without loss of generality, let $e_3 \neq e_5$. Let $X = \{a, b, c\}$. Then $r(x|X) \neq r(y|X)$. It follows that $r(x|W) \neq r(y|W)$. So we may assume that $e_3 = e_5$ and $e_4 = e_6$. Since $n \geq 5$, there exists a vertex of $V(G) \setminus \{v_1, v_2, v_3, v_4\}$, say v_5 such that v_5 is adjacent to one of v_1, v_2, v_3 and v_4 , say v_1 . Let $v_1v_5 = e_m$. Since $W \cap V(A_1) = \emptyset$, by Lemma 5.3, $W \cap V(A_m) \neq \emptyset$. Let $e \in W \cap V(A_m)$. Let $Y = \{a, b, e\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. Thus W_1 is a resolving set of $G(C_k)$.

Theorem 5.6. Let G be a graph of order $n \geq 5$ and $\delta(G) = 1$. If W is a minimum resolving set of $G(C_k)$, then $W \cap V(G) = \emptyset$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ where v_1, v_2, \dots, v_p are the pendant vertices and $E(G) = \{e_1, e_2, \dots, e_m\}$ where e_1, e_2, \dots, e_p are the pendant edges. Let C_i be the edge cycle of e_i and $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$, $e_i = v_{i1}v_{ik}$. Let $V(A_1) = V(C_i) \setminus \{v_{i1}, v_{ik}\}$. Let $W_1 = W \setminus V(G)$. We claim that W_1 is a resolving set of $G(C_k)$. Let x and y be two distinct vertices of $V(G(C_k)) \setminus W_1$. We consider the following three cases.

Case 1: $x, y \in V(C_i)$ for some $1 \leq i \leq p$.

Without loss of generality, let $x, y \in V(C_1)$. Let $e_1 = v_{11}v_{1k}$ and $d(v_{1k}) = 1$ in G . By Lemma 2.7, $W \cap V(A_1) \neq \emptyset$. Let $a \in W \cap V(A_1)$ and $a = v_{1\lfloor \frac{k}{2} \rfloor}$. If $d(x, a) \neq d(y, a)$, then $r(x|W_1) \neq r(y|W_1)$. So we may assume that $d(x, a) = d(y, a)$. Let x lie on $v_{11} - v_{1\lfloor \frac{k}{2} \rfloor}$ path. Then y lies on $v_{1\lfloor \frac{k}{2} \rfloor} - v_{1k}$ path. Clearly, $d(x, w) < d(y, w)$ for all $w \in W_1 \setminus \{a\}$. It follows that $r(x|W_1) \neq r(y|W_1)$.

Case 2: $x \in V(C_i)$ and $y \in V(C_j)$ for some $1 \leq i \leq p$, $1 \leq j \leq m$ and $i \neq j$.

Without loss of generality, let $x \in V(C_1)$ and $y \in V(C_1)$. By Lemma 2.7, $W \cap V(A_1) \neq \emptyset$. Let $a \in W \cap V(A_1)$. Let $d(v_{1k}) = 1$ in G . Let $a = v_{1\lfloor \frac{k}{2} \rfloor}$. If $d(x, a) \neq d(y, a)$, then $r(x|W_1) \neq r(y|W_1)$. So we may assume that $d(x, a) = d(y, a)$. Then clearly, k is even, $x = v_{1k}$ and y is the neighbor of $V(G(C_k)) \setminus \{v_{11}\}$. Since $d(x, v) = d(y, v)$ for all $v \in V(G)$ and W is a minimum resolving set, $d(x, w) \neq d(y, w)$ for some $w \in W_1 \setminus \{a\}$. It follows that $r(x|W_1) \neq r(y|W_1)$.

Case 3: $x, y \notin V(C_i)$ for all $1 \leq i \leq p$.

The proof is similar to proof of Theorem 5.5.

BOUNDS AND EXTREMAL GRAPHS

In this section, we obtain lower and upper bounds for resolving number of $G(C_k)$, when $k \geq 4$ and characterize the extremal graphs.

Definition 6.1. Vertices which are adjacent to pendant vertices are called support vertices. Let p denote the number of pendant vertices of G and s denote the number of support vertices of G .

Theorem 6.2. Let G be a graph of order $n \geq 3$ and size m . If $\delta(G) = 1$ and k is even, then $r(G(C_k)) \leq m + p - s$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$ where v_1, v_2, \dots, v_s are the support vertices of G and $v_{s+1}, v_{s+2}, \dots, v_{s+p}$ are the pendant vertices of G . Let $E(G) = \{e_1, e_2, \dots, e_m\}$ where e_1, e_2, \dots, e_p are the pendant edges. Let $v_{s+1}, v_{s+2}, \dots, v_{2s}$ be the pendant vertices corresponding to the support vertices v_1, v_2, \dots, v_s respectively. For $1 \leq i \leq s$, let $e_i = v_i v_{s+i}$. Let C_i be the edge cycle of e_i and $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$, $1 \leq i \leq m$. For $s + 1 \leq i \leq p$, let $v_i = v_{ik}$. Let $W = \{v_{i\frac{k}{2}} / 1 \leq i \leq m\} \cup \{v_{i(\frac{k}{2}+1)} / s + 1 \leq i \leq p\}$. Then we claim that W is a resolving set of $G(C_k)$. Let x, y be two distinct vertices of $V(G(C_k)) \setminus W$. We consider the following three cases.

Case 1 : $x, y \in V(C_i)$ for some $1 \leq i \leq p$.

If $x, y \in V(C_i)$ for some $s + 1 \leq i \leq p$, then without loss of generality, let $x, y \in V(C_p)$. Let $e_p = v_p v_{pk}$, where $d(v_{pk}) = 1$ in G . Let $Y = \{v_{p\frac{k}{2}}, v_{p(\frac{k}{2}+1)}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. So we may assume that $x, y \notin V(C_i)$ for all $s + 1 \leq i \leq p$. Without loss of generality, let $x, y \in V(C_1)$. If $d(x, v_{1\frac{k}{2}}) \neq d(y, v_{1\frac{k}{2}})$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_{1\frac{k}{2}}) = d(y, v_{1\frac{k}{2}})$. Then x lies on $v_{1\frac{k}{2}}-v_{11}$ path and y lies on $v_{1\frac{k}{2}}-v_{1k}$ path. Since e_1 is a non pendant edge, there exists an edge e_z of $E(G) \setminus \{e_1\}$ such that e_z is incident with v_{11} . Therefore $d(y, v_{z\frac{k}{2}}) = d(x, v_{z\frac{k}{2}}) + 2$. It follows that $r(x|W) \neq r(y|W)$.

Case 2: $x \in V(C_i)$ and $y \in V(C_j)$ for some $1 \leq i \leq p, 1 \leq j \leq m$ and $i \neq j$.

If $x \in V(C_i)$ for some $1 \leq i \leq s$, then without loss of generality, let $x \in V(C_1)$. If $d(x, v_{1\frac{k}{2}}) \neq d(y, v_{1\frac{k}{2}})$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_{1\frac{k}{2}}) = d(y, v_{1\frac{k}{2}})$. Then $x = v_{1k}$ and y is the neighbour of v_{11} in $G(C_k) \setminus V(C_1)$. Without loss of generality, let $y \in V(C_m)$ and $v_{m1} = v_{11}$. Then $d(x, v_{m\frac{k}{2}}) = d(y, v_{m\frac{k}{2}}) + 2$. It follows that $r(x|W) \neq r(y|W)$. If $x \in V(C_i)$ for some $s + 1 \leq i \leq p$, then without loss of generality, let $x \in V(C_p)$. Let $Y = \{v_{p\frac{k}{2}}, v_{p(\frac{k}{2}+1)}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$.

Case 3 : $x, y \notin V(C_i)$ for all $1 \leq i \leq p$.

Without loss of generality, let $x \in V(C_m)$. Since e_m is a non pendant edge, there exist two edges $e_t, e_{t'}$ of $E(G) \setminus \{e_m\}$ such that e_t is incident with v_{m1} and $e_{t'}$ is incident with v_{mk} . Let $v_{m1} = v_{11}$. If $d(x, v_{m\frac{k}{2}}) \neq d(y, v_{m\frac{k}{2}})$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_{m\frac{k}{2}}) = d(y, v_{m\frac{k}{2}})$. If $y \in V(C_m)$, then x lies on $v_{mk}-v_{m\frac{k}{2}}$ path and y lies on $v_{m1}-v_{m\frac{k}{2}}$ path. If $y \notin V(C_m)$, then $x = v_{mk}$ and $y = v_{t2}$ or $y = v_{tk}$. Let $Y = \{v_{t\frac{k}{2}}, v_{t'\frac{k}{2}}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$.

Thus W is a resolving set of $G(C_k)$. Hence $r(G(C_k)) \leq m + p - s$.

Theorem 6.3. Let G be a graph of order $n \geq 3$, size m and $\delta(G) = 1$. If k is even, then $r(G(C_k)) = m + p - s$ if and only if each edge of G is incident with a support vertex of G .

Proof. Let $r(G(C_k)) = m + p - s$.

Let $V(G) = \{v_1, v_2, \dots, v_n\}$ where v_1, v_2, \dots, v_s are the support vertices of G and $v_{s+1}, v_{s+2}, \dots, v_{s+p}$ are the pendant vertices of G . Let $E(G) = \{e_1, e_2, \dots, e_m\}$ where e_1, e_2, \dots, e_p are the pendant edges. Let $v_{s+1}, v_{s+2},$

\dots, v_{2s} be the pendant vertices corresponding to the support vertices v_1, v_2, \dots, v_s respectively. For $1 \leq i \leq s$, let $e_i = v_i v_{s+i}$. Let C_i be the edge cycle of e_i and $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$, $1 \leq i \leq m$. For $1 \leq i \leq p$, let $v_i = v_{ik}$.

Now, we claim that each edge of G is incident with a support vertex of G . Suppose not. Then there exists an edge uv such that u and v are not support vertices. Let $uv = e_m$. Let $W = \{v_{i\frac{k}{2}} / 1 \leq i \leq m-1\} \cup \{v_{i(\frac{k}{2}+1)} / s+1 \leq i \leq p\}$. Let x, y be two distinct vertices of $V(G(C_k)) \setminus W$. If $x, y \in V(C_i)$ for some $1 \leq i \leq p$, then the proof is similar to Case 1 of Theorem 6.2. If $x \in V(C_i), y \in V(C_j)$ for some $1 \leq i \leq p, 1 \leq j \leq m$ and $i \neq j$, then the proof is similar to Case 2 of Theorem 6.2. So we may assume that $x, y \notin V(C_i)$ for all $1 \leq i \leq p$.

If either $x \notin V(C_m)$ or $y \notin V(C_m)$, then without loss of generality, let $x \notin V(C_m)$. Let $x \in V(C_1)$. Since e_1 is a non pendant edge, there exist two distinct edges $e_r, e_{r'}$ of $E(G) \setminus \{e_1\}$ such that e_r is incident with v_{1k} and $e_{r'}$ is incident with v_{11} . Suppose $e_r \neq e_m$ and $e_{r'} \neq e_m$. Let $Y = \{v_{1\frac{k}{2}}, v_{r\frac{k}{2}}, v_{r'\frac{k}{2}}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. Now, we assume that $e_r = e_m$. Let $v_{1k} = v_{rk}, v_{11} = v_{m1}$. Let $Y = \{v_{1\frac{k}{2}}, v_{2\frac{k}{2}}\}$. If $v_{mk} = v_{r1}$, then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. So we may assume that $v_{mk} \neq v_{r1}$. Since e_m is a non pendant edge, there exists an edge e_t of $E(G) \setminus \{e_1, e_r, e_m\}$ such that e_t is incident with v_{21} . Let $Y = \{v_{1\frac{k}{2}}, v_{2\frac{k}{2}}, v_{t\frac{k}{2}}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. Now, we assume that $x, y \in V(C_m)$. Since e_m is a non pendant edge and v_{m1}, v_{mk} are non support vertices in G , let e_{m-1} be incident with v_{m1} and e_{m-2} be incident with v_{mk} . Let $X = \{v_{(m-1)\frac{k}{2}}, v_{(m-2)\frac{k}{2}}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$.

Thus W is a resolving set with cardinality $m + p - s - 1$, which is a contradiction. Hence each edge of G is incident with a support vertex of G .

Conversely, let each edge of G be incident with a support vertex of G . By Theorem 6.2, $r(G(C_k)) \leq m + p - s$. Next, we claim that $r(G(C_k)) \geq m + p - s$. Suppose that $r(G(C_k)) \leq m + p - s - 1$. By Theorem 2.7 and our assumption, there exist either two pendant edges e_1, e_2 are incident with a support vertex v in G such that $|W \cap (V(C_1) \setminus \{v\})| = 1$ and $|W \cap (V(C_2) \setminus \{v\})| = 1$ in $G(C_k)$ or a pendant edge e_1 and a non pendant edge e_2 are incident with a support vertex v in G such that $|W \cap (V(C_1) \setminus \{v\})| = 1$ and $|W \cap (V(C_2) \setminus \{v\})| = \emptyset$ in $G(C_k)$. Thus two neighbors of v in $V(C_1) \cup V(C_2)$ have the same representation, which is a contradiction. Therefore $r(G(C_k)) \geq m + p - s$. Hence $r(G(C_k)) = m + p - s$.

Theorem 6.4. Let G be a graph of order $n \geq 3$, size m and $\delta(G) \geq 2$. If k is even, then $r(G(C_k)) \leq m - 1$.

Proof. Let $E(G) = \{e_1, e_2, \dots, e_m\}$. Let C_i be the edge cycle of e_i . Let $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$ and $v_{i1}v_{ik} = e_i, 1 \leq i \leq m$. We claim that $W = \{v_{i\frac{k}{2}} / 1 \leq i \leq m - 1\}$ is a resolving set of $G(C_k)$. Let x, y be two distinct vertices of $V(G(C_k)) \setminus W$.

If either $x \notin V(C_m)$ or $y \notin V(C_m)$, then without loss of generality, let $x \notin V(C_m)$. Let $x \in V(C_1)$. Since $\delta(G) \geq 2$, there exist two distinct edges $e_r, e_{r'}$ of $E(G) \setminus \{e_1\}$ such that e_r is incident with v_{1k} and $e_{r'}$ is incident with v_{11} . If $e_r \neq e_m$ and $e_{r'} \neq e_m$, then without loss of generality, let $e_r = e_2$ and $e_{r'} = e_3$. Let $Y = \{v_{1\frac{k}{2}}, v_{2\frac{k}{2}}, v_{3\frac{k}{2}}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. Now, we assume that $e_r = e_m$.

Let $e_r = e_2$. Let $v_{1k} = v_{2k}, v_{11} = v_{m1}$. Let $Y = \{v_{\frac{1}{2}k}, v_{\frac{2}{2}k}\}$. If $v_{mk} = v_{21}$, then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. So we may assume that $v_{mk} \neq v_{21}$. Since $\delta(G) \geq 2$, there exists an edge of $E(G) \setminus \{e_1, e_2, e_m\}$, say e_{m-1} such that e_{m-1} is incident with v_{21} . Let $Y = \{v_{\frac{1}{2}k}, v_{\frac{2}{2}k}, v_{\frac{(m-1)}{2}k}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. Now, we assume that $x, y \in V(C_m)$. Since $\delta(G) \geq 2$, let e_1 be incident with v_{m1} and e_2 be incident with v_{mk} . Let $Y = \{v_{\frac{1}{2}k}, v_{\frac{2}{2}k}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. Thus W is a resolving set of $G(C_k)$ and hence $r(G(C_k)) \leq m - 1$.

Theorem 6.5. Let G be a graph of order $n \geq 3$, size m and $\delta(G) \geq 2$. If k is even, then $r(G(C_k)) = m - 1$ if and only if $G \cong C_3$ or C_4 .

Proof. Let $E(G) = \{e_1, e_2, \dots, e_m\}$. Let C_i be the edge cycle of e_i . Let $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$ and $v_{i1}v_{ik} = e_i, 1 \leq i \leq m$. Assume that $r(G(C_k)) = m - 1$. Then we claim that $G \cong C_3$ or C_4 . Suppose G is neither C_3 nor C_4 . Let e_m and e_{m-1} be two non adjacent edges in G . Let $W = \{v_{\frac{i}{2}k} / 1 \leq i \leq m - 2\}$ be a subset of $V(G(C_k))$

with cardinality $m - 2$. Let x, y be two distinct vertices of $V(C_k) \setminus W$. We consider the following three cases.

Case 1: $x, y \in V(C_i)$ for some $1 \leq i \leq m - 2$.

Without loss of generality, let $x, y \in V(C_1)$. If $d(x, v_{\frac{1}{2}k}) \neq d(y, v_{\frac{1}{2}k})$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_{\frac{1}{2}k}) = d(y, v_{\frac{1}{2}k})$. Then x lies on $v_{11}-v_{\frac{1}{2}k}$ path and y lies on $v_{\frac{1}{2}k}-v_{1k}$ path. If e_1 is incident with v_{11} for some $2 \leq i \leq m-2$, then without loss of generality, let e_2 be incident with v_{11} . Clearly, $d(y, v_{\frac{2}{2}k}) = d(x, v_{\frac{2}{2}k}) + 2$. It follows that $r(x|W) \neq r(y|W)$. So we may assume that e_i is not incident with v_{11} for all $2 \leq i \leq m-2$. Since $\delta(G) \geq 2$, e_m or e_{m-1} , say e_m is incident with v_{11} . Let $v_{m1} = v_{11}$. Since $d(v_{mk}) = 2$ in G , let e_2 be incident with v_{mk} . Then $d(x, v_{\frac{2}{2}k}) < d(y, v_{\frac{2}{2}k})$. It follows that $r(x|W) \neq r(y|W)$.

Case 2: $x \in V(C_i)$ for some $1 \leq i \leq m - 2$ and $y \in V(C_m)$.

Without loss of generality, let $x \in V(C_1)$. Since $\delta(G) \geq 2$, there exist two edges $e_t, e_{t'}$ of $E(G) \setminus \{e_m\}$ such that e_t is incident with v_{m1} and $e_{t'}$ is incident with v_{mk} . Let $v_{t'} = v_{m1}$ and $v_{t1} = v_{mk}$. If e_1 is neither incident with v_{m1} nor incident with v_{mk} , then $d(y, v_{\frac{1}{2}k}) > d(x, v_{\frac{1}{2}k})$. It follows that $r(x|W) \neq r(y|W)$. Therefore e_1 is incident with v_{m1} or v_{mk} , say v_{m1} . Let $v_{m1} = v_{11}$. Let $e_{t'} = e_2$. If $e_t \neq e_1$, then $d(x, v_{\frac{1}{2}k}) < d(y, v_{\frac{1}{2}k})$. It follows that $r(x|W) \neq r(y|W)$. So we assume that $e_1 = e_t$ and $d(v_{m1}) = d(v_{mk}) = 2$ in G . Since $G \not\cong C_4, n \geq 5$. Therefore there exist two edges $e_r, e_{r'}$ of $E(G) \setminus \{e_1, e_m, e_{t'}\}$ such that e_r is incident with v_{1k} and $e_{r'}$ is incident with v_{2k} . Thus either $e_r \neq e_m$ or $e_{r'} \neq e_m$. Without loss of generality, let $e_r \neq e_m$ and $e_r = e_3$. Then $d(x, v_{\frac{3}{2}k}) < d(y, v_{\frac{3}{2}k})$. It follows that $r(x|W) \neq r(y|W)$.

Case 3: $x, y \notin V(C_i)$ for all $1 \leq i \leq m - 2$.

Assume that $x, y \in V(C_m)$. Since $\delta(G) \geq 2$, let e_1 be incident with v_{m1} and e_2 be incident with v_{mk} in G . Let $S = \{v_{\frac{1}{2}k}, v_{\frac{2}{2}k}\}$. Then $r(x|S) \neq r(y|S)$. It follows that $r(x|W) \neq r(y|W)$. Now, assume that $x \in V(C_m)$ and $y \in V(C_{m-1})$. Let $e_m = v_1v_2$ and $e_{m-1} = v_2v_3$. Since $\delta(G) \geq 2$, let e_1 be incident with v_{m1} and e_2 be incident with v_{mk} in

G. Let $v_{m1} = v_{11}$ and $v_{mk} = v_{21}$. Since $\delta(G) \geq 2$, there exist two edges $e_t, e_{t'}$ of $E(G) \setminus \{e_m, e_{m-1}\}$ such that e_t is incident with $v_{(m-1)1}$ and $e_{t'}$ is incident with $v_{(m-1)k}$. If either $e_1 \neq e_t$ or $e_2 \neq e_{t'}$, then without loss of generality, let $e_1 \neq e_t$. Let $e_{t'} = e_3$ and $Y = \{v_{\frac{1}{2}k}, v_{\frac{2}{2}k}, v_{\frac{3}{2}k}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. So we may assume that $e_1 = e_t$ and $e_2 = e_{t'}$. If $n = 4$, then since $G \cong C_4$, $v_{1k}v_{2k} \in E(G(C_k))$. Let $v_{m1}v_{2k} = e_3$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. Now, we assume that $n \geq 5$. Let $e_1 = v_1v_2, e_2 = v_3v_4$. Then there exists a vertex of $V(G) \setminus \{v_1, v_2, v_3, v_4\}$, say v_5 such that v_5 is adjacent to one of v_1, v_2, v_3 and v_4 , say v_1 in G . Let $v_1v_5 = e_{m-2}$. Let $v_{(m-2)1} = v_1 = v_{11}$. Then $d(x, v_{\frac{(m-2)k}{2}}) < d(y, v_{\frac{(m-2)k}{2}})$. It follows that $r(x|W) \neq r(y|W)$.

Thus W is a resolving set of $G(C_k)$ with cardinality $m - 2$, which is a contradiction. Hence $G \cong C_3$ or C_4 .

Conversely, let $G \cong C_3$ or C_4 . By Theorem 6.4, $r(G(C_k)) \leq m - 1$. If $G \cong C_3$, then by Lemma 5.3, $r(G(C_k)) \geq 2$. Next, we claim that if $G \cong C_4$, then $r(G(C_k)) \geq 3$. Suppose that $r(G(C_k)) \leq 2$ and $W = \{w_1, w_2\}$ be a resolving set of $G(C_k)$. Let $V(A_i) = V(C_i) \setminus \{v_{i1}, v_{ik}\}$. Let $E(C_4) = \{e_1, e_2, e_3, e_4\}$, where e_1, e_3 are non adjacent edges and e_2, e_4 are non adjacent edges. By Lemma 5.3, let $W \cap V(A_1) = \emptyset, W \cap V(A_2) \neq \emptyset, W \cap V(A_3) = \emptyset$ and $W \cap V(A_4) \neq \emptyset$. Let $v_{11} = v_{4k}, v_{1k} = v_{21}, v_{2k} = v_{31}$ and $v_{3k} = v_{41}$. If either $w_1 \notin \{v_{\frac{2}{2}k}, v_{\frac{2}{2}(k+1)}\}$ or $w_2 \notin \{v_{\frac{4}{2}k}, v_{\frac{4}{2}(k+1)}\}$, then without loss of generality, let $w_1 \notin \{v_{\frac{2}{2}k}, v_{\frac{2}{2}(k+1)}\}$. If $w_1 = v_{\frac{2}{2}(k-1)}$, then $r(v_{2(k-1)}|W) = r(v_{32}|W)$, which is a contradiction. So we may assume that $w_1 \in \{v_{\frac{2}{2}k}, v_{\frac{2}{2}(k+1)}\}$ and $w_2 \in \{v_{\frac{4}{2}k}, v_{\frac{4}{2}(k+1)}\}$. If $W = \{v_{\frac{2}{2}k}, v_{\frac{4}{2}k}\}$, then $r(v_{2k}|W) = r(v_{4k}|W) = (\frac{k}{2}, \frac{k}{2})$, which is a contradiction. If $W = \{v_{\frac{4}{2}k}, v_{\frac{4}{2}(k+1)}\}$, then $r(v_{1(k-1)}|W) = r(v_{2k}|W) = (\frac{k}{2}, \frac{k}{2} + 1)$, which is a contradiction. Thus $r(G(C_k)) \geq m - 1$ and hence $r(G(C_k)) = m - 1$.

Lemma 6.6. Let G be a graph of order $n \geq 3$, size m and $\delta(G) \geq 2$. Then $r(G(C_k)) \geq m - \beta_1(G)$.

Proof. Let $E(G) = \{e_1, e_2, \dots, e_m\}$ and $\beta_1(G) = s$. Let C_i be the edge cycle of e_i and W be a minimum resolving set of $G(C_k)$. Let $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$ and $V(A_i) = V(C_i) \setminus \{v_{i1}, v_{ik}\}$. We claim that $r(G(C_k)) \geq m - s$. Suppose $r(G(C_k)) \leq m - (s + 1)$. Therefore W does not contain any vertex from at least $s + 1$ sets of A_1, A_2, \dots, A_m . Let A_1, A_2, \dots, A_{s+1} be such sets. Then $W \cap V(A_i) = \emptyset$ for all $1 \leq i \leq s + 1$. By Theorem 5.4, e_1, e_2, \dots, e_{s+1} are independent edges of G , which is a contradiction to $\beta_1(G) = s$. By Theorem 5.5, $W \cap V(G) = \emptyset$ and hence $r(G(C_k)) \geq m - \beta_1(G)$.

Theorem 6.7. Let G be a graph of order $n \geq 5$ and size m . If k is odd and $\delta(G) \geq 2$, then $r(G(C_k)) = m - \beta_1(G)$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$, $E(G) = \{e_1, e_2, \dots, e_m\}$ and C_i be the edge cycle of e_i . Let $\beta_1(G) = s$ and $M = \{e_1, e_2, \dots, e_s\}$ be the maximum edge independent set of G . Let $e_i = v_{i1}v_{ik}, 1 \leq i \leq m$. Let $W = \{v_{\lfloor \frac{k}{2} \rfloor} / s + 1 \leq i \leq m\}$. We claim that W is a resolving set of $G(C_k)$. Let x, y be two distinct vertices of $V(G(C_k)) \setminus W$. We consider the following two cases.

Case 1: $x \in V(C_i)$ for some $1 \leq i \leq s$.

Without loss of generality, let $x \in V(C_1)$. Let $e_1 = v_1v_2$. Since $\delta(G) \geq 2$, $d(v_1) \geq 2$ and $d(v_2) \geq 2$. Then there exist two distinct edges $e_r, e_{r'} \in E(G) \setminus \{e_1\}$ such that e_r is incident with v_1 and $e_{r'}$ is incident with v_2 . Since $e_1 \in M$, by Lemma 5.3, $e_r, e_{r'} \notin M$. Without loss of generality, let $e_r = e_m$ and $e_{r'} = e_{m-1}$. Let

$Y = \{v_{m \lfloor \frac{k}{2} \rfloor}, v_{(m-1) \lfloor \frac{k}{2} \rfloor}\}$. If $y \in V(C_1)$, then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$. So we assume that $y \notin V(C_1)$. If either $d(x, v_{m \lfloor \frac{k}{2} \rfloor}) \neq d(y, v_{m \lfloor \frac{k}{2} \rfloor})$ or $d(x, v_{(m-1) \lfloor \frac{k}{2} \rfloor}) \neq d(y, v_{(m-1) \lfloor \frac{k}{2} \rfloor})$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_{m \lfloor \frac{k}{2} \rfloor}) = d(y, v_{m \lfloor \frac{k}{2} \rfloor})$ and $d(x, v_{(m-1) \lfloor \frac{k}{2} \rfloor}) = d(y, v_{(m-1) \lfloor \frac{k}{2} \rfloor})$. Therefore $y \notin V(C_m)$ and $y \notin V(C_{m-1})$.

If $y \in V(C_i)$ for some $s + 1 \leq i \leq m - 2$, then without loss of generality, let $y \in V(C_{m-2})$. Clearly, $d(x, v_{(m-2) \lfloor \frac{k}{2} \rfloor}) > d(y, v_{(m-2) \lfloor \frac{k}{2} \rfloor})$. It follows that $r(x|W) \neq r(y|W)$. So we may assume that $y \in V(C_i)$ for some $2 \leq i \leq s$. Without loss

of generality, let $y \in V(C_2)$. Let $e_2 = v_3v_4$. Since $d(v_3) \geq 2$ and $d(v_4) \geq 2$, there exist two edges $e_t, e_{t'}$ of $E(G) \setminus \{e_1, e_2\}$ such that e_t is incident with v_3 and $e_{t'}$ is incident with v_4 . If either $e_r \neq e_t$ or $e_{r'} \neq e_{t'}$, then without loss of generality, let $e_r \neq e_t$. Let $e_t = e_{m-2}$ and $Y = \{v_{m \lfloor \frac{k}{2} \rfloor}, v_{(m-1) \lfloor \frac{k}{2} \rfloor}, v_{(m-2) \lfloor \frac{k}{2} \rfloor}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that

$r(x|W) \neq r(y|W)$. So we may assume that $e_r = e_t$ and $e_{r'} = e_{t'}$. Since there exists a vertex of $V(G) \setminus \{v_1, v_2, v_3, v_4\}$, say v_5 such that v_5 is adjacent to one of v_1, v_2, v_3 and v_4 , say v_1 in G . Since $e_1 \in M$, by Lemma 5.3, $v_1v_5 \notin M$. Let $v_1v_5 = e_{m-2}$. Let $v_{(m-2)1} = v_1 = v_{11}$. Let $Y = \{v_{m \lfloor \frac{k}{2} \rfloor}, v_{(m-1) \lfloor \frac{k}{2} \rfloor}, v_{(m-2) \lfloor \frac{k}{2} \rfloor}\}$. Then $r(x|Y) \neq r(y|Y)$. It follows that $r(x|W) \neq r(y|W)$.

Case 2: $x \notin V(C_i)$ for all $1 \leq i \leq s$.

Without loss of generality, let $x \in V(C_m)$. If $y \in V(C_i)$ for some $1 \leq i \leq s$, then the proof is similar to Case 1. So we assume that $y \notin V(C_i)$ for all $1 \leq i \leq s$. If $y \notin V(C_m)$, then $d(x, v_{m \lfloor \frac{k}{2} \rfloor}) < d(y, v_{m \lfloor \frac{k}{2} \rfloor})$. It follows that

$r(x|W) \neq r(y|W)$. If $y \in V(C_m)$, then the proof is similar to Subcase 2 of Lemma 5.5. Thus $r(G(C_k)) \leq m - \beta_1(G)$. By Lemma 6.6, $r(G(C_k)) \geq m - \beta_1(G)$. Hence $r(G(C_k)) = m - \beta_1(G)$.

Remark 6.8. If k is odd and $\delta(G) \geq 2$, then $r(C_3(C_k)) = 2$, $r(C_4(C_k)) = r[(K_4 \setminus \{e\})(C_k)] = 3$ and $r(K_4(C_k)) = 4$.

Proof. Let C_i be the edge cycle of the edge e_i and $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$. Let $e_i = v_{i1}v_{ik}$ and $V(A_i) = V(C_i) \setminus \{v_{i1}, v_{ik}\}$. Let $G \cong C_3$. Then by Lemma 5.3, $r(C_3(C_k)) \geq 2$. Then we can easily verify that $\{v_{1 \lfloor \frac{k}{2} \rfloor}, v_{2 \lfloor \frac{k}{2} \rfloor}\}$ is a resolving set of $C_3(C_k)$. Thus $r(C_3(C_k)) = 2$. Let $G \cong K_4 \setminus \{e\}$ or K_4 . Let e_1 and e_2 be two non adjacent edges in G . By Lemma 5.3, $r[(K_4 \setminus \{e\})(C_k)] \geq 3$ and $r(K_4(C_k)) \geq 4$. Let $W_1 = \{v_{3 \lfloor \frac{k}{2} \rfloor}, v_{4 \lfloor \frac{k}{2} \rfloor}, v_{5 \lfloor \frac{k}{2} \rfloor}\}$ and $W_2 = \{v_{3 \lfloor \frac{k}{2} \rfloor}, v_{4 \lfloor \frac{k}{2} \rfloor}, v_{5 \lfloor \frac{k}{2} \rfloor}, v_{6 \lfloor \frac{k}{2} \rfloor}\}$. Then we can easily verify that W_1 and W_2 are the resolving set of $[K_4 \setminus \{e\})(C_k)$ and $K_4(C_k)$ respectively. Thus $r[(K_4 \setminus \{e\})(C_k)] = 3$ and $r(K_4(C_k)) = 4$.

Let $G \cong C_4$. By Lemma 5.3, $r(C_4(C_k)) \geq 2$. But we claim that $r(C_4(C_k)) \geq 3$. Suppose that $r(C_4(C_k)) = 2$. Let $W = \{w_1, w_2\}$ be a resolving set of $C_4(C_k)$. Let $E(C_4) = \{e_1, e_2, e_3, e_4\}$, where e_1, e_3 are non adjacent edges and e_2, e_4 are non adjacent edges. By Lemma 5.3, let $W \cap V(A_1) = \emptyset$, $W \cap V(A_2) \neq \emptyset$, $W \cap V(A_3) = \emptyset$ and $W \cap V(A_4) \neq \emptyset$. Let $v_{11} = v_{4k}, v_{1k} = v_{21}, v_{2k} = v_{31}$ and $v_{3k} = v_{41}$. If either $W = \{v_{2 \lfloor \frac{k}{2} \rfloor}, v_{4 \lfloor \frac{k}{2} \rfloor}\}$ or $W = \{v_{2 \lfloor \frac{k}{2} \rfloor}, v_{4 \lfloor \frac{k}{2} \rfloor}\}$, then there exist two shortest paths between w_1 and w_2 , which is a contradiction to Theorem 2.4. If w_1 lies on $v_{22}v_{2(\lfloor \frac{k}{2} \rfloor - 1)}$ path, then $r(v_{2(k-1)}|W) = r(v_{32}|W)$, which is a contradiction. If $W = \{v_{2 \lfloor \frac{k}{2} \rfloor}, v_{4(\lfloor \frac{k}{2} \rfloor + 1)}\}$, then $r(v_{1(k-1)}|W) = r(v_{2k}|W)$, which is a contradiction. Thus $r(C_4(C_k)) \geq 3$. Let $W = \{v_{1 \lfloor \frac{k}{2} \rfloor}, v_{2 \lfloor \frac{k}{2} \rfloor}, v_{3 \lfloor \frac{k}{2} \rfloor}\}$. Then we can easily verify that W is a resolving set of $C_4(C_k)$ and hence $r(C_4(C_k)) = 3$.

Lemma 6.9. Let G be a graph of order $n \geq 5$, size m and $\delta(G) = 1$. If P denotes the set of all pendant edges of G , then $r(G(C_k)) \geq m - \beta_1(G \setminus P)$.

Proof. By Lemma 2.7, every resolving set of $G(C_k)$ contains at least one non cut vertex from each end block of $G(C_k)$ and using Theorem 5.6 and Lemma 6.6, $r(G(C_k)) \geq m - \beta_1(G \setminus P)$.

Theorem 6.10. Let G be a graph of order $n \geq 5$, size m and $\delta(G) = 1$. Let P denote the set of all pendant edges of G . If k is odd, then $r(G(C_k)) = m - \beta_1(G \setminus P)$.

Proof. Let $V(G) = \{v_1, v_2, \dots, v_n\}$, $E(G) = \{e_1, e_2, \dots, e_m\}$ and C_i be the edge cycle of e_i . Let $\beta_1(G \setminus P) = s$ and $|P| = p$. Let $M = \{e_1, e_2, \dots, e_s\}$ be the maximum edge independent set of

$G \setminus P$ and $P = \{e_{s+1}, e_{s+2}, \dots, e_{s+p}\}$. Let $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$ and $e_i = v_{i1}v_{ik}$, $1 \leq i \leq m$. Let

$W = \left\{ v_{(s+1)\lfloor \frac{k}{2} \rfloor}, v_{(s+2)\lfloor \frac{k}{2} \rfloor}, \dots, v_{m\lfloor \frac{k}{2} \rfloor} \right\}$. We claim that W is a resolving set of $G(C_k)$. Let x, y be two distinct vertices of $V(G(C_k)) \setminus W$.

If $x, y \in V(C_i)$ for some $s+1 \leq i \leq s+p$, then without loss of generality, let $x, y \in V(C_{s+1})$. Let $d(v_{(s+1)1}) = 2$ in $G(C_k)$. If $d(x, v_{(s+1)\lfloor \frac{k}{2} \rfloor}) \neq d(y, v_{(s+1)\lfloor \frac{k}{2} \rfloor})$, then $r(x|W) \neq r(y|W)$. So we may assume that $d(x, v_{(s+1)\lfloor \frac{k}{2} \rfloor}) = d(y, v_{(s+1)\lfloor \frac{k}{2} \rfloor})$. Then x lies on $v_{(s+1)\lfloor \frac{k}{2} \rfloor} - v_{(s+1)1}$ path and y lies on $v_{(s+1)\lfloor \frac{k}{2} \rfloor} - v_{(s+1)k}$ path. Therefore $d(x, w) = d(y, w) + 1$ for all $w \in W \setminus \{v_{(s+1)\lfloor \frac{k}{2} \rfloor}\}$. It follows that $r(x|W) \neq r(y|W)$. If $x \in V(C_{s+1})$, $y \notin V(C_{s+1})$, then $d(x, v_{(s+1)\lfloor \frac{k}{2} \rfloor}) < d(y, v_{(s+1)\lfloor \frac{k}{2} \rfloor})$. It follows that $r(x|W) \neq r(y|W)$. If $x, y \notin V(C_i)$ for all $s+1 \leq i \leq s+p$, then the proof is similar

to Case 1 and Case 2 of Theorem 6.7. Thus $r(G(C_k)) \leq m - \beta_1(G \setminus P)$. By Lemma 6.9, $r(G(C_k)) \geq m - \beta_1(G \setminus P)$. Hence $r(G(C_k)) = m - \beta_1(G \setminus P)$.

Remark 6.11. If k is odd and $\delta(G) = 1$, then $r(P_3(C_k)) = 2$ and $r[(K_{1,3})(C_k)] = r[(K_1 + (K_2 \cup K_1))(C_k)] = 3$.

Proof. Let C_i be the edge cycle of the edge e_i and $V(C_i) = \{v_{i1}, v_{i2}, \dots, v_{ik}\}$. Let $e_i = v_{i1}v_{ik}$. By Lemma 6.9, $r(P_3(C_k)) \geq 2$. Let $E(P_3) = \{e_1, e_2\}$. If $W_1 = \left\{ v_{1\lfloor \frac{k}{2} \rfloor}, v_{2\lfloor \frac{k}{2} \rfloor} \right\}$, then we can easily verify that W_1 is a resolving set of $P_3(C_k)$. Thus $r(P_3(C_k)) \leq 2$ and hence $r(P_3(C_k)) = 2$. If $G \cong K_{1,3}$, then let $E(G) = \{e_1, e_2, e_3\}$. If $G \cong K_1 + (K_2 \cup K_1)$, then let $E(G) = \{e_1, e_2, e_3, e_4\}$ and $e_4 = uv$, where $d(u) = d(v) = 2$. If $W_2 = \left\{ v_{1\lfloor \frac{k}{2} \rfloor}, v_{2\lfloor \frac{k}{2} \rfloor}, v_{3\lfloor \frac{k}{2} \rfloor} \right\}$, then we can easily verify that W_2 is a resolving set of $G(C_k)$. Thus $r(G(C_k)) \leq 3$. By Lemma 6.9, $r(G(C_k)) \geq 3$ and hence $r(G(C_k)) = 3$.

Remark 6.12. Since $s \geq 1$, it follows from **theorems 6.2 and 6.4** that, $r(G(C_k)) \leq m + p - 1$. From Theorem 6.3, equality holds if and only if $G \cong K_{1, n-1}$.

ACKNOWLEDGEMENT

The research work of the second author is supported by the University Grants Commission, New Delhi through Basic Science Research Fellowship (vide Sanction No.F.7-201/2007(BSRF)).

REFERENCES

1. F. Buckley and F. Harary, *Distance in graphs*, Addison Wesley, Reading MA, 1990.
2. G. Chartrand, L. Eroh, M. A. Johnson and O. R. Oellermann, *Resolvability in graphs and metric dimension of a graph*, *Discrete Appl. Math.* 105(2000), 99-113.
3. Gary Chartrand and Ping Zhang, *Introduction to Graph Theory*, Tata McGraw Hill Education Private Limited, New Delhi(2006).
4. F. Harary and R. A. Melter, *On the metric dimension of a graph*, *Ars Combin.* 2(1976), 191-195.
5. M. Jannesari, *The Metric Dimension and Girth of Graphs*, *Bull. Iranian Math. Soc.* Vol 41 No. 3(2015), 633-638.
6. S. Khuller, B. Raghavachari and A. Rosenfeld, *Landmarks in graphs*, *Disc. Appl. Math.* 70(1996), 217-229.
7. J. Paulraj Joseph and N. Shunmugapriya, *Total Resolving Number of a Graph*, *Indian Journal of Mathematics*, Vol 57, No. 3(2015), 323-343.
8. Ping Zhang and Varaporn Saenpholphat, *Connected resolvability of graphs*, *Czechoslovak Mathematical Journal*, Vol 53(2003), No. 4, 827- 840.
9. Ping Zhang and Varaporn Saenpholphat, *On connected resolvability of graphs*, *Australasian Journal of Combinatorics*, Vol 28(2003), 25-37.
10. P. J. Slater, *Leaves of trees*, *Proc. 6th Southeastern Conf. on Combinatorics, Graph Theory and Computing*, Vol 14 of *Congr. Numer.* (1975), 549-559.
11. Sooryanarayana B, *On the metric dimension of a graph*, *Indian. J. Pure Appl. Math* 29(4)(1998), 413-415.



MELLIN AND LAPLACE TRANSFORMS INVOLVING THE PRODUCT OF STRUVE'S FUNCTION AND I-FUNCTION OF TWO VARIABLES

B. Satyanarayana, B.V. Purnima* & Y. Pragathi Kumar**

*Department of Mathematics, Acharya Nagarjuna University, Nagarjuna Nagar, Andhra Pradesh, India.

**Department of Mathematics, College of Natural and Computational Sciences, Adigrat University, Adigrat, Ethiopia

E-mail: drbsn63@yahoo.co.in

ABSTRACT :

The object of this paper is to establish Mellin and Laplace transform involving the product of Struve's function $H_{\nu, \gamma, \mu}^{\lambda, k}[z]$ and I-function of two variables. Some special cases have also been derived.

Keywords: Mellin transforms, Laplace transform, Struve's function and I-function of two variables

1. INTRODUCTION

Recently, The Mellin transform and Laplace transform of product of Struve's function with H-function of two variables [3 , 5] evaluated. In the present paper we establish the same transforms of I-function of two variables with Struve's function.

We shall utilize the following formulae in the present investigation. The I-function of one variable given by Rathie Arjun K [4]

$$I_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,p} \\ (b_j, \beta_j; B_j)_{1,q} \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_L \phi(s) z^s ds \quad (1.1)$$

$$\text{Where } \phi(s) = \frac{\prod_{j=1}^m \Gamma^{B_j}(b_j - \beta_j s) \prod_{j=1}^n \Gamma^{A_j}(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^q \Gamma^{B_j}(1 - b_j + \beta_j s) \prod_{j=n+1}^p \Gamma^{A_j}(a_j - \alpha_j s)}$$

Where $A_j (j = 1, \dots, p)$ and $B_j (j = 1, \dots, q)$ are not in general positive integers

Also (i) $z \neq 0$

(ii) $i = \sqrt{-1}$

(iii) m, n, p, q are integers satisfying $0 \leq m \leq q, 0 \leq n \leq p$

(iv) L is suitable contour in the complex plane

(v) An empty product is interpreted as unity

(vi) $\alpha_j (j = 1, \dots, p); \beta_j (j = 1, \dots, q); A_j (j = 1, \dots, p)$ and $B_j (j = 1, \dots, q)$ are positive numbers

(vii) $a_j (j = 1, \dots, p); b_j (j = 1, \dots, q)$ are complex numbers such that no singularity of

$\Gamma^{B_j}(b_j - \beta_j s)$, $(j = 1, \dots, m)$ coincides with any singularity of $\Gamma^{A_j}(1 - a_j + \alpha_j s)$

$(j = 1, \dots, n)$. In general singularities are not poles.

The detailed conditions can be found in Rathie Arjun K [4]

The I-function of two variables given by Shantha Kumari et al. [6]

$$\begin{aligned}
 I[z_1, z_2] &= I_{P_1, Q_1; P_2, Q_2; P_3, Q_3}^{0, n_1; m_2, n_2; m_3, n_3} \left[\begin{matrix} z_1 \\ z_2 \end{matrix} \left| \begin{matrix} (a_j; a_j, A_j; x_j)_{1, p_1} : (c_j, C_j; U_j)_{1, p_2}; \\ (b_j; b_j, B_j; h_j)_{1, q_1} : (d_j, D_j; V_j)_{1, q_2}; \\ (e_j, E_j; P_j)_{1, p_3} \\ (f_j, F_j; Q_j)_{1, q_3} \end{matrix} \right. \right] \\
 &= \frac{1}{(2\pi i)^2} \int_{L_s} \int_{L_t} \varphi(s, t) \theta_1(s) \theta_2(t) z_1^s z_2^t ds dt \tag{1.2}
 \end{aligned}$$

where

$$\begin{aligned}
 \varphi(s, t) &= \frac{\prod_{j=1}^{n_1} \Gamma^{\xi_j} (1 - a_j + \alpha_j s + A_j t)}{\prod_{j=n_1+1}^{p_1} \Gamma^{\xi_j} (a_j - \alpha_j s - A_j t) \prod_{j=1}^{q_1} \Gamma^{\eta_j} (1 - b_j + \beta_j s + B_j t)} \\
 \theta_1(s) &= \frac{\prod_{j=1}^{n_2} \Gamma^{U_j} (1 - c_j + C_j s) \prod_{j=1}^{m_2} \Gamma^{V_j} (d_j - D_j s)}{\prod_{j=n_2+1}^{p_2} \Gamma^{U_j} (c_j - C_j s) \prod_{j=m_2+1}^{q_2} \Gamma^{V_j} (1 - d_j + D_j s)} \\
 \theta_2(t) &= \frac{\prod_{j=1}^{n_3} \Gamma^{P_j} (1 - e_j + E_j t) \prod_{j=1}^{m_3} \Gamma^{Q_j} (f_j - F_j t)}{\prod_{j=n_3+1}^{p_3} \Gamma^{P_j} (e_j - E_j t) \prod_{j=m_3+1}^{q_3} \Gamma^{Q_j} (1 - f_j + F_j t)}
 \end{aligned}$$

where $n_j, p_j, q_j (j = 1, 2, 3), m_j (j = 2, 3)$ are non negative integers such that $0 \leq n_j \leq p_j, q_1 \geq 0, 0 \leq m_j \leq q_j (j = 2, 3)$ (not all zero simultaneously). $\alpha_j, A_j (j = 1, \dots, p_1); \beta_j, B_j (j = 1, \dots, q_1), C_j (j = 1, \dots, p_2), D_j (j = 1, \dots, q_2), E_j (j = 1, \dots, p_3), F_j (j = 1, \dots, q_3)$ are positive quantities. $a_j (j = 1, \dots, p_1), b_j (j = 1, \dots, q_1), c_j (j = 1, \dots, p_2), d_j (j = 1, \dots, q_2), e_j (j = 1, \dots, p_3)$ and $f_j (j = 1, \dots, q_3)$ are complex numbers. The exponents $\xi_j, \eta_j, U_j, V_j, P_j, Q_j$ may take non integer values.

L_s and L_t are suitable contours of Mellin-Barnes type. More over, the contour L_s is in the complex s-plane and runs from $\sigma_1 - i\infty$ to $\sigma_1 + i\infty$ (σ_1 real), so that all the poles of

$\Gamma^{V_j} (d_j - D_j s) (j = 1, \dots, m_2)$ lie to the right of L_s and all poles of $\Gamma^{U_j} (1 - c_j + C_j s) (j = 1, \dots, n_2), \Gamma^{\xi_j} (1 - a_j + \alpha_j s + A_j t) (j = 1, \dots, n_1)$ lie to the left of L_s . Similar conditions for L_t follows in complex t-plane.

The detailed conditions of this function can be found in Shantha Kumari et al.[6].

According Erdelyi [1 ,p.307]

$$\int_0^{\infty} x^s - 1 \left[\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g(s)x^{-s} ds \right] dx = g(s) \tag{1.3}$$

The Struve's function defined by Kanth [2] as

$$H_{v,y,u}^{\lambda,k} [z] = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{v+2m+1}}{\Gamma(km + y)\Gamma(v + \lambda m + u)}, \tag{1.4}$$

Re(k) > 0, Re(λ) > 0, Re(y) > 0, Re(v + u) > 0

The Mellin transform of the function f(x) is defined as

$$M\{f(x); s\} = \int_0^{\infty} x^{s-1} f(x) dx, \text{ Re}(s) > 0 \tag{1.5}$$

If Laplace transform of f(t) is F(p) and G(s) is Mellin transform of f(t), then

$$F(p) = \sum_{s=0}^{\infty} \frac{(-p)^s}{s!} G(s+1) \tag{1.6}$$

2. MAIN RESULTS

Theorem 1: Prove that

$$\begin{aligned} & M \left\{ \left[H_{v,y,t}^{\lambda,k} \left[ax^\rho \right] I_{p_1,q_1;p_2,q_2;p_3,q_3}^{0,n_1;m_2,n_2;m_3,n_3} \left[\begin{array}{l} z_1 x^{h_1} \left| (a_j; \alpha_j, A_j; \xi_j)_{1,p_1} : (c_j, C_j; U_j)_{1,p_2} ; \right. \right. \\ z_2 x^{h_2} \left| (b_j; \beta_j, B_j; \eta_j)_{1,q_1} : (d_j, D_j; V_j)_{1,q_2} ; \right. \\ \left. \left. (e_j, E_j; P_j)_{1,p_3} \right. \right. \\ \left. \left. (f_j, F_j; Q_j)_{1,q_3} \right. \right. \right] \right] ; s \right\} \\ &= \frac{1}{h_2} (z_2)^{-\left(\frac{s+\Omega(m)}{h_2}\right)} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{a}{2}\right)^{v+2m+1}}{\Gamma(km + y)\Gamma(v + \lambda m + t)} \\ & I_{p_1+p_2+p_3, q_1+q_2+q_3}^{m_2+m_3, n_1+n_2+n_3} \left[(z_2)^{-\left(\frac{h_1}{h_2}\right)} z_1 \left| \begin{array}{l} (c_j, C_j; U_j)_{1,p_2}, \left(e_j + E_j \left(\frac{(s+\Omega(m))}{h_2} \right), -E_j \frac{h_1}{h_2}; P_j \right)_{1,n_3} \\ (d_j, D_j; V_j)_{1,q_2}, \left(f_j + F_j \left(\frac{(s+\Omega(m))}{h_2} \right), -F_j \frac{h_1}{h_2}; Q_j \right)_{1,m_3} \end{array} \right. \right] \end{aligned}$$

$$\left(e_j + E_j \left(\frac{(s + \Omega(m))}{h_2} \right), -E_j \frac{h_1}{h_2}; P_j \right)_{n_3+1, p_3}, \left(a_j + A_j \left(\frac{(s + \Omega(m))}{h_2} \right), \alpha_j - A_j \frac{h_1}{h_2}; \xi_j \right)_{1, n_1},$$

$$\left(f_j + F_j \left(\frac{(s + \Omega(m))}{h_2} \right), -F_j \frac{h_1}{h_2}; Q_j \right)_{m_3+1, q_3}, \dots,$$

$$\left[\begin{array}{l} \left(a_j + A_j \left(\frac{(s + \Omega(m))}{h_2} \right), \alpha_j - A_j \frac{h_1}{h_2}; \xi_j \right)_{n_1+1, p_1} \\ \left(b_j + B_j \left(\frac{(s + \Omega(m))}{h_2} \right), \beta_j - B_j \frac{h_1}{h_2}; \eta_j \right)_{1, n_1} \end{array} \right] \tag{2.1}$$

Where $\Omega(m) = \rho(v+2m+1)$

Provided $h_1 > 0, h_2 > 0, \lambda, a$ are complex numbers

$$\alpha_j - A_j \frac{h_1}{h_2} > 0, j = 1, \dots, p_1$$

$$\beta_j - B_j \frac{h_1}{h_2} > 0, j = 1, \dots, q_1$$

$$|\arg z_1| < (1/2) \pi \Delta_1, |\arg z_2| < (1/2) \pi \Delta_2$$

$$\text{where } \Delta_1 = - \sum_{j=n_1+1}^{p_1} \alpha_j \zeta_j - \sum_{j=1}^{q_1} \beta_j \eta_j + \sum_{j=1}^{m_2} D_j V_j - \sum_{j=m_2+1}^{q_2} D_j V_j + \sum_{j=1}^{n_2} C_j U_j - \sum_{j=n_2+1}^{p_2} C_j U_j$$

$$\Delta_2 = - \sum_{j=n_1+1}^{p_1} A_j \zeta_j - \sum_{j=1}^{q_1} B_j \eta_j + \sum_{j=1}^{m_3} F_j Q_j - \sum_{j=m_3+1}^{q_3} F_j Q_j + \sum_{j=1}^{n_3} E_j P_j - \sum_{j=n_3+1}^{p_3} E_j P_j$$

Proof:

To prove this theorem, take $f(x)$ as

$$H_{v, y, t}^{\lambda, k} \left[\text{ax}^\rho \right] \left[\begin{array}{l} 0, n_1 : m_2, n_2; m_3, n_3 \\ p_1, q_1 : p_2, q_2; p_3, q_3 \end{array} \right] \left[\begin{array}{l} z_1 x^{h_1} \left| (a_j; \alpha_j, A_j; \xi_j)_{1, p_1} : (c_j, C_j; U_j)_{1, p_2}; \right. \\ z_2 x^{h_2} \left| (b_j; \beta_j, B_j; \eta_j)_{1, q_1} : (d_j, D_j; V_j)_{1, q_2}; \right. \\ (e_j, E_j; P_j)_{1, p_3} \\ (f_j, F_j; Q_j)_{1, q_3} \end{array} \right] \tag{in (1.5)}$$

The expression becomes

$$M \left\{ \left[H_{v, y, t}^{\lambda, k} \left[\text{ax}^\rho \right] \left[\begin{array}{l} 0, n_1 : m_2, n_2; m_3, n_3 \\ p_1, q_1 : p_2, q_2; p_3, q_3 \end{array} \right] \left[\begin{array}{l} z_1 x^{h_1} \left| (a_j; \alpha_j, A_j; \xi_j)_{1, p_1} : \right. \\ z_2 x^{h_2} \left| (b_j; \beta_j, B_j; \eta_j)_{1, q_1} : \right. \\ (c_j, C_j; U_j)_{1, p_2}; (e_j, E_j; P_j)_{1, p_3} \\ (d_j, D_j; V_j)_{1, q_2}; (f_j, F_j; Q_j)_{1, q_3} \end{array} \right] \right] ; s \right\}$$

$$= \int_0^{\infty} x^{s-1} H_{v,y,t}^{\lambda,k} \left[ax^{\rho} \right] I_{p_1,q_1;p_2,q_2;p_3,q_3}^{0,n_1;m_2,n_2;m_3,n_3} \left[\begin{matrix} z_1 x^{h_1} (a_j; \alpha_j, A_j; \xi_j)_{1,p_1} : \\ z_2 x^{h_2} (b_j; \beta_j, B_j; \eta_j)_{1,q_1} : \\ (c_j, C_j; U_j)_{1,p_2}; (e_j, E_j; P_j)_{1,p_3} \\ (d_j, D_j; V_j)_{1,q_2}; (f_j, F_j; Q_j)_{1,q_3} \end{matrix} \right] dx$$

Use (1.2) and (1.4) to represent extended general class of polynomials as series and integral form of I-function of two variables in the above integral, of two variables t_1 and t_2 . Put $h_2 t_2 = -u$,

We get

$$\sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{a}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+t)} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \theta_1(t_1) \theta_2\left(\frac{-u}{h_2}\right) \varphi\left(t_1, \frac{-u}{h_2}\right) z_1^{t_1} (z_2)^{-\frac{u}{h_2}} x^{-u} x^{h_1 t_1 + \rho(v+2m+1) + s - 1} \left(\frac{du}{h_2}\right) dt_1 dx$$

Interchange the order of integration, we get

$$= \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{a}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+t)} \frac{1}{h_2} \frac{1}{2\pi i} \int_{L_1} \theta_1(t_1) z_1^{t_1} \left\{ \int_0^{\infty} x^{h_1 t_1 + \rho(v+2m+1) + s - 1} \left[\frac{1}{2\pi i} \int_{L_2} \theta_2\left(\frac{-u}{h_2}\right) \varphi\left(t_1, \frac{-u}{h_2}\right) (z_2)^{-\frac{u}{h_2}} x^{-u} du \right] dx dt_1 \right.$$

Use result (1.3) and (1.1) to get the result. Change of order of integration is justifiable due to convergence of integrals.

Theorem 2: Prove that

$$L \left\{ \left[H_{v,y,t}^{\lambda,k} \left[ax^{\rho} \right] I_{p_1,q_1;p_2,q_2;p_3,q_3}^{0,n_1;m_2,n_2;m_3,n_3} \left[\begin{matrix} z_1 (a_j; \alpha_j, A_j; \xi_j)_{1,p_1} : \\ z_2 (b_j; \beta_j, B_j; \eta_j)_{1,q_1} : \\ (c_j, C_j; U_j)_{1,p_2}; (e_j, E_j; P_j)_{1,p_3} \\ (d_j, D_j; V_j)_{1,q_2}; (f_j, F_j; Q_j)_{1,q_3} \end{matrix} \right] ; s \right. \right. \\ \left. \left. \right\} = \frac{1}{h_2} \sum_{s=0}^{\infty} \frac{(-p)^s}{s!} (z_2)^{-\left(\frac{s+\Omega(m)+1}{h_2}\right)} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{a}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+t)}$$

$$\begin{aligned}
 & I_{P_1 + P_2 + P_3, Q_1 + Q_2 + Q_3}^{m_2 + m_3, n_1 + n_2 + n_3} \left[(z_2)^{-\left(\frac{h_1}{h_2}\right)} z_1 \left| \begin{array}{l} (c_j, C_j; U_j)_{1, p_2}, \left(e_j + E_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), -E_j \frac{h_1}{h_2}; P_j \right)_{1, n_3} \\ (d_j, D_j; V_j)_{1, q_2}, \left(f_j + F_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), -F_j \frac{h_1}{h_2}; Q_j \right)_{1, m_3} \end{array} \right. \right. \\
 & \left. \left(e_j + E_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), -E_j \frac{h_1}{h_2}; P_j \right)_{n_3 + 1, p_3}, \left(a_j + A_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), \alpha_j - A_j \frac{h_1}{h_2}; \xi_j \right)_{1, n_1} \right. \\
 & \left. \left(f_j + F_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), -F_j \frac{h_1}{h_2}; Q_j \right)_{m_3 + 1, q_3} \right. \quad \text{---} \quad \left. \left(a_j + A_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), \alpha_j - A_j \frac{h_1}{h_2}; \xi_j \right)_{n_1 + 1, p_1} \right. \\
 & \left. \left(b_j + B_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), \beta_j - B_j \frac{h_1}{h_2}; \eta_j \right)_{1, n_1} \right] \quad (2.2)
 \end{aligned}$$

Where $\Omega(m) = \rho(v + 2m + 1)$

Proof: Proof of above theorem can be easily obtain by using (1.6)

3. SPECIAL CASES

- (i) Take $\rho = 0, a = 1$ in (2.1) and (2.2), we get Mellin and Laplace transform of I-function of two variables
- (ii) Choose $\xi_j = \eta_j = U_j = V_j = P_j = Q_j = 1$ in (2.1) and (2.2), we get Mellin and Laplace transform of H-function of two variables [] with Struve’s function

$$\begin{aligned}
 & M \left\{ \left[H_{v, y, t}^{\lambda, k} \left[ax^\rho \right] H_{P_1, Q_1 : P_2, Q_2 : P_3, Q_3}^{0, n_1 : m_2, n_2 ; m_3, n_3} \left[z_1 \left(a_j ; \alpha_j, A_j ; 1 \right)_{1, p_1} : (c_j, C_j ; 1)_{1, p_2} ; \right. \right. \right. \\
 & \left. \left. \left. z_2 \left(b_j ; \beta_j, B_j ; 1 \right)_{1, q_1} : (d_j, D_j ; 1)_{1, q_2} ; \right. \right. \right. \\
 & \left. \left. \left. \left(e_j, E_j ; 1 \right)_{1, p_3} \right. \right. \right. \\
 & \left. \left. \left. \left(f_j, F_j ; 1 \right)_{1, q_3} \right. \right. \right. \left. \left. \right. \right] ; s \}
 \end{aligned}$$

$$\begin{aligned}
 & = \frac{1}{h_2} (z_2)^{-\left(\frac{s + \Omega(m)}{h_2}\right)} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{a}{2}\right)^{v + 2m + 1}}{\Gamma(km + y) \Gamma(v + \lambda m + t)} \\
 & H_{P_1 + P_2 + P_3, Q_1 + Q_2 + Q_3}^{m_2 + m_3, n_1 + n_2 + n_3} \left[(z_2)^{-\left(\frac{h_1}{h_2}\right)} z_1 \left| \begin{array}{l} (c_j, C_j)_{1, p_2} \\ (d_j, D_j)_{1, q_2} \end{array} \right. \right. \\
 & \left. \left(e_j + E_j \left(\frac{(s + \Omega(m))}{h_2} \right), -E_j \frac{h_1}{h_2} \right)_{1, n_3}, \left(e_j + E_j \left(\frac{(s + \Omega(m))}{h_2} \right), -E_j \frac{h_1}{h_2} \right)_{n_3 + 1, p_3} \right. \\
 & \left. \left(f_j + F_j \left(\frac{(s + \Omega(m))}{h_2} \right), -F_j \frac{h_1}{h_2} \right)_{1, m_3}, \left(f_j + F_j \left(\frac{(s + \Omega(m))}{h_2} \right), -F_j \frac{h_1}{h_2} \right)_{m_3 + 1, q_3} \right.
 \end{aligned}$$

$$\left[\begin{array}{l} \left(a_j + A_j \left(\frac{(s + \Omega(m))}{h_2} \right), \alpha_j - A_j \frac{h_1}{h_2} \right)_{1, n_1}, \left(a_j + A_j \left(\frac{(s + \Omega(m))}{h_2} \right), \alpha_j - A_j \frac{h_1}{h_2} \right)_{n_1 + 1, p_1} \\ \left(b_j + B_j \left(\frac{(s + \Omega(m))}{h_2} \right), \beta_j - B_j \frac{h_1}{h_2} \right)_{1, n_1} \end{array} \right] \quad (3.1)$$

$$\begin{aligned} & L \left\{ \begin{array}{l} H_{v, y, t}^{\lambda, k} [ax^\rho] H_{p_1, q_1; p_2, q_2; p_3, q_3}^{0, n_1; m_2, n_2; m_3, n_3} \left[\begin{array}{l} z_1 x^{h_1} \left(a_j; \alpha_j, A_j; 1 \right)_{1, p_1} : \\ z_2 x^{h_2} \left(b_j; \beta_j, B_j; 1, q_1 : \right. \\ \left. (c_j, C_j; 1)_{1, p_2}; (e_j, E_j; 1)_{1, p_3} \right. \\ \left. (d_j, D_j; 1)_{1, q_2}; (f_j, F_j; 1)_{1, q_3} \right] ; s \end{array} \right\} \\ & = \frac{1}{h_2} \sum_{s=0}^{\infty} \frac{(-p)^s}{s!} (z_2)^{-\left(\frac{s + \Omega(m) + 1}{h_2} \right)} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{a}{2} \right)^{v + 2m + 1}}{\Gamma(km + y) \Gamma(v + \lambda m + t)} \\ & \quad H_{p_1 + p_2 + p_3, q_1 + q_2 + q_3}^{m_2 + m_3, n_1 + n_2 + n_3} \left[(z_2)^{-\left(\frac{h_1}{h_2} \right)} z_1 \left(\begin{array}{l} (c_j, C_j)_{1, p_2} \\ (d_j, D_j)_{1, q_2} \end{array} \right) \right] \end{aligned}$$

$$\begin{aligned} & \left(e_j + E_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), -E_j \frac{h_1}{h_2} \right)_{1, n_3}, \left(e_j + E_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), -E_j \frac{h_1}{h_2} \right)_{n_3 + 1, p_3}, \\ & \left(f_j + F_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), -F_j \frac{h_1}{h_2} \right)_{1, m_3}, \left(f_j + F_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), -F_j \frac{h_1}{h_2} \right)_{m_3 + 1, q_3}, \\ & \left(a_j + A_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), \alpha_j - A_j \frac{h_1}{h_2} \right)_{1, n_1}, \left(a_j + A_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), \alpha_j - A_j \frac{h_1}{h_2} \right)_{n_1 + 1, p_1} \\ & \left(b_j + B_j \left(\frac{(s + \Omega(m) + 1)}{h_2} \right), \beta_j - B_j \frac{h_1}{h_2} \right)_{1, n_1} \end{aligned} \quad (3.2)$$

Where $\Omega(m) = \rho(v + 2m + 1)$

(iii) Take $(\alpha_{p_1}) = (\beta_{q_1}) = (A_{p_1}) = (B_{q_1}) = (C_{p_2}) = (D_{q_2}) = (E_{p_3}) = (F_{q_3}) = 1$ in (3.1) and (3.2), we get Mellin and Laplace transform for G-function with general class of polynomials

(iv) Write $n_1 = p_1 = q_1 = 0$ and in (3.1) and (3.2), we get

$$M \left\{ H_{v, y, t}^{\lambda, k} [ax^\rho] H_{p_2, q_2}^{m_2, n_2} \left[\begin{array}{l} z_1 x^{h_1} \left(c_j, C_j \right)_{1, p_2} \\ (d_j, D_j)_{1, q_2} \end{array} \right] H_{p_3, q_3}^{m_3, n_3} \left[\begin{array}{l} z_2 x^{h_2} \left(e_j, E_j \right)_{1, p_3} \\ (f_j, F_j)_{1, q_3} \end{array} \right] ; s \right\}$$

$$\begin{aligned}
 &= \frac{1}{h_2} (z_2)^{-\left(\frac{s+\Omega(m)}{h_2}\right)} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{a}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+t)} H_{p_2+p_3, q_2+q_3}^{m_2+m_3, n_2+n_3} \left[(z_2)^{-\left(\frac{h_1}{h_2}\right)} z_1 \left| \begin{matrix} (c_j, C_j)_{1, p_2} \\ (d_j, D_j)_{1, q_2} \end{matrix} \right. \right. \\
 &\left. \left. \left(e_j + E_j \left(\frac{s+\Omega(m)}{h_2} \right), -E_j \frac{h_1}{h_2}; P_j \right)_{1, n_3}, \left(e_j + E_j \left(\frac{s+\Omega(m)}{h_2} \right), -E_j \frac{h_1}{h_2}; P_j \right)_{n_3+1, p_3} \right. \right. \\
 &\left. \left. \left(f_j + F_j \left(\frac{s+\Omega(m)}{h_2} \right), -F_j \frac{h_1}{h_2}; Q_j \right)_{1, m_3}, \left(f_j + F_j \left(\frac{s+\Omega(m)}{h_2} \right), -F_j \frac{h_1}{h_2}; Q_j \right)_{m_3+1, q_3} \right] \quad (3.3) \\
 &L \left\{ H_{\nu, y, t}^{\lambda, k} [ax^\rho] H_{p_2, q_2}^{m_2, n_2} \left[z_1 x^{h_1} \left| \begin{matrix} (c_j, C_j)_{1, p_2} \\ (d_j, D_j)_{1, q_2} \end{matrix} \right. \right] H_{p_3, q_3}^{m_3, n_3} \left[z_2 x^{h_2} \left| \begin{matrix} (e_j, E_j)_{1, p_3} \\ (f_j, F_j)_{1, q_3} \end{matrix} \right. \right]; s \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{h_2} \sum_{s=0}^{\infty} \frac{(-p)^s}{s!} (z_2)^{-\left(\frac{s+\Omega(m)+1}{h_2}\right)} \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{a}{2}\right)^{v+2m+1}}{\Gamma(km+y)\Gamma(v+\lambda m+t)} H_{p_2+p_3, q_2+q_3}^{m_2+m_3, n_2+n_3} \left[(z_2)^{-\left(\frac{h_1}{h_2}\right)} z_1 \left| \begin{matrix} (c_j, C_j)_{1, p_2} \\ (d_j, D_j)_{1, q_2} \end{matrix} \right. \right. \\
 &\left. \left. \left(e_j + E_j \left(\frac{(s+\Omega(m))+1}{h_2} \right), -E_j \frac{h_1}{h_2}; P_j \right)_{1, n_3}, \left(e_j + E_j \left(\frac{(s+\Omega(m))+1}{h_2} \right), -E_j \frac{h_1}{h_2}; P_j \right)_{n_3+1, p_3} \right. \right. \\
 &\left. \left. \left(f_j + F_j \left(\frac{(s+\Omega(m))+1}{h_2} \right), -F_j \frac{h_1}{h_2}; Q_j \right)_{1, m_3}, \left(f_j + F_j \left(\frac{(s+\Omega(m))+1}{h_2} \right), -F_j \frac{h_1}{h_2}; Q_j \right)_{m_3+1, q_3} \right] \quad (3.4)
 \end{aligned}$$

Where $\Omega(m) = \rho(v+2m+1)$

4. CONCLUSION

On specialization of parameters in I-function of two variables, we get various special functions[7]. So, with results of this paper we get Mellin and Laplace transform or various special functions with Struve’s function as special cases.

REFERENCES

1. Erdely A., *Tables of Integral Transforms, Vol I, Mc Graw Hill Book Co., New York(1953),I.*
2. Kanth.B.N., *Jour.Inst.sci,Nepal,3(1)(1980), p.99*
3. Pragathi Kumar Y and Satyanarayana B., *Integral transform involving the product of a generalized Struve's function and H-function of two variables, J.Appl.Math.Anal. and Appl., Vol.6, No. 1-2, pp. 193-198*
4. Rathie Arjun K., *A new generalization of generalized hypergeometric functions, Le Mathematiche, I, LII.Fasc.II(1997), 297-310.*
5. Sahul Hameed K.P., *Investigations in generalized hypergeometric function, Ph.D thesis, Kannur University, India.*
6. Shantha Kumari K, Vasudevan Nambisan T. M and Rathie Arjun K., *A study of I-function of two variables, Le Mathematiche, Vol.69, Issue 1,285-305.*
7. Srivastava H.M, Gupta K.C and Goyal S.P., *The H-function of one and two variables with applications, South Asian Publishers, New Delhi (1982).*



ANALYSIS OF HOT STANDBY DATABASE SYSTEM WITH STANDBY UNIT UNDER CONSTANT OBSERVATION

Amit Manocha*, Sukhvir Singh & Anil Taneja*****

*Department of Mathematics, T.I.T. S., Bhiwani, Haryana, India

**Department of Computer Science, N.C.C.E. Panipat, Haryana, India

***Former Prof., JRE Group of Institution, Greater Noida, (U.P.), India

E-mail: amitmanocha80@yahoo.com, boora_s@yahoo.com, dranil_tan@rediffmail.com

ABSTRACT :

A discrete state space and continuous parameter stochastic model for two unit hot standby database system with standby database under constant observation of database administrator (DBA) has been developed. The primary unit is synchronized with hot standby unit through online transfer of archive redo logs. Standby database unit is always kept under constant observation of DBA to check either its synchronization with primary unit is working properly or not. Failure of database unit either primary or standby is also dealt by of DBA. The system is analyzed by making use of semi-Markov process and regenerative point technique. Mathematical expressions for various performance indicating measures of the system has been obtained and economic analysis has been done. Numerical examples are also discussed on the basis of data collected to illustrate the behavior of model developed. Bounds for various costs pertaining to the profitability of the system have also been obtained.

Keywords : Database system, hot standby, constant observation, semi-Markov processes, regenerative point technique.

1. INTRODUCTION

Data collection, analysis and security are significant issues in today's competitive world. Database systems designed by companies like Oracle, Microsoft, IBM etc. provides solutions to such vital issues. These systems are used to preserve the data for the industries functioning in various sectors like Telecommunication, Automobile, Gas & Oil, Transportation, Education, Medical, Finance, Marketing, Banking, Textile & Garments etc. Any type of operational error in these systems may cause substantial loss of data, resources and revenue as a whole. So, reliability, availability and economic analysis of these automated database systems is really needed in the present scenario. The present paper is an attempt to analyze two unit hot standby database system wherein standby unit is kept under constant observation.

In the literature of reliability, standby systems have been discussed very comprehensively by large number of researchers. El-Said and El-Sherbeny [1] did the profit analysis of two unit cold standby system with preventive maintenance and random change in units. Parasher and Taneja [2] discussed reliability and profit evaluation of PLC hot standby system. Goyal et al. [3] studied a two unit cold standby system working in a sugar mill with operating and rest period. Mahmoud and Moshref [4] analysed two unit cold standby system considering hardware, human error failures and preventive maintenance by considering all the time distributions arbitrary. Mathews et al. [5] carried out reliability analysis of identical two unit parallel CC plant system operative with full installed capacity. Modelling of a deteriorating system with repair satisfying general distribution was done by Yuan and Xu [6]. Jain and Rani [7] discussed availability for repairable system with warm standby, switching failure and reboot

delay. Huang et al. [8] studied a reliability model of warm standby configuration with two identical set of units. Batra and Kumar [9] did the stochastic modeling of printed circuit boards manufacturing system under different conditions. Manocha et al. [10] did the stochastic analysis of two unit hot standby database system. The present study deals with two unit hot standby database system comprised of primary database unit synchronized with hot standby unit through the online transfer of redo log files. However, hot standby unit is kept under the constant observation of DBA.

2. SYSTEM DESCRIPTION AND ASSUMPTIONS

Schematic functioning of proposed two unit hot standby database system is shown in Figure 1. Under normal circumstances, all redo log files created at the primary site are archived at standby site. In case of failure of primary unit, its repair is done immediately by the DBA and pre-synchronized hot standby unit act as standard production unit to assure continuous or smooth run of process. Here the cost of using hot standby unit as primary unit will be recurred by user of system itself. Further, on the failure of primary unit, it is also observed that redo files are not created as well as updated in standby unit. Hence, it may cause serious loss of data. At this juncture, to enhance the reliability of system, standby unit is kept under constant supervision of DBA. In such a situation, probability of non-creation of redo log files in standby unit is almost zero. For availing this facility, the additional incurred cost has to be meted out by the user.

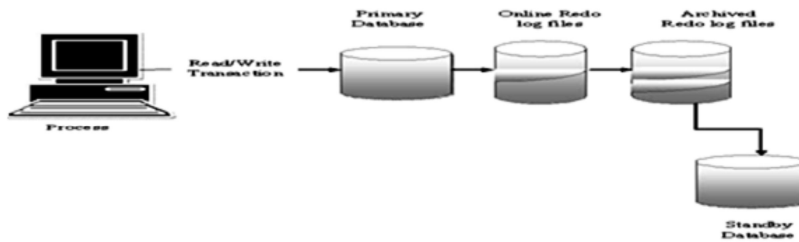


Figure 1 Two Unit Hot Standby Database System

However, non-updation of redo log files in standby unit may be faced but with lower probability. Considering these situations modelling of the system has been done. The proposed system has single DBA facility which performs dual task of repairing the failed unit as well as observing the standby unit. After each repair, the system works as good as new. Time to failure, repair, non-updation of redo log files and updation of redo log files are independent and identically distributed random variables. Time to failure of a unit and time to non-updation of redo log files follows exponential distribution whereas time for repair and updating redo log files follow general distributions.

3. MATERIALS AND METHODS

By using semi-Markov process and regenerative point technique a discrete state space and continuous time stochastic model of a two unit hot standby database system, wherein standby unit is kept under constant observation has been developed. Mathematical expressions for various performance indicating measures of the system such as Mean time to system failure, Mean time to failure of primary database unit, Availability of primary database unit, Busy period of DBA, Expected number of visits by the DBA and profit function has been obtained. Numerical examples are discussed on the basis of data collected. Bounds for various costs which affect the profitability of the system have also been obtained.

4. NOMENCLATURE

- λ / α constant failure rate of primary/ standby database unit
- β rate at which redo log files are not updated in standby database unit
- E/\bar{E} set of regenerative / non-regenerative states
- $M_i(t)$ probability that primary database unit up initially in regenerative state i is up at time t without passing through any other regenerative state or returning to itself through one or more non-regenerative state
- $g(t)/G(t)$ pdf/cdf of the time for repairing the primary database unit
- $g_1(t)/G_1(t)$ pdf/cdf of the time for repairing the standby database unit
- $h(t)/H(t)$ pdf/cdf of the time for updating the redo log files in the standby database unit
- $q_{ij}(t)/ Q_{ij}(t)$ pdf /cdf of time for the system transits from regenerative state i to j
- $q_{ij}^{(k)}(t)/Q_{ij}^{(k)}(t)$ pdf / cdf of time for the system transits from regenerative state i to j via non-regenerative state k
- \otimes / \odot symbol for Stieltjes / Laplace convolution
- $** / *$ symbol for Laplace- Stieltjes/ Laplace Transformation

5. FORMULATION OF MATHEMATICAL MODEL

Symbols for the states of the system are

- P_0 primary database unit is operative
- $H_S / H_r / H_R$ hot standby (database)/ under repair/ repair from previous state
- S_0 hot standby database unit is used as primary database unit
- $S_r / S_w / S_R$ hot standby unit (used as primary database unit) is under repair/ waiting for repair / repair from previous state
- $F_r / F_w / F_R$ failed unit under repair / waiting for repair / repair from previous state
- $H_S \bar{A} \bar{D}$ redo log files were not updated in hot standby database unit

Based on system description and assumptions, the system may be in any of the following states :

- State 0: (P_0, H_S) State 1: $(P_0, H_S \bar{A} \bar{D})$ State 2: (P_0, H_r) State 3: (F_r, S_0) State 4: $(F_w, H_S \bar{A} \bar{D})$
- State 5: (F_w, H_R) State 6: (F_R, S_w) State 7: (P_0, S_r) State 8: (F_w, S_R)

The possible transitions between these states are shown in Figure 2.

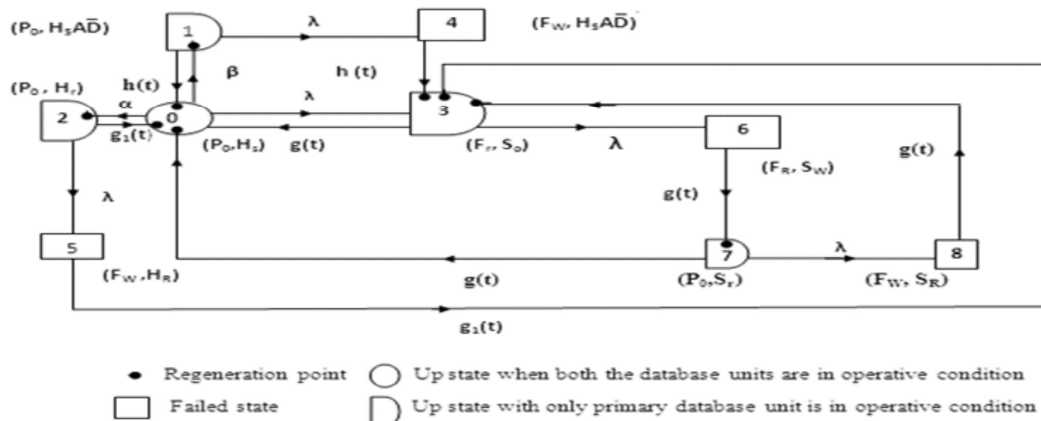


Figure 2 State transition diagram

Here, $E=\{0, 1, 2, 3, 7\}$ is a set of regenerative states whereas $\bar{E} = \{4, 5, 6, 8\}$ is a set of non-regenerative states. States 0, 1, 2, 3 and 7 are up states whereas states 4, 5, 6, and 8 are failed states.

Let $T_0 (\equiv 0), T_1, T_2, \dots$ be the epochs at which system enters any state $i \in E$ and let X_n be the state visited at epoch in T_{n+1} i.e. just after transition at T_n . Then $\{X_n, T_n\}$ is a Markov-renewal process with state space E and $q_{ij} = P\{X_{n+1} = j, T_{n+1} - T_n \leq X_n = i\}$ is the semi-Markov kernel over E . The transition probability matrix (t.p.m.) of the embedded Markov chain is $P = (p_{ij}) = [q_{ij}(\infty) = q(\infty)]$.

By probabilistic arguments, the non-zero elements p_{ij} are,

$$\begin{aligned} p_{01} &= \int_0^\infty \beta e^{-(\lambda+\alpha+\beta)t} dt, & p_{02} &= \int_0^\infty \alpha e^{-(\lambda+\alpha+\beta)t} dt, & p_{03} &= \int_0^\infty \lambda e^{-(\lambda+\alpha+\beta)t} dt, \\ p_{10} &= \int_0^\infty e^{-\lambda t} h(t) dt, & p_{14} &= \int_0^\infty \lambda e^{-\lambda t} \bar{H}(t) dt, & p_{13}^{(4)} &= \int_0^\infty (\lambda e^{-\lambda t} \odot 1) h(t) dt, \\ p_{20} &= \int_0^\infty e^{-\lambda t} g_1(t) dt, & p_{25} &= \int_0^\infty \lambda e^{-\lambda t} \bar{G}_1(t) dt, & p_{23}^{(5)} &= \int_0^\infty (\lambda e^{-\lambda t} \odot 1) g_1(t) dt, \\ p_{30} &= p_{70} = \int_0^\infty e^{-\lambda t} g(t) dt, & p_{36} &= \int_0^\infty \lambda e^{-\lambda t} \bar{G}(t) dt, & p_{37}^{(6)} &= p_{73}^{(8)} = \int_0^\infty (\lambda e^{-\lambda t} \odot 1) g(t) dt \end{aligned}$$

It can be verified that,

$$\begin{aligned} p_{01} + p_{02} + p_{03} &= 1 & p_{10} + p_{14} &= 1, & p_{10} + p_{13}^{(4)} &= 1, & p_{20} + p_{25} &= 1, \\ p_{20} + p_{23}^{(5)} &= 1 & p_{30} + p_{36} &= 1, & p_{30} + p_{37}^{(6)} &= 1, & p_{70} + p_{73}^{(8)} &= 1, \end{aligned}$$

Mean Sojourn time in regenerative state $i \in E$ is given as :

$$\mu_0 = \int_0^\infty e^{-(\lambda+\alpha+\beta)t} dt, \quad \mu_1 = \int_0^\infty e^{-\lambda t} \bar{H}(t) dt, \quad \mu_2 = \int_0^\infty e^{-\lambda t} \bar{G}_1(t) dt, \quad \mu_3 = \mu_7 = \int_0^\infty e^{-\lambda t} \bar{G}(t) dt,$$

The contribution to mean sojourn time when system transit for any regenerative state j when it (time) is counted from the epoch of entrance into state i is given as:

$$\begin{aligned} m_{01} + m_{02} + m_{03} &= \mu_0, & m_{10} + m_{14} &= \mu_1, & m_{20} + m_{25} &= \mu_2, \\ m_{30} + m_{36} &= \mu_3, & m_{10} + m_{13}^{(4)} &= -h^{*'}(0) = K_1 \text{ (say)}, \\ m_{20} + m_{23}^{(5)} &= -g_1^{*'}(0) = K_2 \text{ (say)}, & m_{30} + m_{37}^{(6)} &= m_{70} + m_{73}^{(8)} = -g^{*'}(0) = K_3 \text{ (say)} \end{aligned}$$

6. MEASURES OF SYSTEM EFFECTIVENESS

6.1 Mean Time to System Failure (MTSF)

Let $\pi_i(t)$ be c.d.f. of the first passage time from regenerative state $i \in E$ to a failed state. Regarding the failed state as absorbing state, the recursive relations for ' $\pi_i(t)$ ' are:

$$\pi_0(t) = Q_{01}(t) \otimes \pi_1(t) + Q_{02}(t) \otimes \pi_2(t) + Q_{03}(t) \otimes \pi_3(t)$$

$$\pi_1(t) = Q_{14}(t) + Q_{10}(t) \otimes \pi_0(t)$$

$$\pi_2(t) = Q_{25}(t) + Q_{20}(t) \otimes \pi_0(t)$$

$$\pi_3(t) = Q_{36}(t) + Q_{30}(t) \otimes \pi_0(t)$$

By taking Laplace-Stieljes transformation of the above equations and solving them for $\pi_0^{**}(s)$, We obtain $\pi_0^{**}(s) = N(s)/D(s)$

$$\text{where, } N(s) = Q_{03}^{**}(s) Q_{36}^{**}(s) + Q_{01}^{**}(s) Q_{14}^{**}(s) + Q_{02}^{**}(s) Q_{25}^{**}(s)$$

$$\text{and } D(s) = 1 - Q_{01}^{**}(s) Q_{10}^{**}(s) - Q_{02}^{**}(s) Q_{20}^{**}(s) - Q_{03}^{**}(s) Q_{30}^{**}(s)$$

The reliability of the system at time t is given by,

$$R(t) = L^{-1}[\{1 - \pi_0^{**}(s)\}/s]$$

and the mean time to system failure (MTSF) when the system starts from state '0' is

$$\text{MTSF} = \lim_{s \rightarrow 0} \{1 - \pi_0^{**}(s)\}/s = N/D$$

$$\text{where, } N = (\mu_0 + p_{01}\mu_1 + p_{02}\mu_2 + p_{03}\mu_3) \quad \text{and} \quad D = (1 - p_{01}p_{10} - p_{02}p_{20} - p_{03}p_{30})$$

6.2 Mean Time to Failure of Primary Database Unit (MTFP)

Let $TP_i(t)$ be the c.d.f. of the first passage time of primary database unit from regenerative state $i \in E$ to the state where it is failed. The recursive relations for ' $TP_i(t)$ ' are:

$$TP_0(t) = Q_{03}(t) + Q_{01}(t) \otimes TP_1(t) + Q_{02}(t) \otimes TP_2(t)$$

$$TP_1(t) = Q_{14}(t) + Q_{10}(t) \otimes TP_0(t)$$

$$TP_2(t) = Q_{25}(t) + Q_{20}(t) \otimes TP_0(t)$$

The mean time to failure of primary database unit when the system start from state '0' is given by

$$\text{MTFP} = \lim_{s \rightarrow 0} \{1 - TP_0^{**}(s)\}/s = N_1/D_1$$

$$\text{where, } N_1 = (\mu_0 + p_{01}\mu_1 + p_{02}\mu_2) \quad \text{and} \quad D_1 = (1 - p_{01}p_{10} - p_{02}p_{20})$$

6.3 Availability of Primary Database Unit (AP_0)

Let $AP_i(t)$ be the probability that the primary database unit is in upstate at instant t given that the system entered regenerative state $i \in E$ at $t = 0$. The recursive relations for ' $AP_i(t)$ ' are:

$$AP_0(t) = M_0(t) + q_{01}(t) \odot AP_1(t) + q_{02}(t) \odot AP_2(t) + q_{03}(t) \odot AP_3(t)$$

$$AP_1(t) = M_1(t) + q_{10}(t) \odot AP_0(t) + q_{13}^{(4)}(t) \odot AP_3(t)$$

$$AP_2(t) = M_2(t) + q_{20}(t) \odot AP_0(t) + q_{23}^{(5)}(t) \odot AP_3(t)$$

$$AP_3(t) = q_{30}(t) \odot AP_0(t) + q_{37}^{(6)}(t) \odot AP_7(t)$$

$$AP_7(t) = M_7(t) + q_{70}(t) \odot AP_0(t) + q_{73}^{(8)}(t) \odot AP_3(t)$$

$$\text{where, } M_0(t) = e^{-(\lambda + \alpha + \beta)t}, \quad M_1(t) = e^{-\lambda t} \bar{H}(t), \quad M_2(t) = e^{-\lambda t} \bar{G}_1(t), \quad M_7(t) = e^{-\lambda t} \bar{G}(t)$$

By taking Laplace transform of the above equations and solving them for $AP_0^*(s)$, We obtain

$$AP_0^*(s) = N_2(s)/D_2(s)$$

$$\text{where } N_2(s) = \{M_0^*(s) + q_{01}^*(s)M_1^*(s) + q_{02}^*(s)M_2^*(s)\} \{1 - q_{37}^{(6)*}(s)q_{73}^{(8)*}(s)\} \\ + q_{37}^{(6)*}(s)M_7^*(s) \{q_{01}^*(s)q_{13}^{(4)*}(s) + q_{02}^*(s)q_{23}^{(5)*}(s) + q_{03}^*(s)\}$$

$$\text{and } D_2(s) = \{1 - q_{01}^*(s)q_{10}^*(s) - q_{02}^*(s)q_{20}^*(s)\} \{1 - q_{37}^{(6)*}(s)q_{73}^{(8)*}(s)\} \\ - \{q_{30}^*(s) + q_{70}^*(s)q_{37}^{(6)*}(s)\} \{q_{01}^*(s)q_{13}^{(4)*}(s) + q_{02}^*(s)q_{23}^{(5)*}(s) + q_{03}^*(s)\}$$

In steady-state the availability of primary database unit, is given by

$$AP_0 = \lim_{t \rightarrow \infty} AP_0(t) = \lim_{s \rightarrow 0} sAP_0^*(s) = N_2/D_2$$

$$\text{where, } N_2 = (1 - p_{37}^{(6)}p_{73}^{(8)}) (\mu_0 + p_{01}\mu_1 + p_{02}\mu_2) + (1 - p_{01}p_{10} - p_{02}p_{20}) p_{37}^{(6)}\mu_7$$

$$\text{and } D_2 = (1 - p_{37}^{(6)}p_{73}^{(8)}) (\mu_0 + p_{01}K_1 + p_{02}K_2) + (1 - p_{01}p_{10} - p_{02}p_{20}) (K_3 + p_{37}^{(6)}K_3)$$

Similarly, employing the same argument discussed as above, the mathematical expressions for other performance indicating measures of the system are:

$$\text{Expected time for which standby database unit worked as primary database unit } (S_0) = N_3/D_2$$

$$\text{Expected time for updating the redo log files in standby database unit } (AU_0) = N_4/D_2$$

$$\text{Expected time for repairing primary database unit } (BP_0) = N_5/D_2$$

Expected time for repairing standby database unit $(BH_0) = N_6/D_2$

Expected number of visits by DBA $(V_0) = N_7/D_2$

where, $N_3 = (1 - p_{01}p_{10} - p_{02}p_{20})\mu_3$; $N_4 = p_{01}(1 - p_{37}^{(6)}p_{73}^{(8)})K_1$; $N_5 = (1 - p_{01}p_{10} - p_{02}p_{20})(K_1 + p_{37}^{(6)}K_1)$; $N_6 = (1 - p_{37}^{(6)}p_{73}^{(8)})p_{02}K_2$; $N_7 = (1 - p_{37}^{(6)}p_{73}^{(8)})$

7. COST-BENEFIT ANALYSIS

As the profit is defined as excess of revenue over the cost of production. So, in steady-state, the expected profit per unit time incurred to the system is given by

$$\text{Profit (P)} = C_0AP_0 - C_1S_0 - C_2AU_0 - C_3BP_0 - C_4BH_0 - C_5V_0 - 2CI - K$$

Where, C_0 = Revenue per unit uptime

C_1 = Cost per unit time for which standby database unit worked as primary database unit

C_2 = Cost per unit time for updating the redo log files in standby database unit

C_3 = Cost per unit time for which DBA is busy for repairing primary database unit

C_4 = Cost per unit time for which DBA is busy for repairing standby database unit

C_5 = Cost per visit of DBA

CI = Cost per unit time of Initial Installation

K = Cost per unit time for which standby database unit is kept under constant observation

8. NUMERICAL CALCULATIONS, RESULTS & DISCUSSION

To illustrate the mathematical expressions obtained for the above model with a numerical examples, we consider $h(t) = \gamma e^{-\gamma t}$, $g(t) = \eta e^{-\eta t}$, $g_1(t) = \alpha_1 e^{-\alpha_1 t}$. The data on various rates and costs for primary/standby database unit collected from different users are :

Constant failure rate of primary database unit $(\lambda) = 0.00205$ per hr

Constant repair rate of primary database unit $(\eta) = 0.6529$ per hr

Constant failure rate of standby database unit $(\alpha) = 0.00087$ per hr

Constant repair rate of standby database unit $(\alpha_1) = 0.8533$ per hr

Cost per unit time for which DBA is busy for repairing primary database unit $(C_3) = 7325.58$

Cost per unit time for which DBA is busy for repairing standby database unit $(C_4) = 8750$

and rest of the values are assumed values.

8.1 Effect of rates (β, γ) on MTSF and availability of primary database unit (AP_0)

MTSF and availability of primary database unit (AP_0) are calculated by varying the rate (γ) for different values of rate (β) . The results are shown in Table 1. It is observed that,

- (i). MTSF increase with the increase in the values of rate (γ) and it has higher values for lower values of rate (β) .
- (ii). Availability of primary database unit (AP_0) increase with the increase in the values of rate (γ) and it has higher values for lower values of rate (β) .

γ	MTSF (In hrs)			Availability (AP_0)		
	$\beta = 0.06$	$\beta = 0.07$	$\beta = 0.08$	$\beta = 0.06$	$\beta = 0.07$	$\beta = 0.08$
5	30707.91	27375.01	24707.3	0.996863	0.996862	0.996861
6	34980.74	31341.14	28403.07	0.996864	0.996864	0.996863
7	38851.62	34976.33	31812.98	0.996865	0.996865	0.996864
8	42377.33	38309.9	34967.79	0.996866	0.996865	0.996865
9	45602.07	41396.71	37914.56	0.996866	0.996866	0.996866
10	48571.55	44252.57	40648.3	0.996866	0.996866	0.996866

Table 1 Values of MTSF and AP_0 w.r.t. rate (γ) for different values of rate (β)

8.2 Effect of revenue (C_0) on profit (P) for different values of rate (β)

Figure 3 depicts the behavior of profit (P) with respect to revenue (C_0) for different values of rate (β) by considering other parameters as $\gamma = 12, CI = 5, K = 3, C_1 = 50, C_2 = 500, C_5 = 700$. It is interpreted that profit (P) increases with increase in the revenue (C_0) and has higher values for lower values of rate (β).

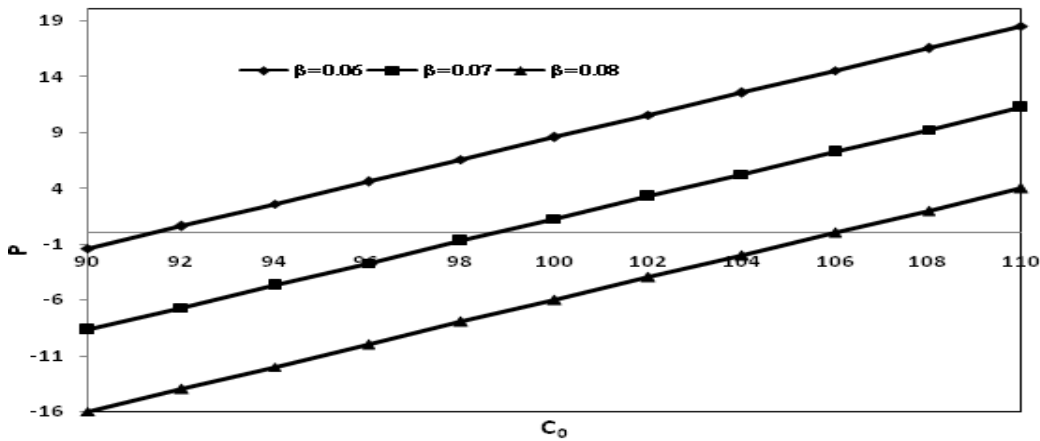


Figure 3 Profit (P) versus Revenue (C_0) for different values of rate (β)

Further,

- i. For $\beta=0.06$, the profit(P) $>$ or $=$ or $<$ 0 according as $C_0 >$ or $=$ or $<$ 91.40. Hence for $\beta=0.06$, the revenue should be more than 91.40 to get the profit.
- ii. For $\beta=0.07$, the profit(P) $>$ or $=$ or $<$ 0 according as $C_0 >$ or $=$ or $<$ 98.72. Hence for $\beta=0.07$, the revenue should be more than 98.72 to get the profit.
- iii. For $\beta=0.08$, the profit(P) $>$ or $=$ or $<$ 0 according as $C_0 >$ or $=$ or $<$ 106.03. Hence for $\beta=0.08$, the revenue should be more than 106.03 to get the profit.

8.3 Effect of Cost (C_2) on profit (P) for different values of cost (C_5)

The profit (P) of the system has been calculated with respect to cost (C_2) for different values of cost (C_5) by fixing another parameters such as $\gamma = 12, \beta = 0.01, CI = 5, K = 2, C_0 = 56, C_1 = 50$, Figure 4 reveals the behavior of profit (P) with respect to cost (C_2) for different values of cost (C_5).

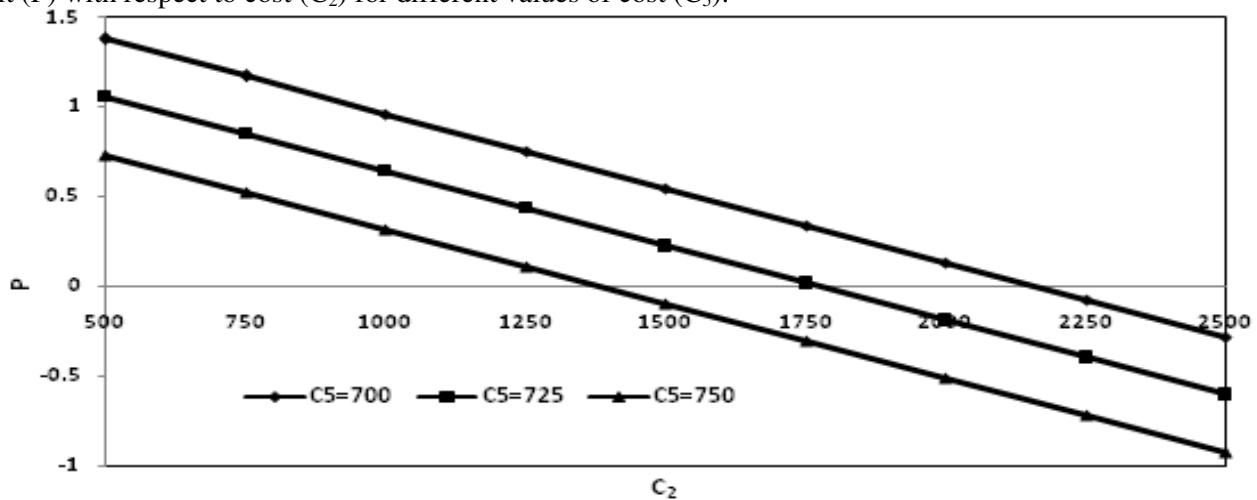


Figure 4 Profit (P) versus Cost (C_2) for different values of cost (C_5)

It can be interpreted that

Profit (P) decreases with increase in the cost (C_2) has higher values for lower value of cost (C_5).

- (i). For $C_5=700$, the profit(P) > or = or < 0 according as $C_2 < \text{or} = > 2160.85$. Hence for $C_5=700$, the cost for updating the redo log files should not be more than 2160.85 to get the profit.
- (ii). For $C_5=725$, the profit(P) > or = or < 0 according as $C_2 \leq \text{or} = > 1773.24$ Hence for $C_5=725$, the cost for updating the redo log files should not be more than 1773.24 to get the profit.
- (iii). For $C_5=750$, the profit(P) > or = or < 0 according as $C_2 \leq \text{or} = > 1385.64$. Hence for $C_5=750$, the cost for updating the redo log files should not be more than 1385.64 to get the profit.

CONCLUSION

Stochastic model for a standby database system in which standby database unit always kept under constant observation of database administrator have been developed. Mathematical expressions for various performance indicating measures of the system are obtained. Numerical analysis is done on the basis of collected data. Bounds (lower/upper) for revenue and cost for updating the redo log files in standby unit have also been obtained.

REFERENCES

1. El-Said, K.M., and M.S. El-sherbeny. Profit Analysis of a Two Unit Cold Standby System with Preventive Maintenance and Random Change in Units. *Journal of Mathematics and Statistics*, 2005; 1(1) : 71-77.
2. Parashar, B., and G. Taneja. Reliability and Profit Evaluation of a Plc Hot Standby System Based on a Master- Slave Concept and Two Types of Repair Facilities. *IEEE Transactions on Reliability*, 2007; 56(3) : 534-539.
3. Goyal, A., G. Taneja , and D.V. Singh. Analysis of a two-unit cold standby system working in a sugar mill with operating and rest period . *Caledonian Journal of Engineering* 2009; 5(1): 1-5.
4. Mahmoud, M.A.W., and M.E. Mosherf. On a two- unit cold standby system considering hardware, human error failures and preventive maintenance. *Mathematical and Computer Modeling*,2010; 51:736-745.
5. Mathews, A.G., S.M. Rizwan, M.C. Majumdar, K.P. Ramchandaran, and G Taneja. Reliability analysis of identical two unit parallel CC plant system operative with full installed capacity. *International Journal of Performability Engineering*, 2011; 7(2):179-187.
6. Yuan, W.Z., and G.Q. Xu. Modelling of a deteriorating system with repair satisfying general distribution. *Applied Mathematics and Computation*, 2012; 218 : 6340-6350.
7. Jain, M.,and S. Rani. Availability Analysis for Repairable System with warm standby, switching failure and reboot delay. *International Journal of Mathematics in Operational Research*, 2013; 5(1) : 19-39.
8. Huang W, J. Lomang, and T. Song. A reliability model of a warm standby configuration with two identical set of units. *Reliability Engineering and System Safety*, 2015; 133 :237-245.
9. Batra, S., and R. Kumar. Stochastic analysis of printed circuit boards manufacturing system considering repairs by internal/external engineers and arbitrary arrival time of external engineers. *International Journal of Pure and Applied Mathematics*, 2016; 106(1) : 59-74.
10. Manocha, A., and G.Taneja. Stochastic and cost-benefit analysis of two unit hot standby database system. *International Journal of Performability Engineering*, 2017; 13(1) : 63-72.



A COMMON FIXED POINT THEOREM USING COMPATIBILITY OF TYPE (A-1) AND WEAKLY COMPATIBILITY

V. Nagaraju*, Bathini Raju**

*Department of Mathematics, University College of Engineering (Autonomous) Osmania University, Hyderabad (Telangana), India

**Department of Mathematics, University Post Graduate College, Secunderabad Osmania University, Hyderabad (Telangana), India

Email: viswanag2007@gmail.com*, bathiniraju107@gmail.com**

ABSTRACT :

The purpose of this paper is to establish a common fixed point theorem in a metric space using the weaker conditions such as compatibility of type (A-1), weakly compatibility and weakly reciprocal continuity.

Keywords: Fixed point, compatible mappings of type (A-1), reciprocally continuous maps, weakly compatible maps and associated sequence.

AMS (2010) Mathematics Classification: 54H25, 47H10

1. INTRODUCTION

G.Jungck [1] introduced the concept of compatibility which is weaker than weakly commuting maps. In 1993, Jungck and Cho [7] introduced the concept of compatible mappings of type (A) by generalizing the definition of weakly uniformly contraction maps. Pathak and Khan [10] introduced the concepts of A-compatibility and S-compatibility by splitting the definition of compatible mapping of type(A). Pathak et.al [8] renamed A-compatibility and S-compatibility as compatible mappings of type(A-1) and compatible mappings of type(A-2) respectively. In 1998, Jungck and Rhoades[4] defined weaker class of maps known as weakly compatible mappings.

R.P.Pant [2] introduced a new notion of continuity namely reciprocal continuity for a pair of self maps and proved some common fixed point theorems. In this paper we prove a common fixed point theorems for four self maps in which one pair is weakly reciprocally continuous, compatible mapping of type (A-1) and other pair is weakly compatible.

2. DEFINITIONS AND PRELIMINARIES

2.1 Compatible mappings

Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n \rightarrow \infty} d(STx_n, TSx_n) = 0$,

whenever $\langle x_n \rangle$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

2.2 Weakly compatible mappings

Two self maps S and T of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence point. i.e., if $Su = Tu$ for some $u \in X$ then $STu = TSu$.

2.3 Reciprocally continuous mappings

Two self maps S and T of a metric space (X,d) are said to be reciprocally continuous, if $\lim_{n \rightarrow \infty} TSx_n = Tt$ and $\lim_{n \rightarrow \infty} STx_n = St$ whenever $\langle x_n \rangle$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

2.4 Weakly reciprocally continuous mappings

Two self maps S and T of a metric space (X,d) are said to be reciprocally continuous, if $\lim_{n \rightarrow \infty} TSx_n = Tt$ or $\lim_{n \rightarrow \infty} STx_n = St$ whenever $\langle x_n \rangle$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

2.5 Compatible mappings of type (A)

Two self maps S and T of a metric space (X,d) are said to be compatible mappings of type (A) if $\lim_{n \rightarrow \infty} d(STx_n, TTx_n) = 0$ and $\lim_{n \rightarrow \infty} d(TSx_n, SSx_n) = 0$ whenever $\langle x_n \rangle$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

2.6 Compatible mappings of type (A-1)

Two self maps S and T of a metric space (X,d) are said to be compatible mappings of type(A-1) if $\lim_{n \rightarrow \infty} d(TSx_n, SSx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t$ for some $t \in X$.

2.7 Associated Sequence: Suppose P, Q, S and T are self maps of a metric space (X,d) such that $S(X) \subset Q(X)$ and $T(X) \subset P(X)$. Then for an arbitrary $x_0 \in X$ there is a point x_1 in X such that $Sx_0 = Qx_1$ and for this point x_1 , there is a point x_2 in X such that $Tx_1 = Px_2$ and so on. Proceeding in this way, we can obtain a sequence $\langle y_n \rangle$ in X such that $y_{2n} = Sx_{2n} = Qx_{2n+1}$ and $y_{2n+1} = Px_{2n+2} = Tx_{2n+1}$ for $n \geq 0$. We shall call this sequence as an ‘‘associated sequence of x_0 ’’ relative to the four self maps P,Q,S and T.

2.8 Lemma: Let P, Q, S and T be self maps from a complete metric space (X,d) into itself satisfying the conditions

$$S(X) \subset Q(X) \text{ and } T(X) \subset P(X) \tag{2.8.1}$$

$$\text{and } d(Sx, Ty) \leq \alpha \frac{d(Qy, Ty)[1 + d(Px, Sx)]}{[1 + d(Px, Qy)]} + \beta d(Px, Qy)$$

$$\text{for all } x, y \text{ in } X \text{ where } \alpha, \beta \geq 0, \alpha + \beta < 1. \tag{2.8.2}$$

Then the associated sequence $\{y_n\}$ relative to the four self maps P,Q,S and T is a Cauchy sequence in X.

Proof: From the definition (2.7) and (2.8.2), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha \frac{d(Qx_{2n+1}, Tx_{2n+1})[1 + d(Px_{2n}, Sx_{2n})]}{[1 + d(Px_{2n}, Qy_{2n+1})]} + \beta d(Px_{2n}, Qy_{2n+1}) \\ &= \alpha \frac{d(y_{2n}, y_{2n+1})[1 + d(y_{2n-1}, y_{2n})]}{[1 + d(y_{2n-1}, y_{2n})]} + \beta d(y_{2n-1}, y_{2n}) \\ &= \alpha d(y_{2n}, y_{2n+1}) + \beta d(y_{2n-1}, y_{2n}) \text{ implies that} \end{aligned}$$

$$(1 - \alpha)d(y_{2n}, y_{2n+1}) \leq \beta d(y_{2n-1}, y_{2n}) \text{ implies that}$$

$$d(y_{2n}, y_{2n+1}) \leq \frac{\beta}{(1-\alpha)} d(y_{2n-1}, y_{2n}) = hd(y_{2n-1}, y_{2n}), \text{ where } h = \frac{\beta}{1-\alpha}.$$

That is, $d(y_{2n}, y_{2n+1}) \leq h(y_{2n-1}, y_{2n})$ (2.8.3)

Similarly, we can prove that $d(y_{2n+1}, y_{2n+2}) \leq hd(y_{2n}, y_{2n+1})$. (2.8.4)

Hence, from (2.8.3) and (2.8.4), we have

$$d(y_n, y_{n+1}) \leq hd(y_{n-1}, y_n) \leq h^2d(y_{n-2}, y_{n-1}) \leq \dots \leq h^n d(y_0, y_1) . \tag{2.8.5}$$

Now for any positive integer p, we have

$$\begin{aligned} d(y_n, y_{n+p}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d(y_{n+p-1}, y_{n+p}) \\ &\leq h^n d(y_0, y_1) + h^{n+1}d(y_0, y_1) + \dots + h^{n+p-1}d(y_0, y_1) \\ &= (h^n + h^{n+1} + \dots + h^{n+p-1})d(y_0, y_1) \\ &= h^n (1 + h + h^2 + \dots + h^{p-1})d(y_0, y_1) \\ &< \frac{h^n}{1-h} d(y_0, y_1) \rightarrow 0 \text{ as } n \rightarrow \infty, \text{ since } h < 1. \end{aligned}$$

Thus the sequence $\{y_n\}$ is a Cauchy sequence in X. Since X is a complete metric space, the sequence $\{y_n\}$ converges to some point z in X.

2.9 Remark: The converse of the above Lemma is not true. That is, if P,Q,S and T are self maps of a metric space (X, d) satisfying (2.8.1) , (2.8.2) and even if for any x_0 in X and for any associated sequence of x_0 converges, then the metric space (X, d) need not be complete.

2.10 Example: Let $X = (0, 1]$ with $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a metric space. Define the self maps S,T,P and Q of X by

$$Sx = Tx = \begin{cases} \frac{1}{3} & \text{if } 0 < x < \frac{1}{2} \\ \frac{1}{2} & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} \text{ and } Px = Qx = \begin{cases} \frac{1}{4} & \text{if } 0 < x < \frac{1}{2} \text{ and if } x = 1 \\ 1-x & \text{if } \frac{1}{2} \leq x < 1 \end{cases}$$

Then $S(X) = T(X) = \left\{ \frac{1}{3}, \frac{1}{2} \right\}$ while $P(X) = Q(X) = \left(0, \frac{1}{2} \right]$.

Clearly $S(X) \subset Q(X)$ and $T(X) \subset P(X)$. Also the inequality (2.8.2) can easily be verified for appropriate

values of $\alpha, \beta \geq 0, \alpha + \beta < 1$. Moreover if we take $x_n = \frac{1}{2} + \frac{1}{2n}$ for $n \geq 1$, then the sequence

$$Sx_0, Tx_1, Sx_2, Tx_3, \dots, Sx_{2n}, Tx_{2n+1}, \dots \text{ converges to } \frac{1}{2} \in X . \text{ But X is not a complete metric space.}$$

Now we generalize the result of P.C.Lohani and V.H.Badshah [6] as follows.

3. MAIN RESULT

3.1 Theorem: Let P,Q,S and T be self maps of a metric space (X, d) satisfying the conditions

$$S(X) \subset Q(X) \text{ and } T(X) \subset P(X) \tag{3.1.1}$$

$$d(Sx, Ty) \leq \alpha \frac{d(Qy, Ty)[1 + d(Px, Sx)]}{[1 + d(Px, Qy)]} + \beta d(Px, Qy) \tag{3.1.2}$$

for all x, y in X where $\alpha, \beta \geq 0, \alpha + \beta < 1$.

The pair (S,P) is compatible mapping of type (A-1) , weakly reciprocally continuous and the pair (Q,T) is weakly compatible and (3.1.3)

an associated sequence $\langle x_n \rangle$ of a point $x_0 \in X$ relative to four self maps P, Q, S and T such that the sequence $Sx_0, Tx_1, Sx_2, Tx_3, \dots, Sx_{2n}, Tx_{2n+1}, \dots$ converges to some point $z \in X$. (3.1.4)

Then z is a unique common fixed point of S,P,Q and T.

Proof: By (3.1.4), we have

$$Sx_{2n} \rightarrow z, Qx_{2n+1} \rightarrow z, Tx_{2n+1} \rightarrow z \text{ and } Px_{2n} \rightarrow z \text{ as } n \rightarrow \infty . \tag{3.1.5}$$

$$\text{Since the pair (S,P) is compatible mappings of type(A-1), } \lim_{n \rightarrow \infty} PSx_{2n} = \lim_{n \rightarrow \infty} SSx_{2n} \tag{3.1.6}$$

Also since the pair (S,P) is weakly reciprocally continuous,

$$SPx_{2n} \rightarrow Sz, PSx_{2n} \rightarrow Pz \text{ as } n \rightarrow \infty . \tag{3.1.7}$$

We shall now prove that $Sz = Pz = Qz = Tz = z$.

To prove $Pz = z$, put $x = Sx_{2n}$ and $y = x_{2n+1}$ in (3.1.2), we get

$$d(SSx_{2n}, Tx_{2n+1}) \leq \alpha \frac{d(Qx_{2n+1}, Tx_{2n+1})[1 + d(PSx_{2n}, SSx_{2n})]}{[1 + d(PSx_{2n}, Qx_{2n+1})]} + \beta d(PSx_{2n}, Qx_{2n+1})$$

Letting $n \rightarrow \infty$ and using (3.1.5), (3.1.6) and (3.1.7) in the above inequality, we get

$$\begin{aligned} d(Pz, z) &\leq \alpha \frac{d(z, z)[1 + d(Pz, Pz)]}{[1 + d(Pz, z)]} + \beta d(Pz, z) \\ &= \beta d(Pz, z) \\ &< d(Pz, z), \text{ a contradiction, since } \alpha, \beta \geq 0 \text{ and } \alpha + \beta < 1. \end{aligned}$$

$$\text{Thus we have } (Pz, z) = 0 \text{ which implies that } Pz = z . \tag{3.1.8}$$

To prove $Sz = z$, put $x = z$ and $y = x_{2n+1}$ in (3.1.2), we get

$$d(Sz, Tx_{2n+1}) \leq \alpha \frac{d(Qx_{2n+1}, Tx_{2n+1})[1 + d(Pz, Sz)]}{[1 + d(Pz, Qx_{2n+1})]} + \beta d(Pz, Qx_{2n+1})$$

Letting $n \rightarrow \infty$ and using (3.1.5) and (3.1.8) in the above inequality, we get

$$\begin{aligned} d(Sz, z) &\leq \alpha \frac{d(z, z)[1 + d(z, Sz)]}{[1 + d(z, z)]} + \beta d(z, z) \\ &\leq 0 \text{ which gives that } d(Sz, z) = 0 . \end{aligned}$$

Thus we have $d(Sz, z) = 0$ which implies that $Sz = z$.

Therefore $Pz = Sz = z$.

Since $S(X) \subset Q(X)$, there exists $u \in X$ such that $z = Sz = Qu$.

To prove $Tu = z$, put $x = z$ and $y = u$ in (3.1.2), we have

$$d(Sz, Tu) \leq \alpha \frac{d(Qu, Tu)[1 + d(Pz, Sz)]}{[1 + d(Pz, Qu)]} + \beta d(Pz, Qu)$$

$$d(z, Tu) \leq \alpha \frac{d(z, Tu)[1 + d(z, z)]}{[1 + d(z, z)]} + \beta d(z, z),$$

$$= \alpha d(z, Tu)$$

That is, $d(z, Tu) \leq \alpha d(z, Tu)$.

$< d(z, Tu)$, a contradiction, since $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$.

Thus we have $d(z, Tu) = 0$ which implies that $Tu = z$.

Therefore $Tu = Qu = z$.

Since the pair (Q,T) is weakly compatible, $QTu = TQu$ which gives $Qz = Tz$. (3.1.9)

Finally to prove $Tz = z$, put $x = x_{2n}$ and $y = z$ in (3.1.2), we have

$$d(Sx_{2n}, Tz) \leq \alpha \frac{d(Qz, Tz)[1 + d(Px_{2n}, Sx_{2n})]}{[1 + d(Px_{2n}, Qz)]} + \beta d(Px_{2n}, Qz)$$

Letting $n \rightarrow \infty$ and using (3.1.5) and (3.1.9) in the above inequality, we get

$$d(z, Tz) \leq \alpha \frac{d(Tz, Tz)[1 + d(z, z)]}{[1 + d(z, Tz)]} + \beta d(z, Tz)$$

$$= \beta d(z, Tz)$$

$< d(z, Tz)$, a contradiction, since $\alpha, \beta \geq 0$ and $\alpha + \beta < 1$.

Thus we have $d(z, Tz) = 0$ which implies that $Tz = z$.

Therefore $Tz = Qz = z$.

Hence $Sz = Pz = Qz = Tz = z$, showing that z is a common fixed point of P,Q,S and T.

Uniqueness: Let z and w be two common fixed points of P,Q,S and T. Then we have $z = Sz = Pz = Qz = Tz$ and $w = Sw = Pw = Qw = Tw$.

Put $x = z$ and $y = w$ in (3.1.2), we get

$$d(z, w) \leq \alpha \frac{d(w, w)[1 + d(z, z)]}{[1 + d(z, w)]} + \beta d(z, w)$$

$$= \beta d(z, w)$$

$< d(z, w)$, a contradiction.

Thus we have $d(z, w) = 0$ which implies that $z = w$.

Hence z is a unique common fixed point of S,P,Q and T.

3.2 Remark: From the example (2.8.1), clearly $S(X) \subset Q(X)$, $T(X) \subset P(X)$ and it can easily be verified that the pair (S,P) is weakly reciprocally continuous and compatible mapping of type (A-1) and the pair (Q,T) weakly compatible as they commute at their coincidence point $\frac{1}{2}$. Also, if we take $x_n = \frac{1}{2} + \frac{1}{2n}$ for $n \geq 1$, then the sequence $Sx_0, Tx_1, Sx_2, Tx_3, \dots, Sx_{2n}, Tx_{2n+1}, \dots$ converges to $\frac{1}{2} \in X$. Moreover, the rational inequality holds for the values of $\alpha, \beta \geq 0, \alpha + \beta < 1$. It may be noted that ' $\frac{1}{2}$ ', is the unique common fixed point of P, Q, S and T.

REFERENCES

1. G.Jungck, *Compatible mappings and common fixed points*, *Internat.J. Math. &Math.Sci.*, 9(1986), 771-778.
2. R.P.Pant, *A Common fixed point theorem under a new condition*, *Indian J.of Pure and App.Math.*,30(2) (1999), 147-152.
3. Jungck, *Compatible mappings and common fixed points (2)*, *Internat.J.Math.&Math.Sci.*, 11(1988),285-288.
4. G.Jungck and B.E. Rhoades, *Fixed point for set valued functions without continuity*, *Indian J.Pure.Appl.Math*, 29(3) (1998), 227-238.
5. V.Srinivas and Ravi S, *Generation of a common fixed point theorem using A-compatible and B-compatible mappings of type(E)*,*Global Journal of Pure and Applied Mathematics*, 13(6) (2017), 1735-1744.
6. P.C.Lohani and V.H.Badshah , *Compatible mappings and common fixed point for four mappings*,*Bull.Cal.Math.Soc.*, 90(1998) , 301-308.
7. Jungck G, Murthy P.P and Cho Y.J., *Compatible mappings of type (A) and common fixed points*, *Math.Japan*, 38(1993), 381-391.
8. Khan M.S, Pathak H.K and George Reny,*Compatible mappings of type (A-1) and type(A-2) and common fixed points in fuzzy metric spaces*, *International Math.Forum*,2(11)(2007),515-524.
9. V.Nagaraju and V.Srinivas, *Common fixed point theorem on compatible mappings of type (P)*, *Gen.Math.Notes*, 21(2) (2014), 87-94.
10. Pathak H.K. and Khan M.S., *A comparison of various types of compatible maps and common fixed points*, *Indian J.Pure appl.Math*, 28(4) (1997), 477-485.
11. S.Ravi and V.Srinivas, *A fixed point theorem using weakly Semicompatible mappings in metric space*, *International Journal of Mathematical Analysis*, 11(15) (2017), 695-706.
12. Pant R.P., Bisht R.K and Arora D, *weak reciprocal continuity and fixed point theorems*, *Ann Univ Ferrara (Springer)*, 57, (2011), 181-190.



ON LOWER SEPERATION AXIOMS VIA IDEALS

Nitakshi Goyal*, Navpreet Singh Noorie**

*&** Department of Mathematics, Punjabi University Patiala, Punjab, India

**Research Supervisor and Corresponding Author

E-mail: goyal.nishu53@gmail.com, noorie@pbi.ac.in

ABSTRACT :

In the present paper, we introduce S_1 and S_2 spaces with respect to an ideal containing the class of all S_1 and S_2 spaces. We shall also give various characterizations of these spaces.

*Key Words and Phrases : $S_1 \text{ mod } \mathfrak{I}$, $S_2 \text{ mod } \mathfrak{I}$, ideal, *-topology. 2000 MSC: 54D10.*

1. INTRODUCTION

In [8], Shanin introduced the concept of R_0 topological spaces. In [2], Davis studied this week separation axiom and also introduced R_1 space. In [4], Dorsett investigate some further properties of R_0 and R_1 spaces and also give characterizations of these spaces in terms of nets and closures. In [5], Dunham introduced the concept of weekly Hausdorff space and proved that it is equivalent to R_1 space. In [1], Császár, call these R_0 and R_1 spaces as S_1 and S_2 spaces respectively and give various properties of S_1 and S_2 spaces. On the other hand the study of separation axioms via ideals is also a well researched topic in the literature. Ideals in topological spaces have been used to study topological properties introduced by Kuratowski[7] and Vaidyanathaswamy[9], where an ideal \mathfrak{I} on a topological space (X, τ) is a collection of subsets of X which is closed under finite unions and closed downwards i.e. every subset of a member of \mathfrak{I} is in \mathfrak{I} . Further a new topology $\tau^*(\mathfrak{I}, \tau)$ called the *-topology is given which is generally finer than the original topology and the corresponding kuratowski closure operator for the *-topology is given by $cl^*(A) = A \cup A^*(\mathfrak{I}, \tau)$ [10], where $A^*(\mathfrak{I}, \tau) = \{x \in X : U \cap A \notin \mathfrak{I} \text{ for every open subset } U \text{ of } x \text{ in } X \text{ called a local function of } A \text{ with respect to } \mathfrak{I} \text{ and } \tau. \text{ We will write } A^* \text{ for } A^*(\mathfrak{I}, \tau) \text{ and } \tau^* \text{ for } \tau^*(\mathfrak{I}, \tau).$

In this paper, we will give idealization of S_1 and S_2 spaces, which we call $S_1 \text{ mod } \mathfrak{I}$ and $S_2 \text{ mod } \mathfrak{I}$ spaces respectively containing the class of all S_1 and S_2 spaces. The relationship of these spaces among themselves and with the known spaces are investigated. We also give characterization of $S_1 \text{ mod } \mathfrak{I}$ spaces in terms of closure of a point in the given topology and its *-topology (Theorem 3.3 and 3.4 below) and also in terms of convergence of filter (Theorem 3.6 below). Further, characterizations of $S_2 \text{ mod } \mathfrak{I}$ spaces in terms of closure of a point and Kernel of a point (Theorem 4.7 and 4.9 below) and in terms of \mathfrak{I} -convergence of a filter (Theorem 4.8 below) are given.

2. PRELIMINARIES

The following section contains some definitions and results that will be used in our further sections.

Given an ideal topological space (X, τ, \mathfrak{I}) , the collection $\beta = \{V - I : V \in \tau \text{ and } I \in \mathfrak{I}\}$ will form a basis [10] for the *-topology τ^* . Also for any subset A of X , $\mathfrak{I}_A = \{I \in \mathfrak{I} : I \subset A\}$ is an ideal on A and $(\tau|A)^*(\mathfrak{I}_A, \tau|A) = \tau^*|A$ [6]. And an ideal \mathfrak{I} is said to be codense [3] if $\tau \cap \mathfrak{I} = \phi$.

Definition 2.1. Let (X, τ) be a topological space. A filter \mathcal{F} on X is a collection of non-empty subsets of X such that (a) $\phi \notin \mathcal{F}$ (b) $A \in \mathcal{F}$ and $B \in \mathcal{F}$ implies $A \cup B \in \mathcal{F}$ (c) $A \in \mathcal{F}$ and $A \subset B$ implies $B \in \mathcal{F}$.

Also for any point x of X , \mathcal{F} is said to be convergent to x written $\mathcal{F} \rightarrow x$ if for every open subset U of x in X , $U \in \mathcal{F}$. The collection of all such points is denoted by $\lim \mathcal{F}$.

Definition 2.2.[4] Let (X, τ) be a topological space, then for any point x of X , $\text{Ker}\{x\} = \bigcap \{ G : G \text{ is open subset of } x \}$.

Definition 2.3. A topological space (X, τ) is said to be

(a) S_2 space $[1](R_1$ in sense of [2] [4] [5]) if for every pair of distinct points x and y , whenever $\text{cl}\{x\} \neq \text{cl}\{y\}$ then there exist disjoint nhds. containing them.

(b) S_1 space $[1](R_0$ in sense of [2] [4] [5] [8]) if for every pair of distinct points x and y , whenever x has a nhd. not containing y , then y has a nhd. not containing x .

Throughout this paper, (X, τ) will denote the topological space. If \mathfrak{I} is an ideal on X , then (X, τ, \mathfrak{I}) is known as ideal topological space. By an open subset of X , we always mean open subset in the topological space (X, τ) . For a subset A of X , $\text{cl}(A)$ and $\text{cl}^*(A)$ denote the closure of A in (X, τ) and (X, τ^*) respectively and A^c will denote the complement of A in X .

3. S_1 MOD \mathfrak{I} SPACES:

Definition 3.1. An ideal space (X, τ, \mathfrak{I}) is said to be S_1 mod \mathfrak{I} if for every pair of distinct points x and y in X , whenever x has a τ -open subset not containing y , y has a τ^* -open subset not containing x .

Every S_1 space is S_1 mod \mathfrak{I} , since $\phi \in \mathfrak{I}$. Also it can be seen easily that (X, τ^*) is S_1 implies (X, τ, \mathfrak{I}) is S_1 mod \mathfrak{I} . But the converse is not true as can be seen from the following examples.

Example 3.1. Let $X = \{a, b, c\}$, $\tau = \{ \phi, \{a\}, \{b\}, \{a, b\}, X \}$ and $\mathfrak{I} = \{ \phi, \{a\}, \{b\}, \{a, b\} \}$. So $\tau^* = \emptyset(X)$. Then X is S_1 mod \mathfrak{I} obviously, but not S_1 . Since a has a open subset not containing c but c has no open subset not containing a .

Example 3.2. Let $X = \{a, b, c\}$, $\tau = \{ \phi, \{a\}, \{b, c\}, X \}$ and $\mathfrak{I} = \{ \phi, \{c\} \}$. So $\tau^* = \{ \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X \}$. Then X is S_1 and hence also S_1 mod \mathfrak{I} . But (X, τ^*) is not S_1 . Since b has a τ^* -open subset not containing c but c has no τ^* -open subset not containing b .

Remark 3.1. From Example 3.1 it can be easily checked that (X, τ^*) is S_1 but (X, τ) is not S_1 . Also Example 3.2 shows that (X, τ) is S_1 but (X, τ^*) is not S_1 .

Therefore, the relationship of this separation axiom with respect to topological spaces can be seen below.

- a) (X, τ) is $S_1 \Rightarrow (X, \tau, \mathfrak{I})$ is S_1 mod $\mathfrak{I} \not\Rightarrow (X, \tau)$ is S_1 .
- b) (X, τ^*) is $S_1 \Rightarrow (X, \tau, \mathfrak{I})$ is S_1 mod $\mathfrak{I} \not\Rightarrow (X, \tau^*)$ is S_1 .
- c) (X, τ) is $S_1 \Leftrightarrow (X, \tau^*)$ is S_1 .

Theorem 3.1. If an ideal space (X, τ, \mathfrak{I}) is S_1 mod \mathfrak{I} and $\mathfrak{I} \subset \mathcal{J}$, where \mathcal{J} is an ideal. Then (X, τ, \mathcal{J}) is S_1 mod \mathcal{J} .

Proof. Follows from the fact that if $\mathfrak{I} \subset \mathcal{J}$ then $\tau^*(\mathfrak{I}) \subset \tau^*(\mathcal{J})$.

The following theorem shows that subspace of S_1 mod \mathfrak{I} space is S_1 mod \mathfrak{I} .

Theorem 3.2. Let (X, τ, \mathfrak{I}) be S_1 mod \mathfrak{I} space and $A \subset X$. Then $(A, \tau_A, \mathfrak{I}_A)$ is also S_1 mod \mathfrak{I} space.

Proof. Let x and y be two distinct points of A such that x has a τ_A -open subset G not containing y . So $G = U \cap A$, where U is open subset of X containing x but not y . Therefore, (X, τ, \mathfrak{I}) is S_1 mod \mathfrak{I} implies that there exists τ^* -open subset V containing y but not x and so $V \cap A$ is $(\tau_A)^*$ -open subset containing y but not x . Hence $(A, \tau_A, \mathfrak{I}_A)$ is S_1 mod \mathfrak{I} .

Further we will give various characterizations of S_1 mod \mathfrak{I} spaces. The following theorem characterize S_1 mod \mathfrak{I} space in terms of τ^* -closed sets.

Theorem 3.3. Let (X, τ, \mathfrak{I}) be an ideal space. Then the following are equivalent:

- (X, τ, \mathfrak{I}) is $S_1 \text{ mod } \mathfrak{I}$.
- If $x \in \text{cl}^*\{y\}$, then $y \in \text{cl}\{x\}$.
- If $G \in \tau$ and $x \in G$, then $\text{cl}^*\{x\} \subset G$ i.e. every open subset is a union of τ^* -closed sets.

Proof. (a) \Rightarrow (b): Obvious from the definition of $S_1 \text{ mod } \mathfrak{I}$.

(b) \Rightarrow (c): Let G be open subset of X and $y \in \text{cl}^*\{x\}$ for some $x \in G$. Then (b) implies $x \in \text{cl}\{y\}$. This implies that every open subset containing x also contains y , so $y \in G$. Therefore $\text{cl}^*\{x\} \subset G$. Hence (c) holds.

(c) \Rightarrow (a): Let U be an open subset and $x \in U$. If $y \notin U$, then by (c) $\text{cl}^*\{x\} \subset U$ implies $y \notin \text{cl}^*\{x\}$. This implies that U has a τ^* -open subset not containing x . Hence (a) holds.

Corollary 3.1. An ideal space (X, τ, \mathfrak{I}) is $S_1 \text{ mod } \mathfrak{I}$ if and only if any subset A of X is a union of τ^* -closed sets, whenever A^C is union of closed sets.

Proof. Proof is obvious and hence is omitted.

The following theorem characterize $S_1 \text{ mod } \mathfrak{I}$ space in terms of closure of a point in given topology and its τ^* -topology.

Theorem 3.4. Let (X, τ, \mathfrak{I}) be an ideal space. Then the following are equivalent:

- (X, τ, \mathfrak{I}) is $S_1 \text{ mod } \mathfrak{I}$.
- For any distinct points x, y of X , either $\text{cl}\{x\} = \text{cl}\{y\}$ or $\text{cl}^*\{x\} \cap \text{cl}\{y\} = \emptyset$ or $\text{cl}\{x\} \cap \text{cl}^*\{y\} = \emptyset$.

Proof. (a) \Rightarrow (b): Let x, y be two distinct points of X such that $\text{cl}\{x\} \neq \text{cl}\{y\}$. So either $x \notin \text{cl}\{y\}$ or $y \notin \text{cl}\{x\}$. Let $x \notin \text{cl}\{y\}$, so $x \in (\text{cl}\{y\})^C$. Since $(\text{cl}\{y\})^C$ is open subset of X , so by the equivalence of (a) and (c) of Theorem 3.3, we have $\text{cl}^*\{x\} \subset (\text{cl}\{y\})^C$. Therefore, $\text{cl}^*\{x\} \cap \text{cl}\{y\} = \emptyset$. Also $y \notin \text{cl}\{x\}$ implies $\text{cl}\{x\} \cap \text{cl}^*\{y\} = \emptyset$. Hence (b) holds.

(b) \Rightarrow (a): Let U be open subset of X and $x \in U$. Let $y \in \text{cl}^*\{x\} - \{x\}$, then by (b), $\text{cl}\{x\} = \text{cl}\{y\}$. This implies that every open subset containing x also contains y , so $y \in U$. Therefore, $\text{cl}^*\{x\} \subset U$. Hence (a) holds by Theorem 3.3.

Theorem 3.5. Let (X, τ, \mathfrak{I}) be an ideal space. Then the following are equivalent:

- (X, τ, \mathfrak{I}) is $S_1 \text{ mod } \mathfrak{I}$.
- For any $x \in X$, $\text{cl}^*\{x\} \subset \text{Ker}\{x\}$.
- Any closed set F in X expressed as $F = \bigcap \{ G : F \subset G, G \text{ is } \tau^*\text{-open} \}$.
- Any open set G in X expressed as $G = \bigcup \{ F : F \subset G, F \text{ is } \tau^*\text{-closed} \}$.
- For any non-empty set A and open subset G in X such that $A \cap G \neq \emptyset$ there exists a τ^* -closed set F such that $A \cap F \neq \emptyset$ and $F \subset G$.
- For any closed set F in X , $x \notin F$ implies $F \cap \text{cl}^*\{x\} = \emptyset$.

Proof. (a) \Rightarrow (b): Let $y \in \text{cl}^*\{x\}$, then by using the equivalence of (a) and (b) of Theorem 3.3, we have $x \in \text{cl}\{y\}$. This implies that every open subset containing x also contains y . So $y \in \text{Ker}\{x\}$. Hence (b) holds.

(b) \Rightarrow (c): Let F be any closed set in X . Then obviously $F \subset \bigcap \{ G : F \subset G \text{ and } G \text{ is } \tau^*\text{-open} \}$. Conversely, let $x \notin F$, so $x \in F^C$ and so $\text{Ker}\{x\} \subset F^C$. Therefore, by (b), we have $\text{cl}^*\{x\} \subset \text{Ker}\{x\} \subset F^C$. This implies that $F \subset (\text{cl}^*\{x\})^C$. Since, $(\text{cl}^*\{x\})^C$ is τ^* -open such that $F \subset (\text{cl}^*\{x\})^C$ and $x \notin (\text{cl}^*\{x\})^C$. Therefore, $x \notin \bigcap \{ G : F \subset G, G \text{ is } \tau^*\text{-open} \}$. Hence (c) holds.

(c) \Rightarrow (d): is obvious.

(d) \Rightarrow (e): Let A be non-empty subset and G is open subset of X such that $A \cap G \neq \emptyset$. Then by (d), $A \cap G = A \cap (\bigcup \{ F : F \subset G, F \text{ is } \tau^*\text{-closed} \}) = \bigcup \{ A \cap F : F \subset G, F \text{ is } \tau^*\text{-closed} \} \neq \emptyset$. Therefore, there exists τ^* -closed set F such that $F \subset G$ and $A \cap F \neq \emptyset$. Hence (e) holds.

(e) \Rightarrow (f): Let F be any closed set in X and $x \in X$ such that $x \notin F$. So $x \in F^C$ and so $\{x\} \cap F^C \neq \emptyset$. Therefore by (e), there exists τ^* -closed set W such that

$\{x\} \cap W \neq \emptyset$ and $W \subset F^C$. So $x \in W$ and W is τ^* -closed implies $cl^*\{x\} \subset W$ and so $F \subset (cl^*\{x\})^C$. Hence $cl^*\{x\} \cap F = \emptyset$. Hence (f) holds.

(f) \Rightarrow (a): Let x, y be two distinct points of X and U be open subset in X containing x but not y . So $x \notin U^C$ and U^C is closed in X and so by (f), $cl^*\{x\} \cap U^C = \emptyset$. Therefore, $cl^*\{x\} \subset U$ implies that $y \notin cl^*\{x\}$. Hence there exists τ^* -open subset G of y not containing x . Hence (a) holds.

The following theorem gives the characterization of $S_1 \text{ mod } \mathfrak{T}$ space in terms of convergence of a filter.

Theorem 3.6. Let (X, τ, \mathfrak{T}) be an ideal space. Then the following are equivalent:

- a) (X, τ, \mathfrak{T}) is $S_1 \text{ mod } \mathfrak{T}$.
- b) If $x, y \in X$, then $y \in cl^*\{x\}$ implies for all filter \mathcal{F} , $\mathcal{F} \rightarrow y$ implies $\mathcal{F} \rightarrow x$.
- c) For any $x, y \in X$, $y \in cl^*\{x\}$ implies $cl\{x\} = cl\{y\}$.

Proof. (a) \Rightarrow (b): Let $x, y \in X$ such that $y \in cl^*\{x\}$, then by the equivalence of (a) and (b) of Theorem 3.3, $x \in cl\{y\}$. Now, consider a filter \mathcal{F} such that $\mathcal{F} \rightarrow y$. So every open subset containing y is a member of \mathcal{F} . But $x \in cl\{y\}$ implies every open subset containing x also contains y . Therefore, every open subset containing x also a member of \mathcal{F} . Hence $\mathcal{F} \rightarrow x$.

(b) \Rightarrow (c): Let $x, y \in X$ such that $y \in cl^*\{x\}$ and consider the nhd. filter \mathcal{N}_y . So $\mathcal{N}_y \rightarrow y$. Therefore, by (b) $\mathcal{N}_y \rightarrow x$. This implies that every open subset containing x is a member of \mathcal{N}_y , so every open subset containing x also contains y and so $x \in cl\{y\}$. Also $y \in cl^*\{x\} \subset cl\{x\}$. Hence $cl\{x\} = cl\{y\}$.

(c) \Rightarrow (a): follows from Theorem 3.3.

The Example below shows that the reverse implication does not hold in Theorem 3.5(b) and Theorem 3.6(b), (c).

Example 3.3. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\mathfrak{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. So $\tau^* = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Then $Ker\{c\} = X$, but $cl^*\{c\} = \{b, c\}$. So $Ker\{c\} \not\subset cl^*\{c\}$.

Also $cl\{b\} = cl\{c\}$, but $c \notin cl^*\{b\}$. And for any filter \mathcal{F} , $\mathcal{F} \rightarrow c$ as well as $\mathcal{F} \rightarrow b$, since X is the only open subset containing b and c . Therefore for all filter \mathcal{F} , $\mathcal{F} \rightarrow c$ implies $\mathcal{F} \rightarrow b$, but $c \notin cl^*\{b\}$.

4. $S_2 \text{ MOD } \mathfrak{T} \text{ SPACES:}$

Definition 4.1. An ideal space (X, τ, \mathfrak{T}) is said to be $S_2 \text{ mod } \mathfrak{T}$ if for every pair of distinct points x and y in X , whenever x has a τ -open subset not containing y , there exist open nhds. U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathfrak{T}$.

Theorem 4.1. Every $S_2 \text{ mod } \mathfrak{T}$ space is $S_1 \text{ mod } \mathfrak{T}$.

Proof. Let (X, τ, \mathfrak{T}) be an ideal space and x, y be two distinct elements of X such x has a open subset not containing y . Then X is $S_2 \text{ mod } \mathfrak{T}$ implies there exist open nhds. U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathfrak{T}$. Now if $x \notin V$ then y has a open subset V (and hence τ^* -open) not containing x . And if $x \in V$, then $U \cap V \in \mathfrak{T}$ implies $\{x\} \in \mathfrak{T}$. Therefore, $V - \{x\}$ is the required τ^* -open subset containing y but not x . Hence X is $S_1 \text{ mod } \mathfrak{T}$.

It can be seen easily that every S_2 space is $S_2 \text{ mod } \mathfrak{T}$, since $\emptyset \in \mathfrak{T}$, but the converse is not true as can be seen from the example below:

Example 4.1. Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\mathfrak{T} = \{\emptyset, \{a\}\}$. Then X is $S_2 \text{ mod } \mathfrak{T}$ but not S_2 .

Theorem 4.2. Let (X, τ, \mathfrak{T}) be an ideal space and \mathfrak{T} is codense, then X is S_2 if and only if X is $S_2 \text{ mod } \mathfrak{T}$.

Proof. Proof follows from the fact that \mathfrak{T} is codense implies $\tau \cap \mathfrak{T} = \emptyset$. So for any open nhds. U and V in X , $U \cap V \in \mathfrak{T}$ implies $U \cap V = \emptyset$.

Theorem 4.3. Let (X, τ, \mathfrak{I}) be an ideal space, then (X, τ^*) is S_2 implies (X, τ, \mathfrak{I}) is $S_2 \text{ mod } \mathfrak{I}$.

Proof. Let x, y be two distinct elements of X such that x has an open subset U not containing y . Then (X, τ^*) is S_2 implies there exist disjoint τ^* -open nhds. G and H such that $x \in G$ and $y \in H$. So there exist open subsets V and W and $I_1, I_2 \in \mathfrak{I}$ such that $x \in V - I_1 \subset G$ and $y \in W - I_2 \subset H$. Therefore, $(V \cap W) - (I_1 \cup I_2) = (V - I_1) \cap (W - I_2) \subset G \cap H = \phi$, which implies that $V \cap W = I_1 \cup I_2$, so $V \cap W \in \mathfrak{I}$, since $I_1 \cup I_2 \in \mathfrak{I}$. Hence (X, τ, \mathfrak{I}) is $S_2 \text{ mod } \mathfrak{I}$.

The following example shows that the converse of above theorem need not be true.

Example 4.2. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ and $\mathfrak{I} = \{\phi, \{c\}\}$. So $\tau^* = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then (X, τ, \mathfrak{I}) is $S_2 \text{ mod } \mathfrak{I}$, but (X, τ^*) is not S_2 . Since b has a τ^* -open subset $\{a, b\}$ not containing c , but there does not exist any disjoint τ^* -open subsets containing b and c .

Remark 4.1. From Example 4.1 it can be easily checked that (X, τ^*) is S_2 but (X, τ) is not S_2 . Also Example 4.2 shows that (X, τ) is S_2 but (X, τ^*) is not S_2 .

Therefore, the relationship of this separation axiom with respect to topological spaces can be seen below.

- a) (X, τ) is $S_2 \Rightarrow (X, \tau, \mathfrak{I})$ is $S_2 \text{ mod } \mathfrak{I} \not\Rightarrow (X, \tau)$ is S_2 .
- b) (X, τ^*) is $S_2 \Rightarrow (X, \tau, \mathfrak{I})$ is $S_2 \text{ mod } \mathfrak{I} \not\Rightarrow (X, \tau^*)$ is S_2 .
- c) (X, τ) is $S_2 \Leftrightarrow (X, \tau^*)$ is S_2 .

Theorem 4.4. If an ideal space (X, τ, \mathfrak{I}) is $S_2 \text{ mod } \mathfrak{I}$ and $\mathfrak{I} \subset \mathcal{J}$, where \mathcal{J} is an ideal. Then (X, τ, \mathcal{J}) is $S_2 \text{ mod } \mathcal{J}$.

Proof. Proof is obvious and hence is omitted.

Theorem 4.5. Let (X, τ, \mathfrak{I}) be $S_2 \text{ mod } \mathfrak{I}$ space and $A \subset X$. Then $(A, \tau_A, \mathfrak{I}_A)$ is also $S_2 \text{ mod } \mathfrak{I}$ space.

Proof. Proof is obvious and hence is omitted.

Theorem 4.6. Every finite $S_1 \text{ mod } \mathfrak{I}$ space is $S_2 \text{ mod } \mathfrak{I}$.

Proof: Let (X, τ, \mathfrak{I}) be $S_1 \text{ mod } \mathfrak{I}$ space, where X is finite. Let $x, y \in X$ and G be open subset in X such that $x \in G$ and $y \notin G$. Therefore, by the equivalence of (a) and (c) of Theorem 3.3, we have $\text{cl}^*\{x\} \subset G$, so $G = \cup \{ \text{cl}^*\{x\} : x \in G \}$ and so G is τ^* -closed, since X is finite. Hence there exist open subset G and τ^* -open subset G^c such that $x \in G$, $y \in G^c$ and $G \cap G^c = \phi$. Hence (X, τ, \mathfrak{I}) is $S_2 \text{ mod } \mathfrak{I}$ space.

The following example shows that if X is not finite then $S_1 \text{ mod } \mathfrak{I}$ space need not be $S_2 \text{ mod } \mathfrak{I}$.

Example 4.3. Let X is set of real numbers and $\tau = \{ G \subset X : G^c \text{ is finite} \}$. Take $\mathfrak{I} = \{\phi\}$. Then X is $S_1 \text{ mod } \mathfrak{I}$, but not $S_2 \text{ mod } \mathfrak{I}$, since X has no disjoint open subsets.

Further we will give various characterizations of $S_2 \text{ mod } \mathfrak{I}$ spaces.

Theorem 4.7. An ideal space (X, τ, \mathfrak{I}) is $S_2 \text{ mod } \mathfrak{I}$ if and only if for each $x, y \in X$, one of the following holds:

- a) $\text{cl}\{x\} = \text{cl}\{y\}$.
- b) there exist open subsets U and V , $x \in U$, $y \in V$, $U \cap V \in \mathfrak{I}$.

Proof. Proof is obvious and hence is omitted.

Further we will give characterization of $S_2 \text{ mod } \mathfrak{I}$ space in terms of convergence of a filter with respect to an ideal. Before this firstly, we will define the convergence of filter with respect to an ideal.

Definition 4.2. Let (X, τ, \mathfrak{I}) be ideal space. Consider the filter \mathcal{F} on X such that $\mathcal{F} \cap \mathfrak{I} = \phi$. For any $x \in X$, \mathcal{F} is said to be convergent to x with respect to an ideal (written as \mathfrak{I} -convergence) if for every open subset U of x there exists a member F of \mathcal{F} such that $F - U \in \mathfrak{I}$. By \mathfrak{I} -lim \mathcal{F} , we mean the collection of points to which \mathcal{F} converges with respect to an ideal.

Theorem 4.8. An ideal space (X, τ, \mathfrak{I}) is $S_2 \text{ mod } \mathfrak{I}$ if and only if for $x, y \in X$, $\text{cl}\{x\} = \text{cl}\{y\}$ whenever there is a filter \mathcal{F} not containing the members of \mathfrak{I} such that $x, y \in \mathfrak{I}$ -lim \mathcal{F} .

Proof. Let (X, τ, \mathfrak{I}) be $S_2 \text{ mod } \mathfrak{I}$ space and $x, y \in X$. Let \mathcal{F} be a filter such that $\mathcal{F} \cap \mathfrak{I} = \phi$ and $x, y \in \mathfrak{I}$ -lim \mathcal{F} . If $\text{cl}\{x\} \neq \text{cl}\{y\}$, then by Theorem 4.7, there exist subsets U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathfrak{I}$. Since x, y

$\in \mathfrak{I}\text{-lim } \mathcal{F}$, so there exist $F \in \mathcal{F}$ such that $F \cap U \in \mathfrak{I}$ and $F \cap V \in \mathfrak{I}$ and so $F \cap (U \cap V) \in \mathfrak{I}$. Therefore, $F \in \mathfrak{I}$, contradicting $\mathcal{F} \cap \mathfrak{I} = \emptyset$. Thus $\text{cl}\{x\} = \text{cl}\{y\}$.

Conversely, suppose there does not exist any open subsets U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathfrak{I}$. Thus we can define a filterbase $\mathcal{F}(B) = \{U \cap V : x \in U \text{ and } y \in V\}$ and let \mathcal{F} be filter generated by $\mathcal{F}(B)$. Then $x, y \in \mathfrak{I}\text{-lim } \mathcal{F}$ and thus $\text{cl}\{x\} = \text{cl}\{y\}$. Hence by Theorem 4.7, (X, τ, \mathfrak{I}) is $S_2 \text{ mod } \mathfrak{I}$.

Next result characterizes $S_2 \text{ mod } \mathfrak{I}$ space in terms of Kernel of points.

Theorem 4.9. An ideal space (X, τ, \mathfrak{I}) is $S_2 \text{ mod } \mathfrak{I}$ if and only if for $x, y \in X$ such that $\text{Ker}\{x\} \neq \text{Ker}\{y\}$, there exist open subsets U and V such that $x \in U$, $y \in V$ and $U \cap V \in \mathfrak{I}$. In fact, the sets U and V are such that $\text{Ker}\{x\} \subset U$, $\text{Ker}\{y\} \subset V$.

Proof. Proof is obvious and hence is omitted.

REFERENCES

1. A. Császár, *General Topology*, A. Hilger Ltd., Bristol, 1978.
2. A.S. Davis, "Indexed systems of neighbourhoods for general topological spaces", *Amer. Math. Monthly*, 68(1961), 886-893.
3. J. Dontchev, M. Ganster and D. Rose, *Ideal Resolvability, Topology and its Applications*, 93(1)(1999), 1-16.
4. C. Dorsett, R_0 and R_1 topological spaces, *Mat. Vesnik*, 2(15)(30)(1978), 117-122.
5. W. Dunham, *Weakly Hausdorff Spaces*, *Kyungpook Mathematical Journal*, 15(1975), 41-50.
6. D. Janković and T.R. Hamlett, *New topologies from old via ideals*, *The American Mathematical Monthly* 97(4) (1990) 295-310.
7. K. Kuratowski, *Topology, volume I*, Academic Press, New York, 1966.
8. N. A. Shanin, "On separation in topological spaces", *Dokl. Akad. Nauk. SSSR*, 38(1943), 110-113.
9. R. Vaidyanathaswamy, *The localisation Theory in Set Topology*, *Proc. Indian Acad. Sci.*, 20(1945), 51-61.
10. *Set Topology*, Chelsea Publishing Company, New York, 1946.



RELIABILITY AND PROFIT ANALYSIS OF A SYSTEM WITH INSTRUCTION, REPLACEMENT AND TWO OF THE THREE TYPES OF REPAIR POLICY

Rashmi Gupta

Dept. of Mathematics, Vaish College of Engineering, Rohtak (Haryana)
E-mail : rgupta2450@gmail.com

ABSTRACT :

The present paper introduces the instruction time and the possibility that ordinary repairman may damage the unit to the extent that : (i) it rather goes to more degraded stage but repairable (ii) it may become irreparable and hence replaced. Two-unit cold standby system is examined and two has been analyzed by making use of Semi-Markov Processes and regenerative point technique. Various measures of system effectiveness including profit incurred have been evaluated. Various conclusions have been drawn through graphical study for a particular case.

INTRODUCTION

In order to increase the reliability, concept of redundancy is used by the users of various systems. As a result, two-unit standby systems have widely been studied in the field of reliability. Concept of two types of repairman has been considered in some of these studies including [3-6] wherein one of the repairman had been taken as an ordinary and the other as an expert. The ordinary repairman may not be able to do some complex repairs and then an expert comes. Long stay of the expert with the system may be costly and hence idea of instruction time was introduced by Kumar et al. [7].

There may also be situations when the ordinary repairman even after getting the instruction may damage the failed unit during his try for repair. This leads to the unit in more degraded stage and sometimes to a stage where we are left with no other option but to replace it by a new one.

The purpose of the present study is :

- (i) to introduce redundancy
- (ii) to introduce a new type of repair policy which is defined as : **“when the ordinary repairman makes the unit damaged and leads it to more degraded stage due to mishandling, it is undertaken by the expert at much earlier stage than the stage at which its repair has been started by the ordinary repairman”**
- (iii) to make the replacement when the failed unit is made no more repairable by the ordinary repairman
- (iv) to reduce the stay of the expert.

The present paper, therefore, investigates two-unit cold standby system introducing the aforesaid repair policy together with instruction and replacement. It is assumed that if at the time of completion of the repair of a failed unit by the expert, the second unit is found in failed state, it is also repaired by the expert. Other assumptions are as usual. The system has been analyzed by making use of Semi-Markov Processes and regenerative point technique.

Various measures of system effectiveness including profit incurred have been evaluated. Various conclusions have been drawn through graphical study for a particular case.

NOTATIONS

λ	:	constant failure rate of a unit
p_1	:	probability that the ordinary repairman is able to complete the repair
q_1	:	probability that the ordinary repairman is unable to complete the repair
a	:	probability that resume repair policy is adopted
b_2	:	probability that unit is damaged but repairable
b_3	:	probability that the unit is damaged but irreparable
$g(t), G(t)$:	p.d.f. and c.d.f. of the repair time of the ordinary repairman
$g_1(t), G_1(t)$:	p.d.f. and c.d.f. of repair time of the expert repairman when resume repair policy is adopted
$g_2(t), G_2(t)$:	p.d.f. and c.d.f. of repair time of the expert repairman when repeat repair policy (type-I) is adopted
$g_3(t), G_3(t)$:	p.d.f. and c.d.f. of repair time of the expert repairman when repeat repair policy (type-II) is adopted
$g_4(t), G_4(t)$:	p.d.f. and c.d.f. of replacement time
$i(t), I(t)$:	p.d.f. and c.d.f. of time when expert gives instruction to ordinary repairman

Symbols for the state of system are :

o	:	operative unit
cs	:	cold standby unit
F_r	:	failed unit under repair of ordinary repairman
F_R	:	repair of the failed unit by the ordinary repairman is continuing from previous state
F_{re_1}	:	failed unit under repair of the expert repairman when resume repair policy is adopted
F_{Re_1}	:	repair of the failed unit by the expert repairman is continuing from the previous state under resume repair policy
F_{re_2}	:	failed unit under repair of the expert repairman when repeat repair policy (type-I) is adopted
F_{Re_2}	:	repair of the failed unit by the expert repairman is continuing from the previous state under repeat repair policy (type-I)
F_{re_3}	:	failed unit under repair of the expert repairman when repeat repair policy (type-II) is adopted
F_{Re_3}	:	repair of the failed unit by the expert repairman is continuing from the previous state under repeat repair policy (type-II)
F_{rep}	:	failed unit under replacement
F_{Rep}	:	replacement of the failed unit continuing from the previous state
F_{ei}	:	failed unit under instructions given by expert repairman to the ordinary repairman

Transition Probabilities and Mean Sojourn Times

The transition diagram showing the various states of the system is shown as in **Fig. 1**. The epochs of entry into states 0, 1, 2, 3, 4, 5, 10, 11, 12 and 13 are regeneration points and thus these states are regenerative states. States 6, 7, 8, 9, 10, 11, 12 and 14 are failed states.

The non-zero elements $p_{ij} = \lim_{s \rightarrow 0} q_{ij}^*(s)$ are :

$$\begin{aligned}
 p_{01} = p_{12} = 1 & \quad ; & \quad p_{20} = p_1 g^*(\lambda) & \quad ; & \quad p_{23} = q_1 a g^*(\lambda) \\
 p_{24} = q_1 b_2 g^*(\lambda) & \quad ; & \quad p_{25} = q_1 b_3 g^*(\lambda) & \quad ; & \quad p_{26} = 1 - g^*(\lambda) \\
 p_{21}^{(6)} = p_1 (1 - g^*(\lambda)) & \quad ; & \quad p_{2,10}^{(6)} = q_1 a (1 - g^*(\lambda)) & \quad ; & \quad p_{2,11}^{(6)} = q_1 b_2 (1 - g^*(\lambda))
 \end{aligned}$$

$$\begin{aligned}
 p_{2,12}^{(6)} = q_1 b_3 (1 - g^*(\lambda)) & \quad ; & \quad p_{30} = g_1^*(\lambda) & \quad ; & \quad p_{37} = p_{3,13}^{(7)} = 1 - g_1^*(\lambda) \\
 p_{40} = g_3^*(\lambda) & \quad ; & \quad p_{48} = p_{4,13}^{(8)} = 1 - g_3^*(\lambda)
 \end{aligned}$$

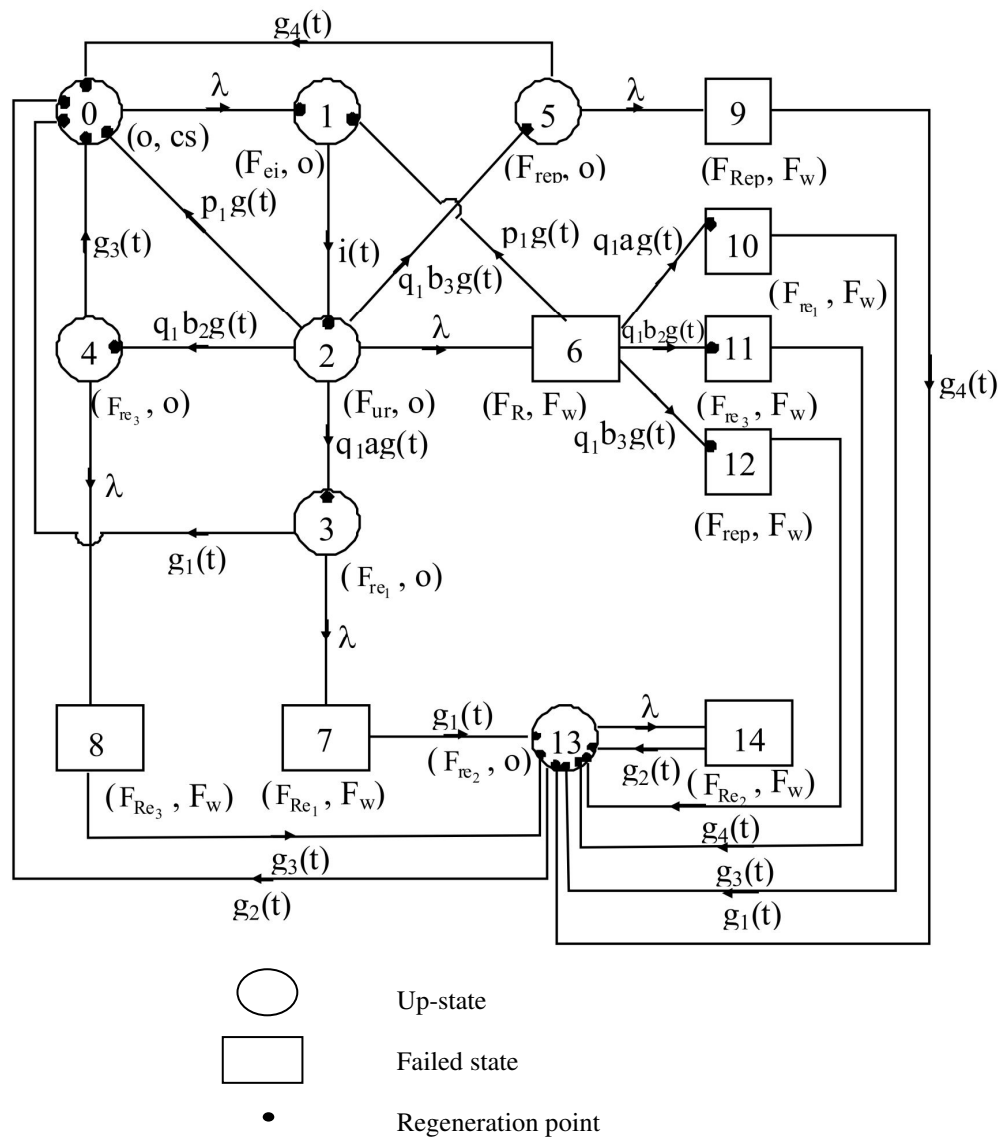


Fig. 1

$$\begin{aligned}
 p_{50} &= g_4^*(\lambda) & ; & & p_{59} &= p_{5,13}^{(9)} = 1 - g_4^*(\lambda) \\
 p_{10,13} &= p_{11,13} = p_{12,13} = 1 \\
 p_{13,0} &= g_2^*(\lambda) & ; & & p_{13,14} &= p_{13,13}^{(14)} = 1 - g_2^*(\lambda) \quad \dots(1-19)
 \end{aligned}$$

By these transition probabilities, it can be verified that

$$\begin{aligned}
 p_{01} &= p_{12} = p_{10,13} = p_{11,13} = p_{12,13} = 1 \\
 p_{20} + p_{23} + p_{24} + p_{25} + p_{26} &= 1 \\
 p_{20} + p_{23} + p_{24} + p_{25} + p_{21}^{(6)} + p_{2,10}^{(6)} + p_{2,11}^{(6)} + p_{2,12}^{(6)} &= 1 \\
 p_{30} + p_{37} &= p_{30} + p_{3,13}^{(7)} = 1 \\
 p_{40} + p_{48} &= p_{40} + p_{4,13}^{(8)} = 1 \\
 p_{50} + p_{59} &= p_{50} + p_{5,13}^{(9)} = 1 \\
 p_{13,0} + p_{13,14} &= p_{13,0} + p_{13,13}^{(14)} = 1 \quad \dots(20-26)
 \end{aligned}$$

The mean sojourn time (μ_i) in state i are :

$$\begin{aligned}
 \mu_0 &= \frac{1}{\lambda}, & \mu_1 &= -i^{*'}(0), & \mu_2 &= \frac{1 - g^*(\lambda)}{\lambda} \\
 \mu_3 &= \frac{1 - g_1^*(\lambda)}{\lambda}, & \mu_4 &= \frac{1 - g_3^*(\lambda)}{\lambda}, & \mu_5 &= \frac{1 - g_4^*(\lambda)}{\lambda} \\
 \mu_{10} &= -g_1^{*'}(0), & \mu_{11} &= -g_3^{*'}(0), & \mu_{12} &= -g_4^{*'}(0) \\
 \mu_{13} &= \frac{1 - g_2^*(\lambda)}{\lambda} \quad \dots(27-36)
 \end{aligned}$$

The unconditional mean time taken by the system to transit for any state j when it is counted from epoch of entrance into state i is mathematically stated as

$$m_{ij} = \int_0^{\infty} tq_{ij}(t)dt = -q_{ij}^{*'}(0) \quad \dots(37)$$

Thus,

$$\begin{aligned}
 m_{01} &= \mu_0 & ; & & m_{12} &= \mu_1 \\
 m_{20} + m_{23} + m_{24} + m_{25} + m_{26} &= \mu_2 \\
 m_{20} + m_{23} + m_{24} + m_{25} + m_{21}^{(6)} + m_{2,10}^{(6)} + m_{2,11}^{(6)} + m_{2,12}^{(6)} &= k_1 \text{ (say)} \\
 m_{30} + m_{37} &= \mu_3 & ; & & m_{30} + m_{3,13}^{(7)} &= \mu_{10} \\
 m_{40} + m_{48} &= \mu_4 & ; & & m_{40} + m_{4,13}^{(8)} &= \mu_{11} \\
 m_{50} + m_{59} &= \mu_5 & ; & & m_{50} + m_{5,13}^{(9)} &= \mu_{12} \\
 m_{10,13} &= \mu_{10} & ; & & m_{11,13} &= \mu_{11} & ; & m_{12,13} &= \mu_{12} \\
 m_{13,0} + m_{13,14} &= \mu_{13} & ; & & m_{13,0} + m_{13,13}^{(14)} &= k_2 \text{ (say)} \quad \dots(38-52)
 \end{aligned}$$

MEAN TIME TO SYSTEM FAILURE

By probabilistic arguments, we obtain the following recursive relations for $\phi_i(t)$:

$$\begin{aligned}
 \phi_0(t) &= Q_{01}(t) \textcircled{S} \phi_1(t) \\
 \phi_1(t) &= Q_{12}(t) \textcircled{S} \phi_2(t)
 \end{aligned}$$

$$\begin{aligned}\phi_2(t) = & Q_{20}(t) \otimes \phi_0(t) + Q_{23}(t) \otimes \phi_3(t) + Q_{24}(t) \otimes \phi_4(t) \\ & + Q_{25}(t) \otimes \phi_5(t) + Q_{26}(t)\end{aligned}$$

$$\phi_3(t) = Q_{30}(t) \otimes \phi_0(t) + Q_{37}(t)$$

$$\phi_4(t) = Q_{40}(t) \otimes \phi_0(t) + Q_{48}(t)$$

$$\phi_5(t) = Q_{50}(t) \otimes \phi_0(t) + Q_{59}(t)$$

$$\phi_{13}(t) = Q_{13,0}(t) \otimes \phi_0(t) + Q_{13,14}(t) \quad \dots(53-59)$$

Taking Laplace-Steiltjes Transforms (L.S.T.) of these relations and solving then for $\phi_0^{**}(s)$, the mean time to system failure (MTSF) when the system starts from the state '0' is

$$T_0 = \lim_{s \rightarrow 0} \frac{1 - \phi_0^{**}(s)}{s} = \frac{N}{D} \quad \dots(60)$$

where

$$N = \mu_0 + \mu_1 + \mu_2 + p_{23} \mu_3 + p_{24} \mu_4 + p_{25} \mu_5$$

$$D = 1 - p_{20} - p_{23} p_{30} - p_{24} p_{40} - p_{25} p_{50} \quad \dots(61-62)$$

AVAILABILITY ANALYSIS

Using the arguments of the theory of regenerative processes, the availability $A_i(t)$ is seen to satisfy the following recursive relations :

$$A_0(t) = M_0(t) + q_{01}(t) \otimes A_1(t)$$

$$A_1(t) = M_1(t) + q_{12}(t) \otimes A_2(t)$$

$$\begin{aligned}A_2(t) = & M_2(t) + q_{20}(t) \otimes A_0(t) + q_{23}(t) \otimes A_3(t) + q_{24}(t) \otimes A_4(t) \\ & + q_{25}(t) \otimes A_5(t) + q_{21}^{(6)}(t) \otimes A_1(t) + q_{2,10}^{(6)}(t) \otimes A_{10}(t) \\ & + q_{2,11}^{(6)}(t) \otimes A_{11}(t) + q_{2,12}^{(6)}(t) \otimes A_{12}(t)\end{aligned}$$

$$A_3(t) = M_3(t) + q_{30}(t) \otimes A_0(t) + q_{3,13}^{(7)}(t) \otimes A_{13}(t)$$

$$A_4(t) = M_4(t) + q_{40}(t) \otimes A_0(t) + q_{4,13}^{(8)}(t) \otimes A_{13}(t)$$

$$A_5(t) = M_5(t) + q_{50}(t) \otimes A_0(t) + q_{5,13}^{(9)}(t) \otimes A_{13}(t)$$

$$A_{10}(t) = q_{10,13}(t) \otimes A_{13}(t)$$

$$A_{11}(t) = q_{11,13}(t) \otimes A_{13}(t)$$

$$A_{12}(t) = q_{12,13}(t) \otimes A_{13}(t)$$

$$A_{13}(t) = M_{13}(t) + q_{13,0}(t) \otimes A_0(t) + q_{13,13}^{(14)}(t) \otimes A_{13}(t) \quad \dots(63-72)$$

where

$$\begin{aligned}M_0(t) = e^{-\lambda t} & \quad ; \quad M_1(t) = \bar{I}(t) & \quad ; \quad M_2(t) = e^{-\lambda t} \bar{G}(t) \\ M_3(t) = e^{-\lambda t} \bar{G}_1(t) & \quad ; \quad M_4(t) = e^{-\lambda t} \bar{G}_3(t) & \quad ; \quad M_5(t) = e^{-\lambda t} \bar{G}_4(t) \\ M_{13}(t) = e^{-\lambda t} \bar{G}_2(t) & & \quad \dots(73-79)\end{aligned}$$

Taking Laplace transforms of the above equations and solving them for $A_0^*(s)$, in steady-state, the availability of the system is given by

$$A_0 = \lim_{s \rightarrow 0} (s A_0^*(s)) = \frac{N_1}{D_1} \quad \dots(80)$$

where

$$\begin{aligned}
 N_1 &= [\mu_0(1 - p_{21}^{(6)}) + \mu_1 + \mu_2 + p_{23}\mu_3 + p_{24}\mu_4 + p_{25}\mu_5]p_{13,0} \\
 &\quad + \mu_{13}[p_{23} p_{3,13}^{(7)} + p_{24} p_{4,13}^{(8)} + p_{25} p_{5,13}^{(9)} + p_{2,10}^{(6)} + p_{2,11}^{(6)} + p_{2,12}^{(6)}] \\
 D_1 &= [\mu_0(1 - p_{21}^{(6)}) + \mu_1 + k_1 + \mu_{10}(p_{23} + p_{2,10}^{(6)}) + \mu_{11}(p_{24} + p_{2,11}^{(6)}) \\
 &\quad + \mu_{12}(p_{25} + p_{2,12}^{(6)})] p_{13,0} + \mu_{13} [p_{23} p_{3,13}^{(6)} + p_{24} p_{4,13}^{(8)} + p_{25} p_{5,13}^{(9)} \\
 &\quad + p_{2,10}^{(6)} + p_{2,11}^{(6)} + p_{2,12}^{(6)}] \qquad \dots(81-82)
 \end{aligned}$$

BUSY PERIOD ANALYSIS OF THE ORDINARY REPAIRMAN

By probabilistic arguments, we have the following recursive relations for $B_i(t)$:

$$\begin{aligned}
 B_0(t) &= q_{01}(t) \odot B_1(t) \\
 B_1(t) &= q_{12}(t) \odot B_2(t) \\
 B_2(t) &= W_2(t) + q_{20}(t) \odot B_0(t) + q_{21}^{(6)}(t) \odot B_1(t) + q_{23}(t) \odot B_3(t) \\
 &\quad + q_{24}(t) \odot B_4(t) + q_{25}(t) \odot B_5(t) + q_{2,10}^{(6)}(t) \odot B_{10}(t) \\
 &\quad + q_{2,11}^{(6)}(t) \odot B_{11}(t) + q_{2,12}^{(6)}(t) \odot B_{12}(t) \\
 B_3(t) &= q_{30}(t) \odot B_0(t) + q_{3,13}^{(7)}(t) \odot B_{13}(t) \\
 B_4(t) &= q_{40}(t) \odot B_0(t) + q_{4,13}^{(8)}(t) \odot B_{13}(t) \\
 B_5(t) &= q_{50}(t) \odot B_0(t) + q_{5,13}^{(9)}(t) \odot B_{13}(t) \\
 B_{10}(t) &= q_{10,13}(t) \odot B_{13}(t) \\
 B_{11}(t) &= q_{11,13}(t) \odot B_{13}(t) \\
 B_{12}(t) &= q_{12,13}(t) \odot B_{13}(t) \\
 B_{13}(t) &= q_{13,0}(t) \odot B_0(t) + q_{13,13}^{(14)}(t) \odot B_{13}(t) \qquad \dots(83-91)
 \end{aligned}$$

where

$$W_2(t) = \overline{G}(t) \qquad \dots(92)$$

Taking L.T. of the above equations and solving them for $B_0^*(s)$, in steady-state, the total fraction of the time for which the system is under repair of the ordinary repairman is given by

$$B_0 = \lim_{s \rightarrow 0} (s B_0^*(s)) = \frac{N_2}{D_1} \qquad \dots(93)$$

where

$$N_2 = k_1 p_{13,0} \qquad \dots(94)$$

and D_1 is already specified.

BUSY PERIOD ANALYSIS OF THE EXPERT REPAIRMAN(REPAIR TIME ONLY)

By probabilistic arguments, we have the following recursive relations for $B_i^c(t)$:

$$\begin{aligned}
 B_0^c(t) &= q_{01}(t) \odot B_1^c(t) \\
 B_1^c(t) &= q_{12}(t) \odot B_2^c(t) \\
 B_2^c(t) &= q_{20}(t) \odot B_0^c(t) + q_{21}^{(6)}(t) \odot B_1^c(t) + q_{23}(t) \odot B_3^c(t) + q_{24}(t) \odot B_4^c(t) \\
 &\quad + q_{25}(t) \odot B_5^c(t) + q_{2,10}^{(6)}(t) \odot B_{10}^c(t) + q_{2,11}^{(6)}(t) \odot B_{11}^c(t)
 \end{aligned}$$

$$\begin{aligned}
 & + q_{2,12}^{(6)}(t) \odot B_{12}^e(t) \\
 B_3^e(t) &= W_3(t) + q_{30}(t) \odot B_0^e(t) + q_{3,13}^{(7)}(t) \odot B_{13}^e(t) \\
 B_4^e(t) &= W_4(t) + q_{40}(t) \odot B_0^e(t) + q_{4,13}^{(8)}(t) \odot B_{13}^e(t) \\
 B_5^e(t) &= q_{50}(t) \odot B_0^e(t) + q_{5,13}^{(9)}(t) \odot B_{13}^e(t) \\
 B_{10}^e(t) &= W_{10}(t) + q_{10,13}(t) \odot B_{13}^e(t) \\
 B_{11}^e(t) &= W_{11}(t) + q_{11,13}(t) \odot B_{13}^e(t) \\
 B_{12}^e(t) &= q_{12,13}(t) \odot B_{13}^e(t) \\
 B_{13}^e(t) &= W_{13}(t) + q_{13,0}(t) \odot B_0^e(t) + q_{13,13}^{(14)}(t) \odot B_{13}^e(t) \quad \dots(95-104)
 \end{aligned}$$

where

$$\begin{aligned}
 W_3(t) &= W_{10}(t) = \overline{G}_1(t) \\
 W_4(t) &= W_{11}(t) = \overline{G}_3(t) \\
 W_{13}(t) &= \overline{G}_2(t) \quad \dots(105-107)
 \end{aligned}$$

Taking L.T. of the above equations and solving them for $B_0^{e*}(s)$, in steady-state, the total fraction of the time for which the system is under repair of the expert repairman is given by

$$B_0^e = \lim_{s \rightarrow 0} (s B_0^{e*}(s)) = \frac{N_3}{D_1} \quad \dots(108)$$

where

$$\begin{aligned}
 N_3 &= \mu_{10}(p_{23} + p_{2,10}^{(6)}) + \mu_{11}(p_{24} + p_{2,11}^{(6)}) \\
 & + k_2[p_{23} p_{3,13}^{(7)} + p_{24} p_{4,13}^{(8)} + p_{25} p_{5,13}^{(9)} + p_{2,10}^{(6)} + p_{2,11}^{(6)} + p_{2,12}^{(6)}] \quad \dots(109)
 \end{aligned}$$

and D_1 is already specified.

EXPECTED INSTRUCTION TIME

By probabilistic arguments, we have the following recursive relations for $IT_i(t)$:

$$\begin{aligned}
 IT_0(t) &= q_{01}(t) \odot IT_1(t) \\
 IT_1(t) &= W_1(t) + q_{12}(t) \odot IT_2(t) \\
 IT_2(t) &= q_{20}(t) \odot IT_0(t) + q_{21}^{(6)}(t) \odot IT_1(t) + q_{23}(t) \odot IT_3(t) \\
 & + q_{24}(t) IT_4(t) + q_{25}(t) \odot IT_5(t) + q_{2,10}^{(6)}(t) \odot IT_{10}(t) \\
 & + q_{2,11}^{(6)}(t) \odot IT_{11}(t) + q_{2,12}^{(6)}(t) \odot IT_{12}(t) \\
 IT_3(t) &= q_{30}(t) \odot IT_0(t) + q_{3,13}^{(7)}(t) \odot IT_{13}(t) \\
 IT_4(t) &= q_{40}(t) \odot IT_0(t) + q_{4,13}^{(8)}(t) \odot IT_{13}(t) \\
 IT_5(t) &= q_{50}(t) \odot IT_0(t) + q_{5,13}^{(9)}(t) \odot IT_{13}(t) \\
 IT_{10}(t) &= q_{10,13}(t) \odot IT_{13}(t) \\
 IT_{11}(t) &= q_{11,13}(t) \odot IT_{13}(t) \\
 IT_{12}(t) &= q_{12,13}(t) \odot IT_{13}(t) \\
 IT_{13}(t) &= q_{13,0}(t) \odot IT_0(t) + q_{13,13}^{(14)}(t) \odot IT_{13}(t) \quad \dots(110-119)
 \end{aligned}$$

where

$$W_1(t) = \bar{I}(t) \tag{120}$$

Taking L.T. of the above equations and solving them for $IT_0^*(s)$, in steady-state, the total fraction of the time for which the expert is busy in giving the instructions to the ordinary repairman is given by

$$IT_0 = \lim_{s \rightarrow 0} (s IT_0^*(s)) = \frac{N_4}{D_1} \tag{121}$$

where

$$N_4 = \mu_1 p_{13,0} \tag{122}$$

and D_1 is already specified.

EXPECTED NUMBER OF VISITS BY THE ORDINARY REPAIRMAN

By probabilistic arguments, we have the following recursive relations for $V_i(t)$:

$$V_0(t) = Q_{01}(t) \otimes V_1(t)$$

$$V_1(t) = Q_{12}(t) \otimes [1 + V_2(t)]$$

$$V_2(t) = Q_{20}(t) \otimes V_0(t) + Q_{21}^{(6)}(t) \otimes V_1(t) + Q_{23}(t) \otimes V_3(t) \\ + Q_{24}(t) \otimes V_4(t) + Q_{25}(t) \otimes V_5(t) + Q_{2,10}^{(6)}(t) \otimes V_{10}(t) \\ + Q_{2,11}^{(6)}(t) \otimes V_{11}(t) + Q_{2,12}^{(6)}(t) \otimes V_{12}(t)$$

$$V_3(t) = Q_{30}(t) \otimes V_0(t) + Q_{3,13}^{(7)}(t) \otimes V_{13}(t)$$

$$V_4(t) = Q_{40}(t) \otimes V_0(t) + Q_{4,13}^{(8)}(t) \otimes V_{13}(t)$$

$$V_5(t) = Q_{50}(t) \otimes V_0(t) + Q_{5,13}^{(9)}(t) \otimes V_{13}(t)$$

$$V_{10}(t) = Q_{10,13}(t) \otimes V_{13}(t)$$

$$V_{11}(t) = Q_{11,13}(t) \otimes V_{13}(t)$$

$$V_{12}(t) = Q_{12,13}(t) \otimes V_{13}(t)$$

$$V_{13}(t) = Q_{13,0}(t) \otimes V_0(t) + Q_{13,13}^{(14)}(t) \otimes V_{13}(t) \tag{123-132}$$

Taking L.S.T. of the above equations and solving them for $V_0^{**}(s)$, in steady-state, the number of visits per unit time by the ordinary repairman is given by

$$V_0 = \lim_{t \rightarrow \infty} \left[\frac{V_0(t)}{t} \right] = \lim_{s \rightarrow 0} [s V_0^{**}(s)] = \frac{N_5}{D_1} \quad \dots(133)$$

where

$$N_5 = p_{13,0} \quad \dots(134)$$

and D_1 is already specified.

EXPECTED NUMBER OF VISITS BY THE EXPERT REPAIRMAN

By probabilistic arguments, we have the following recursive relations for $V_i^e(t)$:

$$\begin{aligned} V_0^e(t) &= Q_{01}(t) \odot [1 + V_1^e(t)] \\ V_1^e(t) &= Q_{12}(t) \odot V_2^e(t) \\ V_2^e(t) &= Q_{20}(t) \odot V_0^e(t) + Q_{23}(t) \odot [1 + V_3^e(t)] + Q_{24}(t) \odot [1 + V_4^e(t)] \\ &\quad + Q_{25}(t) \odot V_5^e(t) + Q_{21}^{(6)}(t) \odot [1 + V_1^e(t)] \\ &\quad + Q_{2,10}^{(6)}(t) \odot [1 + V_{10}^e(t)] + Q_{2,11}^{(6)}(t) \odot [1 + V_{11}^e(s)] \\ &\quad + Q_{2,12}^{(6)}(t) \odot V_{12}^e(t) \\ V_3^e(t) &= Q_{30}(t) \odot V_0^e(t) + Q_{3,13}^{(7)}(t) \odot V_{13}^e(t) \\ V_4^e(t) &= Q_{40}(t) \odot V_0^e(t) + Q_{4,13}^{(8)}(t) \odot V_{13}^e(t) \\ V_5^e(t) &= Q_{50}(t) \odot V_0^e(t) + Q_{5,13}^{(9)}(t) \odot V_{13}^e(t) \\ V_{10}^e(t) &= Q_{10,13}(t) \odot V_{13}^e(t) \\ V_{11}^e(t) &= Q_{11,13}(t) \odot V_{13}^e(t) \\ V_{12}^e(t) &= Q_{12,13}(t) \odot V_{13}^e(t) \\ V_{13}^e(t) &= Q_{13,0}(t) \odot V_0^e(t) + Q_{13,13}^{(14)}(t) \odot V_{13}^e(t) \end{aligned} \quad \dots(135-144)$$

Taking L.S.T. of the above equations and solving them for $V_0^{**}(s)$, in steady-state, the number of visits per unit time by the expert is given by

$$V_0^e = \lim_{s \rightarrow 0} [s V_0^{**}(s)] = \frac{N_6}{D_1} \quad \dots(145)$$

Where

$$N_6 = p_{13,0} [1 + p_{23} + p_{24} + p_{2,10}^{(6)} + p_{2,11}^{(6)}] \quad \dots(146)$$

and D_1 is already specified.

BUSY PERIOD ANALYSIS OF REPAIRMAN [REPLACEMENT TIME ONLY]

By probabilistic arguments, we have the following recursive relations for $B_i^R(t)$:

$$\begin{aligned} B_0^R(t) &= q_{01}(t) \odot B_1^R(t) \\ B_1^R(t) &= q_{12}(t) \odot B_2^R(t) \\ B_2^R(t) &= q_{20}(t) \odot B_0^R(t) + q_{21}^{(6)}(t) \odot B_1^R(t) + q_{23}(t) \odot B_3^R(t) \\ &\quad + q_{24}(t) \odot B_4^R(t) + q_{25}(t) \odot B_5^R(t) + q_{2,10}^{(6)}(t) \odot B_{10}^R(t) \\ &\quad + q_{2,11}^{(6)}(t) \odot B_{11}^R(t) + q_{2,12}^{(6)}(t) \odot B_{12}^R(t) \end{aligned}$$

$$\begin{aligned}
 B_3^R(t) &= q_{30}(t) \odot B_0^R(t) + q_{3,13}^{(7)}(t) \odot B_{13}^R(t) \\
 B_4^R(t) &= q_{40}(t) \odot B_0^R(t) + q_{4,13}^{(8)}(t) \odot B_{13}^R(t) \\
 B_5^R(t) &= W_5(t) + q_{50}(t) \odot B_0^R(t) + q_{5,13}^{(9)}(t) \odot B_{13}^R(t) \\
 B_{10}^R(t) &= q_{10,13}(t) \odot B_{13}^R(t) \\
 B_{11}^R(t) &= q_{11,13}(t) \odot B_{13}^R(t) \\
 B_{12}^R(t) &= W_{12}(t) + q_{12,13}(t) \odot B_{13}^R(t) \\
 B_{13}^R(t) &= q_{13,0}(t) \odot B_0^R(t) + q_{13,13}^{(14)}(t) \odot B_{13}^R(t) \quad \dots(147-156)
 \end{aligned}$$

where

$$W_5(t) = W_{12}(t) = \overline{G}_4(t) \quad \dots(157)$$

Taking L.T. of the above equations and solving them for $B_0^{R*}(s)$, in steady-state, the total fraction of the time for which the system is under replacement is given by

$$B_0^R = \lim_{s \rightarrow 0} (s B_0^{R*}(s)) = \frac{N_7}{D_1} \quad \dots(158)$$

where

$$N_7 = \mu_{12}(p_{25} + p_{2,12}^{(6)}) p_{13,0} \quad \dots(159)$$

and D_1 is already specified.

EXPECTED NUMBER OF REPLACEMENTS

By probabilistic arguments, we have the following recursive relations for $RP_1(t)$:

$$\begin{aligned}
 RP_0(t) &= Q_{01}(t) \odot RP_1(t) \\
 RP_1(t) &= Q_{12}(t) \odot RP_2(t) \\
 RP_2(t) &= Q_{20}(t) \odot RP_0(t) + Q_{23}(t) \odot RP_3(t) + Q_{24}(t) \odot RP_4(t) \\
 &\quad + Q_{25}(t) \odot [1 + RP_5(t)] + Q_{21}^{(6)}(t) \odot RP_1(t) + Q_{2,10}^{(6)}(t) \odot RP_{10}(t) \\
 &\quad + Q_{2,11}^{(6)}(t) \odot RP_{11}(t) + Q_{2,12}^{(6)}(t) \odot [1 + RP_{12}(t)] \\
 RP_3(t) &= Q_{30}(t) \odot RP_0(t) + Q_{3,13}^{(7)}(t) \odot RP_{13}(t) \\
 RP_4(t) &= Q_{40}(t) \odot RP_0(t) + Q_{4,13}^{(8)}(t) \odot RP_{13}(t) \\
 RP_5(t) &= Q_{50}(t) \odot RP_0(t) + Q_{5,13}^{(9)}(t) \odot RP_{13}(t) \\
 RP_{10}(t) &= Q_{10,13}(t) \odot RP_{13}(t) \\
 RP_{11}(t) &= Q_{11,13}(t) \odot RP_{13}(t) \\
 RP_{12}(t) &= Q_{12,13}(t) \odot RP_{13}(t) \\
 RP_{13}(t) &= Q_{13,0}(t) \odot RP_0(t) + Q_{13,13}^{(14)}(t) \odot RP_{13}(t) \quad \dots(160-169)
 \end{aligned}$$

Taking L.S.T. of the above equations and solving them for $RP_0^{**}(s)$, in steady-state, the total number of expected replacement is given by

$$RP_0 = \lim_{s \rightarrow 0} [s RP_0^{**}(s)] = \frac{N_8}{D_1} \quad \dots(170)$$

where

$$N_8 = p_{13,0}(p_{25} + p_{2,12}^{(6)}) \quad \dots(171)$$

and D_1 is already specified.

PROFIT ANALYSIS

The expected total profit incurred to the system in steady-state is given by

$$P = C_0A_0 - C_1B_0 - C_2B_0^e - C_4V_0 - C_5V_0^e - C_6B_0^R - C_7RP_0 - C_8IT_0 \quad \dots(172)$$

where

C_0 = revenue per unit up time of the system

C_1 = cost per unit time for which the ordinary repairman is busy for repairing the failed unit

C_2 = cost per unit time for which the expert repairman is busy for repairing the unit

C_4 = cost per visit of the ordinary repairman

C_5 = cost per visit of the expert repairman

C_6 = cost per unit time for which the repairman is busy for replacing the unit

C_7 = cost per replacement.

C_8 = cost per unit time for which expert repairman is busy in giving the instruction to the ordinary repairman.

PARTICULAR CASE

For graphical interpretation, the following particular case is considered :

$$\begin{aligned} g(t) &= \alpha e^{-\alpha t} & ; & & g_1(t) &= \alpha_1 e^{-\alpha_1 t} \\ g_2(t) &= \alpha_2 e^{-\alpha_2 t} & ; & & g_3(t) &= \alpha_3 e^{\alpha_3 t} \\ g_4(t) &= \alpha_4 e^{-\alpha_4 t} & ; & & i(t) &= \gamma e^{-\gamma t} \end{aligned}$$

On the basis of the numerical values taken as :

$$\begin{aligned} p_1 &= 0.5, q_1 = 0.5, a = 0.5, b_2 = 0.45, b_3 = 0.05, \gamma = 10, \\ \alpha &= 1, \alpha_1 = 2.5, \alpha_2 = 2, \alpha_3 = 1, \alpha_4 = 5, \lambda = 0.05 \end{aligned}$$

The values of various measures of system effectiveness are obtained as :

Mean time to system failure (MTSF) = 340.4345

Availability (A_0) = 0.9964026

Busy period of the ordinary repairman (B_0) = 0.0476797

Busy period of the expert repairman (repair time only) (B_0^e) = 0.0168343

Expected instruction time (IT_0) = 0.0047679

Expected number of visits by the ordinary repairman (V_0) = 0.047679

Expected number of visits by the expert repairman (V_0^e) = 0.0703276

Busy period of the expert repairman(replacement time only)(B_0^R) = 0.000238399

Expected number of replacements (RP_0) = 0.00119199

Graphical Interpretation

The above particular case is considered from the graphical interpretation.

Fig. 2 reveals the pattern of the profit with respect to failure rate (λ) for different values of repair rate (α). The profit decreases as the failure rate increases and is higher for higher values of repair rate (α). Following inferences can be made through the graph :

For $\alpha = 1, 1.5$ and 2 ; the system is profitable only if $\lambda < 0.103, 0.113$ and 0.119 respectively.’

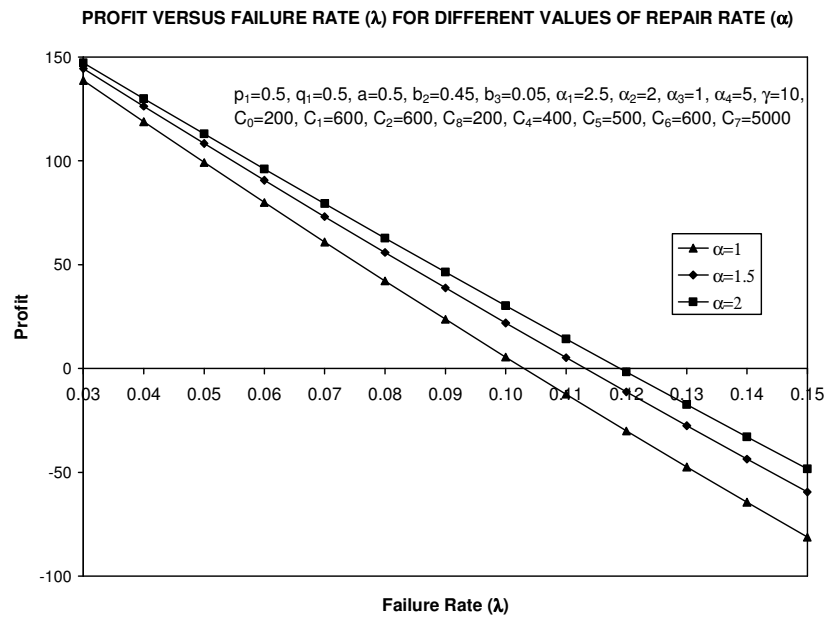


Fig. 2

So, the companies can be suggested to purchase only those systems which do not have failure rates greater than those mentioned above.

Fig. 3 depicts the behaviour of the profit with respect to revenue (C_0) for different values of cost (C_2). The profit increases as C_0 increases and becomes lower for higher values of cost (C_2). Following is also observed from the graph :

For $C_2 = 4000, 5000$ and 6000 the system is profitable only if $C_0 > 157.8, 174.7$ and 191.6 respectively.

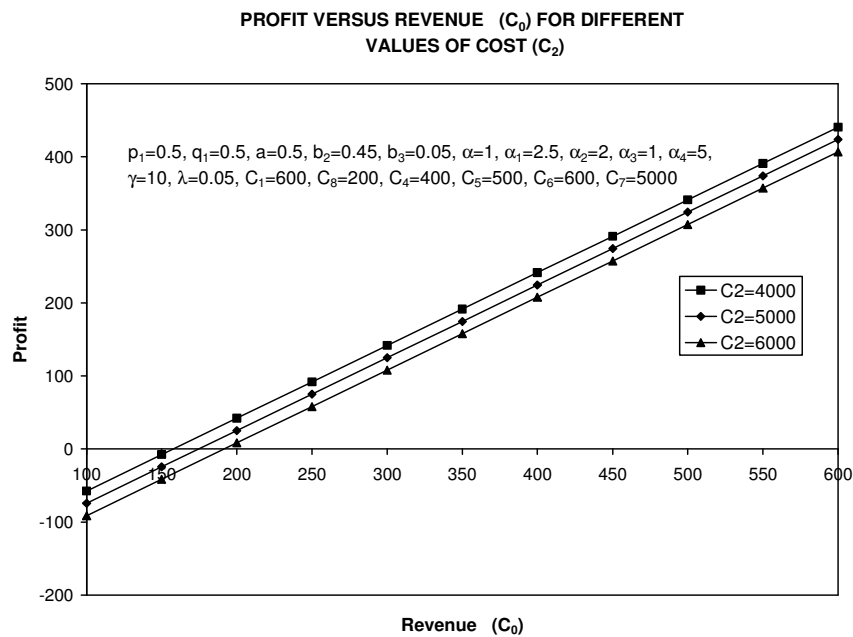


Fig. 3

Fig. 4 depicts the behaviour of the profit with respect to replacement cost (C_7) for different values of repair rate (α_4). The profit decreases as the cost (C_7) increases and is higher for higher values of repair rate (α_4). Following is also observed from the graph :

For $\alpha_4 = 1, 2$ and 10 , the system is profitable only if $C_7 < 4280.15, 4541.65$ and 4749.4 accordingly.

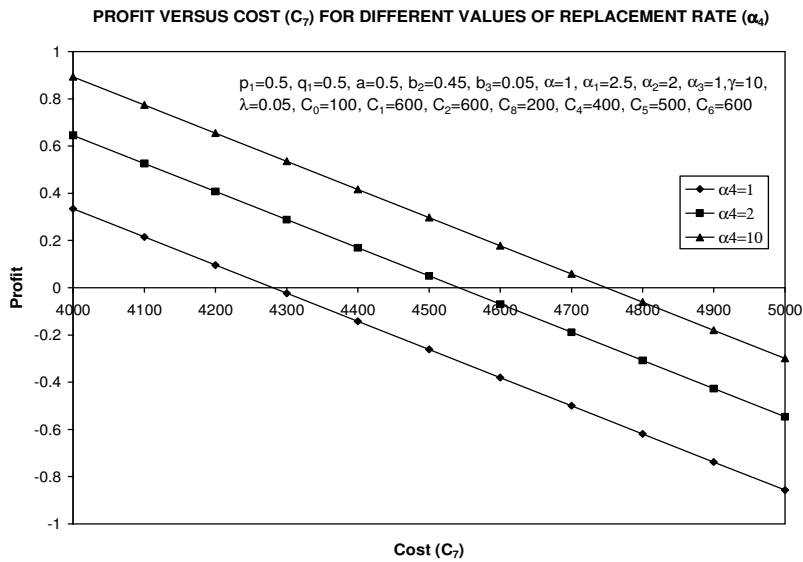


Fig. 4

Fig. 5 shows the behavior of the profit with respect to probability (p_1) for different values of probability (a). The profit increases as p increases and is higher for higher values of probability (a). Following is also observed from the graph :

For $a = 0.2, 0.4$ and 0.6 , the system is profitable only if $p_1 > 0.357, 0.326$ and 0.271 respectively.

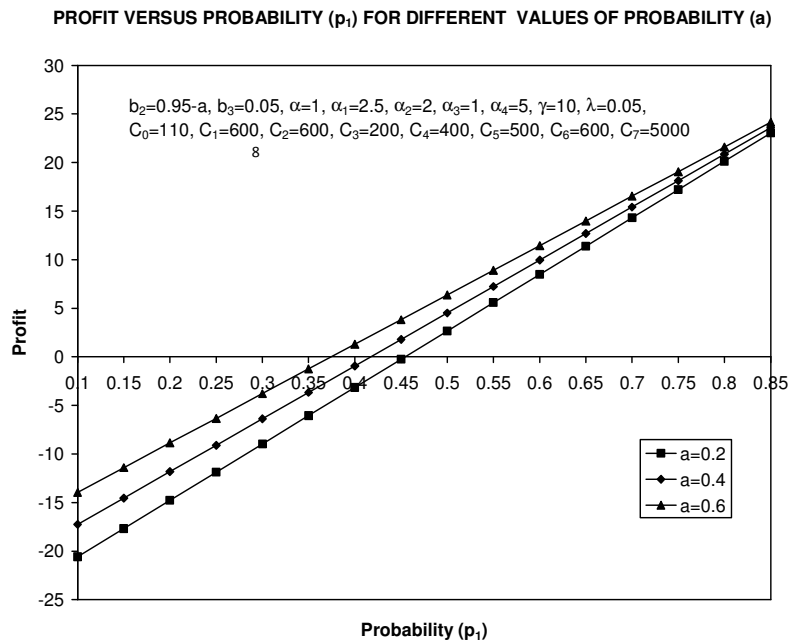


Fig. 5

REFERENCES

1. Gupta Kumar, Suresh and Gupta, Rashmi, *Economic Analysis of a reliability model for two unit cold standby system with three types of repair policy*, *Aryabhata Journal of Mathematics and Informatics*, Vol. 8, No. 2, 243-252 (2016).
2. Gupta Kumar, Suresh and Gupta, Rashmi, *Analysis of a reliability model with instruction and two of the three types of repair policy*, *AIPQR transitions*, Vol. 41, No. 2, 139-153, (2017).
3. Goyal, V. and Murari, K., *Cost analysis of a two-unit standby system with two types of repairman* *Microelectron. Reliab.*, **24**, 849-855 (1984).
4. Gupta, K.C. and Taneja, A.K., *Comparative study of two models for a two-unit standby system with instructions and two types of repairman*, *J. Indian Soc. Stat. Oper. Res.*, **Vol. XXIII(1-4)**, 23-39 (2002).
5. Kumar, A., Gupta, S.K. and Taneja, G., *Comparative study of the profit of a two server system including patience time and instruction time*. *Microelectron. Reliab.*, **36(10)**, 1595-1601(1996).
6. Tuteja, R.K. and Taneja, G., *Cost-benefit analysis of a two-server, two-unit, warm standby system with different types of failure*, *Microelectron. Reliab.*, **32**, 1353-1359(1992).
7. Kumar, A., Gupta, S.K. and Taneja, G., *Probabilistic analysis of a two-unit cold standby system with instructions at need*. *Microelectron. Reliab.*, **35(5)**, 829-832(1995).
8. Rander, M.C., Kumar, S. and Kumar, A., *Cost analysis of two dissimilar cold standby system with preventive maintenance and replacement of standby*, *Microelectron. Reliab.*, **34(7)**, 171-174(1994).



WEAK ISOMORPHISM ON BIPOLAR TOTAL FUZZY GRAPH

M. Vijaya*, V. Mekala**

*Research Advisor, Department of Mathematics, Marudupandiyar college, Thanjavur (Tamilnadu) India

**Research Scholar, Department of Mathematics, Marudupandiyar college, Thanjavur (Tamilnadu) India

E-mail : mathvijaya79@yahoo.com, mekalaeverwins1@gmail.com

ABSTRACT :

In this paper bipolar total fuzzy graph $BT(G):(\sigma_{BT}, \mu_{BT})$ of bipolar fuzzy graph $G:(\sigma, \mu)$ is defined. Properties of bipolar total fuzzy Graph and the weak isomorphism between the bipolar subdivision fuzzy graph and bipolar total fuzzy graph, bipolar middle fuzzy graph and bipolar total fuzzy graph are discussed.

Keywords : Bipolar subdivision fuzzy graph, Bipolar middle fuzzy graph, isomorphism bipolar fuzzy graphs.

1. INTRODUCTION

Graph theory has numerous applications to problems in computer science, electrical engineering, system analysis, operations research, economics, networking routing, transportation etc. In 1965, Zadeh [8] introduced the notion of a fuzzy subset of a set. Since then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including medical and life sciences, management sciences, social sciences, engineering, statistics, graph theory, artificial intelligence, expert systems, decision making and automata theory. In 1994, Zhang [9] initiated the concept of bipolar fuzzy sets as a generalization of fuzzy sets. A bipolar fuzzy set is an extension of Zadeh's fuzzy set theory whose membership degree range in $[-1, 1]$. In this paper order, size of the nodes and edges of the bipolar total fuzzy graphs are discussed. weak isomorphism between bipolar subdivision fuzzy graph and bipolar total fuzzy graph, bipolar middle fuzzy graph and total fuzzy graph is proved.

2. PRELIMINARIES

In this section, we introduce some basic definitions that are required in the sequel.

Definition: 2.1

Let $G:(\sigma, \mu)$ be a fuzzy graph with its underlying set V and crisp graph $G^* :(\sigma^*, \mu^*)$ The pair $T(G):(\sigma_T, \mu_T)$ of G is defined as follows. Let the vertex set of $T(G)$ be $V \cup E$.

The fuzzy subset σ_T is defined on $V \cup E$ as

$$\begin{aligned}\sigma_T(x) &= \sigma(x) \text{ if } x \in V \\ &= \mu(e) \text{ if } e \in E\end{aligned}$$

The fuzzy relation μ_T is defined as

$$\begin{aligned}\mu_T(x, y) &= \mu(x, y) && \text{if } x, y \in V \\ \mu_T(x, e) &= \sigma(x) \wedge \mu(e) && \text{if } x \in V, e \in E, \text{ and the node} \\ &&& \text{'x' lies on the edge 'e'}. \\ &= 0 && \text{otherwise} \\ \mu_T(e_i, e_j) &= \mu(e_i) \wedge \mu(e_j) && \text{if the edges } e_i \text{ and } e_j \text{ have a node in} \\ &&& \text{common between them} \\ &= 0 && \text{otherwise}\end{aligned}$$

By the definition $\mu_T(u, v) \leq \sigma_T(u) \wedge \sigma_T(v)$ for all u, v in $V \cup E$. Hence μ_T is a fuzzy relation on the fuzzy subset σ_T . Hence the pair $T(G): (\sigma_T, \mu_T)$ is a fuzzy graph and is termed as total fuzzy graph of G .

Definition: 2.2

A bipolar fuzzy graph with an underlying set V is defined to be a pair $G:(\sigma, \mu)$, where $\sigma = (m_\sigma^P, m_\sigma^N)$, $\mu = (m_\mu^P, m_\mu^N)$ and crisp graph $G^* : (\sigma^*, \mu^*)$. The pair $BT(G): (\sigma_{BT}, \mu_{BT})$ of G where $\sigma_{BT} = (m_{\sigma_{BT}}^P, m_{\sigma_{BT}}^N)$, $\mu_{BT} = (m_{\mu_{BT}}^P, m_{\mu_{BT}}^N)$ we call σ_{BT} the bipolar total vertex set of V , μ_{BT} the bipolar fuzzy edge set of E respectively. The bipolar fuzzy subset σ_{BT} is defined on $V \cup E$ as

$$\begin{aligned} \sigma_{BT}^P(x) &= \sigma^P(x) && \text{if } x \in V \\ &= \mu^P(e) && \text{if } e \in V \\ \sigma_{BT}^N(x) &= \sigma^N(x) && \text{if } x \in V \\ &= \mu^N(e) && \text{if } e \in V \end{aligned}$$

The bipolar fuzzy relation μ_{BT} is defined as

$$\begin{aligned} \mu_{BT}^P(x, y) &= \mu^P(x, y) && \text{if } x, y \in V \\ \mu_{BT}^N(x, y) &= \mu^N(x, y) && \text{if } x, y \in V \\ \mu_{BT}^P(x, e) &= \sigma(x) \wedge \mu(e) && \text{if } x \in V, e \in E \text{ and the vertex 'x' lies on the edge 'e'} \\ &= 0 && \text{otherwise} \\ \mu_{BT}^N(x, e) &= \sigma(x) \vee \mu(e) && \text{if } x \in V, e \in E \text{ and the vertex 'x' lies on the edge 'e'} \\ &= 0 && \text{otherwise} \\ \mu_{BT}^P(e_i, e_j) &= \mu(e_i) \wedge \mu(e_j) && \text{if the edges } e_i \text{ and } e_j \text{ have a node in} \\ &= 0 && \text{common between them} \\ &= 0 && \text{otherwise} \\ \mu_{BT}^N(e_i, e_j) &= \mu(e_i) \vee \mu(e_j) && \text{if the edges } e_i \text{ and } e_j \text{ have a node in} \\ &= 0 && \text{common between them} \\ &= 0 && \text{otherwise} \end{aligned}$$

By the definition $\mu_{BT}^P(x, y) \leq \sigma_{BT}^P(x) \wedge \sigma_{BT}^P(y)$, $\mu_{BT}^N(x, y) \geq \sigma_{BT}^N(x) \vee \sigma_{BT}^N(y)$ for all x, y in $V \cup E$. Hence μ_{BT} is a bipolar fuzzy relation on the bipolar fuzzy subset σ_{BT} . Hence

The pair $BT(G): (\sigma_{BT}, \mu_{BT})$ is a bipolar fuzzy graph, and is termed as bipolar total fuzzy graph of G .

Example: 2.3

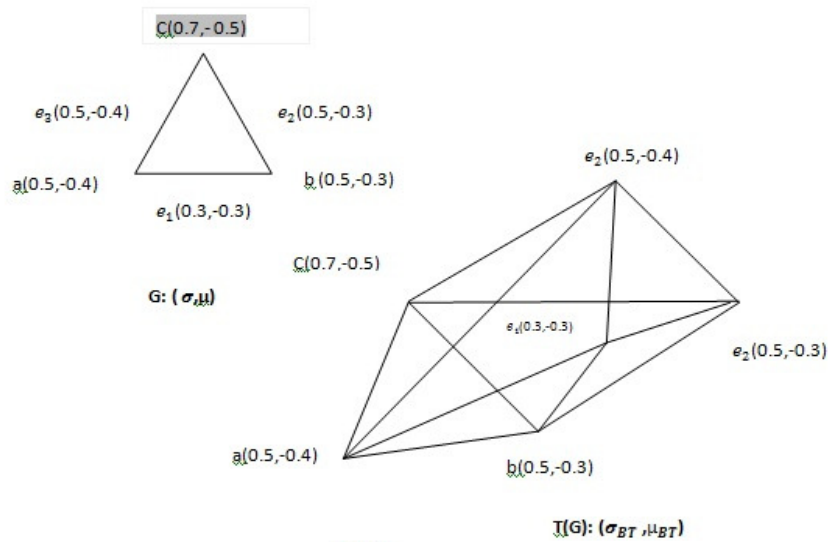


Fig: (i)

Definition : 2.4

A bipolar fuzzy graph with the underlying crisp graph $G^* : (\sigma^*, \mu^*)$ is defined to be a pair $G : (\sigma, \mu)$ where $\sigma = (m_\sigma^P, m_\sigma^N)$, $\mu = (m_\mu^P, m_\mu^N)$. Let G^* be (V, E) . The nodes and edges of G are taken together as node set, of the pair $\text{Bsd}(G) : (\sigma_{\text{Bsd}}, \mu_{\text{Bsd}})$, where $\sigma_{\text{Bsd}} = (m_{\sigma_{\text{Bsd}}}^P, m_{\sigma_{\text{Bsd}}}^N)$, $\mu_{\text{Bsd}} = (m_{\mu_{\text{Bsd}}}^P, m_{\mu_{\text{Bsd}}}^N)$. In $\text{Bsd}(G)$ each edge 'e' in G is replaced by a new vertex and that vertex is made as a neighbour of those vertices which lie on 'e' in G . Hence σ_{Bsd} is a bipolar fuzzy subset defined on $V \cup E$ as

$$\begin{aligned} \sigma_{\text{Bsd}}^P(x) &= \sigma^P(x) && \text{if } x \in V \\ &= \mu^P(e) && \text{if } e \in V \\ \sigma_{\text{Bsd}}^N(x) &= \sigma^N(x) && \text{if } x \in V \\ &= \mu^N(e) && \text{if } e \in V \end{aligned}$$

The bipolar fuzzy relation μ_{Bsd} on $V \cup E$ is defined as $\mu_{\text{Bsd}}^P(x, e) \leq \sigma_{\text{Bsd}}^P(x) \wedge \sigma_{\text{Bsd}}^P(e)$, $\mu_{\text{Bsd}}^N(x, e) \geq \sigma_{\text{Bsd}}^N(x) \vee \sigma_{\text{Bsd}}^N(e)$ for all x, e in $V \cup E$, μ_{Bsd} is a bipolar fuzzy relation of σ_{Bsd} and hence the pair $\text{Bsd}(G) : (\sigma_{\text{Bsd}}, \mu_{\text{Bsd}})$ is a bipolar fuzzy graph. This pair is termed as bipolar subdivision of fuzzy graph G .

Definition: 2.5

A bipolar fuzzy graph with an underlying set V is defined to be a Pair $G : (\sigma, \mu)$, where $\sigma = (m_\sigma^P, m_\sigma^N)$, $\mu = (m_\mu^P, m_\mu^N)$ and crisp graph $G^* : (\sigma^*, \mu^*)$. The Pair $\sigma_{\text{BM}} = (m_{\sigma_{\text{BM}}}^P, m_{\sigma_{\text{BM}}}^N)$, $\mu_{\text{BM}} = (m_{\mu_{\text{BM}}}^P, m_{\mu_{\text{BM}}}^N)$. we call σ_{BM} the bipolar middle fuzzy vertex set of V , μ_{BT} the bipolar middle fuzzy edge set of E respectively. The bipolar fuzzy subset σ_{BM} is defined on $V \cup E$ as

$$\begin{aligned} \sigma_{\text{BM}}^P(x) &= \sigma^P(x) && \text{if } x \in V \\ &= \mu^P(e) && \text{if } e \in E \\ \sigma_{\text{BM}}^N(x) &= \sigma^N(x) && \text{if } x \in V \\ &= \mu^N(e) && \text{if } e \in E \end{aligned}$$

The bipolar fuzzy relation μ_{BM} is defined as

$$\begin{aligned} \mu_{\text{BM}}^P(e_i, e_j) &= \mu(e_i) \wedge \mu(e_j) && \text{if } e_i, e_j \in \mu^* \text{ are adjacent in } G^* \\ &= 0 && \text{otherwise} \\ \mu_{\text{BM}}^N(e_i, e_j) &= \mu(e_i) \vee \mu(e_j) && \text{if } e_i, e_j \in \mu^* \text{ are adjacent in } G^* \\ &= 0 && \text{otherwise} \\ \mu_{\text{BM}}^P(v_i, v_j) &= 0 && \text{if } v_i, v_j \text{ are in } \sigma^* \\ \mu_{\text{BM}}^N(v_i, v_j) &= 0 && \text{if } v_i, v_j \text{ are in } \sigma^* \\ \mu_{\text{BM}}^P(v_i, e_j) &= \mu(e_j) && \text{if } v_i \text{ in } \sigma^* \text{ lies on the edge } e_j \in \mu^* \\ &= 0 && \text{otherwise} \\ \mu_{\text{BM}}^N(v_i, e_j) &= \mu(e_j) && \text{if } v_i \text{ in } \sigma^* \text{ lies on the edge } e_j \in \mu^* \\ &= 0 && \text{otherwise} \end{aligned}$$

By the definition $\mu_{\text{BM}}^P(x, y) \leq \sigma_{\text{BM}}^P(x) \wedge \sigma_{\text{BM}}^P(y)$, $\mu_{\text{BM}}^N(x, y) \geq \sigma_{\text{BM}}^N(x) \wedge \sigma_{\text{BM}}^N(y)$, for all x, y in $V \cup E$. Hence μ_{BM} is a bipolar fuzzy relation on the bipolar fuzzy subset σ_{BM} . Hence the pair $\text{BM}(G) : (\sigma_{\text{BM}}, \mu_{\text{BM}})$ is a bipolar fuzzy graph and is termed as bipolar middle fuzzy graph of G .

Definition: 2.6

Let G_1 and G_2 be the bipolar fuzzy graphs. A homomorphism f from G_1 to G_2 is a mapping $f : V_1 \rightarrow V_2$ which satisfies the following conditions.

$$\begin{aligned} \text{(a)} \quad \mu_{A_1}^P(x_1) &\leq \mu_{A_2}^P(f(x_1)) && \text{for all } x_1 \in V_1 \\ \mu_{A_1}^N(x_1) &\geq \mu_{A_2}^N(f(x_1)) && \text{for all } x_1 \in V_1 \end{aligned}$$

$$(b) \quad \begin{aligned} \mu_{B_1}^P(x_1y_1) &\leq \mu_{B_2}^P(f(x_1)f(y_1)) \\ \mu_{B_1}^N(x_1y_1) &\geq \mu_{B_2}^N(f(x_1)f(y_1)) \end{aligned}$$

Definition: 2.7

Let G_1 and G_2 be the bipolar fuzzy graphs. An isomorphism f from G_1 to G_2 is a bijective mapping $f:V_1 \rightarrow V_2$ which satisfies the following conditions.

$$(a) \quad \begin{aligned} \mu_{A_1}^P(x_1) &= \mu_{A_2}^P(f(x_1)) && \text{for all } x_1 \in V_1 \\ \mu_{A_1}^N(x_1) &= \mu_{A_2}^N(f(x_1)) && \text{for all } x_1 \in V_1 \end{aligned}$$

$$(b) \quad \begin{aligned} \mu_{B_1}^P(x_1y_1) &= \mu_{B_2}^P(f(x_1)f(y_1)) && \text{for all } x_1y_1 \in E_1 \\ \mu_{B_1}^N(x_1y_1) &= \mu_{B_2}^N(f(x_1)f(y_1)) && \text{for all } x_1y_1 \in E_1 \end{aligned}$$

We denote $G_1 \cong G_2$ if there is an isomorphism from G_1 to G_2 .

Definition: 2.8

Let G_1 and G_2 be the bipolar fuzzy graphs. Then, a weak isomorphism f from G_1 to G_2 is a bijective mapping $f:V_1 \rightarrow V_2$ which satisfies the following conditions,

$$(a) \quad f \text{ is homomorphism}$$

$$(b) \quad \begin{aligned} \mu_{A_1}^P(x_1) &= \mu_{A_2}^P(f(x_1)) && \text{for all } x_1 \in V_1 \\ \mu_{A_1}^N(x_1) &= \mu_{A_2}^N(f(x_1)) && \text{for all } x_1 \in V_1 \end{aligned}$$

3. PROPERTIES OF BIPOLAR TOTAL FUZZY GRAPH

Theorem: 3.1

$$\text{Order BT}(G) = \text{Order}(G) + \text{Size}(G) = \text{Order Bsd}(G)$$

Proof:

As the node set of $\text{BT}(G)$ is $V \cup E$ and the bipolar fuzzy subset σ_{BT} on $V \cup E$ is defined as

$$\begin{aligned} (\sigma_{BT}^P(x), \sigma_{BT}^N(x)) &= (\sigma^P(x), \sigma^N(x)) && \text{if } x \in V \\ &= (\mu^P(e), \mu^N(e)) && \text{if } e \in E \\ \text{Ordr BT}(G) &= (\sum_{x \in V \cup E} \sigma_{BT}^P(x), \sum_{x \in V \cup E} \sigma_{BT}^N(x)) \\ &= (\sum_{x \in v} \sigma_{BT}^P(x), \sum_{x \in v} \sigma_{BT}^P(x)) + (\sum_{x \in E} \sigma_{BT}^N(x), \sum_{x \in v} \sigma_{BT}^N(x)) \\ &= \sum_{x \in \sigma^*} \sigma(x) + \sum_{x \in \mu^*} \mu(x) \\ &= \text{Order}(G) + \text{Size}(G) \\ &= \text{Order Bsd}(G) \end{aligned}$$

Theorem:3.2

$$\text{Size BT}(G) = 3\text{Size}(G) + \sum_{e_i e_j \in \mu^*} \mu(e_i) \wedge \mu(e_j)$$

Proof:

$$\begin{aligned} \text{Size BT}(G) &= (\sum_{x,y \in V \cup E} \mu_{BT}^P(x,y), \sum_{x,y \in V \cup E} \mu_{BT}^N(x,y)) \\ &= (\sum_{x,y \in V} \mu_{BT}^P(x,y), \sum_{x,y \in V} \mu_{BT}^N(x,y)) + (\sum_{x,y \in E} \mu_{BT}^P(x,y), \sum_{x,y \in E} \mu_{BT}^N(x,y)) \\ &\quad + (\sum_{x \in V, y \in E} \mu_{BT}^P(x,y), \sum_{x \in V, y \in V} \mu_{BT}^N(x,y)) \\ &= \text{Size}(G) + \sum_{x \in V, y \in E} \sigma(x) \wedge \mu(y) + \sum_{y_i, y_j \in E} \mu(y_i) \wedge \mu(y_j) \\ &= \text{Size}(G) + 2 \sum_{y \in E} \mu(y) + \sum_{y_i, y_j \in E} \mu(y_i) \wedge \mu(y_j) \end{aligned}$$

(Since two vertices lie on each edge &
Its weight is less than the weight of the vertices)

Hence,

$$\begin{aligned} \text{Size BT(G)} &= \text{Size (G)} + 2 \text{ Size(G)} + \sum_{y_i, y_j \in E} \mu(y_i) \wedge \mu(y_j) \\ &= 3\text{Size(G)} + \sum_{y_i, y_j \in E} \mu(y_i) \wedge \mu(y_j) \end{aligned}$$

4. WEAK ISOMORPHISM ON BIPOLAR TOTAL FUZZY GRAPH

Theorem: 4.1

If G is a bipolar fuzzy graph then Bsd(G) is weak isomorphic to BT(G).

Proof:

A bipolar fuzzy graph with the underlying crisp graph $G^* : (\sigma^*, \mu^*)$ is defined to be a pair

$G : (\sigma, \mu)$ where $\sigma = (m_\sigma^P, m_\sigma^N)$, $\mu = (m_\mu^P, m_\mu^N)$. Let G^* be (V, E) . By the definition of the bipolar subdivision fuzzy graph Bsd(G) : $(\sigma_{Bsd}, \mu_{Bsd})$ where

$\sigma_{Bsd} = (m_{\sigma_{Bsd}}^P, m_{\sigma_{Bsd}}^N)$, $\mu_{Bsd} = (m_{\mu_{Bsd}}^P, m_{\mu_{Bsd}}^N)$ and σ_{Bsd} on $V \cup E$ is defined as

$$\left\{ \begin{array}{ll} \sigma_{Bsd}^P(x) = \sigma_A^P(x) & \text{if } x \in V \text{ and} \\ = \mu_B^P(x) & \text{if } x \in E \\ \sigma_{Bsd}^N(x) = \sigma_A^N(x) & \text{if } x \in V \text{ and} \\ = \mu_B^N(x) & \text{if } x \in E \end{array} \right. \quad (1)$$

The fuzzy relation μ_{Bsd} on $V \cup E$ is a fuzzy relation on σ_{Bsd} , defined as,

$$\begin{aligned} \mu_{Bsd}^P(x, e) &= \sigma_{Bsd}(x) \wedge \sigma_{Bsd}(e) & \text{if } x \in V, e \in E \text{ and } x \text{ lies on } e \\ &= 0 & \text{otherwise} \\ \mu_{Bsd}^N(x, e) &= \sigma_{Bsd}(x) \vee \sigma_{Bsd}(e) & \text{if } x \in V, e \in E \text{ and } x \text{ lies on } e \\ &= 0 & \text{otherwise} \end{aligned}$$

Using eqn (1) in the above equation

$$\begin{aligned} \text{ie) } \mu_{Bsd}^P(x, e) &= \sigma(x) \wedge \mu(e) & \text{if } x \in V, e \in E \text{ and } x \text{ lies on } e \\ &= 0 & \text{otherwise} \\ \mu_{Bsd}^N(x, e) &= \sigma(x) \vee \mu(e) & \text{if } x \in V, e \in E \text{ and } x \text{ lies on } e \\ &= 0 & \text{otherwise} \end{aligned}$$

Define a map 'h' from sd(G) to T(G), as the identity map $h: V \cup E \rightarrow V \cup E$, h will be a bijection satisfying $\sigma_{BT}^P(h(x)) = \sigma_{BT}^P(x) = \sigma^P(x) = \sigma_{Bsd}^P(x)$, $\sigma_{BT}^N(h(x)) = \sigma_{BT}^N(x) = \sigma^N(x) = \sigma_{Bsd}^N(x)$ if $u \in V$ (by the definition of BT(G) and Bsd(G))

$$\left\{ \begin{array}{ll} \sigma_{BT}^P(h(x)) = \sigma_{BT}^P(x) = \mu^P(x) = \sigma_{Bsd}^P(x) & \text{if } x \in E \\ \sigma_{BT}^N(h(x)) = \sigma_{BT}^N(x) = \mu^N(x) = \sigma_{Bsd}^N(x) & \text{if } x \in E \end{array} \right.$$

$$\begin{aligned} \text{ie) } \sigma_{BT}^P(h(x)) &= \sigma_{Bsd}^P(x) & \text{for all } x \text{ in } V \cup E \\ \sigma_{BT}^N(h(x)) &= \sigma_{Bsd}^N(x) & \text{for all } x \text{ in } V \cup E \end{aligned} \quad (2)$$

Case : 1

If $x, y \in V$, $\mu_{BT}^P(h(x), h(y)) = \mu_{BT}^P(x, y) = \mu^P(x, y)$ if x & $y \in V$, $(x, y) \in \mu^*$

$$\mu_{BT}^N(h(x),h(y)) = \mu_{BT}^N(x,y) = \mu^N(x,y) \quad \text{if } x \text{ \& } y \in V, (x,y) \in \mu^*$$

By definition of Bsd(G),

$$\begin{aligned} \mu_{Bsd}^P(x,y) &= 0 & \text{if } x,y \in V, \\ \mu_{Bsd}^N(x,y) &= 0 & \text{if } x,y \in V \end{aligned}$$

$$\begin{aligned} \mu_{Bsd}^P(x,y) &\leq \mu_{BT}^P(h(x),h(y)) & \text{if } x,y \in V \\ \mu_{Bsd}^N(x,y) &\geq \mu_{BT}^N(h(x),h(y)) & \text{if } x,y \in V \end{aligned} \tag{3}$$

Case : 2

If $x \in V, y = e \in E$, then

$$\begin{aligned} \mu_{BT}^P(h(x),h(e)) = \mu_{BT}^P(x,e) &= \sigma(x) \wedge \mu(e) & \text{if } x \in V, e \in E \text{ and } x \text{ lies on } e \\ &= 0 & \text{Otherwise} \end{aligned}$$

$$\begin{aligned} \mu_{BT}^N(h(x),h(e)) = \mu_{BT}^N(x,e) &= \sigma(x) \vee \mu(e) & \text{if } x \in V, e \in E \text{ and } x \text{ lies on } e \\ &= 0 & \text{Otherwise} \end{aligned}$$

ie) $\mu_{Bsd}^P(x,e) = \mu_{BT}^P(h(x),h(e))$ if $x \in V, e \in E$
 $\mu_{Bsd}^N(x,e) = \mu_{BT}^N(h(x),h(e))$ if $x \in V, e \in E$ (by definition of Bsd(G))

Case : 3

If $x = e_i, y = e_j \in E$, then

$$\begin{aligned} \mu_{BT}^P(e_i,e_j) = \mu(e_i) \square \mu(e_j) & \text{if the edges } e_i \text{ and } e_j \text{ have a node in common between them} \\ &= 0 & \text{Otherwise} \end{aligned}$$

$$\begin{aligned} \mu_{BT}^N(e_i,e_j) = \mu(e_i) \square \mu(e_j) & \text{if the edges } e_i \text{ and } e_j \text{ have a node in common between them} \\ &= 0 & \text{Otherwise} \end{aligned}$$

ie) $\mu_{Bsd}^P(e_i,e_j) = 0$ if $e_i, e_j \in E$
 $\mu_{Bsd}^N(e_i,e_j) = 0$ if $e_i, e_j \in E$
 $\mu_{Bsd}^P(e_i,e_j) \leq \mu_{BT}^P(e_i,e_j)$ if $e_i, e_j \in E$
 $\mu_{Bsd}^N(e_i,e_j) \geq \mu_{BT}^N(e_i,e_j)$ if $e_i, e_j \in E$

From (3),(4) & (5)

$$\begin{aligned} \mu_{Bsd}^P(x,y) &\leq \mu_{BT}^P(h(x),h(y)) & \text{for all } x,y \in V \cup E \\ \mu_{Bsd}^N(x,y) &\geq \mu_{BT}^N(h(x),h(y)) & \text{for all } x,y \in V \cup E \end{aligned}$$

Hence by (2) & (5) h: Bsd(G) → BT(G) is a weak isomorphism.

Theorem: 4.2

BM(G) is weak isomorphic with BT(G)

Proof:

Consider the identity map h: BM(G) → BT(G) as h: V ∪ E → V ∪ E

By the definition of σ_{BT} in BT(G) and σ_{BM} in BM(G)

We have

$$\begin{cases} \sigma_{BM}^P(x) = \sigma_{BT}^P(h(x)) & \text{for all } x \in V \cup E \\ \sigma_{BM}^N(x) = \sigma_{BT}^N(h(x)) & \text{for all } x \in V \cup E \end{cases} \tag{1}$$

Case :1

$$\mu_{BM}^P(e_i,e_j) = \mu(e_i) \square \mu(e_j) \quad \text{if } e_i, e_j \in \mu^* \text{ and are adjacent in } G^*$$

$$\begin{aligned}
 &= 0 && \text{Otherwise} \\
 \mu_{BM}^N(e_i, e_j) &= \mu(e_i) \square \mu(e_j) && \text{if } e_i, e_j \in \mu^* \text{ and are adjacent in } G^* \\
 &= 0 && \text{Otherwise} \\
 \mu_{BT}^P(e_i, e_j) &= \mu(e_i) \square \mu(e_j) && \text{if the edges } e_i \text{ and } e_j \text{ are adjacent } G^* \\
 &= 0 && \text{Otherwise} \\
 \mu_{BT}^N(e_i, e_j) &= \mu(e_i) \square \mu(e_j) && \text{if the edges } e_i \text{ and } e_j \text{ are adjacent } G^* \\
 &= 0 && \text{Otherwise}
 \end{aligned}$$

Hence

$$\begin{cases}
 \mu_{BM}^P(e_i, e_j) = \mu_{BT}^P(e_i, e_j) = \mu_{BT}^P(h(e_i), h(e_j)) & \text{if } e_i, e_j \in \mu^* \\
 \mu_{BM}^N(e_i, e_j) = \mu_{BT}^N(e_i, e_j) = \mu_{BT}^N(h(e_i), h(e_j)) & \text{if } e_i, e_j \in \mu^*
 \end{cases}$$

Case: 2
$$\begin{cases}
 \mu_{BM}^P(x, y) = 0 & \text{if } x, y \text{ are in } \sigma^* \\
 \mu_{BM}^N(x, y) = 0 & \text{if } x, y \text{ are in } \sigma^*
 \end{cases}$$

$$\begin{cases}
 \mu_{BT}^P(x, y) = \mu^P(x, y) & \text{if } x, y \in \sigma^* \\
 \mu_{BM}^N(x, y) = \mu^N(x, y) & \text{if } x, y \in \sigma^*
 \end{cases}$$

$$\begin{cases}
 \mu_{BM}^P(x, y) = 0 \leq \mu^P(x, y) = \mu_{BT}^P(x, y) & \text{if } x, y \text{ are in } \sigma^* \\
 \mu_{BM}^N(x, y) = 0 \geq \mu^N(x, y) = \mu_{BT}^N(x, y) & \text{if } x, y \text{ are in } \sigma^*
 \end{cases}$$

Case: 3

$$\begin{cases}
 \mu_{BM}^P(y_i, e_j) = \mu^P(e_j) & \text{if } y_i \text{ in } \sigma^* \text{ lies on the edge } e_j \in \mu^* \\
 = 0 & \text{Otherwise} \\
 \mu_{BM}^N(y_i, e_j) = \mu^P(e_j) & \text{if } y_i \text{ in } \sigma^* \text{ lies on the edge } e_j \in \mu^* \\
 = 0 & \text{Otherwise}
 \end{cases}$$

So,

$$\begin{aligned}
 \mu_{BM}^P(y_i, e_j) &= \mu_{BT}^P(y_i, e_j) = \mu_{BT}^P(h(y_i), h(e_j)), && \text{if } y_i \in V, e_j \in E \\
 \mu_{BM}^N(y_i, e_j) &= \mu_{BT}^N(y_i, e_j) = \mu_{BT}^N(h(y_i), h(e_j)), && \text{if } y_i \in V, e_j \in E
 \end{aligned}$$

From all the three cases,

$$\begin{cases}
 \mu_{BM}^P(x, y) \leq \mu_{BT}^P(h(x), h(y)) & \text{for all } x, y \in V \cup E \\
 \mu_{BM}^N(x, y) \geq \mu_{BT}^N(h(x), h(y)) & \text{for all } x, y \in V \cup E
 \end{cases} \tag{2}$$

‘h’ being a bijection and from equations (1) & (2) BM(G) Is weak isomorphic with BT(G).

5. CONCLUSION :

In this paper new concepts bipolar total fuzzy graph is introduced. It is found that Bsd(G) is weak isomorphic to BT(G), BM(G) is weak isomorphic to BT(G). it is proved the weak isomorphism between the bipolar subdivision fuzzy graph and bipolar total fuzzy graph, bipolar middle fuzzy graph and bipolar Total fuzzy graph.

REFERENCES:

1. T. Akiyama, T.Hamada, I.Yoshimura, Graph equations for line graphs and middle graphs, TRU math 12(2) (1976).

2. W.R.Zhang, *Bipolar fuzzy sets*, *Proc. of fuzzy-IEEE (1998)*,835-840.
3. D.Michalak, *On Middle and Total graphs with coarseness number Equal 1,Lecture notes in Mathematics, Vol 1018: Graph Theory, Springer - Verlag,Berlin,139-150(1983)*.
4. A.Rosenfeld, *Fuzzy graphs, Fuzzy sets and Applications (L.A.Zadeh , K.S.Fu, M.Shimura, Eds.)*, *Academi Press, New York,(1975)* 77-95.
5. A.A.Talebi, H.Rashmanlou and N.Mehdipoor, *Isomorphism on Vague graphs, Annals of Fuzzy Mathematics and Informatics, Vol 6, No.3, (2013) PP.575-588*.
6. Hossein Rashmanlou, Young Bae jun, R.A.Borzoeei, *More results on highly irregular bipolar fuzzy graphs, Annals of Fuzzy Mathematics and Information, Vol X, No.X, (Month 201y), pp.1-XX*.
7. F.Harary, *Graph Theory, Addition-wesley, Reading, Mass., 1969*.
8. L.A. Zadeh *Fuzzy Sets, Information and control* 8,338-353(1965).
9. W.R.Zhang, *Bipolar fuzzy sets and relations: a Computational Framework for cognitive Modeling and Multi agent decision analysis, Proc of IEEE Conf. (1994). 305-309*.



APPLICATIONS OF STABLE SET PROBLEMS IN HYPERGRAPH

A. Ramesh Kumar*, G. Kavitha**

*Head, Department of Mathematics, Srimad Andavan Arts & Science College, Trichy, Tamilnadu, India

**Asst. Professor, Kongunadu College of Engineering and Technology, Tholurpatti, Thottiam, Tamilnadu, India

E-mail : andavanmathsramesh@gmail.com*, kavipraba18@gmail.com**

ABSTRACT :

This paper considers the stable set of hypergraphs and presents several new results and algorithms using the semi- tensor product of matrices. By the definitions of an incidence matrix of a hypergraph and characteristic logical vector of a vertex subset, an equivalent algebraic condition is established for hypergraph stable sets, as well as a new algorithm, which can be used to search all the stable sets of any hypergraph.

1. INTRODUCTION

A hypergraph $H = (V, E)$ is composed of a finite set and a collection E of nonempty subsets of V , in which V is called the vertex set of H and E is called the edge set of H . Thus, graphs are a special kind of hypergraphs with two vertices in each edge. One of the basic problems about hypergraph theory is the stable set problem, which has been widely applied in many research fields like network coding^{[1],[2]}. Another basic problem about hypergraph theory is the coloring problem, which is one of NP-complete problems. There are various forms of hypergraph coloring such as vertex coloring, good coloring of edges, strong coloring, and equitable coloring. Graph coloring has been widely used in many real-life areas including scheduling and timetabling in engineering, register allocation in compilers, and air traffic flow management and frequency assignment in mobile^{[3],[4],[5],[6]}. The coloring problems of a special kind of graphs have been widely discussed in^{[7],[8],[9]}. In recent years, there have been some references considering hypergraph theory, such as^{[10],[11]}. It has been successfully applied to many different areas such as Markov decision process^[12], complete simple games^[13], linear programming^[14], and cooperation structures in games^[15]. And a few references have analyzed the colorability of different kinds of hypergraphs. However, there are no proper algebraic algorithms for stable set and coloring problems of hypergraphs. Thus, they are still open problems and it is necessary for us to establish new formulations and algorithms.

In recent years, Cheng et al.^{[16],[17]} have proposed an effective tool, called the semitensor product (STP) of matrices. Via STP, Boolean networks can be converted into an algebraic form and many problems of Boolean networks, such as controllability and observability^[18], fixed points and cycles, and control design problems, have been investigated.

we investigate the stable set and vertex coloring problems of hypergraphs and present some new results and algorithms via STP. By incidence matrix and characteristic logical vector (CLV), a necessary and sufficient condition, as well as a new algorithm, is established for hypergraph stable sets. Then, we study the vertex coloring problem. An algebraic equivalent condition and an algorithm for coloring problem are obtained. With the two algorithms, we can calculate all the stable sets and coloring schemes with the given colors for any hypergraph. The results we obtained in this paper are feasible and clear, illustrated by an example and a practical application to the storing problems. Compared to^[19], which has considered the stable set and coloring problems of graphs by STP, the results we obtained seem to be the generalization of^[19]. However, just applying the results about graphs in^[19] to

hypergraphs, we cannot get the similar results about hypergraphs. In fact, there are many differences. We use incidence matrix of hypergraphs, while Wang et al. in [19] have used adjacent matrix of graphs. The derivations are completely different since the fundamental techniques used are not the same.

2. NOTATIONS

- (i) $M_{m \times n}$ is the set of $m \times n$ real matrices.
- (ii) δ_n^i is the i^{th} column of the identity matrix I_n .
- (iii) $\Delta_n := \{ \delta_n^i \mid i = 1, \dots, n \}$, $\Delta_2 := \Delta$.
- (iv) $D := \{0,1\}$. Identify $1 \sim \delta_2^1, 0 \sim \delta_2^2$; then, $D \sim \Delta$.
- (v) $A \in M_{m \times n}$ is called a Boolean matrix, if all its entries are either 0 or 1. The set of $m \times n$ Boolean matrices is denoted by $B_{m \times n}$.
- (vi) A matrix $L \in M_{n \times r}$ is called a logical matrix if the columns of L , denoted by $\text{Col}(L)$, belong to Δ_n . That is, $\text{Col}(L) \subseteq \Delta_n$. And $\text{Col}_i(L)$ means the i th column of L . Denote the set of $n \times r$ logical matrices by $L_{n \times r}$.
- (vii) If $L \in L_{n \times r}$, by definition it can be expressed as $L = [\delta_n^{i_1} \delta_n^{i_2} \dots \delta_n^{i_r}]$. Briefly, we denote it by $L = \delta_n [i_1 \ i_2 \ \dots \ i_r]$.
- (viii) For $A = (a_{ij}), B = (b_{ij}) \in M_{m \times n}$, $A \geq (\leq, >, <) B$ means $a_{ij} \geq (\leq, >, <) b_{ij}$, for all i, j .
- (ix) For a set S , $|S|$ is the cardinality of S .
- (x) $A = \text{Diag}\{A_1, A_2, \dots, A_r\}$ is a block-diagonal matrix with A_i in the (i, i) th position ($1 \leq i \leq r$).
- (xi) Let $A = (a_{ij}) \in M_{m \times n}, B \in M_{p \times q}$. The Kronecker product of matrices and is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \dots \dots \dots (1)$$

3. PRELIMINARIES

In this section, we shall give some necessary preliminaries on STP and hypergraph theory, which will be used later.

3.1. Definition

Let $A \in M_{m \times n}$ and $B \in M_{p \times q}$. The STP of matrices A and B , denoted by $A \times B$, is defined as $A \times B = (A \otimes I_{s/n})(B \otimes I_{s/p}) \dots (2)$

where $s = \text{lcm}\{n, p\}$ is the least common multiple of n and p .

3.2. Definition

A swap matrix $W[m, n]$ is an $mn \times mn$ matrix, defined as follows: label its columns by $(11, 12, \dots, 1n, \dots, m1, m2, \dots, mn)$; label its rows by $(11, 21, \dots, m1, \dots, 1n, 2n, \dots, mn)$ and then the element at the position $[(I, J), (i, j)]$ is

$$w_{(I,J),(I,j)} = \delta_{ij}^{I,J} = \begin{cases} 1, & I = i, J = j, \\ 0, & \text{Otherwise.} \end{cases} \dots (3)$$

3.3. Definition

Let $A = (a_{ij}), B = (b_{ij}) \in M_{m \times n}$. The Hadamard product of A and B is defined as $A \odot B = (a_{ij}b_{ij}) \in M_{m \times n}$. . . (4)

3.4. Definition

Let $V = \{v_1, v_2, \dots, v_n\}$ be a finite set, and let $E = \{E_1, E_2, \dots, E_m\}$ be a family of subsets of V ; that is, $E_j \subseteq V, j = 1, 2, \dots, m$. The family E is said to be a hypergraph on V denoted by $H = (V, E)$, if $E_j \neq \emptyset, j = 1, 2, \dots, m$, and $\bigcup_{j=1}^m E_j = V$. The elements v_1, v_2, \dots, v_n are called the vertices (hypervertices) and the sets E_1, E_2, \dots, E_m are called the edges (hyperedges).

The incidence matrix of hypergraph $H = (V, E)$ is a matrix $A = (a_{ij})$ with m rows that represent the edges of H and n columns that represent the vertices of H , such that

$$a_{ij} = \begin{cases} 1, & v_j \in E_i, \\ 0, & v_j \notin E_i. \end{cases} \quad \dots (5)$$

3.5. Definition

Given a hypergraph $H = (V, E)$, a set $S \subseteq V$ is called a stable set if it contains no edge E_i with $|E_i| > 1$. Furthermore, S is called a maximum stable set, if any vertex subset strictly containing S is not a stable set. A stable set is called an absolutely maximum stable set if $|S|$ is the largest among all of the stable sets of H . The stable number of H , denoted by $\alpha(H)$, is defined to be the maximum cardinality of all the stable sets of H .

3.6. Definition

A q -coloring is defined to be a partition of V into q stable sets S_1, S_2, \dots, S_q , each corresponding to a color. A hypergraph for which there exists a q -coloring is said to be q -colorable.

4. STABLE SET PROBLEM

In the section, we investigate the stable set problem of hypergraphs using the STP method and present algebraic equivalent conditions, as well as an algorithm.

Given a hypergraph $H = (V, E)$ with n vertices $V = \{v_1, v_2, \dots, v_n\}$ and m edges $E = \{E_1, E_2, \dots, E_m\}$, assume that the incidence matrix of H is $A = (a_{ij})_{m \times n}$. Denote the i th row of A of A by $a_i, i = 1, 2, \dots, m$; then $A = [a_1^T, a_2^T, \dots, a_m^T]^T$. Assume that S is a subset of V . Then, in the following, we will discuss under what conditions the subset is a stable set. First, we define some vectors.

The CLV of S , denoted by $V_S = [x_1, x_2, \dots, x_n]$, is denoted as

$$x_i = \begin{cases} 1, & v_i \in S, \\ 0, & v_i \notin S. \end{cases} \quad \dots (6)$$

And then denote

$$y_{ij} = \begin{bmatrix} a_{ij} \\ 1 - a_{ij} \end{bmatrix}, \quad y_j = \begin{bmatrix} x_{ij} \\ 1 - x_j \end{bmatrix}, \quad \dots (7)$$

$i = 1, 2, \dots, m; j = 1, 2, \dots, n;$

It is easy to see that V_S is a Boolean vector and $y_{ij}, y_j \in \Delta$. Then, we can present the following results.

4.1. Theorem : Consider the hypergraph $H = (V, E)$ expressed as above. Then S is a stable set of H if and only if the last row of matrix \bar{M} has atleast one zero element, where

$$\bar{M} = \left(\bigotimes_{j=1}^n M_j \right) \left(\hat{a}_{k=1}^{n-1} \left(I_{2^k} \otimes W_{[2,2^k]} \right) \right) \left(\sum_{l=1}^m Y_l \right), \dots (8)$$

$$M = M_n M_i (I_2 \otimes M_c) M_r, Y_l = \hat{a}_{t=1}^n y_{lt}$$

Proof. Let $\bar{E}_l = E \setminus S$ with CLV $\bar{a}_l = [\bar{a}_{l1}, \bar{a}_{l2}, \dots, \bar{a}_{ln}]$, $l = 1, 2, \dots, m$. Denote

$$\bar{y}_{lt} = \begin{bmatrix} \bar{a}_{lt} \\ 1 - \bar{a}_{lt} \end{bmatrix}, l = 1, 2, \dots, m; t = 1, 2, \dots, n; \dots (9)$$

Then, S is a stable set if and only if, for every $l \in \{1, 2, \dots, m\}$, $\bar{E}_l \neq \emptyset$; that is $\bar{a}_l \neq 0_{l \times n}$. Since $\bar{a}_l \neq 0_{l \times n}$ if and only if $\hat{a}_{t=1}^n \bar{y}_{lt} \neq \delta_{2^n}$ if and only if the last element of $\hat{a}_{t=1}^n \bar{y}_{lt} \neq \delta_{2^n}$ if and only if the last element of $\hat{a}_{t=1}^n \bar{y}_{lt}$ is 0, we just need to prove that, for every $l \in \{1, 2, \dots, m\}$, the last element of $\hat{a}_{t=1}^n \bar{y}_{lt}$ is 0 if and only if the last row of matrix \bar{M} has one zero component at least.

Let $J_1^T = \delta_{2^n}$. If, for every $l \in \{1, 2, \dots, m\}$, the last element of $\hat{a}_{t=1}^n \bar{y}_{lt}$ is 0, then, we get, for every l , $J_1 \hat{a}_{t=1}^n \bar{y}_{lt} = 0$. Thus, \bar{y}_{lt} satisfies

$$J_1 \sum_{l=1}^m \hat{a}_{t=1}^n \bar{y}_{lt} = 0 \dots (10)$$

Since $\bar{E}_l = E \setminus S$, $\bar{a}_{lt} = a_{lt} - a_{lt} \wedge x_t$. Hence,

$$\begin{aligned} \bar{y}_{lt} &= y_{lt} - y_{lt} \wedge y_t = \neg(y_{lt} \rightarrow (y_{lt} \wedge y_t)) \\ &= M_n M_i y_{lt} M_c y_{lt} y_t = M_n M_i (I_2 \otimes M_c) M_r y_{lt} y_t \dots (11) \end{aligned}$$

□ $M y_{lt} y_t$.

So (10) can be expressed as

$$J_1 \sum_{l=1}^m \hat{a}_{t=1}^n M y_{lt} y_t = 0 \dots (12)$$

By Definition 3.5, we have

$$\begin{aligned} \hat{a}_{t=1}^n M y_{lt} y_t &= \left(\bigotimes_{j=1}^n M_j \right) \left(\hat{a}_{t=1}^n y_{lt} y_t \right), \\ \hat{a}_{t=1}^n y_{lt} y_t &= y_{l1} y_{l2} y_{l3} \dots y_{ln} y_n \\ &= y_{l1} W_{[2,2]} y_{l2} y_{l1} y_{l2} \dots y_{ln} y_n \\ &= \left(I_2 \otimes W_{[2,2]} \right) y_{l1} y_{l2} y_{l1} y_{l2} \dots y_{ln} y_n = \dots \\ &= \hat{a}_{k=1}^{n-1} \left(I_{2^k} \otimes W_{[2,2^k]} \right) Y_1 Y \dots (13) \end{aligned}$$

where $Y = \hat{a}_{k=1}^{n-1} y_t \in \Delta_2$ and Y_1 is described in (8). Therefore, (10) becomes

$$J_1 \sum_{l=1}^m \left(\bigotimes_{j=1}^n M_j \right) \hat{a}_{k=1}^{n-1} \left(I_{2^k} \otimes W_{[2,2^k]} \right) Y_1 Y = 0 \dots (14)$$

That is,

$$J_1 \bar{M} Y = 0 \dots (15)$$

Noticing that $Y \in \Delta_{2^n}$, we have that (15) having solution Y is equivalent to the last row of \bar{M} having one zero element at least. The necessity is proved.

4.2. Algorithm

Consider a hypergraph H . The following steps are given to find all the stable sets of H .

(1) Calculate the matrix \bar{M} given in (8).

(2) Denote the last row of \bar{M} by $\beta = [b_1, b_2, \dots, b_{2^n}]$. If $b_i \neq 0$, for every $i \in [1, 2, \dots, 2^n]$, then, H has no stable set and stop. Otherwise, find out all the zero elements of β : $b_{i_1} = b_{i_2} = \dots = b_{i_p} = 0$. Then, $b_{i_k} = 0$ corresponds to a solution, $Y = \delta_{2^n}^{i_k}$, of (15) and so does a stable set of H .

(3) From $\hat{a}_{j=1}^n y_j = \delta_{2^n}^{i_k}$, we can retrieve y_t as $y_t = S_t^n \hat{a}_{j=1}^n y_j = S_t^n \delta_{2^n}^{i_k}$, $t = 1, 2, \dots, n$, where S_t^n , $t = 1, 2, \dots, n$, are defined as follows [13]:

$$\begin{aligned} S_1^n &= \delta_2 \left[\underbrace{1 \dots 1}_{2^{n-1}} \underbrace{2 \dots 2}_{2^{n-1}} \right], \\ S_2^n &= \delta_2 \left[\underbrace{1 \dots 1}_{2^{n-2}} \underbrace{2 \dots 2}_{2^{n-2}} \underbrace{1 \dots 1}_{2^{n-2}} \underbrace{2 \dots 2}_{2^{n-2}} \right], \\ S_n^n &= \delta_2 \left[\underbrace{1 \quad 2 \quad \dots \quad 1 \quad 2}_{2^{n-1}} \right], \end{aligned} \quad \dots (16)$$

By the definition of (12), we obtain the stable set corresponding to $\delta_{2^n}^{i_k}$:

$$S(i_k) = \{v_t \mid y_t = \delta_2^1, 1 \leq t \leq n\} \quad \dots (17)$$

Thus, all the stable sets of H are $\{S(i_k) \mid k = 1, 2, \dots, p\}$.

4.3. Example : Consider the hypergraph $H = (V, E)$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$, $E = \{E_1, E_2, E_3, E_4\}$, $E_1 = \{v_1, v_2, v_3\}$, $E_2 = \{v_3, v_4, v_5\}$, $E_3 = \{v_2, v_3, v_4\}$, and $E_4 = \{v_2, v_4, v_5\}$.

By the definition of the incidence matrix of the hypergraph H , the incidence matrix of is

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \quad \dots (18)$$

By MATLAB toolbox, we easily get

$$\begin{aligned} M &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \\ Y_1 &= \delta_{2^5}^4, & Y_2 &= \delta_{2^5}^{25}, \\ Y_3 &= \delta_{2^5}^{18}, & Y_4 &= \delta_{2^5}^{21}, \end{aligned} \quad \dots (19)$$

Thus,

$$\sum_{l=1}^4 Y_l = \delta_{2^5}^4 + \delta_{2^5}^{25} + \delta_{2^5}^{18} + \delta_{2^5}^{21}$$

$$= [0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0]^T$$

Then, we obtain the last row of \bar{M} as

$$\beta = [4\ 2\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 3\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0] \dots (20)$$

and the indexes of zero elements in β are

$$P = \{i_k \mid \beta(i_k) = 0\}$$

$$= [6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 19, 20, 22, 23, 24, 26, 27, 28, 29, 30, 31, 32] \dots (21)$$

For each index $i_k \in P$, let $\hat{a}_{i=1}^5 y_i = \delta_{2^n}^{i_k}$, via computing $y_i = S_1^5 \delta_{2^5}^{i_k} = 1, 2, \dots, 5$, we have all the stable sets of as follows :

- $i_1 = 6 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (1, 1, 0, 1, 0) \sim S_1 = \{v_1, v_2, v_4\},$
- $i_2 = 7 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (1, 1, 0, 0, 1) \sim S_2 = \{v_1, v_2, v_5\},$
- $i_3 = 8 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (1, 1, 0, 0, 0) \sim S_3 = \{v_1, v_2\}$
- $i_4 = 10 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (1, 0, 1, 1, 0) \sim S_4 = \{v_1, v_3, v_4\},$
- $i_5 = 11 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (1, 0, 1, 0, 1) \sim S_5 = \{v_1, v_3, v_5\},$
- $i_6 = 12 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (1, 0, 1, 0, 0) \sim S_6 = \{v_1, v_3\},$
- $i_7 = 13 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (1, 0, 0, 1, 1) \sim S_7 = \{v_1, v_4, v_5\},$
- $i_8 = 14 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (1, 0, 0, 1, 0) \sim S_8 = \{v_1, v_4\},$
- $i_9 = 15 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (1, 0, 0, 0, 1) \sim S_9 = \{v_1, v_5\},$
- $i_{10} = 16 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (1, 0, 0, 0, 0) \sim S_{10} = \{v_1\},$
- $i_{11} = 19 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (0, 1, 1, 0, 1) \sim S_{11} = \{v_2, v_3, v_5\},$
- $i_{12} = 20 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (0, 1, 1, 0, 0) \sim S_{12} = \{v_2, v_3\},$
- $i_{13} = 22 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (0, 1, 0, 1, 0) \sim S_{13} = \{v_2, v_4\},$
- $i_{14} = 23 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (0, 1, 0, 0, 1) \sim S_{14} = \{v_2, v_5\},$
- $i_{15} = 24 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (0, 1, 0, 0, 0) \sim S_{15} = \{v_2\},$
- $i_{16} = 26 \sim (x_1, x_2, x_3, x_4, x_5)$
 $= (0, 0, 1, 1, 0) \sim S_{16} = \{v_3, v_4\},$

Then, the index set Q of j satisfying $\text{Col}_j(M) \ll b$, is

$$Q = \{j \mid \text{Col}_j(M) \ll b\} \dots (25)$$

$$= \{6, 7, 10, 11, 13, 14, 19, 20, 22, 23, 26, 27\}$$

For each $j \in Q$, let $\hat{a}_{i=1}^5 x_i = \delta_{2^5}^j$. By computing $x_i = S_{i,2}^5 \delta_{2^5}^j$, $i = 1, 2, \dots, 5$, we have

$$j = 6, \quad \delta_{2^5}^6 \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[1 \ 1 \ 0 \ 1 \ 0],$$

$$j = 7, \quad \delta_{2^5}^7 \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[1 \ 1 \ 0 \ 0 \ 0],$$

$$j = 10, \quad \delta_{2^5}^{10} \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[1 \ 0 \ 1 \ 1 \ 0],$$

$$j = 11, \quad \delta_{2^5}^{11} \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[1 \ 0 \ 1 \ 0 \ 1],$$

$$j = 13, \quad \delta_{2^5}^{13} \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[1 \ 0 \ 0 \ 1 \ 1],$$

$$j = 14, \quad \delta_{2^5}^{14} \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[1 \ 0 \ 0 \ 1 \ 0],$$

$$j = 19, \quad \delta_{2^5}^{19} \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[0 \ 1 \ 1 \ 0 \ 0],$$

$$j = 20, \quad \delta_{2^5}^{20} \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[0 \ 1 \ 0 \ 1 \ 0],$$

$$j = 22, \quad \delta_{2^5}^{22} \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[0 \ 1 \ 0 \ 0 \ 1],$$

$$j = 23, \quad \delta_{2^5}^{23} \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[0 \ 1 \ 0 \ 0 \ 1],$$

$$j = 26, \quad \delta_{2^5}^{26} \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[0 \ 0 \ 1 \ 1 \ 0],$$

$$j = 27, \quad \delta_{2^5}^{27} \sim [x_1, x_2, x_3, x_4, x_5] = \delta_2[0 \ 0 \ 1 \ 0 \ 1], \dots (26)$$

from which we obtain the following 12 coloring schemes :

- Scheme 1 : $S_{c_1}^6 = \{v_1, v_2, v_4\}$ (Red),
 $S_{c_2}^8 = \{v_3, v_5\}$ (Blue) ;
- Scheme 2 : $S_{c_1}^7 = \{v_1, v_2, v_5\}$ (Red),
 $S_{c_2}^8 = \{v_3, v_4\}$ (Blue) ;
- Scheme 3 : $S_{c_1}^{10} = \{v_1, v_3, v_4\}$ (Red),
 $S_{c_2}^8 = \{v_2, v_5\}$ (Blue) ;
- Scheme 4 : $S_{c_1}^{11} = \{v_1, v_3, v_5\}$ (Red),
 $S_{c_2}^8 = \{v_2, v_4\}$ (Blue) ;
- Scheme 5 : $S_{c_1}^{13} = \{v_1, v_3, v_5\}$ (Red),
 $S_{c_2}^8 = \{v_2, v_4\}$ (Blue) ;
- Scheme 6 : $S_{c_1}^{14} = \{v_1, v_4\}$ (Red),
 $S_{c_2}^8 = \{v_2, v_3, v_5\}$ (Blue) ;
- Scheme 7 : $S_{c_1}^{19} = \{v_2, v_3, v_5\}$ (Red),
 $S_{c_2}^8 = \{v_1, v_4\}$ (Blue) ;
- Scheme 8 : $S_{c_1}^{20} = \{v_2, v_3\}$ (Red),
 $S_{c_2}^8 = \{v_1, v_4, v_5\}$ (Blue) ;
- Scheme 9 : $S_{c_1}^{22} = \{v_2, v_4\}$ (Red),

$$\begin{aligned}
& S_{c_2}^8 = \{v_1, v_3, v_5\} \text{ (Blue) ;} \\
\text{Scheme 10 : } & S_{c_1}^{23} = \{v_2, v_5\} \text{ (Red),} \\
& S_{c_2}^8 = \{v_1, v_3, v_4\} \text{ (Blue) ;} \\
\text{Scheme 11 : } & S_{c_1}^{26} = \{v_3, v_4\} \text{ (Red),} \\
& S_{c_2}^8 = \{v_1, v_2, v_5\} \text{ (Blue) ;} \\
\text{Scheme 12 : } & S_{c_1}^{27} = \{v_3, v_5\} \text{ (Red),} \\
& S_{c_2}^8 = \{v_1, v_2, v_4\} \text{ (Blue) ;} \quad \dots (27)
\end{aligned}$$

Thus, there are totally 12 kinds of storing methods.

6. CONCLUSION

In this paper, the stable set and vertex coloring problems of hypergraphs have been revised. Several new results and algorithms have been presented via a method of STP. By defining the incidence matrix of hypergraph and CLV of a vertex subset, one equivalent condition has been established for hypergraph stable set. And a new algorithm to find out all the stable sets and all the absolutely maximum stable sets has been obtained. Furthermore, we have considered the vertex coloring problem and got a necessary and sufficient condition in the form of algebraic inequality, by which an algorithm has been derived to search all the coloring schemes and minimum coloring partitions with the given colors for any hypergraph. Finally, the illustrative example and the application to storing problem have shown that the results presented in this paper are very effective. In papers, the scheduling jobs can induce a mixed graph coloring^[20], not a hypergraph coloring. Thus, the mixed graph coloring problem will be interesting to be discussed by STP in the future.

REFERENCES

1. D. Traskov, M. Heindlmaie, M. Medard, R. Koetter, and D. S. Lun, "Scheduling for network coded multicast: a conflict graph formulation," in *Proceedings of the IEEE GLOBECOM Workshops*, pp. 1–5, New Orleans, La, USA, December 2008.
2. M. Yang and Y. Yang, "A hypergraph approach to linear network coding in multicast networks," *IEEE Transactions on Parallel and Distributed Systems*, vol. 21, no. 7, pp. 968–982, 2010.
3. N. Barnier and P. Brisset, "Graph coloring for air traffic flow management," *Annals of Operations Research*, vol. 130, no. 1–4, pp. 163–178, 2004.
4. M. W. Carter, G. Laporte, and S. Y. Lee, "Examination timetabling: algorithmic strategies and applications," *Journal of the Operational Research Society*, vol. 47, no. 3, pp. 373–383, 1996.
5. G. J. Chaitin, M. A. Auslander, A. K. Chandra, J. Cocke, M. E. Hopkins, and P. W. Markstein, "Register allocation via coloring," *Computer Languages*, vol. 6, no. 1, pp. 47–57, 1981.
6. A. Gamst, "Some lower bounds for a class of frequency assignment problems," *IEEE Transactions on Vehicular Technology*, vol. 35, no. 1, pp. 8–14, 1986.
7. B. Wang and J.-L. Wu, "Total colorings of planar graphs without intersecting 5-cycles," *Discrete Applied Mathematics*, vol. 160, no. 12, pp. 1815–1821, 2012.
8. B. Wang and J.-L. Wu, "Total colorings of planar graphs with maximum degree seven and without intersecting 3-cycles," *Discrete Mathematics*, vol. 311, no. 18-19, pp. 2025–2030, 2011.

9. X. Zhang and J.-L. Wu, "On edge colorings of 1-planar graphs," *Information Processing Letters*, vol. 111, no. 3, pp. 124–128, 2011.
10. H. Shen, S. Lv, X. Dong, J. Deng, X. Wang, and X. Zhou, "Hypergraph modeling and approximation algorithms for the minimum length link scheduling in multiuser MIMO net-works," *Journal of Applied Mathematics*, vol. 2013, Article ID 982713, 9 pages, 2013.
11. F. M. Malvestuto, "Decomposable convexities in graphs and hypergraphs," *ISRN Combinatorics*, vol. 2013, Article ID 453808, 9 pages, 2013.
12. L. R. Nielsen and A. R. Kristensen, "Finding the K best policies in a finite-horizon Markov decision process," *European Journal of Operational Research*, vol. 175, no. 2, pp. 1164–1179, 2006.
13. J. Freixas and M. A. Puente, "Dimension of complete simple games with minimum," *European Journal of Operational Research*, vol. 188, no. 2, pp. 555–568, 2008.
14. C. Bentz, D. Cornaz, and B. Ries, "Packing and covering with linear programming: a survey," *European Journal of Operational Research*, vol. 227, no. 3, pp. 409–422, 2013.
15. A. Jimenez-Losada, J. R. Fernandez, M. Ordoñez, and M. Grabisch, "Games on fuzzy communication structures with Choquet players," *European Journal of Operational Research*, vol. 207, no. 2, pp. 836–847, 2010.
16. D. Cheng and H. Qi, *Semi-Tensor Product of Matrices: Theory and Applications*, Science Press, Beijing, China, 2007.
17. D. Cheng, H. Qi, and Z. Li, *Analysis and Control of Boolean Networks: A Semi-Tensor Product Approach*, Communications and Control Engineering Series, Springer, London, UK, 2011.
18. D. Cheng and H. Qi, "Controllability and observability of Boolean control networks," *Automatica*, vol. 45, no. 7, pp. 1659–1667, 2009.
19. Y. Wang, C. Zhang, and Z. Liu, "A matrix approach to graph maximum stable set and coloring problems with application to multi-agent systems," *Automatica*, vol. 48, no. 7, pp. 1227–1236, 2012.
20. Y. N. Sotskov, V. S. Tanaev, and F. Werner, "Scheduling problems and mixed graph colorings," *Optimization*, vol. 51, no. 3, pp. 597–624, 2002.



A SURVEY OF E – SUPER VERTEX MAGIC LABELINGNESS AND V – SUPER VERTEX MAGIC LABELINGNESS

Seema Mehra* and Neelam Kumari**

* & ** Department of Mathematics, M. D. University, Rohtak (Haryana) India
 E-mail : sberwal2007@gmail.com, neelammorgill@gmail.com

ABSTRACT :

In this paper, we studied some basic properties of V – super vertex magic labeling and E – super vertex magic labeling and established E – super vertex magic labeling of some families of graphs. In this survey, we have collected studies on the V – super vertex magic labeling and E – super vertex magic labeling of the graph C_n and the union graph of mutually non- intersecting C_n , the graph mC_n and give some results.

Keywords : Vertex magic labelings, super vertex magic labeling and vertex magic constants.

1. INTRODUCTION

All graphs in this paper are finite, simple and undirected. For more details and graph theoretic notations, see [10, 11].

Graph labeling traces its origin to the famous conjecture that all trees are graceful presented by A. Rosa [3] in 1966. Graph labeling is a mapping that maps the graph elements into an integer set. In recent years, many graph labelings have been evolved, and an excellent survey of graph labeling can be found in Gallian's paper [4]. Sedlacek [12,13] introduced vertex magic labeling and MacDougall et. al [5, 6] introduced the notion of vertex magic total labeling in 1999. If G is a finite, simple and undirected graph with p vertices and q edges, then a vertex magic total labeling is a bijection f from $V(G) \cup E(G)$ to the integers $1, 2, \dots, p+q$ with the property that for every vertex $u \in V(G)$, the sum $f(u) + \sum_{uv \in E(G)} f(uv)$ is a constant. They proved that the following graphs have vertex magic total labeling: C_n, P_n for $n > 2$; K_m, n for $m > 1$; and K_n for odd n . MacDougall et. al [5] further introduced the concept of super vertex magic total labeling. They called a vertex magic total labeling super if $f(V(G)) = \{1, 2, \dots, p\}$ i.e. the smallest labels are assigned to vertices. Swaminathan and Jayanti [7] introduced another super vertex magic total labeling and called it E- super vertex magic total labeling. They called a vertex magic total labeling E-super if $f(E(G)) = \{1, 2, \dots, q\}$ Marimuthu and Balakrishanan[8] studied some basic properties of such vertex labeling and established E- super vertex magic total labeling for some families of graphs.

Definition 1.1 Let $G(p, q)$ be a finite graph. A one – one mapping $f : V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$ is called vertex-magic total (VMT) labeling if all vertex weight are same either, $w(u) = f(u) + \sum_{v \in N(u)} f(uv) = k$, for every $u \in V(G)$ and k is a constant.

Definition 1.2 Let $G(V, E)$ be a graph with a vertex magic total labeling. A vertex total labeling is called V- super vertex magic if $f(V(G)) = \{1, 2, \dots, p\}$. Here G is called a V- super vertex magic graph.

Definition 1.3 Let $G(V, E)$ be a graph with a vertex magic total labeling. A vertex total labeling is called E- super vertex magic if $f(E(G)) = \{1, 2, \dots, q\}$. Here G is called a E- super vertex magic graph.

Definition 1.4 A prism graph, denoted γ_n and sometimes also called a circular ladder graph is a graph corresponding to the skeleton of an n-prism. Prism graphs are therefore both planar and polyhedral. An n-prism graph has $2n$ nodes and $3n$ edges.

Let C_n denotes the cycle on n vertices; P_n path on n vertices and $m C_n$ denotes the graph obtained from any m copies of C_n which have no common vertex.

2. SOME RESULTS

2.1. Some Results on V – Super Vertex Magic Labeling and E – Super Vertex Magic Labeling of graphs

As to the super vertex magic total labeling of some family of graphs, we have the following results.

In 2003, Swaminathan and Jayanti [7] proved the following results:

Theorem 1. If G has a V – super vertex magic total labeling, then

$$k = \frac{(p+q)(p+q+1)}{p} - \frac{p+1}{2}.$$

Theorem 2. If G has a E – super vertex magic total labeling, then

$$\bar{k} = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}.$$

Corollary 3. If G has a super vertex magic total labeling, then $p / q(q+1)$ if p is odd and $p / 2q(q+1)$ if p is even.

Corollary 4. If G has V – super vertex magic total labeling, then

(i). $p \equiv 0 \pmod{8}$ and $q \equiv 0$ or $3 \pmod{4}$, or

(ii). $p \equiv 4 \pmod{8}$ and $q \equiv 1$ or $2 \pmod{4}$.

2.2 Results on E-Super Vertex Magic Labeling of Graphs

As to the E – super vertex magic labeling of some family of graphs, we have following theorems.

The following theorem is useful in finding classes of graphs that are not E – super vertex magic.

Theorem 5.[8] Let G be a graph with p vertices and q edges. If p is even and $q = p - 1$ or p , then G is not E – super vertex magic.

Corollary 6. If p is even, then every tree is not E – super vertex magic.

Theorem 7. A path P_n is E – super vertex magic if and only if n is odd and $n \geq 3$.

Proof. For even n , P_n cannot admits E – super vertex magic by theorem 5.

Let $V(P_n) = \{v_1, v_2, \dots, v_n\}$. Let $f: V \cup E \rightarrow \{1, 2, \dots, p+q\}$ be defined as

$f(v_n) = 2n-1$, $f(v_i) = n + i - 2$, $i = 1, 2, \dots, n-1$. Label the edges $v_i v_{i+1}$ with

$$f(v_i v_{i+1}) = \begin{cases} \frac{n-1}{2} & \text{if } i \text{ is odd,} \\ n - \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

It can be easily checked that f is E – super vertex magic. The magic constant \bar{k} is given by $\frac{5n-3}{2}$.

Corollary 8. A star graph $K_{1, n}$ is E – super vertex magic if and only if $n = 2$.

Proof. When $n = 2$, $K_{1, n}$ is P_3 which is E – super vertex magic by theorem 7.

Theorem 9. A cycle C_n is E – super vertex magic if and only if n is odd.

Proof. We denote the vertices of C_n with v_1, v_2, \dots, v_n . Choose a vertex v_1 . Begin at edge $v_1 v_2$ and label the first edge 1, then label $v_3 v_4, v_5 v_6, \dots, v_n v_1$ with numbers $2, 3, \dots, \frac{n+1}{2}$ and the remaining edges $v_2 v_3, v_4 v_5, \dots, v_{n-1} v_n$ with the remaining consecutive integers $\frac{n+1}{2} + 1, \dots, n$. When all edges have been labeled, then place the integer $(n+1)$ on v_n and move in anti-clockwise direction labeling each vertex with the next consecutive integer. In this labeling the E – super vertex magic constant is $\frac{5n+3}{2}$.

Example 1. E – Super vertex magic labeling for P_5 and C_5 .

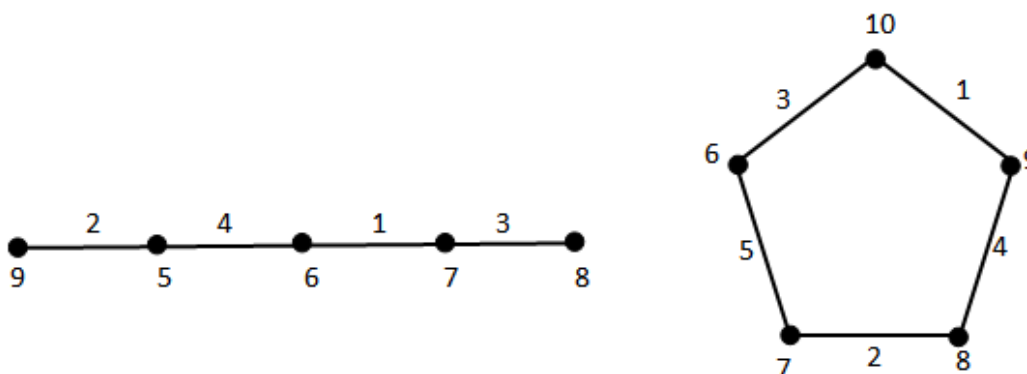


Figure 1 : P_5 and C_5 with magic constants 11 and 14.

Definition 10 (Fan graph). Fan graph F_n is obtained from wheel graph by deleting one edge of cycle C_n . The number of vertices of F_n is $n + 1$ and number of edges is $2n-1$.

Theorem 11. The fan graph F_n has no E – super vertex magic labeling except $n = 2$.

Proof. When $n = 2$, F_n is C_3 which is E – super vertex magic by theorem 9. Now suppose F_n has a E – super vertex magic labeling then by theorem 2

$$\begin{aligned} \bar{k} &= 2n-1 + \frac{(n+2)}{2} + \frac{(2n-1)2n}{2n-1} \\ &= \frac{13n^2+n}{(2n+2)}. \end{aligned}$$

which is not an integer for all n except 2, 3, 11. Suppose, there exist a E – super vertex magic labeling for F_{11} , with magic constant \bar{k} . Then $\bar{k} = 66$, but if the edges incident with the central vertex u receives the minimum labels 1 to 11, then the sum of the edge labels at $u = 66$. So the vertex weight of u will be greater than 66, the magic constant. Thus, F_{11} is not E – super vertex magic.

A (n, t) kite graph consists a cycle C_n with a t -edge path (tail) attached to one vertex of the cycle C_n . Kite graph $K_{n, t}$ consists number of vertices $n+t$ and number of edges is also $n+t$.

Theorem 12. The kite graph $K_{n, t}$ has a E -super vertex magic labelings if and only if $n+t$ is odd.

Proof. Suppose $K_{n, t}$ has E -super vertex magic labeling then

To find E -super vertex magic constant,

$$\bar{k} = q + \frac{(p+1)}{2} + \frac{q(q+1)}{p}$$

$$\bar{k} = (n+t) + \frac{(n+t+1)}{2} + \frac{(n+t)(n+t+1)}{n+t}$$

$$= \frac{5(n+t)+3}{2}$$

This is an integer if and only if $n+t$ is odd.

2.3 Results on V-Super Vertex Magic Labeling of Graphs

As to the V – super vertex magic labeling of some family of graphs, we have following theorems.

In 2004, MacDougall, Miller [5, 6] gave the following results and presented a conjecture as follows :

Conjecture 13. A sufficient condition of the graph K_n is V -super vertex magic is $n \equiv 0 \pmod{4}; n > 4$.

In 2007, Gomez proved the above conjecture.

Theorem 14. If graph G has a pendant vertex, then G is not V – super vertex magic.

Corollary 15. No tree, path P_n and kite $K_{n, t}$ admit V – super vertex magic labeling.

Theorem 16. A cycle C_n has a V – super vertex magic if and only if n is odd.

Proof. We denote the vertices of C_n with v_1, v_2, \dots, v_n . Choose a vertex v_1 . Begin at vertex v_1 and label the first vertex 1, then label v_2, v_3, \dots, v_n with consecutive numbers 2, 3, ..., n . When all the vertices has been labeled then label the edges $v_i v_{i+1}$ with

$$f(v_i v_{i+1}) = \begin{cases} 2n - \frac{i-1}{2} & \text{if } i \text{ is odd,} \\ 2n - \frac{n-1}{2} - \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$$

This labeling f clearly has the V – super vertex magic labeling. By theorem 1 magic constant is

$$k = \frac{(p+q)(p+q+1)}{p} - \frac{p+1}{2}$$

$$= \frac{2n(2n+1)}{n} - \frac{(n+1)}{2}$$

$$= \frac{(7n+3)}{2}$$

$k = \frac{7n+3}{2}$ is not an integer for even n . Therefore C_n has no V -super vertex magic labeling for even n . So cycle C_n has a V – super vertex magic if and only if n is odd.

In 2002, MacDougall, Miller and Wallis [9] proved that wheel graph W_n for $n > 11$, fan graph F_n for $n > 10$, complete bipartite graph $K_{m, n}$ for $m \neq n, m, n \geq 2$ has no vertex magic total labeling. They proved that only complete bipartite graph $K_{m, m}$ and $K_{m, m+1}$ for $m > 1$ admits vertex magic total labeling. Thus, these are the only complete bipartite graph that could admit a V – super vertex magic labeling. Let if possible then by theorem 2

$$k = \frac{(p+q)(p+q+1)}{p} - \frac{p+1}{2}$$

$$k = \frac{(2m+m^2)(2m+m^2+1)}{2m} - \frac{2m+1}{2}.$$

This is not an integer either m is odd or even. Thus, we have the following theorem.

Theorem 17. No complete bipartite graph is V- super vertex magic.

2.4. Some Results on Graphs having both V – Super Vertex Magic Labeling and E – Super Vertex Magic Labeling

As to the V – Super vertex magic total labeling and E – Super vertex magic total labeling of regular graphs, we have the following results.

In 2002, MacDougall, Miller [6] give results as follows,

Theorem 18. If a r- regular graph G of order p has a V – Super vertex magic total labeling then p and r have following properties and

- (i). $p \equiv 0(\text{mod } 8)$ and $q \equiv 0 (\text{mod } 4)$.
- (ii). $p \equiv 4(\text{mod } 8)$ and $q \equiv 2 (\text{mod } 4)$.

The only regular graphs with $r = 2$ are cycles or disjoint union of cycles. For cycles, Swaminathan and Jayanti gave the complete answer.

Theorem 18. The cycle C_n admits E-super vertex magic total labeling and V- super vertex magic total labeling if and only if n is odd.

Theorem 19. $m C_n$ admits E-super vertex magic total labeling and V- super vertex magic total labeling if and only if m and n both are odd.

Theorem 20. The prism graph γ_n has a super vertex magic total labeling.

References :

1. P. Jeyanthi, Studies in Graph Theory – Magic Labeling and Related Concepts, Lambert Academic Publishing, USA (2012).
2. G. Chartrand, L. Lesniak, Graph & Diagraph, Chapman & Hall/CRC, 1996.
3. A. Rosa, On certain valuation of the vertices of a graph, In: Theory of Graphs, Proc. Internat. Symposium, Rome (July 1966), 349-355.
4. J. A. Gallian, A dynamic survey of graph labeling, Electron. J. Combin. 16(2009) #DS6.
5. J.A. MacDougall, M. Miller, Slamir and W.D.Wallis, Vertex magic total labeling of graphs, utilitas Math., 61(2002), 3-21.
6. J.A. MacDougall, M. Miller, K.A. Sugeng, Super vertex magic total labeling of graphs, In: Proc. Of the 15th Australian Workshop on Combinatorial Algorithm 2004, pp. 222-229.
7. V.Swaminathan and P.Jayanti, Super vertex magic total labeling, Indian J. Pure Appl. Math. 34(6) (2003) 935-939.

8. M. Balakrishnan and G. Marimuthu, E – Super vertex magic labeling of graphs, Discrete Appl. Math. 160 (2012), 1766 – 1774.
9. M. MacDougall, M. Miller and W.D. Wallis, Vertex magic total labeling of wheel and related graphs, Utilitas Math. 62(2002), 175 -183.
10. Harary F., Graph theory, Addison Wesley, Reading MA, (1969).
11. Wallis W.D., Magic Graphs, Birkhauser, Boston – Basek – Berlin, 2001.
12. Sedlacek J, On magic graphs, Math , Slovaca, 26 (1976) 329-335.
13. Sedlacek J. Problem 27, In: Theory and Its Application, Proc. Symposium. Smolenice, June (1963), 163-169.



q –HYPERCONVEXITY IN T_0 –ULTRA–QUASI–METRIC SPACES AND EXISTENCE OF FIXED POINT THEOREMS

Qazi Aftab Kabir*, Ramakant Bhardwaj, Rizwana Jamal*** & Masroor Mohammad******

* & **** Department of Mathematics, Saifia Science College Bhopal, Madhya Pradesh, India.

** Professor, Department of Mathematics, Technocrats Institutes of Technology, group of institutes Bhopal, (M.P) India.

*** Professor and Head, Department of Mathematics, Saifia Science College Bhopal, (M.P) India.

Email : quziaftabkabir@gmail.com

ABSTRACT :

In this article the existence of fixed point theorems for generalized nonexpansive mappings in q -hyperconvex T_0 -ultra -quasi-metric spaces has been proved.

Keywords : fixed points, q -hyperconvexity, T_0 -quasi-metric, T_0 -ultra-quasi-metric space.

1. INTRODUCTION

A metric space (Z, d) is said to be hyperconvex if each gathering of closed balls $\{B(z_\alpha, r_\alpha)\}_{\alpha \in A} \leq r_\alpha + r_\beta$ has non-empty intersection $\bigcap_\alpha B(z_\alpha, r_\alpha) \neq \emptyset$. In 1956 Aronszajn and Panitchpakdi [1], presented the thought of hyperconvex spaces, who demonstrated that they are the same as injective metric spaces. Q –hyperconvex-quasi-metric space is another currentline of research in hyperconvexity. This concept was presented by Kunzi and Otafudu in [10]. We first outline a portion of the meanings of the hyperconvex metric spaces which were examined by many authors see ([4], [5], [6][8], [9], [14],[16] and [17]). We also begin an examination of the properties of q –hyperconvex-quasi-metric spaces.

Further work about q -hyperconvexity can be found in [7], [11], [12]. Recently, Kunzi and Otafudu [10] presented and examined the idea of q -hyperconvexity in T_0 –ultra-quasi-metric spaces and contracting maps with certain fixed point theorems for nonexpansive maps on q -hyperconvex quasi-metric space. In this article we study this concept by generalizing and by showing that an non-expansive mappings in a q -hyperconvex T_0 –ultra-quasi-metric space has a fixed points.

2. DEFINITIONS AND PRELIMINARIES

This section recalls some elementary definitions and example from the asymmetric topology which are necessary for a good understanding of the work below.

Definition 2.1. Let Z be a set and $d: Z \rightarrow Z \rightarrow [0, \infty)$ be a function mapping into the set $[0, \infty)$ of non-negative reals. Then d is an ultra-quasi-pseudometric on Z if

(a) $d(z, z) = 0$ for all $z \in Z$, and

(b) $d(z, w) \leq \max\{d(z, u), d(u, z)\}$ whenever $z, u, w \in Z$.

We shall say that d also satisfies the following condition (known as the T_0 –condition)

(c) for any $z, u \in Z, d(z, u) = 0 = d(u, z)$ implies that $z = u$, then d is called a T_0 –ultra –quasi-metric space on Z .

Definition 2.2. Let (Z, d) be an T_0 –ultra-quasi-metric space and let $\mathcal{FP}(Z, d)$ be the set of all pairs $f = (f_1, f_2)$ of functions where $f_i: Z \rightarrow [0, \infty)$ ($i = 1, 2$) for any such pairs (f_1, f_2) and (g_1, g_2) we set

$$N((f_1, f_2), (g_1, g_2)) = \max\{\sup_{z \in Z} n(f_1(z), g_1(z)), \sup_{z \in Z} n(g_2(z), f_2(z))\}.$$

It is obvious that N is an extended T_0 –ultra-quasi-metric on the set $\mathcal{FP}(Z, d)$ of these function pairs.

Definition 2.3. A map $f: Z \rightarrow Z$ where (Z, d) is an ultra-quasi-pseudometric space is called nonexpansive if

$$d(f(z), f(u)) \leq d(z, u),$$

Whenever $z, u \in Z$.

Definition 2.4. A map $f: Z \rightarrow Z$ where (Z, d) is an ultra-quasi-pseudometric space is called generalized nonexpansive if for each $z, u \in Z$ with $d(z, u) > 0$, we have that

$$d(f(z), f(u)) \leq \max\{d(z, u), d(f(z), z), d(u, f(u))\}.$$

3. Q-SPHERICAL COMPLETENESS OF HYPERCONVEX

In this section we shall recall some results about q-spherical completeness of hyperconvex belonging mainly to [12]

Let (Z, d) be an q-hyperconvex ultra-quasi-pseudometric space and for each $z \in Z$ and $r, \in [0, \infty)$, whenever $i, j \in I$. Let

$$\bigcap_{i, j \in I} (C_d(z_i, r) = \{u_j \in Z : d(z_i, u_j) \leq r\})$$

Be the $\tau(d^{-1})$ –closed ball of radius r at z .

Lemma 3.1. Let (Z, d) be an q-hyperconvex ultra-quasi-pseudometric space.

Whenever $i, j \in I$, moreover let $z_i, z_j \in Z$ and $r_i, s_j \geq 0$.

Then

$$\bigcap_{i, j \in I} (C_d(z_i, r_i) \cap C_{d^{-1}}(z_j, s_j)) \neq \emptyset.$$

If and only if $d(z_i, z_j) \leq \max\{r_i, s_j\}$

Definition 3.2. A quasi-pseudometric space (Z, d) is called q –hyperconvex provided that for each family $(z_i)_{i \in I}$ of points in Z and families of nonnegative real numbers $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ the following condition holds: if

$$d(z_i, z_j) \leq r_i + s_j$$

Whenever $i, j \in I$, then

$$\bigcap_{i \in I} (C_d(z_i, r_i) \cap C_{d^{-1}}(z_i, s_i)) \neq \emptyset.$$

Definition 3.3. Let (Z, d) be an ultra –quasi-pseudometric space. Let $(z_i)_{i \in I}$ be a family of points in Z and let $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be families of non-negative real numbers. We shall say that the family $(C_d(z_i, r_i), C_{d^{-1}}(z_i, s_i))_{i \in I}$ has a mixed binary intersection property provided that

$$d(z_i, z_j) \leq \max\{r_i, s_j\}$$

Whenever $i, j \in I$.

We say that (Z, d) is q –hyperconvexity provided that each family $(C_d(z_i, r_i), C_{d^{-1}}(z_i, s_i))_{i \in I}$ possessing the mixed binary intersection property also satisfies $\bigcap_{i \in I} (C_d(z_i, r_i) \cap C_{d^{-1}}(z_i, s_i)) \neq \emptyset$.

Remark 3.4. if d and d^{-1} are identical and $r_i = s_i$ for $i \in I$, then $(C_d(z_i, r_i))$ and $(C_{d^{-1}}(z_i, s_i))$ coincide and then we recover the well-known definition of hyper convexity due to Aronszajn and Panitchpakdi [1].

Example 3.5. Let the set \mathbb{R} of the reals be equipped with the T_0 –quasi-metric $x(z, u) = \max\{z - u, 0\}$ whenever $z, u \in \mathbb{R}$. Then (\mathbb{R}, x) is q –hyperconvex.

Example 3.6.(see [13])The q -hyperconvex ultra-quasi-metric space $([0, \infty), n)$ is q -spherically complete.

4. MAIN RESULTS

Theorem 4.1. Let (Z, d) be a q –hyperconvex T_0 -ultra-quasi-metric space. If $f: Z \rightarrow Z$ is a nonexpansive mappings, then f has a fixed point.

Proof. Let $r \in Z$ and denote by $C_r^d = C_d(r, d(f(r), r))$ and $C_r^{d^{-1}} = C_{d^{-1}}(r, d(r, f(r)))$

The closed balls with centers at $r \in Z$ and radius $d(f(r), r)$ and $d(r, f(r))$ respectively such that $d(r, f(r)) = d(f(r), r)$. Put

$$C_r = C_r^d \cap C_r^{d^{-1}}$$

Let \mathcal{A} be the collection of all such closed balls C_r such that r runs over Z .

Define \preceq on \mathcal{A} by

$C_r \preceq C_s$ if and only if $C_s \subseteq C_r$.

It can be proved easily that (\mathcal{A}, \preceq) is a partially ordered set. Let \mathcal{A}_1 be a nonempty chain in \mathcal{A} . Then by q –hyperconvexity of (Z, d) , we have that

$$\bigcap_{C_r \in \mathcal{A}_1} C_r = C \neq \emptyset.$$

Let $s \in C$ and $C_r \in \mathcal{A}_1$. Then we have

$$d(r, s) \leq d(f(r), r) \text{ and } d(s, r) \leq d(r, f(r)).$$

Let now $z \in C_s$. Then

$$d(s, z) \leq d(f(s), s)$$

And

$$d(z, s) \leq d(s, f(s)).$$

$$d(s, z) \leq d(f(s), s)$$

$$\leq \max\{d(f(s), f(r)), d(f(r), r), d(r, s)\}$$

$$= \max\{d(f(s), f(r)), d(f(r), r)\}$$

If $d(f(s), f(r)) \leq d(f(r), r)$, then we have

$$d(s, z) \leq d(f(r), r).$$

If on the other hand we have

$$d(f(s), f(r)) > d(f(r), r),$$

Then

$$d(s, z) < \max\{d(f(s), s), d(r, f(r))\} = d(r, f(r))$$

Thus in the both case, we have

$$d(s, z) \leq d(f(r), r).$$

From the above inequality, we have now that

$$\begin{aligned} d(r, z) &\leq \max\{d(r, s), d(s, z)\} \\ &\leq \max\{d(f(r), r), d(f(r), r)\} \\ &= d(f(r), r) \end{aligned}$$

Which means that $z \in C_d(r, d(f(r), r))$. We have thus shown that

$$C_d(s, d(f(s), s)) \subseteq C_d(r, d(f(r), r)). \tag{4.1.1}$$

By a similar computation, one can show that

$$C_{d^{-1}}(s, d(s, f(s))) \subseteq C_{d^{-1}}(r, d(r, f(r))). \tag{4.1.2}$$

equations (4.1.1) and (4.1.2) imply that for all $C_r \in \mathcal{A}_1, C_s \subseteq C_r$. In other words. We say that $C_r \preceq C_s$ for all $C_r \in \mathcal{A}_1$. Thus C_s is an upper bond in \mathcal{A} for the chain \mathcal{A}_1 . We therefore conclude by zorn’s lemma that \mathcal{A} has a maximal element, say $C_x, x \in Z$. We shall prove that $f(x) = x$. We do this by contradiction.

Suppose on the contrary that $d(x, f(x)) > 0$.

Let $b \in C_{f(x)}$, then

$$d(f(x), b) \leq d(f(f(x)), f(x)) < d(x, f(x))$$

And

$$\begin{aligned} d(b, f(x)) &\leq d(f(x), f(f(x))) < d(f(x), x). \\ d(b, x) &\leq \max\{d(b, f(x)), d(f(x), x)\} \\ &< \max\{d(x, f(x)), d(f(x), x)\} \\ &= d(x, f(x)) \end{aligned}$$

Similarly, we can prove that $d(x, b) \leq d(f(x), x)$.

The last two inequalities imply that $b \in C_x$. Therefore $C_{f(x)} \subseteq C_x$. Indeed, we have that $x \notin C_{f(x)}$. This follows from the following two inequalities:

$$\begin{aligned} d(f(x), f(f(x))) &< \max\{d(f(x), x), d(x, f(x)), d(f(x), f(f(x)))\} \\ &= d(f(x), x) \end{aligned}$$

And

$$\begin{aligned} d(f(f(x)), f(x)) &< \max\{d(f(f(x)), f(x)), d(f(x), x), d(x, f(x))\} \\ &= d(x, f(x)) \end{aligned}$$

This however contradicts the maximality of C_x . Hence we must have that $f(x) = x$.

Hence proved.

Theorem 4.2. Let (Z, d) be a q –hyperconvex T_0 -ultra-quasi-pseudometric space and $F: Z \rightarrow Z$ is a generalized nonexpansive mappings. Then either F has atleast one fixed point or there exists a closed ball B radius r such that $F: B \rightarrow B$. Moreover, $d(r, Fr) = d(Fr, r) = r$ for each $r \in B$.

Proof. Let $r \in Z$. Let us denote by

$$C_d^r = C_d(r, d(Fr, r)) \text{ and } C_{d^{-1}}^r = C_{d^{-1}}(r, d(r, Fr)),$$

With $d(Fr, r) = d(r, Fr)$. Set

$$C^r = C_d^r \cap C_{d^{-1}}^r$$

And $\mathcal{A} := \{C^r, r \in Z\}$. Define the relation $C^r \preceq C^s$ on \mathcal{A} by

$$C^r \preceq C^s \text{ if and only if } C^s \subseteq C^r.$$

Then (\mathcal{A}, \preceq) is a partially ordered set. With their relation and the zorns lemma, that \mathcal{A} has a maximal element C^w .

Consider now such maximal element C^w . For any $s \in C^w$, we have

$$d(s, Fs) \leq \max\{d(s, w), d(w, Fw), d(Fw, Fs)\} = d(w, Fw),$$

And

$$d(Fs, s) \leq \max\{d(Fs, w), d(w, Fw), d(Fw, s)\} = d(Fw, w).$$

Therefore, we conclude that for any $u \in C^s$, $d(w, u) \leq d(w, Fw)$ and $d(u, w) \leq d(Fw, w)$, which entails that $C^s \leq C^w$ and then $Fs \in C^w$.

Now if we assume that

$$d(s, Fs) < d(w, Fw)$$

Then

$$d(s, w) = d(w, Fw) > d(s, Fs)$$

This implies that $w \in C_d^w$ but $w \notin C_d^s$ which is impossible from the maximality of C^w . Thus

$$d(s, Fs) = d(w, Fw) = r \quad \text{for any } s \in C^w$$

Similarly, if we assume that $d(Fs, s) < d(Fw, w)$ then

$$d(w, s) = d(Fw, w) > d(Fs, s)$$

This implies that $w \in C_{d-1}^w$ but $w \notin C_{d-1}^s$ which is impossible from the maximality of C^w . Thus

$$d(Fs, s) = d(Fw, w) \text{ for any } s \in C^w$$

Hence

$$d(s, Fs) = d(w, Fw) = d(Fw, w) = d(Fs, s) = r$$

For any $s \in C^w$.

CONCLUSION:

In this paper, we proved some new results which extend the uniformity and generalization of several results related to fixed point theorems spherically completeness of q-hyperconvexity in T_0 -quasi-metric space.

REFERENCES :

1. Aronszajn, N., and Panitchpakdi, P., *Extension of uniformly continuous transformations and hyperconvex metric spaces*, *Pacific J. Math.* 6(1956), 405– 439.
2. Bonsangue, M. M., Breugel, F. Van and Rutten, J. J. M. M., *Generalized Ultrametric Spaces, completion, topology, and power domains via the Yoneda embedding*, *CWI reports/AP/CS-R9560*, (1995).
3. Espinola, R. & Khamsi, M. A., *Introduction to hyperconvex spaces*, in: *Handbook of metric fixed point theory*, Kluwer, Dordrecht, (2001), 391–435.
4. Goebel, K. and Kirk, W. A., *A fixed point theorem for asymptotically nonexpansive mappings*, *Proc. Amer. Math. Soc.* 35(1) (1972), 171-174.
5. Isbell, J. R., *Six theorems about injective metric spaces*, *Comment. Math. Helvetic.* 39(1964), 65–76.
6. Kabir, Q. A., Mohammad, M., Jamal, R and Bhardwaj, R., *Unique fixed points and mappings in hyperconvex metric spaces*, *International journal of Mathematics and its Applications.* 5(3) (2017), 171-177.
7. Kemajou, E., Kunzi, H. P and Otafudu, O. O., *The Isbell-hull of a di-space*, *Topology Appl.* 159 (2012), 2463–2475.
8. Khamsi, M. A., Knaust, H., Nguyen, N. T. and O'Neill, M. D., *Lambda-hyperconvexity in metric spaces*, *Nonlinear Anal.* 43(2000), 21-31.

9. Khamsi, M. A., & Kirk, W. A., *An Introduction to Metric Spaces and Fixed Point Theory*, John Wiley, New York, (2001).
10. Khamsi, M. A., *On asymptotically nonexpansive mappings in hyperconvex metric spaces*, *Proc. Amer. Math. Soc.* 132(2004), 365-373.
11. Kunzi, H. P., & Otafudu, O. O., *q-Hyperconvexity in quasi-pseudo-metric spaces and fixed point theorems*, *J. Function spaces and Applications*, (2012), 765903.
12. Kunzi, H. P., & Otafudu, O. O., *The ultra-quasi-metrically injective hull of a T_0 -ultra-quasi-metric space*, *Appl. Categor. Struct.* 21(2013), 651-670.
13. Mishra, S.N. & Otafudu, O.O., *On asymptotically nonexpansive mappings in q-hyperconvex T_0 -quasi-metric spaces*, *Advances in fixed point theory.* 3(2013), 135-140.
14. Salbany, S., "Injective objects and morphisms," in *Topology and Its Relation to Analysis, Algebra and Combinatorics*, World Scientific, Teaneck, NJ, USA, pp. (1989), 394–409.
15. Sine, R. C., *On linear contraction semigroups in sup norm spaces*, *Nonlinear Anal.* 3(1979), 885-890.
16. Sine, R. C., *Hyperconvexity and approximate fixed points*, *Nonlinear Anal.* 13(1989), 863–869.
17. Sine, R. C., *Hyperconvexity and nonexpansive multifunctions*. *Trans. Am. Math. Soc.* 315 (1989), 755–767.
18. Soardi, P., *Existence of fixed points of nonexpansive mappings in certain Banach lattices*, *Proc. Amer. Math. Soc.* 73(1979), 25–29.



COMPUTATION OF AUGMENTED ZAGREB INDEX AND THEIR POLYNOMIAL OF CERTAIN CLASS OF WINDMILL GRAPHS

B. Chaluvvaraju and S. A. Diwakar

Department of Mathematics, Bangalore University, Jnana Bharathi Campus, Bangalore-560 056, Karnataka, INDIA
 E-mail: {bchaluvvaraju/dsatumkur}@ gmail.com

ABSTRACT :

The augmented Zagreb index of a graph $G=(V, E)$ is defined by $AZI(G) = \sum_{uv \in E(G)} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3$. In this paper, we compute the augmented Zagreb index and their polynomials of certain classes of windmill graphs like French windmill graph, Dutch windmill graph, Kulli cycle windmill graph and Kulli path windmill graph.

2010 Mathematics Subject Classification: 05C05, 05C07, 05C35.

Key words and phrases: Augmented-Zagreb indices; French windmill graph; Dutch windmill graph; Kulli cycle windmill graph and Kulli path wind mill graph.

1. INTRODUCTION

Throughout this paper, we considered only simple graphs and, the collection of vertices and the set of edges are denoted by $V(G)$ and $E(G)$, respectively. The number of vertices of a graph adjacent to v is the degree of a vertex v of G , denoted by $d_G(v)$. The reader may refer [9], for undefined notations and terminologies.

Chemical graph theory is a branch of Mathematical chemistry which has an important effect on the development of the chemical sciences. A single number that can be used to characterize some property of the graph of a molecular is called a topological index for that graph. There are numerous topological descriptors that have found some applications in theoretical chemistry, especially in QSPR/QSAR research.

The first two Zagreb indices was introduced by Gutman and Trinajstic [8] to take account of the contributions of pairs of adjacent vertices. In [3], Furtula et al., introduced augmented Zagreb index (AZI), defined as $AZI(G) = \sum_{uv \in E(G)} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3$ and is a degree based topological invariant to a well established for its better correlation properties and we refer [5]. Recently many other indices were studied, for example, in [1], [2], [3], [7], [12], [13].

In the following, we make use of some necessary calculations for computing the Zagreb indices and their polynomials of G , we make use of the vertex set partition $V_a = \{v \in V: d_G(v) = a\}$ and edge set partitions $E_b = \{e = uv \in E: d_G(u) + d_G(v) = b\}$ and $E_c^* = \{e = uv \in E: d_G(u)d_G(v) = c\}$.

2. FRENCH WINDMILL GRAPH

The French windmill graph $F_n^{(m)}$ is the graph by taking $m \geq 2$ copies of the complete graph K_n ; $n \geq 2$ with a vertex in common. This graph is shown in Figure-1. The French windmill graph $F_2^{(m)}$ is called a star graph, the

French windmill graph $F_3^{(m)}$ is called a friendship graph and the French windmill graph $F_3^{(2)}$ is called a butterfly graph. Further note that $F_3^{(m)}$ is same as $D_3^{(m)}$. For more details, we refer [6].

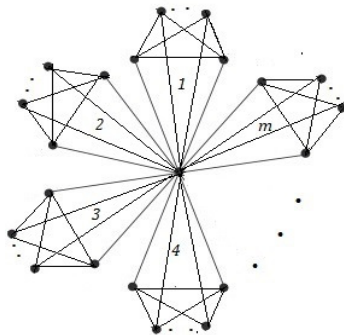


Figure 1. French windmill graph $F_n^{(m)}$.

Let $G = F_n^{(m)}$, where $F_n^{(m)}$ is a French windmill graph. By algebraic method, we get $|V(G)| = m(n - 1) + 1$ and $|E(G)| = \frac{mn(n-1)}{2}$.

We have two partitions of the vertex set $V(G)$ as follows:

$$V_{n-1} = \{v \in V(G) : d_G(v) = n - 1\}; |V_{n-1}| = m(n - 1), \text{ and}$$

$$V_{(n-1)m} = \{v \in V(G) : d_G(v) = (n - 1)m\}; |V_{(n-1)m}| = 1.$$

Also we have two partitions of the edge set $E(G)$ as follows:

$$E_{2(n-1)} = E_{(n-1)^2}^* = \{uv \in E(G) : d_G(u) = d_G(v) = n - 1\};$$

$$|E_{2(n-1)}| = |E_{(n-1)^2}^*| = m \left[\frac{n(n-1)}{2} - (n - 1) \right] = \frac{m(n^2 - 3n + 2)}{2}, \text{ and}$$

$$E_{(n-1)(m+1)} = E_{m(n-1)^2}^* = \{uv \in E(G) : d_G(u) = n - 1, d_G(v) = (n - 1)m\} \quad ; \quad |E_{(n-1)(m+1)}| = |E_{m(n-1)^2}^*| = (n - 1)m.$$

Theorem 2. 1. An augmented Zagreb index and their polynomial of French windmill graph are

$$AGI \left(F_n^{(m)} \right) = m(n - 1)^7 \left[\frac{1}{16(n-2)^2} + \frac{m^3}{[(n-1)(m+1)-2]^3} \right] \text{ and}$$

$$AGI \left(F_n^{(m)}, x \right) = \frac{m(n-1)}{2} \left[(n - 2)x^{\left[\frac{(n-1)^2}{2(n-2)} \right]^3} + 2x^{\left[\frac{m(n-1)^2}{(n-1)(m+1)-2} \right]^3} \right].$$

Proof. Let $G = F_n^{(m)}$ be a French windmill graph. Consider

$$AGI \left(F_n^{(m)} \right) = \sum_{uv \in E(G)} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3$$

$$= \sum_{E_{2(n-1)}} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 + \sum_{(n-1)(m+1)} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3$$

$$= \frac{1}{2} m(n^2 - 3n + 2) \left[\frac{(n-1)(n-1)}{(n-1)+(n-1)-2} \right]^3 + m(n - 1) \left[\frac{(n-1)(n-1)m}{(n-1)+(n-1)m-2} \right]^3$$

$$= \frac{1}{2} m(n - 1)(n - 2) \left[\frac{(n-1)^6}{8(n-2)^3} \right] + m(n - 1) \left[\frac{(n-1)^6 m^3}{[(n-1)(m+1)-2]^3} \right]$$

$$= \frac{m(n-1)^7(n-2)}{16(n-2)^3} + \frac{m^3(n-1)^7}{[(n-1)(m+1)-2]^3}$$

$$= m(n - 1)^7 \left[\frac{1}{16(n-2)^2} + \frac{m^3}{[(n-1)(m+1)-2]^3} \right].$$

Now, for augmented-Zagreb polynomial of $F_n^{(m)}$, we have

$$AGI \left(F_n^{(m)}, x \right) = \sum_{uv \in E} x^{\left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3}$$

$$\begin{aligned}
 &= \sum_{E_{2(n-1)}} x^{\left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2}\right]^3} + \sum_{E_{(n-1)(m+1)}} x^{\left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2}\right]^3} \\
 &= \frac{1}{2}m(n^2 - 3n + 2)x^{\left[\frac{(n-1)^2}{2(n-2)}\right]^3} + m(n-1)x^{\left[\frac{m(n-1)^2}{(n-1)(m+1)-2}\right]^3} \\
 &= \frac{1}{2}m(n-1) \left[(n-2)x^{\left[\frac{(n-1)^2}{2(n-2)}\right]^3} + 2x^{\left[\frac{m(n-1)^2}{(n-1)(m+1)-2}\right]^3} \right].
 \end{aligned}$$

Corollary 2.1. An augmented-Zagreb index and their polynomial of a friendship graph $F_3^{(m)}$ are $AZI(F_3^{(m)}) = 24m$ and $AZI(F_3^{(m)}, x) = 3mx^8$.

Corollary 2.2. An augmented-Zagreb index and their polynomial of a butterfly graph $F_3^{(2)}$ are $AZI(F_3^{(2)}) = 48$ and $AZI(F_3^{(2)}, x) = 6x^8$.

3. DUTCH WINDMILL GRAPH

The Dutch windmill graph $D_n^{(m)}$ is the graph obtained by taking m copies of the cycle C_n with a vertex in common. This graph is shown in Figure-2. The Dutch windmill graph $D_n^{(m)} = F_n^{(m)}$ is called a friendship graph. For more details on windmill graph, see [6].

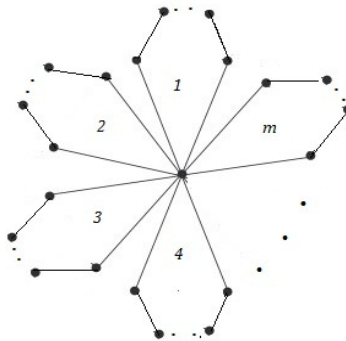


Figure 2. Dutch windmill graph $D_n^{(m)}$.

Let $G = D_n^{(m)}$, where $D_n^{(m)}$ is a dutch windmill graph. By algebraic method, we get $|V(G)| = m(n-1) + 1$ and $|E(G)| = mn$.

We have two partitions of the vertex set $V(G)$ as follows:

$$V_2 = \{v \in E(G) : d_G(v) = 2\}; |V_2| = (n-1)m, \text{ and}$$

$$V_{2m} = \{v \in V(G) : d_G(v) = 2m\}; |V_{2m}| = 1.$$

Also we have two partitions of the edge set $E(G)$ as follows:

$$E_4 = E_4^* = \{uv \in E(G) : d_G(u) = d_G(v) = 2\}; |E_4| = |E_4^*| = (n-2)m, \text{ and}$$

$$E_{2m+2} = E_{2(2m)}^* = \{uv \in E(G) : d_G(u) = 2, d_G(v) = 2m\}; |E_{2m+2}| = |E_{2(2m)}^*| = 2m.$$

Theorem 3.1. An augmented-Zagreb index and their polynomial of dutch windmill graph are

$$AZI(D_n^{(m)}) = 8m[2m + n - 2] \text{ and } AZI(D_n^{(m)}, x) = mn x^8.$$

Proof. Let $G = D_n^{(m)}$ be a dutch windmill graph. Consider

$$AZI(D_n^{(m)}) = \sum_{uv \in E} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3$$

$$\begin{aligned}
 &= \sum_{E_4} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 + \sum_{E_{2m+2}} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 \\
 &= (n-2)m \left[\frac{2 \times 2}{2+2-2} \right]^3 + 2m \left[\frac{2 \times 2m}{2+2m-2} \right]^3 \\
 &= 8m(n-2) + 16m \\
 &= 8m[2m + n - 2]
 \end{aligned}$$

Now, for an augmented-Zagreb polynomial of a $D_n^{(m)}$, we have

$$\begin{aligned}
 AZI \left(D_n^{(m)}, x \right) &= \sum_{uv \in G} x^{\left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3} \\
 &= \sum_{E_4} x^{\left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3} + \sum_{E_{2m+2}} x^{\left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3} \\
 &= (n-2)m x^8 + 2m x^8 \\
 &= mn x^8
 \end{aligned}$$

4. KULLI CYCLE WINDMILL GRAPH

The Kulli cycle windmill graph $C_{n+1}^{(m)}$ is the graph obtained by taking m copies of the graph $K_1 + C_n$ for $n \geq 3$ with a vertex K_1 in common. This graph is shown in Figure-3. This type of windmill graph is initiated by Kulli et al., in [10].

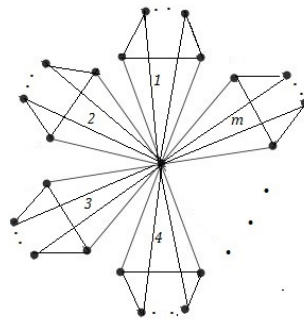


Figure 3. Kulli cycle windmill graph $C_{n+1}^{(m)}$.

Let $G = C_{n+1}^{(m)}$, where $C_{n+1}^{(m)}$ is a Kulli cycle windmill graph. By algebraic method, we get $|V(G)| = mn + 1$ and $|E(G)| = 2mn$.

We have two partitions of the vertex set $V(G)$ as follows:

$$V_3 = \{v \in V(G) : d_G(v) = 3\}; |V_3| = mn, \text{ and}$$

$$V_{mn} = \{v \in V(G) : d_G(v) = mn\}; |V_{mn}| = 1.$$

Also we have two partitions of the edge set $E(G)$ as follows:

$$E_6 = \{uv \in E(G) : d_G(u) = d_G(v) = 3\}; |E_6| = mn, \text{ and}$$

$$E_{mn+3} = \{uv \in E(G) : d_G(u) = mn, d_G(v) = 3\}; |E_{mn+3}| = mn.$$

Theorem 4.1. An augmented-Zagreb index and their polynomial of a Kulli cycle windmill graph are

$$AZI \left(C_{n+1}^{(m)} \right) = 27mn \left[\frac{27}{64} + \frac{(mn)^3}{(mn+1)^3} \right] \text{ and}$$

$$AZI \left(C_{n+1}^{(m)}, x \right) = mn x^{\frac{729}{64}} + mn x^{\frac{27mn^3}{(mn+1)^3}}.$$

Proof. Let $G = C_{n+1}^{(m)}$, where $C_{n+1}^{(m)}$ is a Kulli cycle windmill graph. Consider

$$AZI \left(C_{n+1}^{(m)} \right) = \sum_{uv \in E} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3$$

$$\begin{aligned}
 &= \sum_{E_6} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 + \sum_{E_{mn+3}} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 \\
 &= mn \left[\frac{3 \times 3}{3+3-2} \right]^3 + mn \left[\frac{3mn}{mn+3-2} \right]^3 \\
 &= mn \left[\frac{9}{4} \right]^3 + mn \left[\frac{3mn}{mn+1} \right]^3 \\
 &= 27mn \left[\frac{27}{64} + \frac{(mn)^3}{(mn+1)^3} \right]
 \end{aligned}$$

Now, for an augmented-Zagreb polynomial of a $C_{n+1}^{(m)}$, we have

$$\begin{aligned}
 AZI \left(C_{n+1}^{(m)}, x \right) &= \sum_{E_6} x \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 + \sum_{E_{mn+3}} x \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 \\
 &= mn x^{\frac{729}{64}} + mn x^{\frac{27(mn)^3}{(mn+1)^3}}
 \end{aligned}$$

5. KULLI PATH WINDMILL GRAPH

The Kulli path windmill graph $P_{n+1}^{(m)}$ is the graph obtained by taking m copies of the graph $K_1 + P_n$ for $n \geq 2$ with a vertex K_1 in common. This graph is shown in Figure-4. This type of windmill graph is initiated by Kulli etal., in [11].

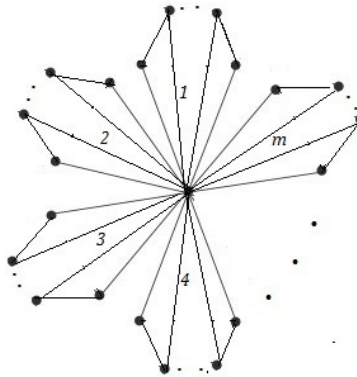


Figure 4. Kulli path windmill graph $P_{n+1}^{(m)}$.

Let $G = P_{n+1}^{(m)}$, where $P_{n+1}^{(m)}$ is a Kulli path windmill graph with $m \geq 2$ and $n \geq 4$. By algebraic method, we get $|V(G)| = mn + 1$ and $|E(G)| = 2mn - m$.

We have three partitions of the vertex set $V(G)$ as follows:

$$\begin{aligned}
 V_2 &= \{v \in V(G): d_G(v) = 2\}; |V_2| = 2m, \\
 V_3 &= \{v \in V(G): d_G(v) = 3\}; |V_3| = mn - 2m, \text{ and} \\
 V_{mn} &= \{v \in V(G): d_G(v) = mn\}; |V_{mn}| = 1.
 \end{aligned}$$

Also we have four partitions of the edge set $E(G)$ as follows:

$$\begin{aligned}
 E_5 &= \{uv \in E(G): d_G(u) = 2, d_G(v) = 3\}; |E_5| = 2m, \\
 E_6 &= \{uv \in E(G): d_G(u) = 3, d_G(v) = 3\}; |E_6| = mn - 3m, \\
 E_{mn+2} &= \{uv \in E(G): d_G(u) = mn, d_G(v) = 2\}; |E_5| = 2m, \text{ and} \\
 E_{mn+3} &= \{uv \in E(G): d_G(u) = mn, d_G(v) = 3\}; |E_5| = mn - 2m.
 \end{aligned}$$

Theorem 5.1. An augmented-Zagreb index and their polynomial of a Kulli path windmill graph are

$$\begin{aligned}
 AZI \left(P_{n+1}^{(m)} \right) &= 32m + (mn - 3m) \frac{729}{64} + (mn - 2m) \left[\frac{3mn}{mn+1} \right]^3 \text{ and} \\
 AZI \left(P_{n+1}^{(m)}, x \right) &= 4mx^8 + (mn - 3m)x^{\frac{729}{64}} + (mn - 2m)x^{\left[\frac{3mn}{mn+1} \right]^3}.
 \end{aligned}$$

Proof. Let $G = P_{n+1}^{(m)}$, where $P_{n+1}^{(m)}$ is a Kulli cycle windmill graph with $m \geq 2$ and $n \geq 4$. Consider

$$\begin{aligned} AZI \left(P_{n+1}^{(m)} \right) &= \sum_{uv \in E} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 \\ &= \sum_{E_5} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 + \sum_{E_6} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 \\ &\quad + \sum_{E_{mn+2}} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 + \sum_{E_{mn+3}} \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 \\ &= 2m \left[\frac{2 \times 3}{3+2-2} \right]^3 + (mn-3m) \left[\frac{3 \times 3}{3+3-2} \right]^3 \\ &\quad + 2m \left[\frac{2mn}{mn+2-2} \right]^3 + (mn-2m) \left(\frac{3mn}{mn+3-2} \right)^3 \\ &= 32m + (mn-3m) \frac{729}{64} + (mn-2m) \left(\frac{3mn}{mn+1} \right)^3. \end{aligned}$$

Now, for an augmented-Zagreb polynomial of a $P_{n+1}^{(m)}$, we have

$$\begin{aligned} AZI \left(P_{n+1}^{(m)}, x \right) &= \sum_{uv \in E} x \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 \\ &= \sum_{E_5} x \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 + \sum_{E_6} x \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 \\ &\quad + \sum_{E_{mn+2}} x \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 + \sum_{E_{mn+3}} x \left[\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2} \right]^3 \\ &= 4mx^8 + (mn-3m)x^{\frac{729}{64}} + (mn-2m)x^{\left[\frac{3mn}{mn+1} \right]^3}. \end{aligned}$$

Acknowledgement

Thanks are due to Prof. V. R. Kulli for his help and valuable suggestions in the preparation of this paper.

REFERENCES

1. B. Chaluvareju, H. S. Boregowda and S. A. Diwakar, Zagreb indices and their polynomials of certain classes of windmill graphs, submitted (2017).
2. B. Chaluvareju, H. S. Boregowda and S. A. Diwakar, Hyper-Zagreb indices and their polynomials of some special kinds of windmill graphs, *Int. J. Adv. Math.* 4 (2017) 21-32.
3. G. H. Fath-Tabar, Zagreb Polynomial and PI Indices of some Nano Structures, *Digest Journal of Nanomaterials and Biostructures*, 4(1) (2009) 189-191.
4. G. H. Fath-Tabar, Old and New Zagreb indices of graphs, *MATCH communications in Mathematical and in Computer Chemistry*, 65 (2011) 79-84.
5. B. Furtula, A. Graovac, D. Vukicevic, Augmented Zagreb Index, *J. Math. Chem.* 48 (2010) 370-380.
6. J. A. Gallian, Dynamic Survey DS6: Graph Labeling, *Electronic J. Combin.*, DS6, (2007), 1-58.
7. I. Gutman, V. R. Kulli, B. Chaluvareju and H. S. Boregowda, On Banhatti and Zagreb Indices. *J. Int. Math. Virtual Inst.* 7 (2017), 53-67.
8. I. Gutman and N. Trinajstic, Graph Theory and molecular orbitals, Total π -electron energy of alternant hydrocarbons, *Chem. Phys. Lett.*, 17 (1972) 535-538.
9. V. R. Kulli, *College Graph Theory*, Vishwa Internat. Pub., Gulbarga, India (2012).
10. V. R. Kulli, B. Chaluvareju and H. S. Boregowda, Some degree based connectivity indices of Kulli cycle windmill graph, *South Asain J. Maths.* 6 (6), (2016), 263-268.
11. V. R. Kulli, B. Chaluvareju and H. S. Boregowda, Computation of connectivity indices of Kulli path windmill graph, *TWMS J. Appl. Eng. Math.*, to appear (2017).
12. M. Randic, On characterization of molecular branching, *J. Am. Chem. Soc.*, 97 (1975) 6609-6615.]
13. R. Todeschini, V. Consonni, *Handbook of Molecular Descriptors*, Wiley-VCH, Weinheim, 2000.



POWER GRAPH OF SOME FINITE GROUPS Z_n AND C_n OF PRIME ORDER

Ajay Kumar*, Manju Pruthi**

* & ** Department of Mathematics, Indira Gandhi University, Meerpur-Rewari, Haryana
 E-mail : ajay61286@gmail.com, manju.pruthi@yahoo.com

ABSTRACT :

There are a variety of ways to associate directed or undirected graphs to a group. It may be interesting to investigate the relations between the structure of these graphs and characterizing certain properties of the group in term of some properties of the associated graphs. The Power graph $\Gamma_p(G)$, of a group G is the graph whose vertex set is the group element and two elements are adjacent if one is the power of other. In this paper, we investigate some properties of the power graph of Z_n and C_n of prime order.

Keywords: Finite graph, complete graph, connected graph, power graph, regular graph.

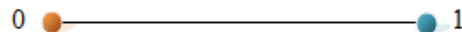
1. INTRODUCTION

The investigation of graphs related to group of prime order of Z_n and C_n , is an important topic in algebraic structure. This paper is devoted to the study of power graph which were introduced by Kelarev Quinn. Let us review some facts of power graph. In this paper, we represent finite group in the form of a graph such that one is power of other, then these graphs are called power graph. In this paper, we shall study power graphs of Z_n and C_n of prime order.

2. POWER GRAPH OF Z_n OF PRIME ORDER

2.1 Power graph of group Z_2 :

Let $G = Z_2 = \{0, 1\}$ be the group under addition modulo 2. Then the power graph of Z_2 is:

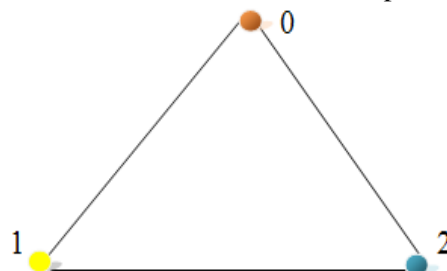


We have some following properties about this graph:-

1. This power graph is finite graph.
2. This power graph is connected graph.
3. This power graph is regular graph.
4. This power graph is complete graph.
5. The chromatic number of this graph is 2.

2.2 Power graph of group Z_3 :

Let $G = Z_3 = \{0, 1, 2\}$ be the group under addition modulo 3. Then the power graph of Z_3 is:-

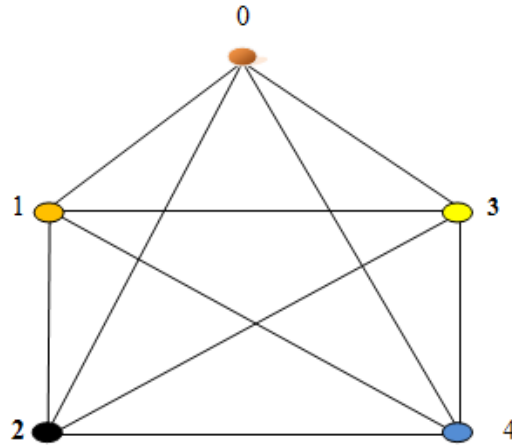


We have some following properties about this graph:-

1. This power graph is finite graph.
2. This power graph is connected graph.
3. This power graph is regular graph.
4. This power graph is complete graph.
5. This power graph is unicyclic graph.
6. The chromatic number of this graph is 3.

2.3 Power graph of group Z_5 :

Let $G = Z_5 = \{0, 1, 2, 3, 4\}$ be the group under addition modulo five. Then the power graph of the group Z_5 is:-

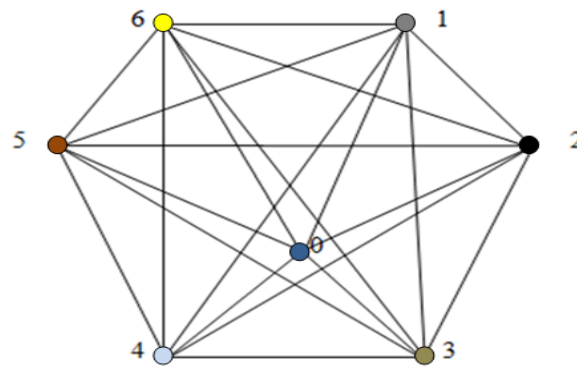


We have some following properties about this graph:-

1. This power graph is finite graph.
2. This power graph is connected graph.
3. This power graph is regular graph.
4. This power graph is complete graph.
5. The chromatic number of this graph is 5.

2.4 Power graph of group Z_7 :

Let $G = Z_7 = \{0, 1, 2, 3, 4, 5, 6\}$ the group under addition modulo 7. Then the power graph of the group Z_7 is:-



We have some following properties about this graph:-

1. This power graph is finite graph.
2. This power graph is connected graph.
3. This power graph is regular graph.
4. This power graph is complete graph.
5. The chromatic number of this graph is 7.

3. POWER GRAPH OF C_N OF PRIME ORDER

3.1 Power graph of group C_2 :

Let $G = C_2 = \{g \mid g^2 = I\}$ the cyclic group of order 2 under multiplication. Then the power graph of C_2 is:-
We have, $G = C_2 = \{I, g\}$

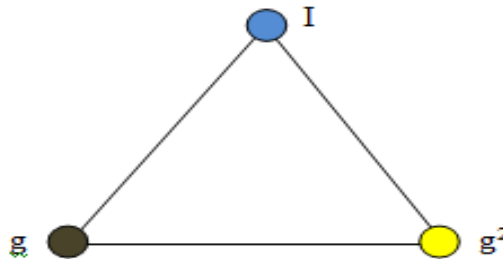


We have some following properties about this graph:-

1. This power graph is finite graph.
2. This power graph is connected graph.
3. This power graph is regular graph.
4. This power graph is complete graph.
5. The chromatic number is 2.

3.2 Power graph of cyclic group C_3 :

Let $G = C_3 = \{g \mid g^3 = 1\}$ be the cyclic group of order 3 under multiplication. Then the power graph of $C_3 = \{I, g, g^2\}$ is:-



We have some following properties about this graph:-

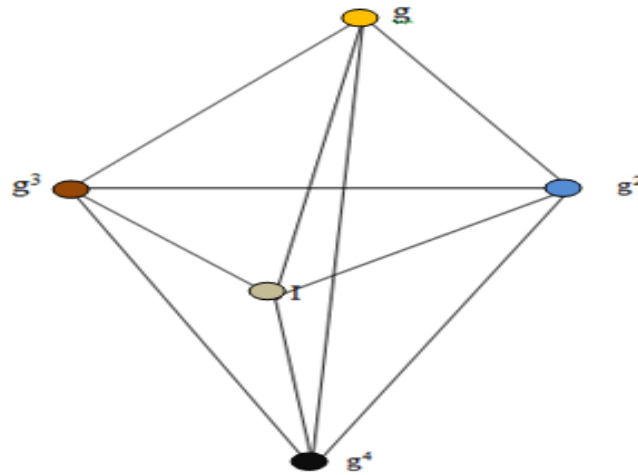
1. This power graph is finite graph.
2. This power graph is connected graph.
3. This power graph is regular graph.
4. This power graph is complete graph.
5. This power graph is unicyclic graph.
6. The chromatic number of this graph is 3.

3.3 Power graph of cyclic group C_5 :

Let $G = C_5 = \{g \mid g^5 = 1\}$ be the cyclic group of order 5 under multiplication. Then the power graph of C_5 is:-
We have, $G = C_5 = \{I, g, g^2, g^3, g^4\}$

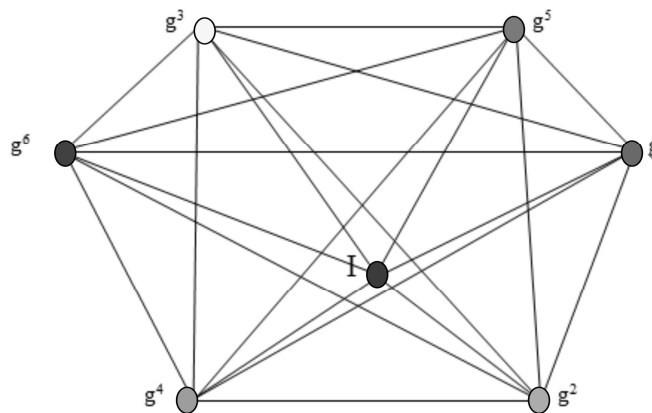
We have some following properties about this graph:-

1. This power graph is finite graph.
2. This power graph is connected graph.
3. This power graph is regular graph.
4. This power graph is complete graph.
5. The chromatic number is 5.



3.4 Power graph of cyclic group C_7 :

Let $G = C_7 = \{g \mid g^7 = 1\}$ be the cyclic group of order 7 under multiplication. Then the power graph of C_7 is:-



We have some following properties about this graph:-

1. This power graph is finite graph.
2. This power graph is connected graph.
3. This power graph is regular graph.
4. This power graph is complete graph.
5. The chromatic number of this graph is 7.

4. CONCLUSION :

From the above power graphs of Z_n and C_n , we conclude that the power graph of finite group of prime order of Z_n and C_n satisfies the properties of finite graph, connected graph, regular graph and complete graph. And using graph we also find out their chromatic number. Thus, we can say that power graph of finite group Z_n and C_n of prime order is finite, connected, regular, complete and the chromatic number is according to prime number ‘n’.

REFERENCES:

1. Andrasafi, B., *Introductory Graph theory Adam Hilger*, Bristol, tr. A. Recksi (1977).
2. Beck I., *Colouring of a commutative ring*, J. Algebra, 116, 208-226 (1988).
3. Birkhoff, G.D., *A determinant formula for the number of ways of coloring a map* Ann of Math. (2) 14 (1912/13) No. 1-4, 42-46.
4. Bollobas, B., *Modern graph theory*, Springer-verlag, New York (1998).
5. Bondy, J.A. and Murty, U.S.R., “*Graph Theory with Applications*,”North Hollan Elsevier, New York, 1976.
6. G. Chartrand and P. Zhang, *Introduction to graph theory*, Tata McGraw-Hill,2006.
7. Jensen, T.R. and Toft, B., *Graph coloring problems*, Wiley, New York, 1995.
8. Satyanaryan, B. and Prasad, K.S. “*Discrete Mathematics and Graph Theory.*”



APPROXIMATION OF THE CUBIC FUNCTIONAL EQUATION IN RANDOM NORMED SPACES : DIRECT AND FIXED POINT METHOD

Nawneet Hooda^{*}, Shalini Tomar^{}**

^{*}Department of mathematics, DCR University of Science & Technology, Murthal, Sonapat, Haryana

^{**} Department of mathematics, Kanya Mahavidyalaya, Kharkhoda, Sonapat, Haryana, India

E-mail : nawneethooda@gmail.com*, s_saroha30@yahoo.com**

ABSTRACT :

In this paper, the stability of cubic functional equation $f(kx+y) - f(x+ky) = (k-1)(k+1)^2[f(x) - f(y)] - k(k-1)f(x-y)$ (where k is a positive integer greater than 2) using direct and fixed point method in random normed spaces has been proved.

Keywords : Hyers-Ulam-Rassias stability, cubic functional equation and random normed spaces.

Mathematical subject classification- 39B72, 47H09.

1. INTRODUCTION

In 1940 Ulam [3] first raised a question of stability of group homomorphisms, which is as following: "When is it true that a function which approximately satisfies a functional equation F must be close to an exact solution of F ?". If the problem accepts a solution, then the equation F is said to be stable.

D. H. Hyers [13] answered the problem of Ulam by assuming the groups as Banach spaces. Then Th. M. Rassias [22] gave a generalized version of the theorem of Hyers for approximately linear mappings. Gavruta [11] proved a generalization of Rassias theorem by introducing a general control function $\phi(x, y)$.

The functional equation $f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x)$ is said to be the *cubic functional equation* since cx^3 is its solution. Every solution of the cubic functional equation is said to be cubic mapping. The stability problem for the cubic functional equation was proved by Jun and Kim [14] for mappings $f : X \rightarrow Y$ where X is a real normed space and Y is Banach space. They proved that a function f between X and Y is solution of above cubic functional equation if and only if there exists a unique function $C : X \times X \times X \rightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$ and C is symmetric for every fixed one variable and additive for fixed two variables.

In this paper, we introduce the following cubic functional equation

$$f(kx+y) - f(x+ky) = (k-1)(k+1)^2[f(x) - f(y)] - k(k-1)f(x-y) \quad (1)$$

And our main aim is to prove the generalized Hyers-Ulam-Rassias stability of it in random normed spaces, where k is a positive integer greater than 2.

2. PRELIMINARIES

In this paper, we will use the usual terminologies, notations and conventions of the theory of random normed spaces as in [1,16,17,23,24]. The space of all probability distribution functions is denoted by

$\Delta^+ = \{F : R \cup \{-\infty, +\infty\} \rightarrow [0, 1] : F \text{ is left-continuous and non-decreasing on } R \text{ and } F(0) = 0, F(+\infty) = 1\}$, the subset $D^+ \subseteq \Delta^+$ is the set

$$D^+ = \{F \in \Delta^+ : l^-F(+\infty) = 1\},$$

where $l^-F(x)$ denotes the left limit of the function f at the point x that is $l^-f(x) = \lim_{t \rightarrow x^-} f(t)$. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in R$. For all $a \geq 0$, any element of Δ^+ is the distribution function ϵ_a given by

$$\epsilon_a(t) = \begin{cases} 0, & \text{if } t \leq a \\ 1, & \text{if } t > a \end{cases}$$

and we can easily see that $\epsilon_0(t)$ is its maximal element.

2.1 Definition [16]

A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a *continuous triangular norm* (briefly, a continuous t-norm) if T satisfies the following conditions:

- T is commutative and associative;
- T is continuous;
- $T(a, 1) = a$ for all $a \in [0, 1]$;
- $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Some examples of continuous t-norms are $T_L(x, y) = \max\{x + y - 1, 0\}$ (the Lukasiewicz t-norm) $T_p(x, y) = xy$ and $T_M(x, y) = \min(x, y)$. We know that, if T is a t-norm and $\{x_n\}$ are given numbers in $[0, 1]$, then $T_{i=1}^n x_i$ is defined recursively by $T_{i=1}^1 x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$. $T_{i=1}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$. Also, for the Lukasiewicz t-norm the following implication holds:

$$\lim_{n \rightarrow \infty} (T_L)_{i=1}^\infty x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^\infty (1 - x_n) < \infty$$

2.2 Definition [24]

A *random normed space* (RN-space) is a triplet (X, Φ, T) , where X is a vector space, T is a continuous t-norm and $\Phi : X \rightarrow D^+$ is a mapping such that the following conditions hold:

- $\Phi_x(t) = \epsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
- $\Phi_{ax}(t) = \Phi_x(t/|a|)$ for all $a \in R, a \neq 0, x \in X$ and $t \geq 0$;
- $\Phi_{x+y}(t+s) \geq T(\Phi_x(t), \Phi_y(s))$, for all $x, y \in X$ and $t, s \geq 0$.

For every normed space $(X, \|\cdot\|)$ we can define a random normed space (X, Φ, T_M) where

$$\Phi_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$. This space is called the *induced random normed space*.

2.3 Definition [23]

Let (X, Φ, T) be an RN-space.

- A sequence $\{x_n\}$ in X is said to be *convergent* to $x \in X$ if for all $\epsilon > 0$ and $\lambda > 0$, there exist positive integer N such that $\Phi_{x_n-x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.
- A sequence $\{x_n\}$ in X is said to be *cauchy sequence* in X if for all $\epsilon > 0$ and $\lambda > 0$, there exist positive integer N such that $\Phi_{x_n-x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

- The RN-space (X, Φ, T) is said to be *complete* if every Cauchy sequence in X is convergent.

2.4 Theorem [23]

If (X, Φ, T) is RN-space and $\{x_n\}$ is a sequence such that $x_n \rightarrow x$, then $\lim_{n \rightarrow \infty} \Phi_{x_n}(t) = \Phi_x(t)$.

2.5 Definition[15]

Let X be a non-empty set. A function $d : X \times X \rightarrow [0, \infty]$ is called a *complete generalized metric* on X if d satisfies the following conditions:

- $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$
- Every d -Cauchy sequence in X is d -convergent, i.e. $\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0$ for a sequence $x_n \in X$ ($n = 1, 2, \dots$) implies the existence of an element $x \in X$ with $\lim_{n \rightarrow \infty} d(x, x_n) = 0$.

The ordered pair (X, d) is called *complete generalized metric space*. It differs from the usual complete metric space by the fact that not every two points in X have necessarily a finite distance.

2.6 Theorem [6]

Let (X, d) be a complete generalized metric space and $J : X \rightarrow X$ a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that

- $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
- the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;
- $d(y, y^*) \leq (1/(1 - L))d(y, Jy)$ for all $y \in Y$.

The generalised Hyers-Ulam-Rassias stability of cubic functional equation in random normed spaces have been broadly studied by various authors in [4,5,8,10]. There are many spaces namely Non-Archimedean spaces, Quasi-Banach spaces, fuzzy normed spaces etc. which attracts authors [7,9,12-14,17,18,20,22,25] to establish stability results of different functional equations. In this paper we work with cubic functional equation (1) in random normed spaces.

3. STABILITY OF FUNCTIONAL EQUATION (1) IN RANDOM NORMED SPACE : A DIRECT METHOD

In this section, using direct method, we will prove the generalized Hyers-Ulam-Rassias stability of cubic functional equation (1) in random normed spaces. Let us define a function D_f as follows:

$$D_f(x, y) = f(kx + y) - f(x + ky) - (k - 1)(k + 1)^2[f(x) - f(y)] + k(k - 1)f(x - y).$$

3.1 Theorem

Let X be a real linear space, (Z, Φ, T) be a random normed space and $\phi : X^2 \rightarrow Z$ be a function such that for some $0 < \mu < k^3$

$$\Phi_{\phi(kx, 0)}(t) \geq \Phi_{\mu\phi(x, 0)}(t) \tag{2}$$

and

$$\lim_{n \rightarrow \infty} \Phi_{\phi(k^n x, k^n y)}(k^{3n} t) = 1 \tag{3}$$

for all $x \in X, t > 0$. If (Y, ζ, T) be a complete random normed space and $f : X \rightarrow Y$ is a mapping with $f(0)=0$ such that for all $x, y \in X$ and $t > 0$

$$\zeta_{D_f(x,y)}(t) \geq \Phi_{\phi(x,y)}(t) \tag{4}$$

then the limit $C(x) = \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^{3n}}$ exists for all $x \in X$ and defines a unique cubic mapping $C : X \rightarrow Y$ such that

$$\zeta_{f(x)-C(x)}(t) \geq T_{m=0}^{n-1}(\Phi_{\phi(x,0)}((k^3 - \mu)t)). \tag{5}$$

Proof: Existence- By putting $y = 0$ in (4) we get that,

$$\zeta_{\frac{f(kx)}{k^3}-f(x)}(t) \geq \Phi_{\phi(x,0)}(k^3 t) \tag{6}$$

for all $x \in X$. Replacing x by $k^n x$ in (6) and using (2), we get

$$\zeta_{\frac{f(k^{n+1}x)}{k^{3n+3}}-\frac{f(k^n x)}{k^{3n}}}(t) \geq \Phi_{\phi(k^n x,0)}(k^{3n+3}t) \geq \Phi_{\phi(x,0)}\left(\frac{k^{3n+3}t}{\mu^n}\right) \tag{7}$$

Since,

$$\frac{f(k^n x)}{k^{3n}} - f(x) = \sum_{m=0}^{n-1} \left(\frac{f(k^{m+1}x)}{k^{3m+3}} - \frac{f(k^m x)}{k^{3m}} \right) \tag{8}$$

therefore we can say that

$$\begin{aligned} \zeta_{\frac{f(k^n x)}{k^{3n}}-f(x)}\left(\sum_{m=0}^{n-1} \frac{\mu^m t}{k^{3m+3}}\right) &\geq T_{m=0}^{n-1}\left(\zeta_{\frac{f(k^{m+1}x)}{k^{3m+3}}-\frac{f(k^m x)}{k^{3m}}}\left(\frac{\mu^m t}{k^{3m+3}}\right)\right) \\ &\geq T_{m=0}^{n-1}(\Phi_{\phi(x,0)}(t)). \end{aligned} \tag{9}$$

This implies that

$$\zeta_{\frac{f(k^n x)}{k^{3n}}-f(x)}(t) \geq T_{m=0}^{n-1}\left(\Phi_{\phi(x,0)}\left(\frac{t}{\sum_{m=0}^{n-1} \frac{\mu^m}{k^{3m+3}}}\right)\right). \tag{10}$$

Replacing x by $k^p x$ in (10), we get

$$\begin{aligned} \zeta_{\frac{f(k^{n+p}x)}{k^{3(n+p)}}-\frac{f(k^p x)}{k^{3p}}}(t) &\geq T_{m=0}^{n-1}\left(\Phi_{\phi(k^p x,0)}\left(\frac{t}{\sum_{m=0}^{n-1} \frac{\mu^m}{k^{3(m+p+1)}}}\right)\right) \\ &\geq T_{m=0}^{n-1}\left(\Phi_{\phi(x,0)}\left(\frac{t}{\sum_{m=0}^{n-1} \frac{\mu^{m+p}}{k^{3(m+p+1)}}}\right)\right) \\ &= T_{m=0}^{n-1}\left(\Phi_{\phi(x,0)}\left(\frac{t}{\sum_{m=p}^{n+p-1} \frac{\mu^m}{k^{3(m+1)}}}\right)\right). \end{aligned} \tag{11}$$

Since,

$$\lim_{p,n \rightarrow \infty} \Phi_{\phi(x,0)} \left(\frac{t}{\sum_{m=p}^{n+p-1} \frac{\mu^m}{k^{3(m+1)}}} \right) = 1, \quad (12)$$

therefore $\left\{ \frac{f(k^n x)}{k^{3n}} \right\}$ is a Cauchy sequence in complete random normed space (Y, ζ, T) , so there exists some point $C(x) \in Y$ such that

$$\lim_{n \rightarrow \infty} \left\{ \frac{f(k^n x)}{k^{3n}} \right\} = C(x).$$

Fix $x \in X$ and put $p = 0$ in (11). Then we get

$$\zeta_{\frac{f(k^n x)}{k^{3n}} - f(x)}(t) \geq T_{m=0}^{n-1} \left(\Phi_{\phi(x,0)} \left(\frac{t}{\sum_{m=0}^{n-1} \frac{\mu^m}{k^{3m+3}}} \right) \right) \quad (13)$$

and therefore, for each $\varepsilon > 0$, we can say

$$\begin{aligned} \zeta_{C(x)-f(x)}(t+\varepsilon) &\geq T \left(\zeta_{\frac{f(k^n x)}{k^{3n}} - f(x)}(\varepsilon), \zeta_{\frac{f(k^n x)}{k^{3n}} - f(x)}(t) \right) \\ &\geq T \left(\zeta_{\frac{f(k^n x)}{k^{3n}} - f(x)}(\varepsilon), T_{m=0}^{n-1} \left(\Phi_{\phi(x,0)} \left(\frac{t}{\sum_{m=0}^{n-1} \frac{\mu^m}{k^{3m+3}}} \right) \right) \right). \end{aligned} \quad (14)$$

Taking the limit as $n \rightarrow \infty$, we get

$$\zeta_{C(x)-f(x)}(t+\varepsilon) \geq T_{m=0}^{n-1} \left(\Phi_{\phi(x,0)}((k^3 - \mu)t) \right). \quad (15)$$

Since ε is arbitrary, by taking $\varepsilon \rightarrow 0$ in (15), we get

$$\zeta_{C(x)-f(x)}(t) \geq T_{m=0}^{n-1} \left(\Phi_{\phi(x,0)}((k^3 - \mu)t) \right). \quad (16)$$

Replacing x by $k^n x$ and y by $k^n y$ in (4), we get

$$\zeta_{\frac{D_f(k^n x, k^n y)}{k^{3n}}}(t) \geq \Phi_{\phi(k^n x, k^n y)}(k^{3n} t) \quad (17)$$

for all $x, y \in X, t > 0$. Therefore, by taking $n \rightarrow \infty$ in (17) and using (3) we have

$$C(kx + y) - C(x + ky) = (k-1)(k+1)^2 [C(x) - C(y)] - k(k-1)C(x-y).$$

Uniqueness: To prove the uniqueness of the mapping C , we suppose that $D: X \rightarrow Y$ is another mapping which satisfies (5). Since f is a cubic mapping, therefore C and D are also cubic. Therefore for all $n \in \mathbb{N}$ and every $x \in X$, $C(k^n x) = k^{3n} C(x)$ and $D(k^n x) = k^{3n} D(x)$. Thus, we have for all $t > 0$,

$$\begin{aligned} \zeta_{C(x)-D(x)}(t) &= \lim_{n \rightarrow \infty} \zeta_{\frac{C(k^n x)}{k^{3n}} - \frac{D(k^n x)}{k^{3n}}}(t) \\ &\geq \lim_{n \rightarrow \infty} \left\{ T \left(\zeta_{\frac{C(k^n x)}{k^{3n}} - \frac{f(k^n x)}{k^{3n}}}\left(\frac{t}{2}\right), \zeta_{\frac{D(k^n x)}{k^{3n}} - \frac{f(k^n x)}{k^{3n}}}\left(\frac{t}{2}\right) \right) \right\} \end{aligned}$$

$$\begin{aligned} &\geq \lim_{n \rightarrow \infty} \{T_{m=0}^{n-1}(\Phi_{\phi(k^n x, 0)}(\frac{(k^3 - \mu)k^{3n}t}{2})), T_{m=0}^{n-1}(\Phi_{\phi(k^n x, 0)}(\frac{(k^3 - \mu)k^{3n}t}{2}))\} \\ &\geq \lim_{n \rightarrow \infty} \{T_{m=0}^{n-1}(\Phi_{\phi(x, 0)}(\frac{(k^3 - \mu)k^{3n}t}{2\mu^n})), T_{m=0}^{n-1}(\Phi_{\phi(x, 0)}(\frac{(k^3 - \mu)k^{3n}t}{2\mu^n}))\} = 1. \end{aligned}$$

So, $C(x) = D(x)$ for all $x \in X$. Hence the proof.

3.2 Corollary

Let X be a real linear space, (Z, Φ, T_M) be a random normed space and $\phi : X^2 \rightarrow Z$ be a function such that for some $0 < \mu < k^3$,

$$\Phi_{\phi(kx, 0)}(t) \geq \Phi_{\mu\phi(x, 0)}(t) \tag{18}$$

and

$$\lim_{n \rightarrow \infty} \Phi_{\phi(k^n x, k^n y)}(k^{3n}t) = 1 \tag{19}$$

for all $x, y \in X, t > 0$. If (Y, ζ, T_M) is a complete random normed space and $f : X \rightarrow Y$ is a mapping with $f(0)=0$ such that for all $x, y \in X$ and $t > 0$

$$\zeta_{D_f(x, y)}(t) \geq \Phi_{\phi(x, y)}(t) \tag{20}$$

then the limit $C(x) = \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^{3n}}$ exists for all $x \in X$ and defines a unique cubic mapping $C : X \rightarrow Y$ such that

$$\zeta_{f(x)-C(x)}(t) \geq \Phi_{\phi(x, 0)}((k^3 - \mu)t). \tag{21}$$

Proof: The proof can be easily generated by taking T-norm as minimum T-norm in above theorem.

3.3 Corollary

Let X be a real linear space, (Z, Φ, T_M) be a random normed space and (Y, ζ, T_M) be a complete random normed space. Let $p \in (0, 1)$ and $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping with $f(0) = 0$ satisfying

$$\zeta_{D_f(x, y)}(t) \geq \Phi_{(\|x\|^p + \|y\|^p)z_0}(t) \tag{22}$$

for all $x, y \in X$ and $t > 0$. Then the limit $C(x) = \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^{3n}}$ exists for all $x \in X$ and defines a unique cubic mapping $C : X \rightarrow Y$ such that

$$\zeta_{f(x)-C(x)}(t) \geq \Phi_{\|x\|^p z_0}((k^3 - k^{3p})t) \tag{23}$$

for all $x \in X$ and $t > 0$.

Proof: Let $\phi : X^2 \rightarrow Z$ be defined as $\phi(x, y) = (\|x\|^p + \|y\|^p)z_0$. Then the proof follows from corollary (3.2) by taking $\mu = k^{3p}$.

3.4 Corollary

Let X be a real linear space, (Z, Φ, T_M) be a random normed space and (Y, ζ, T_M) be a complete random normed space. Let $z_0 \in Z$ and $f : X \rightarrow Y$ is a mapping with $f(0) = 0$ satisfying

$$\zeta_{D_f(x, y)}(t) \geq \Phi_{\tilde{\alpha}z_0}(t) \tag{24}$$

for all $x, y \in X$ and $t > 0$. Then the limit $C(x) = \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^{3n}}$ exists for all $x \in X$ and defines a unique cubic mapping $C : X \rightarrow Y$ such that

$$\zeta_{f(x)-C(x)}(t) \geq \Phi_{\alpha_0}((k^3 - 1)t) \tag{25}$$

for all $x \in X$ and $t > 0$.

Proof: Let $\phi : X^2 \rightarrow Z$ be defined as $\phi(x, y) = \alpha_0$. Then the proof follows from corollary 3.2 by taking $\mu = 1$.

3.5 Theorem

Let X be a real linear space, (Z, Φ, T) be a random normed space, and $\phi : X^2 \rightarrow Z$ be a function such that, for some $0 < \mu < \frac{1}{k^3}$

$$\Phi_{\phi(\frac{x}{k}, 0)}(t) \geq \Phi_{\mu\phi(x, 0)}(t) \tag{26}$$

and

$$\lim_{n \rightarrow \infty} \Phi_{(\frac{x}{k^n}, \frac{y}{k^n})}(\frac{t}{k^{3n}}) = 1 \tag{27}$$

for all $x, y \in X, t > 0$. If (Y, ζ, T) be a complete random normed space and $f : X \rightarrow Y$ is a mapping with $f(0) = 0$ and satisfying (4) for all $x, y \in X$ and $t > 0$, then the limit $C(x) = \lim_{n \rightarrow \infty} k^{3n} f(\frac{x}{k^n})$ exists for all $x \in X$ and defines a unique cubic mapping $C : X \rightarrow Y$ such that

$$\zeta_{f(x)-C(x)}(t) \geq \Phi_{\phi(x, 0)} T_{m=0}^{n-1}(\frac{(1-\mu k^3)}{\mu} t). \tag{28}$$

Proof: Existence- putting $y = 0$ in (4), we get that

$$\zeta_{f(kx)-k^3 f(x)}(t) \geq \Phi_{\phi(x, 0)}(t) \tag{29}$$

Replacing x by $\frac{x}{k^{(n+1)}}$ and using (26), we get

$$\begin{aligned} \zeta_{k^{3n} f(\frac{x}{k^n}) - k^{3n+3} f(\frac{x}{k^{n+1}})}(t) &\geq \Phi_{\phi(\frac{x}{k^{(n+1)}}, 0)}(\frac{t}{k^{3n}}) \\ &\geq \Phi_{\phi(x, 0)}(\frac{t}{\mu^{(n+1)} k^{3n}}). \end{aligned} \tag{30}$$

Since,

$$k^{3n} f(\frac{x}{k^n}) - f(x) = \sum_{m=0}^{n-1} (k^{3m+3} f(\frac{x}{k^{m+1}}) - k^{3m} f(\frac{x}{k^m})) \tag{31}$$

therefore we can say that

$$\begin{aligned} \zeta_{k^{3n} f(\frac{x}{k^n}) - f(x)}(\sum_{m=0}^{n-1} \mu^{(m+1)} t k^{3m}) &\geq T_{m=0}^{n-1}(\zeta_{k^{3m+3} f(\frac{x}{k^{m+1}}) - k^{3m} f(\frac{x}{k^m})}(\mu^{(m+1)} t k^{3m})) \\ &\geq T_{m=0}^{n-1}(\Phi_{\phi(x, 0)}(t)). \end{aligned} \tag{32}$$

This implies that

$$\zeta_{k^{3n}f(\frac{x}{k^n})-f(x)}(t) \geq T_{m=0}^{n-1}(\Phi_{\phi(x,0)}(\frac{t}{\sum_{m=0}^{n-1} \mu^{(m+1)} k^{3m}})). \tag{33}$$

Replacing x by $\frac{x}{k^p}$ in (33), we get

$$\begin{aligned} \zeta_{k^{3(n+p)}f(\frac{x}{k^{n+p}})-k^{3p}f(\frac{x}{k^p})}(t) &\geq T_{m=0}^{n-1}(\Phi_{\phi(\frac{x}{k^p},0)}(\frac{t}{\sum_{m=0}^{n-1} \mu^{(m+1)} k^{3(m+p)}})) \\ &\geq T_{m=0}^{n-1}(\Phi_{\phi(x,0)}(\frac{t}{\sum_{m=0}^{n-1} \mu^{(m+p+1)} k^{3(m+p)}})). \end{aligned} \tag{34}$$

Since,

$$\lim_{p,n \rightarrow \infty} \Phi_{\phi(x,0)}(\frac{t}{\sum_{m=p}^{\infty} \mu(\frac{\mu}{1})^m \frac{1}{k^3}}) = 1, \tag{35}$$

therefore $\{k^{3n}f(\frac{x}{k^n})\}$ is a Cauchy sequence in complete random normed space (Y, ζ, T) , so there exists some point $C(x) \in Y$ such that

$$\lim_{n \rightarrow \infty} \{k^{3n}f(\frac{x}{k^n})\} = C(x).$$

Fix $x \in X$ and put $p = 0$ in (34). Then we get

$$\zeta_{k^{3n}f(\frac{x}{k^n})-f(x)}(t) \geq T_{m=0}^{n-1}(\Phi_{\phi(x,0)}(\frac{t}{\sum_{m=0}^{n-1} \mu^{(m+1)} k^{3m}})) \tag{36}$$

and therefore for each $\varepsilon > 0$, we can say

$$\begin{aligned} \zeta_{C(x)-f(x)}(t + \varepsilon) &\geq T(\zeta_{C(x)-k^{3n}f(\frac{x}{k^n})}(\varepsilon), \zeta_{k^{3n}f(\frac{x}{k^n})-f(x)}(t)) \\ &\geq T(\zeta_{C(x)-k^{3n}f(\frac{x}{k^n})}(\varepsilon), T_{m=0}^{n-1}(\Phi_{\phi(x,0)}(\frac{t}{\sum_{m=0}^{n-1} \mu^{(m+1)} k^{3m}}))). \end{aligned} \tag{37}$$

Taking the limit as $n \rightarrow \infty$, we get

$$\zeta_{C(x)-f(x)}(t + \varepsilon) \geq \Phi_{\phi(x,0)}(\frac{(1-\mu k^3)}{\mu} t) \tag{38}$$

Since ε is arbitrary, by taking $\varepsilon \rightarrow 0$ in (38), we get

$$\zeta_{C(x)-f(x)}(t) \geq \Phi_{\phi(x,0)}(\frac{(1-\mu k^3)}{\mu} t). \tag{39}$$

Replacing x by $\frac{x}{k^n}$ and y by $\frac{y}{k^n}$ in (4), we get

$$\zeta_{k^{3n}D_f(\frac{x}{k^n}, \frac{y}{k^n})}(t) \geq \Phi_{\phi(\frac{x}{k^n}, \frac{y}{k^n})}(\frac{t}{k^{3n}}) \quad (40)$$

for all $x, y \in X, t > 0$. Therefore, by taking $n \rightarrow \infty$ in (40) and using (27), we have

$$C(kx + y) - C(x + ky) = (k - 1)(k + 1)^2[C(x) - C(y)] - k(k - 1)C(x - y).$$

The proof of uniqueness of $C(x)$ can be easily generated from proof of theorem (3.1).

3.6 Corollary

Let X be a real linear space, (Z, Φ, T_M) be a random normed space and $\phi: X^2 \rightarrow Z$ be a function such that, for

some $0 < \mu < \frac{1}{k^3}$

$$\Phi_{\phi(\frac{x}{k}, 0)}(t) \geq \Phi_{\mu\phi(x, 0)}(t) \quad (41)$$

and

$$\lim_{n \rightarrow \infty} \Phi_{(\frac{x}{k^n}, \frac{y}{k^n})}(\frac{t}{k^{3n}}) = 1 \quad (42)$$

for all $x, y \in X, t > 0$. If (Y, ζ, T_M) is a complete random normed space and $f: X \rightarrow Y$ is a mapping with $f(0) = 0$ and satisfying (4) for all $x, y \in X$ and $t > 0$, then the limit $C(x) = \lim_{n \rightarrow \infty} k^{3n} f(\frac{x}{k^n})$ exists for all $x \in X$ and defines a unique cubic mapping $C: X \rightarrow Y$ such that

$$\zeta_{f(x) - C(x)}(t) \geq \Phi_{\phi(x, 0)}(\frac{(1 - \mu k^3)t}{\mu}). \quad (43)$$

Proof: The proof can be easily generated by taking T-norm as minimum T-norm in above theorem.

3.7 Corollary

Let X be a real linear space, (Z, Φ, T_M) be a random normed space, and (Y, ζ, T_M) be a complete random normed space. Let $p > 1$ and $z_0 \in Z$. If $f: X \rightarrow Y$ is a mapping with $f(0) = 0$ satisfying (22). Then the limit

$C(x) = \lim_{n \rightarrow \infty} k^{3n} f(\frac{x}{k^n})$ exists for all $x \in X$ and defines a unique cubic mapping $C: X \rightarrow Y$ such that

$$\zeta_{f(x) - C(x)}(t) \geq \Phi_{\|x\|^p z_0}((k^{3p} - k^3)t) \quad (44)$$

for all $x \in X$ and $t > 0$.

Proof: Let $\phi: X^2 \rightarrow Z$ be defined as $\phi(x, y) = (\|x\|^p + \|y\|^p)z_0$. Then the proof follows from corollary 3.6 by taking $\mu = k^{-3p}$.

3.8 Corollary

Let X be a real linear space, (Z, Φ, T_M) be a random normed space and (Y, ζ, T_M) be a complete random normed space. Let $z_0 \in Z$ and $f: X \rightarrow Z$ is a mapping with $f(0) = 0$ satisfying (24). Then the limit

$C(x) = \lim_{n \rightarrow \infty} k^{3n} f(\frac{x}{k^n})$ exists for all $x \in X$ and defines a unique cubic mapping $C: X \rightarrow Y$ such that

$$\zeta_{f(x)-C(x)}(t) \geq \Phi_{\alpha_0}(k^3(k-1)t) \tag{45}$$

for all $x \in X$ and $t > 0$.

Proof: Let $\phi : X^2 \rightarrow Z$ be defined as $\phi(x, y) = \alpha_0$. Then the proof follows from corollary 3.6 by taking $\mu = 1/k^4$.

3.9 Example

Let $(X, \|\cdot\|)$ be a Banach Algebra and

$$\zeta_x(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \max\{1 - \frac{\|x\|}{t}, 0\}, & \text{if } t > 0. \end{cases}$$

for every $x, y \in X$. Let

$$\Phi_{\phi(x,y)}(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \max\{1 - \frac{3k^6(\|x\| + \|y\|)}{t}, 0\}, & \text{if } t > 0. \end{cases}$$

We note that $\Phi_{\phi(x,y)}$ is a distribution function and

$$\lim_{n \rightarrow \infty} \Phi_{(k^n x, k^n y)}(k^{3n}t) = 1$$

for all $x, y \in X$ and all $t > 0$. We assert that (X, ζ, T_L) is an RN-space. We know,

- (i). (for all $t > 0 ; \zeta_x(t) = 1$) \Leftrightarrow (for all $t > 0 ; \frac{\|x\|}{t} = 0$) $\Leftrightarrow (x = 0)$
- (ii). $\zeta_{\lambda x}(t) = 1 - \frac{\|\lambda x\|}{t} = 1 - \frac{|\lambda| \|x\|}{t} = 1 - \frac{\|x\|}{\frac{t}{|\lambda|}} = \zeta_x(\frac{t}{|\lambda|})$, for all $x \in X$ and all $t > 0$.
- (iii). for every $x, y \in X$ and $t, s > 0$ we have

$$\begin{aligned} \zeta_{x+y}(t+s) &= \max\{1 - \frac{\|x+y\|}{t+s}, 0\} = \max\{1 - \frac{\|x\| + \|y\|}{t+s}, 0\} \\ &= \max\{1 - \frac{\|x\|}{t+s} - \frac{\|y\|}{t+s}, 0\} \geq \max\{1 - \frac{\|x\|}{t} - \frac{\|y\|}{s}, 0\} \\ &\geq \max\{1 - \frac{\|x\|}{t}, 1 - \frac{\|y\|}{s}\} = T_L(\zeta_x(t), \zeta_y(s)). \end{aligned}$$

It is also easy to note that (X, ζ, T_L) is complete, since

$$\zeta_{x-y}(t) \geq 1 - \frac{\|x-y\|}{t},$$

for all $x, y \in X$ and $t > 0$ and $(X, \|\cdot\|)$ is complete. Define $f : X \rightarrow X$, $f(x) = x^3 + \|x_0\| x$, where x_0 is a unit vector in X . We can easily calculate that, for all $x, y \in X$

$$\|f(kx+y) - f(x+ky) - (k-1)[(k+1)^2[f(x) - f(y)] + kf(x-y)]\| \leq 3k^6(\|x\| + \|y\|).$$

Therefore, for all $x, y \in X$ and $t > 0$.

$$\zeta_{f(kx+y)-f(x+ky)-(k-1)(k+1)^2[f(x)-f(y)]+k(k-1)f(x-y)}(t) \geq \Phi_{\phi(x,y)}(t).$$

Since,

$$\lim_{p,n \rightarrow \infty} \Phi_{\phi(x,0)} \left(\frac{t}{\sum_{m=p}^{n+p-1} \frac{\mu^m}{k^{3(m+1)}}} \right) = \max \left\{ 1 - \frac{3k^6 \|x\| \left(\sum_{m=p}^{n+p-1} \frac{\mu^m}{k^{3(m+1)}} \right)}{t}, 0 \right\} = 1.$$

Thus all the conditions of Theorem (3.1) hold, thus we obtain a unique cubic mapping $C : X \rightarrow X$ such that for all $x \in X$ and $t > 0$ we have,

$$\zeta_{f(x)-C(x)}(t) \geq \max \left\{ 1 - \frac{3k^6 (\|x\|)}{t(k^3 - \mu)}, 0 \right\}.$$

4. STABILITY OF FUNCTIONAL EQUATION (1) IN RANDOM NORMED SPACE: A FIXED POINT APPROACH

In this section, the generalized Hyers-Ulam-Rassias stability of cubic functional equation (1) in random normed spaces will be proved using fixed point method.

4.1 Theorem

Let X be a real linear space, (Y, ζ, T_M) a complete RN-Space and $\Phi : X^2 \rightarrow D^+$ be a mapping such that for some $0 < \mu < \frac{1}{k^3}$

$$\Phi_{(x,y)}(t) \leq \Phi_{\left(\frac{x}{k}, \frac{y}{k}\right)}(\mu t) \quad (46)$$

for all $x, y \in X$ and $t > 0$. Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ and satisfying

$$\zeta_{D_f(x,y)}(t) \geq \Phi_{(x,y)}(t) \quad (47)$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$, $C(x) = \lim_{n \rightarrow \infty} k^{3n} f\left(\frac{x}{k^n}\right)$ exists and $C : X \rightarrow Y$ is a unique cubic mapping such that

$$\zeta_{f(x)-C(x)}(t) \geq \Phi_{(x,0)}\left(\frac{(1-k^3\mu)t}{\mu}\right) \quad (48)$$

for all $x \in X$ and $t > 0$.

Proof: Putting $y = 0$ in (47) and replacing x by $\frac{x}{k}$, we have

$$\zeta_{f(x)-k^3 f\left(\frac{x}{k}\right)}(t) \geq \Phi_{\left(\frac{x}{k}, 0\right)}(t) \geq \Phi_{(x,0)}\left(\frac{t}{\mu}\right) \quad (49)$$

for all $x \in X$ and $t > 0$. Consider the set

$$S = \{g : X \rightarrow Y; g(0) = 0\} \quad (50)$$

and the generalised metric d in S is defined by

$$d(g, h) = \inf \{c \in [0, \infty] : \zeta_{g(x)-h(x)}(ct) \geq \Phi_{(x,0)}(t) \forall x \in X, t > 0\} \quad (51)$$

where $\inf \emptyset = +\infty$. Then, as in the proof of [19, Lemma 2.1], we can show that (S, d) is a generalised complete metric space. Now, let us define an operator $\Delta : S \rightarrow S$ such that

$$(\Delta h)(x) = k^3 h\left(\frac{x}{k}\right) \quad (52)$$

for all $x \in X$. We assert that Δ is strictly contractive on S .

Given $g, h \in S$, let $c \in [0, \infty]$ be an arbitrary constant with $d(g, h) < c$ that is

$$\zeta_{g(x)-h(x)}(ct) \geq \Phi_{(x,0)}(t) \tag{53}$$

for all $x \in X$ and $t > 0$, and so

$$\begin{aligned} \zeta_{(\Delta g)(x)-(\Delta h)(x)}(k^3 \mu ct) &= \zeta_{k^3 g(\frac{x}{k})-k^3 h(\frac{x}{k})}(k^3 \mu ct) \\ &= \zeta_{g(\frac{x}{k})-h(\frac{x}{k})}(\mu ct) \\ &\geq \Phi_{(\frac{x}{k},0)}(\mu t) \\ &\geq \Phi_{(x,0)}(t) \end{aligned} \tag{54}$$

for all $x \in X$ and $t > 0$. Thus $d(g, h) < c$ implies that

$$d(\Delta g, \Delta h) = d(k^3 g(\frac{x}{k}), k^3 h(\frac{x}{k})) < k^3 \mu c.$$

i.e. $d(\Delta g, \Delta h) \leq k^3 \mu d(g, h)$ (55)

for any $g, h \in S$, where $k^3 \mu$ is lipschitz constant with $0 < k^3 \mu < 1$. Thus Δ is strictly contractive.

It follows from (49) that

$$d(f, \Delta f) = d(f, k^3 f(\frac{x}{k})) \leq \mu \tag{56}$$

By Theorem (2.6), there exists a mapping $C : X \rightarrow Y$ satisfying the following:

(i). C is a fixed point of Δ , that is ,

$$C(\frac{x}{k}) = \frac{1}{k^3} C(x) \tag{57}$$

for all $x \in X$. The mapping C is a unique fixed point of Δ in the set

$$\Omega = \{h \in S : d(g, h) < \infty\} \tag{58}$$

Thus, C is a unique mapping satisfying (57) such that there exist $c \in (0, \infty)$ satisfying

$$\zeta_{f(x)-C(x)}(ct) \geq \Phi_{(x,0)}(t) \tag{59}$$

for all $x \in X$ and $t > 0$.

(ii). $d(\Delta^n f, c) \rightarrow 0$ as $n \rightarrow \infty$, which implies that, for all $x \in X$,

$$\lim_{n \rightarrow \infty} k^{3n} f(\frac{x}{k^n}) = C(x). \tag{60}$$

(iii). $d(f, C) \leq d(f, \Delta f)/(1 - k^3 \mu)$ with $f \in \Omega$, and by using (56) we can say that

$d(f, C) \leq \mu/(1 - k^3 \mu)$ and so

$$\zeta_{f(x)-C(x)}(\frac{\mu t}{1 - k^3 \mu}) \geq \Phi_{(x,0)}(t) \tag{61}$$

for all $x \in X$ and $t > 0$, which proves the inequality (48).

Now, consider

$$\begin{aligned} &\zeta_{k^{3n} \frac{f(kx+y)}{k^n} - k^{3n} \frac{f(x+ky)}{k^n} - k^{3n} (k-1)(k+1)^2 [\frac{f(x)}{k^n} - \frac{f(y)}{k^n}] + k^{3n+1} (k-1) \frac{f(x-y)}{k^n}}(t) \\ &\geq \Phi_{(\frac{x}{k^n}, \frac{y}{k^n})}(\frac{t}{k^{3n}}) \geq \Phi_{(x,y)}(\frac{t}{k^{3n} \mu^n}) \end{aligned} \tag{62}$$

for all $x, y \in X$ and $t > 0$ and $n \geq 1$. Taking limit $n \rightarrow \infty$ in (62), we get

$$\zeta_{C(kx+y)-C(x+ky)-(k-1)(k+1)^2[C(x)-C(y)]+k(k-1)C(x-y)}(t) = 1.$$

since $\lim_{n \rightarrow \infty} \Phi_{(x,y)}\left(\frac{t}{\left(\frac{\mu}{k^3}\right)^n}\right) = 1$. Thus, the mapping $C : X \rightarrow Y$ is cubic. Hence the proof.

4.2 Corollary

Let X be a real linear space, $\vartheta \geq 0$ and $p \in (1, \infty)$. Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ and satisfying

$$\zeta_{D_f(x,y)}(t) \geq \frac{t}{t + \vartheta(\|x\|^p + \|y\|^p)} \tag{63}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$, $C(x) = \lim_{n \rightarrow \infty} k^{3n} f\left(\frac{x}{k^n}\right)$ exists and $C : X \rightarrow Y$ is a unique cubic mapping such that

$$\zeta_{f(x)-C(x)}(t) \geq \frac{(k^{3p} - k^3)t}{(k^{3p} - k^3)t + \vartheta\|x\|^p} \tag{64}$$

for all $x \in X$ and $t > 0$.

Proof: The proof follows from above theorem by assuming

$$\Phi_{(x,y)}(t) = \frac{t}{t + \vartheta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and $t > 0$ and taking $\mu = k^{-3p}$.

4.3 Theorem

Let X be a real linear space, (Y, ζ, T_M) a complete RN-Space and $\Phi : X^2 \rightarrow D^+$ be a mapping such that for some $0 < \mu < k^3$

$$\Phi_{\left(\frac{x}{k}, \frac{y}{k}\right)}(t) \leq \Phi_{(x,y)}(\mu t) \tag{65}$$

for all $x, y \in X$ and $t > 0$. Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ and satisfying (47). Then, for all $x \in X$,

$C(x) = \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^{3n}}$ exists and $C : X \rightarrow Y$ is a unique cubic mapping such that

$$\zeta_{f(x)-C(x)}(t) \geq \Phi_{(x,0)}((k^3 - \mu)t) \tag{66}$$

for all $x \in X$ and $t > 0$.

Proof: Putting $y = 0$ in (47), we have

$$\zeta_{\left(\frac{f(kx)}{k^3} - f(x)\right)}\left(\frac{t}{k^3}\right) \geq \Phi_{(x,0)}(t) \tag{67}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S = \{g : X \rightarrow Y; g(0) = 0\} \tag{68}$$

and the generalised metric d in S is defined by

$$d(g, h) = \inf\{c \in [0, \infty] : \zeta_{g(x)-h(x)}(ct) \geq \Phi_{(x,0)}(t) \forall x \in X, t > 0\} \tag{69}$$

where $\inf \emptyset = +\infty$. Then, as in the proof of [19, Lemma 2.1], we can show that (S, d) is a generalised complete metric space. Now, let us define an operator $\Delta : S \rightarrow S$ such that

$$(\Delta h)(x) = \frac{1}{k^3} h(kx) \tag{70}$$

for all $x \in X$.We assert that Δ is strictly contractive on S .

Given $g, h \in S$, let $c \in [0, \infty]$ be an arbitrary constant with $d(g, h) < c$, that is

$$\zeta_{g(x)-h(x)}(ct) \geq \Phi_{(x,0)}(t) \tag{71}$$

for all $x \in X$ and $t > 0$, and so

$$\begin{aligned} \zeta_{(\Delta g)(x)-(\Delta h)(x)}\left(\frac{\mu ct}{k^3}\right) &= \zeta_{\frac{1}{k^3}g(kx)-\frac{1}{k^3}h(kx)}\left(\frac{\mu ct}{k^3}\right) \\ &= \zeta_{g(kx)-h(kx)}(\mu ct) \\ &\geq \Phi_{(kx,0)}(\mu t) \\ &\geq \Phi_{(x,0)}(t) \end{aligned} \tag{72}$$

for all $x \in X$ and $t > 0$.Thus $d(g, h) < c$ implies that $d(\Delta g, \Delta h) = d\left(\frac{1}{k^3}g(kx), \frac{1}{k^3}h(kx)\right) < \frac{\mu c}{k^3}$. i.e.

$$d(\Delta g, \Delta h) \leq \frac{\mu}{k^3}d(g, h) \tag{73}$$

for any $g, h \in S$,where μ/k^3 is lipschitz constant with $0 < \mu/k^3 < 1$. Thus Δ is strictly contractive.

It follows from (67) that

$$d(f, \Delta f) = d\left(f, \frac{f(kx)}{k^3}\right) \leq \frac{1}{k^3}. \tag{74}$$

By Theorem (2.6),there exists a mapping $C : X \rightarrow Y$ satisfying the following

(i). C is a fixed point of Δ , that is ,

$$C(kx) = k^3 C(x) \tag{75}$$

for all $x \in X$. The mapping C is a unique fixed point of Δ in the set

$$\Omega = \{h \in S : d(g, h) < \infty\}. \tag{76}$$

Thus, C is a unique mapping satisfying (75) such that there exist $c \in (0, \infty)$ satisfying

$$\zeta_{f(x)-C(x)}(ct) \geq \Phi_{(x,0)}(t) \tag{77}$$

for all $x \in X$ and $t > 0$.

(ii) $d(\Delta^n f, c) \rightarrow 0$ as $n \rightarrow \infty$,which implies that, for all $x \in X$.

$$\lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^{3n}} = C(x). \tag{78}$$

(iii). $d(f, C) \leq d(f, \Delta f)/(1 - \mu/k^3)$ with $f \in \Omega$,and by using (74) we can say that $d(f, C) \leq 1/(k^3 - \mu)$ and so

$$\zeta_{f(x)-C(x)}\left(\frac{t}{k^3 - \mu}\right) \geq \Phi_{(x,0)}(t) \tag{79}$$

for all $x \in X$ and $t > 0$,which proves the inequality (66). Rest of the proof can be easily generated from Theorem (4.1).

4.4 Corollary

Let X be a real linear space, $\vartheta \geq 0$ and $p \in (0,1)$.Let $f : X \rightarrow Y$ be a mapping with $f(0) = 0$ and satisfying

(63).Then, for all $x \in X$, $C(x) = \lim_{n \rightarrow \infty} \frac{f(k^n x)}{k^{3n}}$ exists and $C : X \rightarrow Y$ is a unique cubic mapping such that

$$\zeta_{f(x)-C(x)}(t) \geq \frac{(k^3 - k^{3p})t}{(k^p - k^{3p})t + \vartheta \|x\|^p} \quad (80)$$

for all $x \in X$ and $t > 0$.

Proof:The proof follows from above theorem by assuming

$$\Phi_{(x,y)}(t) = \frac{t}{t + \vartheta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and $t > 0$ and taking $\mu = k^{3p}$.

4.5 Remark

In corollaries 4.2 and 4.4 if we assume

$$\Phi_{(x,y)}(t) = \frac{t}{t + \vartheta(\|x\|^p + \|y\|^p)}$$

then we get Ulam-Gavruta-Rassias [7,9,10,25] product stability. Since we put $y=0$ in the functional equation, therefore this stability is obvious.

5. CONCLUSION

In this paper, we proved the generalised Hyers-Ulam-Rassias stability for cubic functional equation (1) in random normed spaces using two different approaches-direct and fixed point method. By using fixed point method we made an interesting connection between fixed point theory, random normed spaces and cubic functional equations.

REFERENCES

1. S. S. Chang, Y. J. Cho, and S. M. Kang, *Nonlinear Operator Theory in Probabilistic Metric Spaces*, Nova Science, Huntington, NY, USA, 2001.
2. S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing, River Edge, NJ, USA, 2002.
3. S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, John Wiley and Sons, New York, NY, USA, 1964.
4. H. Azadi Kenary, *On the stability of a cubic functional equation in random normed spaces*, *J. Math. Ext.*, vol. 4, no. 1(2009), pp. 1 -11.
5. E. Baktash, Y. J. Cho, M. Jalili, R. Saadati, S. M. Vaezpour, *On the stability of cubic mappings and quadratic mappings in random normed spaces*, *J. Inequal. Appl.*, 2008 (2008), article Id 902187, 11 pages.
6. J. B. Diaz, B. Margolis, *A fixed point theorem of the alternative for contraction on a generalized complete metric space*, *Bull. Amer. Math. Soc.*, vol. 74(1968), pp. 305-309.
7. M. Eshaghi Gordji and H. Khodaei, *Stability of Functional Equations*, Lap Lambert Academic Publishing, 2010.
8. M. Eshaghi Gordji and J. M. Rassias and M. B. Savadkouhi, *Approximation of the Quadratic and Cubic functional equations in RN-spaces* *Eur. J. Pure Appl. Math.*, vol. 2(2009), pp. 494-507.
9. M. Eshaghi Gordji, S. Zolfaghari, J. M. Rassias, and M. B. Savadkouhi, *Solution and stability of a mixed type cubic and quartic functional equation in quasi-Banach spaces*, *Abstract and Applied Analysis*, 2009(2009), Article ID 417473, 14 pages.
10. M. Eshaghi Gordji and M. B. Savadkouhi, *Stability of mixed type cubic and quartic functional equations in random normed spaces*, *J. Inequal. Appl.*, 2009(2009), Article ID 527462, 9 pages.

11. P. Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, vol. 184, no. 3(1994), pp. 431-436.
12. D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, Basel, Switzerland, 1998.
13. D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA*, vol. 27, no. 4(1941), pp. 222-224.
14. K.W. Jun and H.M. Kim, The generalized Hyers-Ulam-Rassias stability problem of cubic functional equation, *J. Math. Anal. Appl.*, vol. 27(2002), pp. 867-878.
15. W.A.J. Luxemburg, On the convergence of successive approximation in the theory of ordinary differential equation, *Proc. K. Ned. Akad. Wet., Ser. A., Indag. Math.*, vol. 20, (1958), pp. 540-546.
16. D. Mihet, and V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, *Journal of Mathematical Analysis and Applications*, vol. 343, no. 1(2008), pp. 567 - 572.
17. M.S. Moslehian, G. Sadeghi, Stability of two types of cubic functional equations in non-archimedean spaces, *Real analysis exchange*, vol. 33, no. 2(2007/2008), pp. 375-384.
18. A. Najati, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, *Turkish J. Math.*, vol. 31, no. 4(2007), pp. 395-408.
19. C.G. Park and H.J. Wee, Generalized Hyers-Ulam-Rassias stability of a functional equation in three variables, *J. Chungcheong Math. Soc.*, vol. 18, no. 1 (2005).
20. J. M. Rassias, On approximation of approximately linear mappings by linear mappings, *J. Funct. Anal.*, vol. 46, no. 1(1982), pp. 126-130.
21. J. M. Rassias, Solution of a problem of Ulam, *J. Approx. Theory*, vol. 57, no. 3(1989), pp. 268-273.
22. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, vol. 72, no. 2(1978), pp. 297-300.
23. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland, New York, NY, USA, 1983.
24. A. N. Sherstnev, On the notion of random normed space, *Dokl. Akad. Nauk SSSR*, vol. 149(1963), pp. 280 - 283 (Russian).
25. M. A. Sibaha, B. Bouikhalene, and E. Elqorachi, Ulam-Gavruta-Rassias stability of a linear functional equation, *Int. J. Appl. Math. Stat.*, vol. 7, no. Fe07(2007), pp. 157-166.



CENTRAL TENDENCY OF ANNUAL EXTREMUM OF AMBIENT AIR TEMPERATURE AT DHUBRI

Rinamani Sarmah Bordoloi*, Dhritikesh Chakrabarty**

Research Scholar Department of Statistics, Assam Down Town University , Panikhaiti, Assam, India
 Research Guide, Assam Down Town University and HOD Statistics, Handique Girls College, Dhubri, Assam, India
 E-mail: rinamanisarmah@gmail.com, dhritikesh.c@rediffmail.com

ABSTRACT :

An analytical method has been developed for determining the true value of the central tendency of each of annual maximum and annual minimum of ambient air temperature at a location. Also, the value of central tendency of each of annual maximum and annual minimum of ambient air temperature at Dhubri has been determined by applying the method developed here from the data since the year 1969 onwards. Determination of these two values is based on the assumption that change in temperature over years during the period for which data are available occurs due to change cause only but not due to any assignable cause. The values of these two have been found to be 37.1 and 37.2 Degree Celsius and 8.8 Degree Celsius respectively. Moreover, it has been found that the central tendency of annual minimum of the ambient air temperature at Dhubri cannot be less than 8.7409 Degree Celsius and greater than 8.75 Degree Celsius.

Keywords : Annual maximum, annual minimum, ambient air temperature, Dhubri, analytical method of determination.

1. INTRODUCTION

There are many situations where observations are composed of some parameter and chance error⁷. Change in temperature at a location over temperature periodic year (abbreviated as TPY in this article) corresponds to such a situation.

Temperature at a location attains at a maximum and a minimum and during a TPY^{2, 3, 4 & 5}. The annual extremum (i.e. extremum occurred during a TPY) of temperature at a location is to remain the same provided there is no cause(s) influencing upon the change in temperature at the location other than the chance error which is universal^{3,4 & 5}. For this reason, variation occurs among the observations on annual maximum as well as on annual minimum. Though variation exists, each of annual maximum and of annual minimum temperature has a central tendency. Thus if

X_1, X_2, \dots, X_n

are observations on the annual maximum (or annual minimum) of the ambient air temperature at the location with μ as its central tendency and if the variation among the observations occurs due to chance cause only,

$$X_i = \mu + \varepsilon_i \quad , \quad (i = 1, 2, \dots, n) \tag{1.1}$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are values of the chance error associated to X_1, X_2, \dots, X_n respectively.

The existing statistical methods of estimations namely least squares method, maximum likelihood method, minimum variance unbiased method, method of moment and method of minimum chi-square etc. provide

$$\bar{X} = n^{-1} \sum_{i=1}^n x_i$$

as estimator of the central tendency μ (Kendall & Stuart ⁶; Walker, Helen, & Lev ⁷).

This estimator, however, suffers from an error $e = e(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ given by

$$e = e(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n) = \bar{\varepsilon}_i = n^{-1} \sum_{i=1}^n \varepsilon_i \tag{1.3}$$

Which may not be zero ^{4&5}

In other words, none of these methods can provide the true value of the parameter μ . A method has been developed by ^{3 & 8} for determining almost certain interval for the parameter μ . The method is based on the area property of normal probability distribution ^{9,10,11,12,13,14}. In another study ^{4 & 5}, has developed an analytical method for determining the true value of the parameter μ in the situation where the observations are composed of the parameter itself and chance errors. This method is based on the idea of finding the sufficient shortest interval value for the parameter μ , using order statistics. In this method, it is required to exclude two extreme observations in cumulative manner for computing interval value at very stage in order to obtain the sufficient shortest interval. This method however fails in the situation where insufficient observations are remained after exclusion of the extreme observations at some stage before obtaining the sufficient shortest interval. A method for the same has been developed in order to overcome this inconvenience. This paper is based on the development of this method and on one numerical application of the method in determining the value of the central tendency of each of the annual maximum and the annual minimum of the ambient air temperature at Dhubri. The determination of these two values is based on the assumption that the variation among the observations used in determination occurs due to chance cause only.

The method developed is based on the theory of normal probability distribution discovered by a German mathematician¹⁵ in the year 1809, the credit for which discovery is also given by some authors to a French mathematician^{16 & 17} who published a paper in 1738 that showed the normal distribution as an approximation to the binomial distribution discovered by²⁰ in 1713^{18 & 19}. The normal distribution ^{9,10,11,12,13 & 14} is described by the probability density function

$$f(x; \mu, \sigma) = \{\sigma(2\pi)^{\frac{1}{2}}\}^{-1} \exp[-1/2 \{(x - \mu)/\sigma\}^2], \tag{1.4}$$

$$-\infty < x < \infty, -\infty < \mu < \infty, 0 < \sigma < \infty.$$

Where (i) X is the associated normal variable,

(ii) μ & σ are the two parameters of the distribution

and (iii) Mean of $X = \mu$ & Standard Deviation of $X = \sigma$.

For a normal distribution mean, median and mode are equal. Moreover, the midrange of the distribution coincides with each of them.

2. DEVELOPMENT OF THE METHOD

Let

$$X_1, X_2, \dots, X_n$$

be distinct observations on the annual extremum (maximum or minimum) of the ambient air temperature at a location in the years

1, 2, 3, ……………, n

respectively.

(If the available observations are not distinct, one can extract the distinct observations from them.)

If μ is the central tendency of the annual extremum of the ambient air temperature at the location and if the variation among the observations occurs due to chance cause only,,

$$X_i = \mu + \varepsilon_i \quad , \quad (i = 1, 2, \dots, n) \tag{2.1}$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are values of the chance error variable ε associated to X_1, X_2, \dots, X_n respectively.

It is to be noted that

(1) X_1, X_2, \dots, X_n are known,

(2) $\mu, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are unknown

& (3) the number of linear equations in (2.1) is n with $n + 1$ unknowns implying that the equations are not solvable mathematically.

Reasonable facts /Assumptions regarding ε_i :

(1) $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are unknown values of the variables ε .

(2) The values $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ are very small relative to the respective values X_1, X_2, \dots, X_n .

(3) The variable ε assumes both positive and negative values.

(4) $P(-a - da < \varepsilon < -a) = P(a < \varepsilon < a + da)$ for every real a .

(5) $P(a < \varepsilon < a + da) > P(b < \varepsilon < b + db)$
 & $P(-a - da < \varepsilon < -a) < P(-b - db < \varepsilon < -b)$
 for every real positive $a < b$.

(6) The facts (3), (4) & (5) together imply that ε obeys the normal probability law.

(7) Sum of all possible values of each ε is 0 (zero) which together with the fact (6) implies that $E(\varepsilon) = 0$.

(8) Standard deviation of ε is unknown and small, say σ_ε .

(9) The facts (6), (7) & (8) together imply that ε obeys the normal probability law with mean (expectation) 0 & standard deviation σ_ε . Thus

$$\varepsilon \sim N(0, \sigma_\varepsilon) \tag{2.2}$$

Note (2.1): Since

$\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$

are independently and identically distributed $N(0, \sigma_\varepsilon)$ variates,

their mean defined by

$$\bar{\varepsilon}_l = n^{-1} \sum_{i=1}^n \varepsilon_i$$

is a $N(0, \sigma_\varepsilon/\sqrt{n})$ variate.

3. THE METHOD

Let the observations be arranged in ascending order of magnitude as

$$X_{(1)} < X_{(2)} < \dots < X_{(n)} \tag{2.3}$$

From the model (2.1) satisfied by the observations,

$$X_{(i)} = \mu + \varepsilon_{(i)} \quad , \quad (i = 1, 2, \dots, n) \tag{2.4}$$

where $\varepsilon_{(1)} < \varepsilon_{(2)} < \dots, \varepsilon_{(n)}$

Which implies that $X_{(1)}$ contains the maximum negative error and $X_{(n)}$ contains the maximum positive error among the errors associated to the observations.

Let us construct the n averages defined by

$$\bar{X}_{(i)}(1) = (n - 1)^{-1} \sum_{j=1, j \neq i}^n X_{(j)} \tag{2.5}$$

(i = 1, 2, \dots, n)

Here $\bar{X}_{(1)}(1) > \bar{X}_{(2)}(1) > \dots > \bar{X}_{(n-1)}(1) > \bar{X}_{(n)}(1)$ (2.6)

From the model (2.1),

$$\bar{X}_{(i)}(1) = \mu + \varepsilon_{(i)}(1) \tag{2.7}$$

Where

$$\varepsilon_{(i)}(1) = (n - 1)^{-1} \sum_{j=1, j \neq i}^n \varepsilon_{(j)} \tag{2.8}$$

(i = 1, 2, \dots, n)

By Note (2.1), some of the averages

$$\bar{\varepsilon}_{(1)}(1), \bar{\varepsilon}_{(2)}(1), \dots > \bar{\varepsilon}_{(n-1)}(1), \bar{\varepsilon}_{(n)}(1)$$

will lie above 0 and the others below 0.

Consequently, some of the averages

$$\bar{X}_{(1)}(1), \bar{X}_{(2)}(1), \dots > \bar{X}_{(n-1)}(1), \bar{X}_{(n)}(1)$$

Will lie above μ and the others below μ .

$$\bar{X}_{(1)}(1), \bar{X}_{(2)}(1), \dots, \bar{X}_{(k)}(1)$$

Fall above μ and

$$\bar{X}_{(k+1)}(1), \bar{X}_{(k+2)}(1), \dots, \bar{X}_{(n)}(1)$$

Fall below μ .

Then μ will lie within

$$X_{(k+1)}(1) & X_{(k)}(1) \text{ with} \\ X_{(k+1)}(1) < \mu < X_{(k)}(1) \tag{2.9}$$

Of course, it is trivial that

$$X_{(n)}(1) < \mu < X_{(1)}(1) \tag{2.10}$$

The interval (2.9) can help to determine the true value of μ .

Note that positive error associated to $X_{(i)}(1)$ decreases as i moves from 1 towards some point p and that negative error associated to $X_{(i)}(1)$ decreases as i moves from n towards the point p .

Thus, $X_{(p)}(1)$ is the true value of μ . However, it is still unknown.

It can be thought that it can be possible to detect / determine the true value of μ from an interval value of μ which is of sufficiently small length.

It is to be noted that if one among the large number (n) of observations is excluded and the same method is applied on the remaining observations, one can obtain valid interval for the true value of μ of the type given by (2.9).

Thus, one can obtain a number of such valid intervals for the true value of μ of the type given by (2.9) based on all the observations excluding each one of the available observations.

From the set of these intervals one can obtain the shortest possible interval for the true value of μ . This shortest interval can provide the true value of μ .

4. AMBIENT AIR TEMPERATURE AT DHUBRI

Observations on annual maximum and annual minimum of the ambient air temperature at Dhubri, which have been collected from the metrological department of India, are available from the year 1969 to the year 2013. These have been presented in Table -1 and Table - 4 respectively.

Computation of the Central Tendency of Annual Maximum :

Table – 2 has been constructed for distinct observed values on annual maximum obtained from Table- 1

In Table - 2 has been constructed for distinct observed values on annual maximum arranged in ascending order of magnitude.

In order to determine the value of the central tendency of annual maximum, interval values have been computed by the formula (2.9) from the distinct observations excluding each one of them one after another starting from approximately the middle position and then alternately one from above and from below along with the corresponding shortest interval. These values have been presented in Table - 3.

The shortest interval, obtained, for the central tendency of annual maximum is (37.0364 , 37.3318)

Now, the real number which is strictly greater than 37.0364 and strictly less than 37.3318 are 37.1 and 37.2 (corrected up to one place of decimal).

Computation of the Central Tendency of Annual Minimum :

Table - 5 has been constructed for distinct observed values on annual minimum obtained from Table - 4

Interval values are to be computed from order statistics.

Therefore, Table - 6 has been constructed for distinct observed values on annual minimum arranged in ascending order of magnitude.

In order to determine the value of the central tendency of annual minimum, interval values have been computed by the formula (2.9) from the distinct observations excluding each one of them one after another starting from approximately the middle position and then alternately one from above and from below along with the corresponding shortest interval. These values have been presented in Table - 6.

The shortest interval, obtained, for the central tendency of annual minimum is

(8.7409 , 8.7636)

Now, the real number which is strictly greater than 8.7409 and strictly less than 8.75 is 8.8 (corrected up to one place of decimal).

Hence, the true value of the central tendency of annual minimum of the ambient air temperature at Dhubri is 8.8 Degree Celsius.

TABLES OF DATA, COMPUTATIONS AND RESULTS

Tables of Annual Maximum of ambient air temperature at Dhubri :

TABLE – 1

Observed Value on Highest Maximum Temperature (in Degree Celsius)
occurred during Temperature Periodic Year

Year no	Observed value	Calendar year, Month & Date of occurrence	Year no	Observed value	Calendar year, Month & Date of occurrence
1	36.5	1969, May 21	14	35.8	2003, June 3
2	36.1	1970, April 1	15	35.5	2004, May 8
3	36.2	1971, March 27	16	35.3	2005, September 19
4	35.2	1972, March 27 & July 13	17	36.4	2006, August 11
5	39.6	1973, April 17	18	36.8	2007, August 10
6	35.7	1974, March 18	19	35.2	2008, March 12 & August 9
7	37.8	1975, April 3	20	36.3	2009, April 27
8	38.4	1976, April 13	21	35.5	2010, March 21
9	35.7	1977, April 24	22	35.0	2011, August 31
10	38.7	1979, June 6	23	36.2	2012, April 3
11	37.5	1980, April 16	24	36.0	2013, June 11
12	35.8	1981, June 13			
13	36.1	2001, March 7			

TABLE – 2

Distinct Observed values on highest Maximum temperature (in Degree Celsius)
occurred during temperature periodic year in ascending order.

Serial No	Observed value	Serial No	Observed value	Serial No	Observed value	Serial No	Observed value
1	35.0	6	35.7	11	36.3	16	37.8
2	35.1	7	35.8	12	36.4	17	38.4
3	35.2	8	36.0	13	36.5	18	38.7
4	35.3	9	36.1	14	36.8	19	39.6
5	35.5	10	36.2	15	37.5		

TABLE – 3

Interval values on highest Maximum temperature (in Degree Celsius) occurred during temperature periodic year.

Serial No	Excluded Observation	Interval Yielded	Shortest Interval Yielded
1	Nil	(36.35 , 36.6056)	(36.35 , 36.6056)
2	36.2	(36.3588 , 36.6294)	(36.3588 , 36.6056)
3	36.3	(36.3529 , 36.6353)	(36.3588 , 36.6056)
4	36.1	(36.3647 , 36.6353)	(36.3647 , 36.6056)
5	36.4	(36.3471 , 36.6176)	(36.3647 , 36.6056)
6	36.0	(36.3706 , 36.6412)	(36.3706 , 36.6056)
7	36.5	(36.3412 , 36.6118)	(36.3706 , 36.6056)
8	35.8	(36.3824 , 36.6529)	(36.3824 , 36.6056)
9	36.8	(36.3235 , 36.5941)	(36.3824 , 36.5941)
10	35.7	(36.3882 , 36.6588)	(36.3882 , 36.5941)
11	37.5	(36.2824 , 36.5529)	(36.3882 , 36.5529)
12	35.5	(36.4000 , 36.6706)	(36.4000 , 36.5529)
13	37.8	(36.2647 , 36.5353)	(36.4000 , 36.5353)
14	35.3	(36.4118 , 36.6824)	(36.4118 , 36.5353)
15	38.4	(36.2214 , 36.5)	(36.4118 , 36.5)
16	35.2	(36.4176 , 36.6882)	(36.4176 , 36.5)
17	38.7	(36.2118 , 36.4824)	(36.4176 , 36.4824)
18	35.1	(36.4235 , 36.6941)	(36.4235 , 36.4824)

From this table the shortest interval is (36.4235 , 36.4824) Therefore Maximum temperature at Dhubri is 36.4

Tables for Annual Minimum of Ambient Air Temperature at Dhubri.

TABLE – 4

Observed Value on Lowest Minimum Temperature (in Degree Celsius) occurred during Temperature Periodic Year.

Year no	Observed value	Calendar year, Month & Date of occurrence	Year no	Observed value	Calendar year, Month & Date of occurrence
1	8.1	1969, January 16	13	6.1	2003, January 24
2	7.3	1971, January 31	14	10.0	2004, January 10
3	8.8	1972, February 08	15	9.5	2004, December 27
4	9.2	1973, January 10	16	10.5	2006, January 23
5	9.3	1974, February 08	17	8.9	2007, January 15
6	9.6	1975, January 10	18	9.1	2008, February 02
7	8.6	1976, January 22	19	12.2	2009, January 06
8	7.6	1977, January 30	20	10.0	2009, December 30 & 2010, January 03
9	8.9	1979, January 05	21	9.0	2011, January 14
10	8.4	1980, January 18	22	7.8	2012, January 16
11	9.4	1981, January 10	23	5.8	2013, January 09
12	8.8	2001, January 06			

TABLE – 5
Distinct Observations on Lowest Minimum temperature (in Degree Celsius)
occured during temperature periodic year in ascending order.

Serial No	Observed value	Serial No	Observed value	Serial No	Observed value	Serial No	Observed value
1	5.8	6	8.1	11	9.0	16	9.5
2	6.1	7	8.4	12	9.1	17	9.6
3	7.3	8	8.6	13	9.2	18	10.0
4	7.6	9	8.8	14	9.3	19	10.5
5	7.8	10	8.9	15	9.4	20	12.2

TABLE – 6
Interval values on Lowest Minimum Temperature (in Degree Celsius)
occured during Temperature periodic year.

Serial No	Excluded Observation	Interval Yielded	Shoetest Interval Yielded
1	Nil	(8.5789 , 8.9158)	(8.5789 , 8.9158)
2	9.0	(8.5556 , 8.9111)	(8.5789 , 8.9111)
3	9.1	(8.55 , 8.9056)	(8.5789 , 8.9056)
4	8.9	(8.5611 , 8.9167)	(8.5789 , 8.9056)
5	9.2	(8.5444 , 8.9)	(8.5789 , 8.9)
6	8.8	(8.5667 , 8.922)	(8.5789 , 8.9)
7	9.3	(8.5389 , 8.8944)	(8.5789 , 8.8944)
8	8.6	(8.5778 , 8.933)	(8.5789 , 8.8944)
9	9.4	(8.5333 , 8.8889)	(8.5789 , 8.8889)
10	8.4	(8.5889 , 8.944)	(8.5889 , 8.8889)
11	9.5	(8.5278 , 8.8833)	(8.5889 , 8.8833)
12	8.1	(8.6056 , 8.9611)	(8.6056 , 8.8833)
13	9.6	(8.5222 , 8.8778)	(8.6056 , 8.8778)
14	7.8	(8.6222 , 8.9778)	(8.6222 , 8.8778)
15	10.0	(8.5 , 8.8556)	(8.6222 , 8.8556)
16	7.6	(8.6333 , 8.9889)	(8.6333 , 8.8556)
17	10.5	(8.4722 , 8.8278)	(8.6333 , 8.8278)
18	7.3	(8.65 , 9.0056)	(8.65 , 8.8278)
19	6.1	(8.7167 , 9.0722)	(8.7167 , 8.8278)

From this table the shortest interval is (8.7167 , 8.8278) Therefore Maximum temperature at Dhubri is 8.7 (in one decimal places)

5. CONCLUSION

Each of the existing statistical methods of estimation provides an estimate of the central tendency of annual extremum of the ambient air temperature which suffers from an error though may be small. Moreover, the amount of error involved in this estimate is unknown. The method developed here provides an estimate which is free from error.

The determination of central tendency of the extremum of ambient air temperature at Dhubri is based on the assumption that change in temperature at this location over years during the period for which data are available occurs due to chance cause only but not due to any assignable cause.

Thus if the assumption is true, the values of the central tendency of annual maximum and annual minimum of the ambient air temperature at Dhubri namely 36.4 Degree Celsius and 8.7 Degree Celsius respectively, as obtained in this study, are acceptable. Moreover, one can conclude that

- i) The central tendency of Annual Maximum of the Ambient Air temperature at Dhubri can not be less than 36.4235 and greater than 36.4824 degree celcius and
- ii) The central tendency of annual minimum of the Ambient Air temperature at Dhubri can not be less than 8.7167 and greater than 8.8278 degree celcius.

However, it is yet to examine whether the assumption upon which the current study is based is true.

For a normal distribution mean, median and mode are equal. Each of them is a measure of central tendency. It seems that there exists some method of determination of central tendency in the same situation. Thus, it is a problem for the researchers at this stage to search for whether there exists method for the same based on mean, median and mode as well as to discover the hidden method if exists.

REFERENCES

1. D. Chakrabarty, Analysis of Errors Associated to Observations of Measurement Type, *International Journal of Electronics and Applied Research*, 2014, 1(1), 15 – 28, 2.
2. D. Chakrabarty, Probabilistic Forecasting of Time Series, Report of Post-Doctoral Research Project (2002-2005), University Grants Commission, New Delhi. 2005
3. D.Chakrabarty, Temperature in Assam : Natural Extreme Value, *J. Chem. Bio. Phy. Sci. Sec. C*, 2014, 4(2), 14791-488.
4. D. Chakrabarty, Determination of Parameter from Observations Composed of Itself and Errors, *International Journal of Engineering Science and Innovative Technology*, 2014, 3(2), 304-311.
5. D. Chakarbarty, Observation Composed of a Parameter and Error: An Analytical Method of Determining the Parameter, *International Journal of Electronics and Applied Research*, 2014, 1(2), 20-38.
6. M.G. Kendall & A. Stuart, *Advanced Theory of Statistics*, Vol. 1 & 2, 4th Edition, New York, Hafner Press, 1977.
7. M. Walker, Helen M., & J. Lev, *Statistical Inference*, Oxford & IBH Publishing Company, 1965.

8. D. Chakarabarty, Natural Interval of Monthly Extreme Temperature in the context of Assam, J. Chem. Bio. Phy. Sci. Sec. C, 2014, 4(3), 2424-2433.
9. B. Wlodzimierz, The Normal Distribution: Characterizations with Applications, Springer Verlag, 1995, ISBN 0-387-97990-5.
10. Hazewinkel Michieled, Normal Distribution, Encyclopedia of Mathematics, Springer, 2001, ISBN 978-1-55608-010-4.
11. M. Walker & Helen M, De Moivre on the Law of Normal Probability, in Smith, David Eugene. A Source Book in Mathematics, Dover, 64690-4, 1985.



C-COMPLEX MATRIX COMPLETION PROBLEM

M. S. Ponmudi* & A. Ramesh Kumar**

*Assistant Professor, Department of Mathematics, Srimad Andavan Arts and Science College (Autonomous), T.V.Kovil, Trichy-5.

**Assistant Professor, Department of Mathematics, Srimad Andavan Arts and Science College (Autonomous), T.V.Kovil, Trichy-5.

E-mail : ponmudims@gmail.com, andavanmathsramesh@gmail.com

ABSTRACT :

In this paper we are interested in C-matrix completion problem, when a partial C-matrix has C-matrix completion. An $n \times n$ complex matrix is called a C-matrix if all its principal minors are negative. Here a combinatorially symmetric partial C-matrix has C-matrix completion if the graph of its specified entries is a 1-chordal graph, and then there exists C-matrix completion for a partial C-matrix whose associated graph is an undirected cycle.

Keywords : matrix; Partial matrix; Completion problem; Undirected graph.

MSc code: 15A48

1. INTRODUCTION:

A partial matrix is an array in which some entries are specified, while the remaining entries are free to be chosen. We make the assumption throughout that all diagonal entries are prescribed. A completion of a partial matrix is the conventional matrix resulting from a particular choice of values for the unspecified entries. The completion obtained by replacing all the unspecified entries by zero is called the zero completion and denoted by A_0 . A matrix completion problem asks which partial matrices have completions with a given property.

A natural way to describe an $n \times n$ partial matrix A is via a graph $G_A = (V, E)$, where the set of vertices V is $\{1, 2, \dots, n\}$ and $\{i, j\}, i \neq j$, is an edge or arc if and only if the (i, j) entry is specified as all diagonal entries are specified, we omit loops. A directed graph is specified with a non-combinatorially symmetric partial matrix and, when the partial matrix is combinatorially symmetric, an undirected graph can be used. In this paper we are going to work with combinatorially symmetric partial matrices and therefore we deal only with undirected graphs.

In general, a combinatorially or non combinatorially symmetric partial C -matrix does not have a C -matrix completion. An $n \times n$ partial matrix is said to be combinatorially symmetric if the $(i, j)^{th}$ entry is specified if and only if the $(j, i)^{th}$ entry is; and is said to be sign-symmetric if, for all $i, j \in \{1, 2, \dots, n\}$ such that both $(i, j), (j, i)$ entries are specified.

A path is a sequence of edges $\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{k-1}, i_k\}$ in which all vertices are distinct. A cycle is a closed path, that is, a path in which the first and the last vertices coincide. A chord of the cycle $\{i_1, i_2\}, \{i_2, i_3\}, \dots, \{i_{k-1}, i_k\}, \{i_k, i_1\}$ is an edge $\{i_s, i_t\}$ not in the cycle (with $1 \leq s, t \leq k$).

2. VARIOUS TYPES OF MATRICES

2.1. C-COMPLEX MATRIX

An $n \times n$ complex matrix is called C -matrix if all principal minors are negative.

$$C = \begin{bmatrix} i^2 b_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & i^2 b_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & i^2 b_{nn} \end{bmatrix}_{n \times n}$$

eg. the C -matrix of 3×3 .

$$C_{3 \times 3} = \begin{bmatrix} i^2 b_{11} & a_{12} & a_{13} \\ a_{21} & i^2 b_{22} & a_{23} \\ a_{31} & a_{32} & i^2 b_{33} \end{bmatrix}_{3 \times 3}$$

2.2. PERMUTATION MATRICES:

A permutation matrix P is obtained by interchanging rows on the identity matrix. The Permutation matrix P is then PDP^T .

NOTATIONS: Let $A = (a_{ij})$ be an $n \times n$ C -matrix. then

- (i) If P is a permutation matrix, then PDP^T is a C -matrix.
- (ii) If D is a positive diagonal matrix, then DA, AD are C -matrices.
- (iii) If D is a non-singular diagonal matrix, then DAD^{-1} is a C -matrix.
- (iv) $a_{ij} \neq 0$ and $\text{sign}(a_{ij}) = \text{sign}(a_{ji})$ for all $i, j \in \{1, \dots, n\}$.
- (v) If $a_{i+1} > 0, i = 1, 2, \dots, n-1$, then $A \in g_n$, where $\{A = (a_{ij}) \mid a_{ij} \neq 0 \text{ and } \text{sign}(a_{ij}) = (-1)^{i+j+1}, \text{ for all } i, j \in \{1, \dots, n\}\}$.
- (vi) Any principal sub matrix of A is a C -matrix.

2.3. PARTIAL C-MATRIX:

A partial matrix is said to be a partial C -matrix if every completely specified principal sub matrix is a C -matrix.

Our interest here in the C -matrix completion problem, that is, when a partial C -matrix has an C -matrix completion. Keeping this in mind, It would not make sense to study the existence of C -matrix completion of partial C -matrix with some null entry or of non-sign symmetric partial C -matrices, as the following example illustrate.

Example 2.1

Consider the partial C -matrix

$$A = \begin{bmatrix} i^2 & i^4 & 0 \\ 2 & i^2 & i^4 \\ ? & 2 & i^2 \end{bmatrix} \Rightarrow \begin{vmatrix} i^2 & i^4 \\ 2 & i^2 \end{vmatrix} = \begin{vmatrix} -1 & i^4 \\ 2 & -1 \end{vmatrix} = 1 - 2 < 0, A \text{ has no } C\text{-matrix completion, as any completion}$$

$$A = \begin{bmatrix} i^2 & i^4 & 0 \\ 2 & i^2 & i^4 \\ a & 2 & i^2 \end{bmatrix} \Rightarrow \begin{vmatrix} i^4 & 0 \\ i^2 & i^4 \end{vmatrix} = i^8 = 1 > 0 \text{ of } A \text{ is not } C\text{-matrix by Notations.}$$

Example 2.2

Consider the partial C -matrix

$$A = \begin{bmatrix} ? & ? & 2i^2 \\ ? & i^2 & ? \\ 3 & ? & i^2 \end{bmatrix}$$

A is not sign-symmetric. A admits no C -matrix completion, since any completion

$$A = \begin{bmatrix} a & b & 2i^2 \\ c & i^2 & d \\ 3 & e & i^2 \end{bmatrix} \text{ of } A \text{ is not } C\text{-matrix by Notations.}$$

Proposition 2.1

Let A be a 2×2 sign-symmetric partial C -matrix whose specified entries are all non-zero. A has an C -matrix completion.

Proof

If A is completely unspecified or completely specified, the result is trivial. We denote by U_E the number of unspecified entries of A . Consider the following cases:

a) $U_E = 1$

Using adequate permutation similarities, we can assume that A has either the form $A = \begin{pmatrix} -a_{11} & ? \\ a_{21} & -a_{22} \end{pmatrix}$, with

$a_{11}, a_{22} > 0$, or the form $A = \begin{pmatrix} ? & a_{12} \\ a_{21} & -a_{22} \end{pmatrix}$, With $a_{22} > 0$. In the first case, it suffices to consider a completion

$$A_c = \begin{pmatrix} -a_{11} & c \\ a_{21} & -a_{22} \end{pmatrix} \text{ of } A \text{ such that } a_{21} c > a_{11} a_{22}. \text{ In the second case, consider a completion } A_c = \begin{pmatrix} -c & a_{12} \\ a_{21} & -a_{22} \end{pmatrix},$$

Of A with $0 < c < a_{12} a_{21} / a_{22}$.

b) $U_E > 1$.

In this case, we can complete some entries of matrix A in a way to obtain a partial C -matrix with exactly one unspecified entry and, then, use (a).

Unfortunately a sign-symmetric $n \times n$ partial C -matrix A , with no null specified entries does not admit, in general, C -matrix completions, for $n \geq 3$ when A is non-combinatorially symmetric and for $n \geq 4$ when A is combinatorially symmetric.

Example 2.3

Let $A = (a_{ij})$ be the following non-combinatorially symmetric partial C -matrix

$$A = \begin{bmatrix} i^2 & ? & 3 \\ 2 & i^2 & i^4 \\ ? & 2 & i^2 \end{bmatrix}.$$

Observe that A is partial sign-symmetric and has no specified zero entries. However, $\text{sign}(a_{13}) = 1 \neq (-1)^{1+3+1}$.

Thus, A has no C -matrix completion by proposition 1.1.

By embedding the above matrix as a principal submatrix and putting -1 's on the main diagonal and unspecified entries on the remaining position, that is

$$M = \begin{pmatrix} A & X \\ Y & \bar{I} \end{pmatrix}, \text{ Where } \bar{I} \text{ is a partial matrix with all entries unspecified except for the entries of the main}$$

diagonal that are equal to -1 , and X and Y are completely unspecified submatrices, we produce a partial C -matrix of size $n \times n, n \geq 4$, which has no C -matrix completion.

Example 2.4

Consider the following combinatorially symmetric partial matrix $A = (a_{ij})$

$$A = \begin{bmatrix} i^2 & i^4 & ? & -3 \\ 2 & i^2 & i^4 & ? \\ ? & 2 & i^2 & i^4 \\ -4 & ? & 2 & i^2 \end{bmatrix}.$$

A is sign-symmetric and has no specified null entries. Observe that $\text{sign}(a_{14}) = -1 \neq (-1)^{1+4+1}$. Then, A has no C -matrix completion by notations.

We can extend this result for partial C -matrices of size $n \times n, n \geq 5$, to exclude partial C -matrices like the ones considered in the preceding examples,

We can define the set pg_n to consist of the $n \times n$ partial matrices $A = (a_{ij})$ such that $a_{ij} \neq 0$ and $\text{sign}(a_{ij}) = (-1)^{i+j+1}$, for all $i, j \in \{1, \dots, n\}$ such that the (i, j) entry specified.

When restricting our study to partial C -matrices that belong to pg_n , we are implicitly analyzing the completion problem for partial C -matrices that are permutation or diagonally similar to a partial C -matrix that belongs to pg_n . Take, for instance, a partial C -matrix A with all specified entries negative; it is not difficult to verify that A is diagonally similar to a partial C -matrix $B \in \text{pg}_n$. Therefore, A has an C -matrix completion if and only if B does. Belonging to pg_n is a necessary condition in order to obtain an C -matrix completion of a partial C -matrix. We observe that every combinatorially symmetric partial C -matrix of size 3×3 belongs to pg_3 .

Proposition 2.2

Let A be a 3×3 partial C -matrix. There exists an C -matrix completion A_c of A if and only if $A \in \text{pg}_3$.

Proof.

In light of the preceding remarks we just need to show sufficiency. So let A be in pg_3 . Moreover, as the class of C -matrices is invariant under left and right positive diagonal multiplication, we may take all diagonal entries of A to be -1 . We denote by U_E the number of unspecified entries of A . If $U_E = 0$, A is not a partial matrix and if $U_E = 6$, the result is trivial.

Let us first consider the case in which A has exactly one unspecified entry. By permutation and diagonal similarities, we can assume that this entry is in position $(1,3)$ and that all upper diagonal entries are equal to 1 .

Hence, A has the following form

$$A = \begin{bmatrix} i^2 & i^4 & ? \\ a_{21} & i^2 & i^4 \\ -a_{31} & a_{32} & i^2 \end{bmatrix} \text{ with } a_{21}, a_{32} > 1 \text{ and } a_{31} > 0.$$

Our aim is to prove existence of $c > 0$ such that the completion

$$A_c = \begin{bmatrix} i^2 & i^4 & -c \\ a_{21} & i^2 & i^4 \\ -a_{31} & a_{32} & i^2 \end{bmatrix}$$

Of A is an C -matrix. If $a_{31} > 1$, it suffices to choose $c = 1$. If $a_{31} = 1$, then A_c is an C -matrix for all $c > 1$. In case $a_{31} < 1$, consider the completion A_c with $c = 1/a_{31}^2$.

The formulation of the problem in case $U_E > 1$ reduces to that of $U_E = 1$. In fact, it is possible to complete some adequate unspecified entries in order to obtain a partial C -matrix in pg_3 with a single unspecified entry.

Example 2.5

Let A be the partial matrix $A = \begin{bmatrix} i^2 & i^4 & 11i^2 & ? \\ 2 & i^2 & i^4 & 200i^2 \\ 0.1i^2 & 10 & i^2 & i^4 \\ i^4 & 10i^2 & 1.01 & i^2 \end{bmatrix}$.

It is not difficult to verify that A is a partial C -matrix and $A \in pg_4$. Given $C \in \mathfrak{R}$, the completion

$$A = \begin{bmatrix} i^2 & i^4 & 11i^2 & c \\ 2 & i^2 & i^4 & 200i^2 \\ 0.1i^2 & 10 & i^2 & i^4 \\ i^4 & 10i^2 & 1.01 & i^2 \end{bmatrix}$$
 of A is not an C -matrix. simple calculations show that

$\det A_c[\{1, 2, 4\}] = 1801 - 19c$ and $\det A_c[\{1, 3, 4\}] = -9.89 + 0.899c$. So, these principal minors are both negative if and only if $c > 1801/19$ and $c < 9.89/0.899$, which is impossible.

3. CHORDAL GRAPHS

In order to get started, we recall some very rich clique structure of chordal graphs. A clique in a graph is simply a complete induced subgraph. We also use clique to refer to a complete graph and we denote by k_p a clique on p vertices. A useful view of chordal graph is that they have a tree-like structure in which their maximal clique play the role of vertices. Consider two graph G_1 and G_2 , each of which containing the clique k_p . If we identify the copy of k_p in G_1 with that in G_2 , then the resulting G is called a clique sum of G_1 and G_2 .

If G_1 is the clique k_p and G_2 is any chordal graph containing the clique k_p , $p < q$, then the clique sum of G_1 and G_2 along k_p is also chordal. The cliques that are used to build chordal graphs are the maximal cliques of the resulting chordal graph and the cliques along which the summing takes place are the so-called minimal vertex separators of the resulting chordal graph. If the maximum number of vertices in a minimal vertex separator is p , then the chordal graph is said to be p -chordal.

Theorem 3.1

Let G be an undirected connected 1-chordal graph. Then any partial C -matrix, the graph of whose specified entries is G , has an C -matrix completion.

Proof.

Let A be a partial C -matrix, the graph of whose specified entries is G . The proof is by induction on the number p of maximal cliques in G . For $p = 2$ we obtain the desired completion, if A is an $n \times n$ partial C -matrices,

then the graph of whose specified entries is 1-chordal with two maximal cliques. Then, A admits an C -matrix completion. suppose that the result is true for a 1-chordal graph with $p-1$ maximal cliques and we are going to prove it for p maximal cliques.

Let G_1 be the subgraph induced by two maximal cliques with a common vertex. The submatrix A_1 of A , the graph of whose specified entries is G_1 , and by replacing the obtained completion A_{1c} in A , we obtain a partial C -matrix such that whose associated graph is 1-chordal with $p-1$ maximal cliques. The induction hypothesis allows us obtain the result.

The completion problem for partial C -matrices, the graph of whose specified entries is P -chordal, $P > 1$, is still unresolved. We note here that any P -chordal graph, $P > 1$, contains, as an induced subgraph, a 2 -chordal graph with four vertices. We can assume, without loss of generality, that a 4×4 partial C -matrix, the graph of whose prescribed entries is a 2 -chordal graph, has the form

$$A = \begin{bmatrix} i^2 & i^4 & i^2 a_{13} & x \\ a_{21} & i^2 & i^4 & i^2 a_{24} \\ i^2 a_{21} & a_{32} & i^2 & i^4 \\ y & i^2 a_{42} & a_{43} & i^2 \end{bmatrix}, \text{ with } a_{21}, a_{32}, a_{43} > 1 \text{ and } a_{13} a_{31}, a_{24} a_{42} > 1. \text{ It is easy to prove that}$$

$$\det A = (a_{32} - 1)xy - x \det A_0[\{234\}/\{123\}] - y \det A_0[\{123\}/\{234\}] + \det A_0 \dots \dots \dots (1)$$

From (1) we are going to obtain sufficient conditions for the existence of the desired completion.

If $\det A_0[\{123\}/\{234\}] > 0$, then A admits an C -matrix completion. Consider the completion

$$A_c = \begin{bmatrix} i^2 & i^4 & i^2 a_{13} & c \\ a_{21} & i^2 & i^4 & i^2 a_{24} \\ i^2 a_{21} & a_{32} & i^2 & i^4 \\ d & i^2 a_{42} & a_{43} & i^2 \end{bmatrix},$$

Where $c \in \mathbb{R}$ such that $0 < c < \min\{a_{13}, a_{24}, \det A_0[\{234\}/\{123\}]/(a_{32} - 1)\}$. Now the determinant of any principal submatrix containing position (4,1) is a polynomial in d with negative leading coefficient. Therefore, there exists $M \in \mathbb{R}$ such that A_c is an C -matrix for $d > M$. if $\det A_0[\{234\}/\{123\}] > 0$, choosing

$$0 < d < \min\left\{\frac{\det A_0[\{234\}/\{123\}]}{(a_{32} - 1)}, a_{42} a_{21}, a_{43} a_{31}\right\},$$

We can prove, in an analogous way to previous case, that there exists $H \in \mathbb{R}$ such that A_c is an C -matrix for $c > H$.

Therefore, if $\det A_0[\{123\}/\{234\}] > 0$ or $\det A_0[\{234\}/\{123\}] > 0$, A admits an C -matrix completion.

A partial matrix A is said to be block diagonal if A can be partitioned as

$$A = \begin{bmatrix} A_1 & ? & \dots & ? \\ ? & A_2 & \dots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \dots & A_k \end{bmatrix},$$

Where $?$ indicates a rectangular set of unspecified positions and each A_i is a partial matrix, $i = 1, 2, \dots, k$

Theorem 3.2. If a partial C -matrix A is permutation similar to a block diagonal partial matrix in which each diagonal block has an C -matrix completion, then A admits an C -matrix completion.

Proof. Consider a block diagonal partial matrix

$$A = \begin{bmatrix} A_1 & ? & \dots & ? \\ ? & A_2 & \dots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \dots & A_k \end{bmatrix}$$

Such that A_i is an $n_i \times n_i$ partial C -matrix that admits an C -matrix completion \bar{A}_i , for every $i \in \{1, 2, \dots, k\}$.

Consider the partial C -matrix

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & ? & \dots & ? \\ ? & \bar{A}_2 & \dots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \dots & \bar{A}_k \end{bmatrix}.$$

We can assume, without loss of generality, that each block \bar{A}_i , belongs to g_{n_i} and that all diagonal elements of \bar{A} are equal to -1 .

The proof is by induction on the number of diagonal blocks k .

Firstly, consider the case $k = 2$. \bar{A} can be partitioned as follows

$$\bar{A} = \begin{bmatrix} \tilde{A}_1 & v & ? & ? \\ u^T & i^2 & ? & ? \\ ? & ? & i^2 & w^T \\ ? & ? & z & \tilde{A}_2 \end{bmatrix}, \quad \text{Where} \quad \bar{A}_1 = \begin{bmatrix} \tilde{A}_1 & v \\ u^T & i^2 \end{bmatrix} \text{ and } \bar{A}_2 = \begin{bmatrix} i^2 & w^T \\ z & \tilde{A}_2 \end{bmatrix}.$$

Consider the partial C -matrix \tilde{A} obtained from \bar{A} by specifying the $(n_1, n_1 + 1)$, $(n_1 + 1, n_1)$ entries with $1, 2$, respectively. The principal submatrix $\tilde{A}[\{1, \dots, n_1 + 1\}]$ of \tilde{A} is a partial C -matrix, the graph of whose specified entries is 1 -chordal and connected. we know there exists an C -matrix completion of that submatrix. Let

$$\begin{bmatrix} \tilde{A}_1 & v & x \\ u^T & i^2 & i^4 \\ y^T & 2 & i^2 \end{bmatrix} \text{ be such a completion. The associated graph of the partial matrix}$$

$$\begin{bmatrix} \tilde{A}_1 & v & x & ? \\ u^T & i^2 & 1 & ? \\ y^T & 2 & i^2 & w^T \\ ? & ? & z & \tilde{A}_2 \end{bmatrix} \text{ is } 1\text{-chordal and connected. This guarantees the existence of an } C\text{-matrix completion of}$$

that matrix and, consequently, of A .

We are now in position to prove the result for $k > 2$. Consider the partial C -matrix

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & ? & \dots & ? \\ ? & \bar{A}_2 & \dots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \dots & \bar{A}_k \end{bmatrix} \text{ and use the first part of the proof to obtain, from the diagonal blocks } \bar{A}_1 \text{ and } \bar{A}_2,$$

C -matrix completion B_1 , belonging to $g_{n_1+n_2}$, of the submatrix $\begin{bmatrix} \bar{A}_1 & ? \\ ? & \bar{A}_2 \end{bmatrix}$.

The partial matrix $\begin{bmatrix} B_1 & ? & \dots & ? \\ ? & \bar{A}_3 & \dots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \dots & \bar{A}_k \end{bmatrix}$ is a block diagonal partial C -matrix with $k-1$ blocks. By the induction

hypothesis, such a partial matrix admits an C -matrix completion. Obviously, that C -matrix is also a completion of A .

Since the class of C -matrices is closed under permutation similarity, any partial C -matrix that is permutation similar to a block diagonal partial C -matrix in which each diagonal block has an C -matrix completion can be completed to an C -matrix.

From this result and taking into account that a partial matrix whose graph is non-connected is permutation similar to a block diagonal matrix, we can assume, without loss of generality, that the associated graph of a partial C -matrix is a connected graph.

CONCLUSION:

We conclude that, the partial C -matrix problem is nothing but $n \times n$ C -matrix is a square matrix. The Submatrix of a matrix A of size $n \times n$ lying in row α and column β , $\alpha, \beta \subseteq \{1, 2, \dots, n\}$ is denoted by $A[\alpha, \beta]$. Therefore, a real matrix A of size $n \times n$ is an C -matrix if $\det A[\alpha] < 0$ for all $\alpha \subseteq \{1, 2, \dots, n\}$.

REFERENCES:

1. J.R.S. Blair, B. Peyton, *An introduction to chordal graphs and clique trees, The IMA volumes in Mathematics and its Applications, Springer, New York, 56 (1993) 1–31.*
2. K. Inada, *The production coefficient matrix and the Stolper–Samuelson condition, Econometrica 39 (1971) 219–239.*
3. S.R. Mohan, R. Sridhar, *On characterizing N -matrices using linear complementarity, Linear Algebra Applicat. 160 (1992) 231–245.*
4. S.R. Paranjape, *Simple proofs for the infinite divisibility of multivariate gamma distributions, Sankhya Ser. A 40 (1978) 393–398.*
5. T. Parthasarathy, G. Ravindran, *N -matrices, Linear Algebra Application 139 (1990) 89–102.*
6. R. Saigal, *On the class of complementary cones and Lemke’s algorithm, SIAM J. Appl. Math. 23 (1972) 46–60.*
7. C. Mendes Araújo, Juan R. Torregrosa*, Ana M. Urbano, *N -matrix completion problem, Linear Algebra and its Applications 372 (2003) 111–125.*



A STUDY ON REMANUFACTURING OF USED PRODUCTS IN A VENDOR-BUYER SUPPLY CHAIN

W. Ritha and J. C. Eveline

Department of Mathematics, Holy Cross College (Autonomous), Tiruchirappalli, India.
E-mail : ritha_prakash@yahoo.co.in

ABSTRACT :

Protecting the environment has become a priority for most countries in recent years. Recycling material and remanufacturing used products are inevitable options to reduce waste generation and the exploitation of natural resources. Remanufacturing is often considered as a environmental preferable choice of end of life option in comparison to material recycling or manufacturing of new products. The forward supply chain essentially involves the movement of products from upstream suppliers to the downstream customers while the reverse supply chain involves the movement of used products from customers to upstream suppliers. This paper proposes the study of remanufacturing in the closed loop supply chain consisting of a vendor(manufacturer) and a buyer(retailer).The used products collected by the buyer from the customers are remanufactured by the vendor. The inventory holding cost of collected used products are involved in the model. The optimal lot size of remanufactured products and the collection rate of used products to be remanufactured are obtained which minimize the joint total cost of the supply chain. Finally a numerical example is provided for the described model.

Keywords : *Remanufacturing, EOQ, Vendor-Buyer, Used products, Environment, Reverse logistics.*

Keywords : *Annual maximum, annual minimum, ambient air temperature, Dhubri, analytical method of determination.*

1. INTRODUCTION

Product recovery (repair, refurbishing, remanufacturing) is receiving increasing attention. In the past, engagement in recovery activities was often driven by legislation or by associated environmentally friendly image. But nowadays the main reason for companies to become involved with product recovery is economical. Being active in product recovery reduces the need for virgin materials and thus leads to reduced costs. Recoverable manufacturing systems minimize the environmental impact of industry by reusing materials and reducing energy use. In such systems that are environmentally conscious, products are returned from end users and travel back in the reverse supply chain.

To manufacturers, once a product has been returned to a company, it has several options from which to choose. The first option is to sell the product as a used product if it can be established that it meets sufficient quality levels. The second option is to clean and repair the product to working order. Product repair involves fixing and replacement of failed parts. The third option is to sell the product as a refurbished unit. The product does not lose its identity and is brought back to a specified quality level. The fourth option is to remanufacture. In this option the product will undergo the reverse channel at the fabrication stage where it would be disassembled, remanufactured and reassembled to flow back through the retail outlet back to the consumers as a remanufactured product. The fifth

option is to retrieve one or more valuable parts from the product. The sixth option is to recycle. The main purpose of recycling is reuse materials from used components and products. The seventh option is to recover the energy put in the product through incineration. The last option is disposal. The general goal for any value channel is to keep all materials within the channel and thus minimize any flow into the external environment. The basis of recoverable manufacturing system is remanufacturing. Remanufacturing offers several advantages as a form of waste reduction since it is profitable and environmentally conscious.

Supply chain management has received tremendous attention both from the business world and from academic researchers. Closed loop supply chain consists of both forward supply chain and a reverse supply chain. A forward supply chain is a combination of processes to fulfill customer's requests and includes all possible entities like suppliers, manufacturers, transporters, ware houses, retailers and customers. The management of the reverse flows is an extension of the traditional supply chains with used products or material either returning to reprocessing organizations or being discarded. Reverse supply chain management is defined as the effective and efficient management of the series of activities required to retrieve a product from a customer and either dispose of it or recover value.

The remainder of the paper is organized as follows: Section 2 describes the relevant literature. Section 3 presents the notations and assumptions. The formulation of the model is provided in Section 4. Section 5 illustrates a numerical example. The paper concludes in Section 6. A list of references is also provided.

2. LITERATURE REVIEW

The importance of the repairable/recoverable inventory problem was recognized back in the 1960's. Schradly (1967) determined the optimal procurement and repair quantities for the repairable inventory system of an EOQ model. Mabini et al. (1992) studied the stock out as service level for the repairable system and besides extended to the case of multiple products with limited repair capacity. Fleischmann et al. (2000) explored the design of logistics networks and established general characteristics of product recovery systems. Koh et al. (2002) enquired a joint EOQ and EPQ model to optimally determine EOQ for procurement and inventory level of recoverable products concurrently. Teunter (2004) acquired simple square root formula to determine the optimal production and recovery batch quantities for two classes of policies: (1,R) and (P,1). Inderfurth et al. (2005) included the deteriorating nature of reworkable products into an EPQ-based recoverable system and found the optimal production lot size. A lot of researches addressed the issues of repair and disposal of used products simultaneously. The optimal setup numbers for production and repairs in a collection time interval at a fixed waste disposal rate were derived by Richter (1996a) and assumed waste disposal rate as a decision variable. Richter (1996b) further observed the behaviours of EOQ-related cost factors and/or additional non-EOQ-related cost factors of the repairable system. The EOQ repair and waste disposal problem with integer setup numbers was studied by Richter and Dobos (1999) and showed that the pure strategy for either total repair or total waste disposal is dominant. Teunter (2001) evaluated the recoverable item inventory problem with disposal consideration by involving different holding costs for manufactured and recovered products. Jaber et al. (2014) coped with economic order quantity models where the imperfect items are either sent to an independent repair shop or replaced by good ones from a local supplier. There is growing number of researches addressing remanufacturing issues in a closed-loop supply chain. Guide and Van Wassenhove (2001) showed that the acquisition of used products for remanufacturing is profitable. Heese et al. (2005) developed a quantitative model to investigate the consequences of used products take-back on firms, industry and customers and suggested that a manufacturer can increase both profit margins and sales.

Savaskan et al. (2004) studied four different channel structures in closed-loop supply chains with product remanufacturing and compared these models with respect to return rates, retail prices and channel members' profits. A deterministic mixed integer linear programming model with the network design problem for a closed loop supply chain under uncertainty was developed by Pishvae et al. (2011). Hong and Yeh (2012) concluded that the retailer collection model is better off when the third-party collector is a non-profit organization. Inventory management of produced/remanufactured/repaired and returned items has been receiving increasing attention in recent years. Richter and Weber (2001) extended the Wagner/Whitin model to consider additional variable manufacturing and remanufacturing costs and explored the impact of the disposal excess inventory on the solution. Jayaraman (2006) provided an analytical model for closed-loop supply chains with product recovery and reuse to aid operational decision makers for production planning and control. Chung et al. (2008) developed a closed loop supply chain model with remanufacturing and maximized the joint profits of the supplier, the manufacturer, the third party collector and the retailer. Saadany and Jaber (2008) studied about the coordination of two-level supply chain where the production interruptions are permitted to restore process quality whenever the production process shifts to the out-of-control state. Saadany and Jaber (2010) created and analyzed productions, remanufacture and waste disposal EPQ models. An extended joint economic lot size problem in which the return flow of repairable (remanufacturable) used products was incorporated by Dobos et al. (2011) where the returned products are remanufactured by the vendor.

This paper is an extension of “ **Optimal replenishment quantity of new products and return rate of used products for a retailer** ” by Chih-Chung Lo, Cheng-Kang Chen and Tzu-Chun Weng. In this paper the vendor is engaged in remanufacturing of used products collected by the buyer from the customers. The inventory holding cost for the used products is also included in the model.

3. NOTATIONS AND ASSUMPTIONS

Notations

D	demand of the buyer per time unit,
P_M	manufacturing productivity of the vendor, $P_M > D$,
P_R	remanufacturing productivity of the vendor, $P_R > D$,
K_b	setup cost of an ordering of the buyer,
h_b	holding costs of the new products of the buyer,
u_b	holding costs of the used products of the buyer, $h_b > u_b$,
d_b	disposal costs of the used products of the buyer,
K_v	setup cost of an ordering of the vendor,
h_v	holding costs of the new products of the vendor,
u_v	holding costs of the used products of the vendor, $h_v > u_v$,
C_v	unit purchasing cost of the product of vendor,
Q	lot size of remanufactured products of the system, (decision variable)
τ	return rate of used items to be collected and remanufactured, (decision variable) ($0 < \tau < 1$)
A	The unit cost of collecting, holding and handling a returned product which covers the collecting fee paid by the retailer to consumers.
b	The unit price of a collected used products sold by the retailer to the manufacturer, the salvage value of collected used products $b - A$ is supposed to be positive.

$I(\tau)$ The investment cost of the retailer in collecting used product activities, which is assumed to be a function of return rate τ .

$h_{MR}(\tau)$ holding cost coefficient for manufacturing,

$h_{RM}(\tau)$ holding cost coefficient for remanufacturing,

Assumptions

The assumptions of this model are

1. Single product case.
2. Instantaneous replenishment of the product.
3. Shortage is not permitted.
4. Demand rate is constant and deterministic.
5. Infinite planning horizon.
6. $I(\tau) = C_L\tau^2$ denotes the investment cost function in collecting used products.
7. The cumulative return rate of used products at the current replenishment cycle is expressed by a geometric series as $\sum_{i=1}^{\infty} \tau r^{i-1} = \tau / (1 - r)$, where τ is the initial value and common ratio r , $0 \leq r \leq 1$. It is noted that the total number of collected used products should be non negative and cannot exceed the number of products sold at each replenishment cycle. Hence the constraint $0 \leq \frac{\tau}{1-r} \leq 1$ holds.
8. The vendor collects the used products from the buyer to remanufacture.

4. MODEL FORMULATION

Consider a supply chain consisting of a vendor and a buyer. The buyer is assumed not only to sell the products to public consumers but also to collect those sold used products from them. The collected used products are sent to the vendor for remanufacturing process. The used products sent by the buyer in batches was remanufactured by the vendor.

The joint total cost of the system per cycle is,

$$JTCPC(Q, \tau) = (K_b + K_v) + \frac{Q^2}{2D} [h_b + \tau u_b + h_{MR}(\tau) + h_{RM}(\tau)] + C_v Q + C_L \tau^2 - \frac{(b - A)\tau Q}{1 - r} + d_b(1 - \tau)Q$$

where

$$h_{MR}(\tau) = \tau^2 \cdot \left[h_v \cdot \left(\frac{D}{P_M} - \frac{D}{P_R} \right) - u_v \frac{D}{P_R} \right] - 2\tau \cdot \left[h_v \cdot \left(\frac{D}{P_M} - \frac{D}{P_R} \right) - u_v \right] + h_v \frac{D}{P_M},$$

$$h_{RM}(\tau) = \tau^2 \cdot \left[(h_v - u_v) \cdot \left(\frac{D}{P_R} - \frac{D}{P_M} \right) + u_v \frac{D}{P_M} \right] + 2\tau u_v \cdot \left(1 - \frac{D}{P_M} \right) + h_v \frac{D}{P_M}.$$

The corresponding joint total cost per unit time ($JTCPUT$) can be obtained by dividing the joint total cost per cycle ($JTCPC$) by the cycle length $\frac{Q}{D}$. The objective of the model is to minimize the total cost per unit time, subject to the return rate constraint $0 \leq \frac{\tau}{1-r} \leq 1$. Namely,

Minimize

$$JTCPUT(Q, \tau) = (K_b + K_v) \frac{D}{Q} + \frac{Q}{2} [h_b + \tau u_b + h_{MR}(\tau) + h_{RM}(\tau)] + C_v D + \frac{C_L D \tau^2}{Q} - (b - A) D \frac{\tau}{1-r} + d_b(1 - \tau) D \tag{1}$$

Subject to: $0 \leq \frac{\tau}{1-r} \leq 1$ (2)

In order to solve the proposed non linear programming problem shown in (1), the constraint $0 \leq \frac{\tau}{1-r} \leq 1$ is ignored and the partial derivatives of $JTCPUT(Q, \tau)$ with respect to Q and τ are obtained to find the optimal values.

To find the optimal value of Q for fixed value of τ , the first partial derivative of $JTCPUT(Q, \tau)$ is set to zero and the optimal value of Q is given by,

$$Q^*(\tau) = \frac{\sqrt{2D(K_b + K_v + C_L\tau^2)}}{\sqrt{(h_b + \tau u_b + h_{MR}(\tau) + h_{RM}(\tau))}} \tag{3}$$

Substituting (3) in (1), the $JTCPUT(Q, \tau)$ is expressed as,

$$JTCPUT(\tau) = \sqrt{2D(K_b + K_v + C_L\tau^2)}(h_b + \tau u_b + h_{MR}(\tau) + h_{RM}(\tau)) + C_v D - (b - A)D \frac{\tau}{1 - r} + d_b(1\tau)D \tag{4}$$

Differentiating partially $JTCPUT(\tau)$ with respect to τ and equating to zero, gives the optimal solution of τ .

$$\frac{\partial JTCPUT}{\partial \tau} = \sqrt{2D} \left\{ \frac{\sqrt{(K_b + K_v + C_L\tau^2)}}{2\sqrt{(h_b + \tau u_b + h_{MR}(\tau) + h_{RM}(\tau))}} (u_b + h'_{MR}(\tau) + h'_{RM}(\tau)) + \frac{\sqrt{(h_b + \tau u_b + h_{MR}(\tau) + h_{RM}(\tau))}}{\sqrt{(K_b + K_v + C_L\tau^2)}} (C_L\tau) \right\} - \frac{(b - A)D}{1 - r} - d_b D = 0 \tag{5}$$

5. NUMERICAL EXAMPLE

In this section we establish a numerical example for the above proposed model. The following parameters are used for finding the solution:

$D = 1,000$ piece/year, $P_M = 2,500$ piece/year, $P_R = 1,200$ piece/year, $K_b = 100$ \$/ordering, $h_b = 5$ \$/piece/year, $u_b = 1$ \$/piece/year, $K_v = 1,000$ \$/ordering, $h_v = 3$ \$/piece/year, $u_v = 1$ \$/piece/year, $d_b = 1$ \$/piece, $C_v = 10$, $b = 2$, $A = 1$, $C_L = 50000$ and $r = 0.1$.

From (5) we found $\tau^* = 0.00141492$

From (3) we found $Q^* = 544.921$ and

From (1) we get $JTCPUT^* = 15,034.661$

6. CONCLUSION

This paper studies the remanufacturing of used products in the supply chain comprising of a vendor and a buyer. Remanufacturing is an eco-friendly option as it uses less energy than manufacturing a new product, reduces CO₂ emissions, reduces flow of material to landfill and reduces raw material consumption. The incorporation of remanufacturing in the supply chain reduces the total cost of the system.

REFERENCES

- Chih-Chung Lo., Cheng-Kang Chen , and Tzu-Chun Weng.(2016). “Optimal replenishment quantity of new products for a retailer.” Applied Mathematical Modelling 40(23-24): 9754-9766.
- Chung , S.L., H.M. Wee, and P.C. Yang .(2008). “Optimal policy for a closed-loop supply chain inventory system with remanufacturing”, Math. Comput. Model 48: 867–881.

3. Dobos, I., Gobsch, B., Pakhomova, N., Richter, K.: Remanufacturing of used products in a closed-loop supply chain. In: Csutora, M., Kerekes, S. (eds.) *Accounting for climate change– What and how to measure*, pp. 130–146. Proceedings of the EMAN-EU 2011 Conference, 24–25 January 2011, Budapest, Hungary (2011).
4. El Saadany, A., and M.Y.Jaber.(2008). “Coordinating a two-level supply chain with production interruptions to restore process quality.” *Comput. Ind. Eng* 54: 95–109.
5. El Saadany , A., and M. Y. Jaber.(2010). “A production/remanufacturing inventory model with price and quality dependent return rate.” *Comput. Ind. Eng* 58: 352–362.
6. Fleischmann, M., H.R. Krikke, R. Dekker, and S.D.P. Flapper.(2000). “A characterization of logistics networks for product recovery.” *Omega* 28 : 653–666.
7. Guide, V.D.R., and L.N. Van Wassenhove.(2001). “Managing product returns for remanufacturing.” *Prod. Oper. Manag* 10:142–155.
8. Heese, H.S., K. Cattani, G. Ferrer, W. Gilland, and A.V. Roth.(2005). “Competitive advantage through take-back of used products.” *Eur. J. Oper. Res* 164: 143–157.
9. Hong, I.H., and J.S. Yeh.(2012). “Modeling closed-loop supply chains in the electronics industry : A retailer collection application.” *Transport. Res. E Logist. Transport. Rev* 48: 817–829.



MATHEMATICAL PRINCIPLES IN VEDAS AND PURANAS AND ITS APPLICATIONS

A. K. Sah

Professor of Mathematics, Marwari College, T. M. Bhagalpur University, Bhagalpur (Bihar)
E-mail : arvind.tmbu@gmail.com

ABSTRACT :

In this paper, we first review the work of Vedas and Purana's literature and its classifications. We present the mathematical principles and Sulba –sstras for knowledge of mathematics. We also develop Vedic Mathematical –Sstras and Sub- sstras for multiplication of two numbers with base 10, 100 and 1000 etc.

Keyword: Vedic literature, Rigveda, Yajurveda, Samveda, Atharvaveda, Upanisad.

1. INTRODUCTION

Vedic Mathematics as propounded by Swami Bharti Krishna Tirth. He came across Ganit Sstras which reconstructed and formed 16 Sstras, 13 Up-Sstras (1911-1918 AD), but which got lost the all Sstras. At the last time of his life, he re-wrote an Introductory Volume on the subject but couldn't write further volume on account of his failing health. It is popularly known as Vedic Mathematics which deals with faster computation techniques.

Vedic Mathematics is not only a sophisticated pedagogic and research tool but also an introduction to an ancient civilization. The Vedic Mathematical System is based upon 16 main and many more sub-sstras, which are formulas that can be applied to various mathematical problems. Vedic Mathematics is only one aspect of the entirety of Vedic Culture, which was at one time the original, worldwide culture of the human race for countless millenniums. Vedic literature as such signifies a vast body of sacred and esoteric knowledge concerning eternal spiritual truths revealed to sages (Rishis) during intense meditation. The Vedas are considered full of all kinds of knowledge and an infallible guide for man in his quest for the four goals –Dharma, Artha (material welfare), Kama (pleasure and happiness) and Moksha (Salvation) [see reference 1, 2, 3, and 4].

2. VEDIC LITERATURE AND CLASSIFICATION

There are four kinds of Vedas – Rigveda, Yajurveda, Samaveda and Atharvaveda.

- Rig Veda – Knowledge of Hymns, 10859 verses. “There is only one truth, only men describe it in different ways.”
- Yajur Veda – Knowledge of Liturgy, 3988.
- Sama Veda – Knowledge of Classical Music, 1549 verses
- Atharva veda – Knowledge of Earth

Rigveda was divided into 21 branches and the Yajurveda into 100 branches, the Samaveda into 1,000 branches and the Atharvaveda into 9 branches (Kurma Purana 52.19-20). Every branch has four subdivisions called Samhita (or Mantra), Brahmana (contains mantras and prayers), Aranyaka and Upanisad (both with philosophical contents). So all in all, the Vedas consist of 1130 Samhitas, 1130 Brahmanas, 1130 Aranyakas, and 1130 Upanisads, a total of 4520 titles. By the influence of time, however, many texts have been lost, stolen and destroyed. Some scriptures were so intimate that they have buried and hidden so as not be misused by any one in kali- yuga.

Upavedas: There are four Upaveda - Ayur (medicine), Gandharva (music), Dhanur (martial science), Sthapatya (architecture). Upaveda ("applied knowledge") is used in traditional literature to designate the subjects of certain technical works.

Vedangas ("limbs of Veda"): There are six Vedangas - Siksa (pronunciation), Canda (poetic meter), Nirukta (etymology and lexicology), Vyakarana (grammar), Kalpa (ritual), Jyotisa (astronomy and astrology). First two teach how to speak the Veda, second two teach how to understand the meaning of the Veda and the last two teach how to use the Vedas.

Puranas: These explain the teachings of the four Vedas in story form, making spiritual life simpler. There are eighteen Puranas divided into three groups along with three predominating Deities: sattva (goodness) - Visnu, rajasa (passion) - Brahma and tamasa (ignorance)[Ref.Siva. Padma Purana, Uttara khanda].

There are eighteen Maha Puranas : 1. Brahma, 2. Padma, 3. Vaisnava, 4. Saiva (or Vayu), 5. Bhagavata, 6.Bhavisya, 7.Naradiya, 8.Skanda, 9.Linga, 10.Varaha, 11.Markandeya, 12.Agneya, 13.Brahmavaivarta, 14.Kurma, 15.Matsya, 16.Garuda, 17.Vayaviya and 18.Brahmanda. Garuda Purana 31,43,45,64 also adds: "Bhagavata is the best of all Puranas." They are divided in this way to gradually raise the conditioned soul from ignorance to pure goodness.

The Rigveda Samhita

Rigveda mostly consists of hymns to be sung to the various gods as manifestations of the one Divinity. Varuna, Mitra, Surya, Savitr, Vishnu, Pusan, the Ashvin twins, Agni, Soma, Yama, Parjanya, Indra, Maruts, Rudra, Vishvakarman, Prajapati, Matarishvan, Ushas, Aditi are some of the Gods encountered in the Rg Veda. Varuna - the god of the sky, Indra - the god of war and Agni - the god of fire, are more popular than Vishnu and Rudra (Shiva). Surya, Savitr and Pusan all refer to the solar deity and the Gayatri mantra is addressed to Savitr. Ushas and Aditi are goddesses.

This is the oldest Vedic text, as also the largest. It comprises of 10552 mantras in 1028 hymns (=Suktas).The hymns are altogether attributed to 407 Rishis,or Sages, of which 21 are women Sages (= Rishika).

The Yajurveda Samhitas:

There are two Yajurveda : (1) Shukla Yajurveda (2) Krishna Yajurveda. The extant Shukla Yajurveda Samhitas are Madhyandina and Kanva. The extant Krishna Yajurveda Samhitas are Kathaka, Maitrayaniya, Taittiriya (also called 'Apastambhi' Samhita), Kapishtala (fragmentary) and possibly Charaka. Of the extant Yajurveda Samhitas, the two major ones currently are the Madhyandina and the Taittiriya.

The Yajurveda is a liturgical text, but also contains sacrificial formulas to serve the purpose of ceremonial religion (yajur is derived from the root "yag" to sacrifice). Madhyandina Samhita consists of 40 chapters and is given below:

- Chapters 1-2 deal with Darsapurnamasa rites,
- Chapter 3 with sacrifices performed in the morning and the evenings, sacrifices performed every four months at the start of the three seasons
- Chapters 4-8 with Soma sacrifices
- Chapters 9-10 with Rajasuya and Vajapeya
- Chapters 11-18 with construction of altars for yajnas
- Chapters 19-31 with Sautramani rite
- Chapters 22-25 with the Ashvamedha
- Chapters 26-29 give material supplementary to earlier chapters
- Chapters 30-39 contain mantras for novel and unique rites like the Purushamedha, Sarvamedha, Pitamedha and Pravargya
- Chapter 40 is the Isavasya Upanishad

The Samaveda Samhitas and Melodies:

It is purely a liturgical collection that comprises of 1875 Rks. All these verses are set to melodies, called the Samans. The origins of Indian classical music lie in the Sama Veda. The Samhita is divided into two broad divisions- Purvarchika, on which the Gramageva and the Aranyaka samans are set, and the Uttararchika, on which the Uha and the Uhya chants are set.

The Atharvaveda Samhita:

It is often said that the Atharva Angirasa was originally not given the status of a Veda, but seems to have been later elevated to the position. The main theme of the Atharva Veda is cure for diseases, rites for prolonging life and fulfillment of one's desires, statecraft, penances, magic, charms, spells and sorcery. While the Gods of the Rg Veda are approached with love, the Gods of the Atharva Veda are approached with cringing fear and favor is curried to ward off their wrath.

Sophisticated literary style and high metaphysical ideas mark this Veda. The two extant Samhitas of Atharvaveda are Shaunakiya and Paippalada. The former has 5977 mantras while the latter has approximately 7950 mantras.

The Atharvaveda has numerous names –

- Bhargvangirasa Veda – because of association with Bhrigus and Angirasas
- Atharvangirasa Veda – Because of association with Atharvana and Angirasa priests, and because of a dual nature (sorcery as well as 'shanti-pushti' rites)
- Kshatraveda – because it has several hymns dealing with war, rites of coronation and so on.
- Brahmaveda – because it has several hymns dealing with spirituality

3. VEDIC SHAKHAS AND THEIR GEOGRAPHICAL DISTRIBUTION

The Vedic literature that has come down to our times is attached to various traditional schools of recitation and ritual called the 'shakhas'. All the four Vedas have more than one shakha extant. From various sources, it can be determined that the following geographical distribution of Vedic Shakhas at various intervals of times, and their present state of survival:

Shakala Rigveda: Thrives in Maharashtra, Karnataka, Kerala, Orissa, and Tamil Nadu and to some extent in Uttar Pradesh. Might have existed in Punjab. Nambudiris of Kerala recite even the Brahmana and Aranyaka with accents. Accented manuscripts of Brahmana and Aranyaka are available to this day.

Shankhayana Rigveda: Gujarat and parts of Rajasthan and Maharashtra. Oral tradition extinct, only manuscripts of Samhita are extant. Ritual lives in a very fragmentary condition.

Bashkala Rigveda: Claims have been made about its existence in Kerala, Rajasthan, Bengal and Assam as a living tradition, but have never been verified. The Samhita exists in manuscript. Nambudiris of Kerala are said to follow this Shakha of RV as far as the Samhita is concerned but studies of their oral tradition do not seem to bear this out.

1) **Ashvalayana Rigveda:** Manuscripts of the Samhita have been found in Kashmir, Maharashtra (Ahmadnagar) and Patna (Bihar). In parts of central and eastern India, Shakala RV texts are often attributed to Ashvalayana. For instance, the Aitareya Brahman is often called Ashvalayana Brahmana in West Bengal. Oral traditions extinct although the followers of Shakala Shakha in Maharashtra often term themselves as Ashvalayanas because they follow the Kalpasutra (Shrautasutra + Grhyasutra) of Ashvalayana.

2) **Paingi Rigveda:** It exists in Tamil Nadu and around Andavan & Nikowar. Oral traditions lost but Brahmana texts rumored to exist.

Mandukeya Rigveda: Magadha and eastern and central Uttar Pradesh. Possibly lower Himalayas in Uttarakhand and Himachal Pradesh. No text or oral tradition extant although the Brhaddevata and Rigvidhana might belong to it.

Shaunakiya Atharvaveda: Gujarat, Karnataka, Rajasthan, Coastal Andhra Pradesh, Avadh region in Uttar Pradesh, Himachal Pradesh. Only Gujarat has maintained the oral traditions, and the shakha has been resuscitated in recent times in Tamil Nadu, Karnataka and in Andhra Pradesh.

Staudayana Atharvaveda: According to Majjhima Nikaya, followers of this shakha lived in Koshala (central and eastern Uttar Pradesh). The shakha is completely lost.

Paippalada Atharvaveda: Followers are currently found in parts of Orissa and adjacent areas of Bihar and West Bengal and recite the Samhita in ekasruti (monotone syllable). Epigraphic and literary evidence shows that they once thrived in Karnataka, Kerala, Maharashtra, and parts of Gujarat, East Bengal and in Tamil Nadu as well.

Devadarshi Atharvaveda: According to literary evidence, followers of this Shakha once lived in coastal Andhra Pradesh. Other AV shakhas said to have been prevalent in that region were Shaulkayani and Munjakeshi. The shakha is completely lost.

Charanavaidya and Jajala Atharvaveda: Perhaps existed in Gujarat, Central India and adjacent parts of Rajasthan. According to the Vayu and Brahmanda Puranas, the Samhita of the Charanavaidya shakha had 6026 mantras.

3) Mauda Atharvaveda: According to some scholars, they existed in Kashmir

4) Madhyandina Yajurveda: Currently found all over North India- Uttar Pradesh, Haryana, Punjab, Bihar, Madhya Pradesh, Rajasthan, Gujarat and even Maharashtra (up to Nashik), West Bengal, Assam, Nepal. Along with Taittiriya Yajurveda, it is the most prevalent Vedic shakha. Followers of this school were found in Sindh (Pakistan) in the 19th century but became extinct after Hindus were ethnically cleansed by the Muslim majority after 1947.

5) Kanva Yajurveda: Currently found in Maharashtra, Tamil Nadu, and Andhra Pradesh. In Orissa, the followers of this shakha follow a slightly different text. Epigraphic evidence shows that they were once present all over India, as far as Himachal Pradesh and possibly in Nepal.

6) Charaka Yajurveda: Interior Maharashtra, adjacent parts of Madhya Pradesh, Assam, Gujarat, Uttar Pradesh. Followers of this shakha now follow the Maitrayani YV shakha, having lost their own texts.

7) Maitrayani Yajurveda: In Morvi (Gujarat), parts of Maharashtra (Nashik/Bhadgaon, Nandurbar, Dhule). Earlier, they were spread all the way east up to Allahabad and extended into Rajasthan and possibly into Sindh.

8) Kathaka Yajurveda: The oral traditions became extinct possibly a few decades ago. They were found in central and eastern Punjab, Himachal Pradesh, possibly west Punjab and NWFP. In later times, they got restricted to Kashmir, where all their extant manuscripts have been unearthed. Recently, the entire Hindu minority was cleansed from the Kashmir valley by Islamists, and so the shakha might be deemed extinct completely now.

Charayaniya Katha Yajurveda: It existed in Kashmir.

Kapisthala Katha Yajurveda: Found in West Punjab around the time of the invasion of Alexander. Also in parts of Gujarat. Only a fragmentary Samhita and Grhyasutra text exist, and followers of this shakha are said to exist at the mouths of Narmada and Tapi rivers in Gujarat.

9) Jabala Yajurveda: Central India, around the Narmada region. In Maharashtra, there still exist Shukla-Yajurvedin Brahmins who call themselves 'Jabala Brahmins', but there is no knowledge of the existence of any texts of this shakha.

10) Taittiriya Yajurveda: Buddhist texts and some versions of Ramayana attest their presence in the Gangetic plains but currently they are found all over Southern India. The Taittiriyas are themselves divided into numerous sub-schools. Among these, the followers of Baudhayana and Apastamba were found all over South India (including Maharashtra), while the Hiranyakeshins were found mainly in Konkan and Western Maharashtra. The Vaikhanasas have a more eastern presence- around Tirupati and Chennai. The Vadhulas are present currently in Kerala and earlier in adjacent parts of Tamil Nadu. The Agniveshyas, a subdivision of the Vadhula immigrants from Malabar, are found around Thanjavur in Tamil Nadu. The Apastamba, Hiranyakeshin, Vaikhanasa and Baudhayana schools have survived with all their texts intact. The Vadhulas survive, with most of their texts while the Bharadvajas and Agniveshyas are practically extinct as a living tradition although their fragmentary/dilapidated texts survive.

11) Kauthuma Samaveda Gujarat, Maharashtra, Tamil Nadu (tradition revived with the help of Brahmins from Poona), Kerala, Karnataka, Uttar Pradesh, Bihar (tradition revived a century ago), West Bengal (tradition has been revived recently). There are numerous varieties of Kauthuma chanting. This shakha is the most vibrant tradition of Samaveda.

12) Ranayaniya Samaveda: Orissa (manuscripts available, status of oral tradition not known), Maharashtra, Karnataka (the Havyak <havik> community for instance), Uttar Pradesh (till recently in Bahraich and Mathura), Rajasthan (till recently in Jaipur). The existence of this shakha was endangered till recently, but it has been strengthened with the help of institutions like the Kanchi Kamakoti Matha.

13) Jaiminiya/Talavakara Samaveda: Two distinct sub streams- the Namudiri recitations in Central Kerala, and the recitations of Tamil Nadu Brahmins in districts adjacent to Kerala and in and around Srirangam. The survival of these schools is endangered.

14) Shatyayaniya Samaveda: It has been prevalent in Tamil Nadu and parts of North India. The shakha is no longer extant.

15) Gautama Samaveda: It also exists in Tamil Nadu and in Andhra Pradesh till the 17th cent.C.E. Many followers of the Kauthuma school in Andhra Pradesh still call themselves ‘Gautamas’.

16) Bhallavi Samaveda: It has been prevalent in Karnataka and parts of North India

17) Other Shakhas of Yajurveda: A text called ‘Yajurvedavriksha’ gives the geographical distribution of more than 100 Shakhas of Yajurveda. This description is being left out for brevity.

4. SULBA SUTRA – KNOWLEDGE OF MATHEMATICS

The following Sulba Sutras (Geometrical texts written around 800 BC - 200 BC) exist in print or manuscript

- Apastamba
- Baudhayana
- Manava
- Katyayana
- Maitrayaniya (somewhat similar to Manava text)
- Varaha (in manuscript)
- Vadhula (in manuscript)
- Hiranyakeshin (similar to Apastamba Shulba Sutras)

Ancient Indian Mathematical Texts

Ancient Indian Mathematical texts during the last two millenniums written by, Aryabhata, Bhaskaracharya, Varahamihira, Brahmagupta, etc.

Written by Bhaskaracharya II in 1150 AD - Siddhānta Shiromani, is divided into four parts called

- Lilāvati (arithmetic)
- Bijaganita (algebra)
- Grahaganita (mathematics of the planets) and
- Golādhyāya (spheres)
- Surya Siddhant

Vedic Mathematical Sutras: Consider the following three sutras:

1. “All from 9 and the last from 10,” and its corollary. “Whatever the extent of its deficiency, lessen it still further to that very extent; and also set up the square of that deficiency.”
2. “By one more than the previous one,” and its corollary: “Proportionately.”
3. “Vertically and crosswise,” and its corollary. “The first by the first and the last by the last.”

The first rather cryptic formula is best understood by way of a simple example:

Let us multiply 5 by 9. For this we have to:

1. First, assign as the base for our calculations the power of 10 nearest to the numbers which are to be multiplied. Our base is 10.

2. Write the two numbers to be multiplied on a paper one above the other, and to the right of each write the remainder when each number is subtracted from the base 10. The remainders are then connected to the original numbers with minus signs, signifying that they are less than the base 10.

$$5-5$$

$$9-1$$

3. The answer to the multiplication is given in two parts. The first digit on the left is in multiples of 10 (i.e. the 5 of the answer 45). Although the answer can be arrived at by four different ways, only one is presented here. Subtract the sum of the two deficiencies ($5 + 1 = 6$) from the base (10) and obtain $10 - 6 = 4$ for the left digit (which in multiples of the base 10 is 40).

$$5 - 5$$

$$9 - 1$$

$$4$$

4. Now multiply the two remainder number 5 and 1 to obtain the product 5. This is the right hand portion of the answer which when added to the left hand portion 4 (multiples of 10) produces 45.

$$5 - 5$$

$$9 - 1$$

$$4/5$$

Another method employs cross subtraction. In this example the 1 is subtracted from 5 to obtain the first digit of the answer and the digits 1 and 5 are multiplied together to give the second digit of the answer. This process has been noted by historians as responsible for the general acceptance of the x mark as the sign of multiplication. The algebraically explanation for the first process is

$$(x - a) (x - b) = x (x - a - b) + ab$$

Where x is the base 10, a is the remainder 5 and b is the remainder 1 so that

$$5 = (x - a) = (10 - 5)$$

$$9 = (x - b) = (10 - 1)$$

The equivalent process of multiplying 6 by 8 is then

$$x (x - a - b) + ab \text{ or}$$

$$10(10 - 5 - 1) + 15 = 40 + 5 = 45$$

Illustration: Consider the following cases where 100 has been chosen as the base. Let us multiplying 95 by 78, 91 by 92 and 25 by 98 .

$$95 - 5$$

$$78 - 22$$

$$100-27/110 = 73/110 = 74/10$$

(i)

$$91 - 9$$

$$92 - 8$$

$$83/72$$

(ii)

$$25 - 75$$

$$98 - 2$$

$$23/150 = 24/50$$

(ii)

In example (i), subtract sum of two deficiencies ($5+22= 27$) from the base 100 and obtain 73 which is in left hand portion. Now multiplying remainder numbers 5 and 22 i.e. 110 added in (73×100) 7300 obtain 7410. Similarly process use in other two examples (ii) & (iii)

In the last example we carry the 100 of the 150 to the left and 23 (signifying 23 hundred) becomes 24 hundred. Here in the sutra's words "all from 9 and the last from 10" are shown. The rule is that all the digits of the given original numbers are subtracted from 9, except for the last (the right hand-most one) which should be deducted from 10.

Consider the case when the multiplicand and the multiplier are just above a power of 10. In this case we must cross-add instead of cross subtract. The algebraic formula for the process is:

$$(x + a) (x + b) = x (x + a + b) + ab$$

Illustration 1. We consider multiplication of two numbers 107 above from 100 & 96 below from 100 .

Where 100 has been chosen as base. We have from formulae,

Here $x = 100$, $a=7$, $b = - 4$. Then

$$(100 + 7)(100 - 4) = 100(100 + 7 - 4) + 7(-4) = 100 \times 103 - 28 = 10300 - 28 = 10272.$$

Another ways, we have a combination of subtraction and addition We consider the numbers 107 and 96 for multiplication.

$$\begin{array}{r} 107 + 7 \\ 96 - 4 \\ \hline 103 - 28 = 102 / (100 - 28) = 102 / 72 = 10272 \end{array} \quad \text{and} \quad \begin{array}{r} 11 + 1 \\ 8 - 2 \\ \hline 9 / -2 = 8 / (10 - 2) = 8 / 8 = 88 \end{array}$$

Illustration 2. We consider multiplication of two numbers nearest from 1000. i.e. 992 and 998. Where 1000 as the base .

The multiplication is given in two parts. First subtract the sum of the two deficiencies ($8 + 2 = 10$) from the base 1000 and obtain $1000 - 10 = 990$. This is in left hand portion.

Now multiply the two remainder numbers 8 and 2 to obtain the product 16. This is the right hand portion of the answer which when added to the left hand portion 990×1000 , i.e. 990000

i.e. $990000 + 16 = 990016$

We have, for base 1000,

$$\begin{array}{r} 992 - 8 \\ 998 - 2 \\ \hline \end{array}$$

$$1000 - 10 / 16 = 990 \times 1000 / 16 = 990000 / 16 = 990000 + 16 = 990016$$

i.e. $992 \times 998 = 990016$

Illustration 3. To multiply the numbers 975 and 985, where the base is 1000.

We have, 1000 as base,

$$\begin{array}{r} 975 - 25 \\ 985 - 15 \\ \hline \end{array}$$

$$1000 - 40 / 375 = 960 \times 1000 / 375 = 960000 / 375 = 960375$$

i.e. $975 \times 985 = 960375$

Illustration 4. To multiply the numbers 9985 and 9988, where the base is 10000.

We have, 10000 as base,

$$\begin{array}{r} 9985 - 15 \\ 9975 - 25 \\ \hline \end{array}$$

$$10000 - 40 / 325 = 9960 \times 1000 / 325$$

i.e. $9985 \times 9975 = 99600325$

18) The Sub-Sutras-“Proportionally” Provides for those cases where we wish to use as our base multiples of the normal base of the powers of 10. That is, whenever neither the multiplicand nor the multiplier is sufficiently near a convenient power of 10, which could serve as our base, we simply use a multiple of a power of 10 as our working base, perform our calculations with this working base and then multiply or divide the result proportionally.

Example 1: To multiply 48 by 32, use as our base $50 = 100/2$.

We have, base 50,

$$\begin{array}{r} 48 - 2 \\ 32 - 18 \\ \hline \end{array}$$

$$2 / 30 / 36 \text{ or } (30/2) / 36 = 15/36$$

i.e. $48 \times 32 = 1536$

In this example, subtract sum of two deficiencies (2+18= 20) from the base 50 and obtain (50-20) 30 then divided by 2 (i.e 15) which is in left hand portion. Now multiplying remainders 2 and 18 (i.e 36) which is in right hand portion. Next multiply 15 by 100 and added 36, i.e 1536.

Example 2 : To multiply 41 and 25, use our base 50 = 100/2.

We have, base 50,

$$\begin{array}{r}
 41 - 9 \\
 25 - 25 \\
 \hline
 \end{array}$$

$2 / 16 / 225$ or $(16 / 2) / 225 = 8 / 225 = 10 / 25$

i.e $41 \times 25 = 1025$

Here only the left decimals corresponding to the powers of 10 digits (here 100) are to be effected by the proportional division of 2. These examples show how much easier it is to subtract a few numbers, (especially for more complex calculations) rather than memorize long mathematical tables and perform cumbersome calculations the long way.

Squaring Numbers: The algebraic equivalent of the sutra for squaring a number is:

$$\{a + (-b)\}^2 = a^2 + 2a(-b) + b^2$$

To square 103 we could write as:

$$(100 + 3)^2 = 10,000 + 600 + 9 = 10,609$$

This calculation can easily be done mentally. Similarly, to divide 38,982 by 73, we can write the numerator as $38 \times 3 + 9 \times 2 + 8x + 2$, where x is equal to 10 and the denominator is $7x + 3$. It doesn't take much to figure out that the numerator can also be written as $35 \times 3 + 36 \times 2 + 37x + 12$.

Therefore, $38,982 / 73 = (35 \times 3 + 36 \times 2 + 37x + 12) / (7x + 3) = 5 \times 2 + 3x + 4 = 534$.

CONCLUSION

In this paper we have investigated the multiplication of two numbers by Vedic methods of mathematical sutras and sub- sutras for base on 10,100,1000,10000 which can use in quick solve for multiplication. The algebraic principle involved in the third sutra, “vertically and crosswise”, can be expressed, in one of its applications, as the multiplication of the two numbers represented by $(ax + b)$ and $(cx + d)$, with the answer $acx^2 + x(ad + bc) + bd$.

REFERENCES:

1. Frits Staal. Discovering the Vedas: Origins, Mantras, Rituals, Insights. Penguin Books. pp 3. ISBN 978-0-14-309986-4., 2008.
2. Fisher, Yajnavalkya in the Sruti traditions of the Veda, Acta Orientalia, Volume 45, pp 55–87, 1984.
3. Paul Deussen, Sixty Upanishads of the Veda, Volume 1, Motilal Banarsidass, ISBN 978-8120814684, pp 534-544.
4. The Ancient Vedic Knowledge, Spirituality and Science <http://www.vedandtechs.com/wiki/ancient-vedic-knowledge-spirituality-and-science>



STUDY ON FUZZY γ^* – SEMI INTERIOR AND FUZZY γ^* – SEMI CLOSURE

M. Vijaya*, M. Rajesh**

*Research Advisor, Department of Mathematics, Marudupandiyar college, Thanjavur, (Tamilnadu) India.

**Research Scholar, Department of Mathematics, Marudupandiyar college, Thanjavur, (Tamilnadu) India.

E-mail: mathvijaya79@yahoo.com, rajeshmathi91@gmail.com

ABSTRACT :

*In this paper, we introduce a new classes of sets called fuzzy γ^{**} -semi interior and fuzzy γ^{**} - semi closure sets and its properties are established in fuzzy topological spaces.*

Keywords: Fuzzy γ -semi open, fuzzy γ - semi closed, fuzzy γ - semi interior, fuzzy γ -semi closure, fuzzy γ^ - semi open and fuzzy γ^* - semi closed.*

I. INTRODUCTION

The concept of fuzzy sets operations were first introduced by L.A. Zadeh[3], let X be non empty set and I be the unit interval $[0,1]$, a fuzzy set is a mapping X into I . In 1968 Chang[1] introduced the concept of fuzzy topological space. Azad introduced the notions of fuzzy semi open and fuzzy semi closed sets and T.Noiri and O.R Sayed [2] introduced the notion of γ open sets and γ closed sets. In this paper we introduce fuzzy γ^* semi interior and fuzzy γ^* - semi closure and its properties are established in fuzzy topological spaces.

II. PRELIMINARIES

Through this paper (X, τ) and (Y, σ) denote fuzzy topological spaces. For a fuzzy set A in a fuzzy topological space X or Y . $cl(A)$, $int(A)$, A^c denote the closure, interior, complement of A respectively. By 0_X and 1_X we mean the constant fuzzy sets taking on the values 0 and 1 respectively

Definition: 2.1 (Fuzzy Sets) A fuzzy set is a function with domain X and values in I . That is an element of I^X . let $A \in I^X$. The subset of X in which A assumes non-zero values is known as the support of A for every $x \in X$, $A(x)$ is called the grade of membership of x in A . And X is called carrier of the fuzzy set A . If A takes only 0 and 1, then A is a crisp set in X .

Definition: 2.2 A fuzzy set A of (X, τ) is called

1. Fuzzy semi open (in short Fs open) if $A \leq cl(int(A))$ and a fuzzy semi closed (in short Fs – closed) if $int(cl(A)) \leq A$.
2. Fuzzy preopen (in short Fp-open) if $A \leq Int(cl(A))$ and a fuzzy pre closed (in short Fp-closed) if $cl(Int(A)) \leq A$.
3. Fuzzy strongly semi open (in short Fss –open) if $A \leq int(cl(int A))$ and a fuzzy strongly semi closed (in short Fss-closed) $cl(int(cl(A))) \leq A$.
4. Fuzzy γ -open if $A \leq (int(cl(A))) \vee (cl(int A))$ & fuzzy γ -closed if $cl(int(A)) \wedge int(cl(A)) \leq A$.
5. Fuzzy γ^* - semi open if $int(A) \leq cl(\gamma - int(A))$ and fuzzy γ^* - semi closed if $cl(A) \geq int(\gamma - cl(A))$.

Fuzzy γ^* – semi interior

Definition 2.3. Let (X, τ) be a fuzzy topological space, then for a fuzzy subset A of X , the fuzzy γ^* -semi interior of A (briefly γ^* -sint(A)) is union of all fuzzy γ^* -semi open sets of X contained in A . (i.e) $\gamma^*sint(A) = \vee \{B: B \leq A, B \text{ is fuzzy } \gamma^* \text{-semi open in } X\}$

Proposition 2.4: Let (X, τ) be a fuzzy topological space, then for any fuzzy subsets A and B of a fuzzy topological X we have (i) $\gamma^*sint(A) \leq A$ (ii) A is fuzzy γ^* –semi open $\Leftrightarrow \gamma^*sint(A) = A$.

Proof:

(i) It follows by above definition 2.3

$\gamma^*sint(A)$ is union of all fuzzy γ^* semi open sets of X which contained in A .

ie) $\gamma^*sint(A) = \vee \{B: B \leq A, B \text{ is fuzzy } \gamma^* \text{-semi open in } X\}$, ie $\gamma^*sint(A) = A$.

(ii) Let Assume A be fuzzy γ^* semi open set then $A \leq \gamma^*sint(A)$ by using (i) $\gamma^*sint(A) \leq A$.

Therefore $A = \gamma^*sint(A)$, Conversely, assume that $A = \gamma^*sint(A)$ by using definition 2.3.

ie, $A \leq \gamma^*sint(A)$, therefore A is fuzzy semi open set. (ii) is proved.

Proposition:2.5

i) $\gamma^*sint(\gamma^*sint(A)) = \gamma^*sint(A)$

ii) $A \leq B$ then $\gamma^*sint(A) \leq \gamma^*sint(B)$

Proof

(i) by Proposition 2.4 (i) & (ii) $\gamma^*sint(\gamma^*sint(A)) = \gamma^*sint(A)$. This proves (i).

(ii) Since $A \leq B$, by (i) $\gamma^*sint(A) \leq A$, i.e. $\gamma^*sint(A) \leq A \leq B$, This implies $\gamma^*sint(A) \leq B$

by (i) $\gamma^*sint(\gamma^*sint(A)) \leq \gamma^*sint(B)$, ie $\gamma^*sint(A) \leq \gamma^*sint(B)$. hence proves (ii)

Theorem: 2.6 Let (X, τ) be a fuzzy topological space then two any subset A and B of a fuzzy topological space we have $\gamma^*(sint(A \wedge B)) = (\gamma^*sint(A)) \wedge (\gamma^*sint(B))$

Proof: Since, $A \wedge B \leq B$ and $A \wedge B \leq A$ by using proposition 2.5 (ii), we get $\gamma^*sint(A \wedge B) \leq \gamma^*sint(A)$

and $\gamma^*sint(A \wedge B) \leq \gamma^*sint(B)$ This implies that $\gamma^*(sint(A \wedge B)) \leq (\gamma^*sint(A)) \wedge (\gamma^*sint(B))$ -----(1)

using proposition 2.4 (i) we have $\gamma^*sint(A) \leq A$ and $\gamma^*sint(B) \leq B$ implies that $\gamma^*sint(A) \wedge \gamma^*sint(B) \leq A \wedge B$

Now applying proposition 2.5(ii), we get $\gamma^*sint(\gamma^*sint(A) \wedge \gamma^*sint(B)) \leq \gamma^*sint(A \wedge B)$ by

(i) $\gamma^*sint(\gamma^*sint(A)) \wedge \gamma^*sint(\gamma^*sint(B)) \leq \gamma^*sint(A \wedge B)$, by proposition 2.5

(ii) $\gamma^*sint(A) \wedge \gamma^*sint(B) \leq \gamma^*sint(A \wedge B)$ ----- (2)

from (1) and (2) we have $\gamma^*sint(A \wedge B) = \gamma^*sint(A) \wedge \gamma^*sint(B)$

Theorem: 2.7 Let (X, τ) be a fuzzy topological space then for any fuzzy subset A and B of a fuzzy topological space we have $\gamma^*sint(A \vee B) \geq \gamma^*sint(A) \vee \gamma^*sint(B)$

Proof: Since $A \leq A \vee B$ and $B \leq A \vee B$. using proposition 2.5 (ii) we have $\gamma^*sint(A) \leq \gamma^*sint(A \vee B)$ ----- (1)

and $\gamma^*sint(B) \leq \gamma^*sint(A \vee B)$ ----- (2), from (1) and (2) we have $\gamma^*sint(A \vee B) \geq \gamma^*sint(A) \vee$

$\gamma^*sint(B)$. Hence proved.

Following example shows that the quality need not be hold in theorem 2.7

Example 2.8. let $X = \{a, b, c\}$ and $\tau = \{0, 1, \{a_5, b_3, c_7\}, \{a_2, b_4, c_4\}, \{a_2, b_3, c_4\}, \{a_5, b_4, c_7\}\}$. Then (X, τ) is a fuzzy topological space. The closed set of $\tau^c = \{0, 1, \{a_5, b_7, c_3\}, \{a_8, b_6, c_6\}, \{a_8, b_7, c_6\}, \{a_5, b_6, c_3\}\}$. consider $A = \{a_4, b_3, c_4\}$ and $B = \{a_3, b_7, c_4\}$ then $\gamma^*sint(A) = \{a_3, b_3, c_4\}$ and $\gamma^*sint(B) = \{a_2, b_4, c_4\}$. That implies that $\gamma^*sint(A) \vee \gamma^*sint(B) = \{a_3, b_4, c_4\}$. Now $A \vee B = \{a_4, b_7, c_4\}$, it follows that $\gamma^*sint(A \vee B) = \{a_4, b_5, c_4\}$ then $\gamma^*sint(A \vee B) \not\leq \gamma^*sint(A) \vee \gamma^*sint(B)$ thus $\gamma^*sint(A \vee B) \neq \gamma^*sint(A) \vee \gamma^*sint(B)$

III FUZZY γ^* SEMI CLOSURE

Definition:3.1 Let (X, τ) be a fuzzy topological space then for a fuzzy subset A of X , the fuzzy γ^* -semi closure of A (briefly $\gamma^* - scl(A)$) is the intersection of all fuzzy γ^* -semi closed sets of X contained in A . (i.e) $\gamma^* - scl(A) = \bigwedge \{B: B \geq A, B \text{ is fuzzy } \gamma^* \text{-semi closed set in } X\}$

Proposition: 3.2 Let (X, τ) be a fuzzy topological space, then for any fuzzy subsets A and B of a fuzzy topological X we have (i) $(\gamma^* sint(A))^c = \gamma^* scl(A^c)$ and (ii) $(\gamma^* scl(A))^c = \gamma^* sint(A^c)$.

Proof: By using definition 3.1 $\gamma^* sint(A) = \bigvee \{B: B \leq A, B \text{ is fuzzy } \gamma^* \text{-semi open in } X\}$, taking complement on both sides. We get

$$\begin{aligned} [\gamma^* sint(A)]^c &= (\sup\{B: B \leq A, B \text{ is fuzzy } \gamma^* \text{ semi open}\})^c \\ &= \inf\{B^c: B^c \geq A^c, B^c \text{ is fuzzy } \gamma^* \text{ semi closed}\} \end{aligned}$$

replacing B^c by C , we get $[\gamma^* sint(A)]^c = \bigwedge \{C: C \geq A^c, C \text{ is fuzzy } \gamma^* \text{ semi closed}\}$ by definition $(\gamma^* sint(A))^c = \gamma^* scl(A^c)$. Hence proves (i). By using (i) $[\gamma^* sint(A^c)]^c = \gamma^* scl(A)$

$[\gamma^* sint(A^c)]^c = \gamma^* scl(A)$. Taking complement on both sides we get $(\gamma^* scl(A))^c = \gamma^* sint(A^c)$.

Hence proved by (ii).

Proposition: 3.3

Let (X, τ) be a fuzzy topological space, then for any fuzzy subsets A and B of a fuzzy topological space X we have (i) $A \leq \gamma^* scl(A)$, (ii) A is fuzzy γ^* – semi closed $\Leftrightarrow \gamma^* scl(A) = A$.

Proof:

(i) By using Definition 3.1, γ^* semi closure intersection of all γ^* semi closed sets contained in A . $\gamma^* scl(A) = \bigwedge \{B: B \geq A, B \text{ is fuzzy } \gamma^* \text{ semi closed}\}$ i.e. $A \leq \gamma^* scl(A)$.

(ii) Let A be fuzzy γ^* semi closed subset in X . By using proposition A^c is also γ^* semi open set and again by proposition 3.2(ii), $\gamma^* sint(A^c) = A^c \Leftrightarrow [\gamma^* scl(A)]^c = A^c \Leftrightarrow \gamma^* scl(A) = A$.

Propositions :3.4

Let (X, τ) be a fuzzy topological space, then for any fuzzy subsets A and B of a fuzzy topological X we have (i) $\gamma^* scl(\gamma^* scl(A)) = \gamma^* scl(A)$, (ii) If $A \leq B$ then $\gamma^* scl(A) \leq \gamma^* scl(B)$

Proof: By using 3.3 (ii) $\gamma^* scl(\gamma^* scl(A)) = \gamma^* scl(A)$, by proved (i).

(ii) Suppose $A \leq B$ then $B^c \leq A^c$ by using proposition 3.2(ii) $\gamma^* sint(B^c) \leq \gamma^* sint(A^c)$ taking complement on both sides, We get $[\gamma^* sint(B^c)]^c \geq [\gamma^* sint(A^c)]^c$

by proposition 3.3(ii) Implies $\gamma^* scl(B) \geq \gamma^* scl(A)$

This proves (ii).

Theorem: 3.5 Let (X, τ) be a fuzzy topological space then for any subset A and B of a fuzzy topological space we have $\gamma^* scl(A \vee B) \leq \gamma^* scl(A) \vee \gamma^* scl(B)$

Proof: Since $\gamma^* scl(A \vee B) = [\gamma^* scl(A \vee B)]^c$ by using proposition 3.3(i).

We have $\gamma^* scl(A \vee B) = [\gamma^* sint(A \vee B)]^c = [\gamma^* sint(A^c \wedge B^c)]^c$

by known proposition, we have,

$$\gamma^* scl(A \vee B) = [\gamma^* sint(A^c) \wedge \gamma^* sint(B^c)]^c = [\gamma^* sint(A^c)]^c \vee [\gamma^* sint(B^c)]^c = \gamma^* scl(A^c)^c \vee \gamma^* scl(B^c)^c = \gamma^* scl(A) \vee \gamma^* scl(B)$$

Theorem: 3.6 Let (X, τ) be a fuzzy topological space then for any fuzzy subset A and B of a fuzzy topological space we have $\gamma^* scl(A \wedge B) \leq \gamma^* scl(A) \wedge \gamma^* scl(B)$

Proof : Since $A \wedge B \leq A$ and $A \wedge B \leq B$ by using proposition 3.4 (ii) we get $\gamma^*scl(A \wedge B) \leq \gamma^*scl(A)$ and $\gamma^*scl(A \wedge B) \leq \gamma^*scl(B)$. This implies $\gamma^*scl(A \wedge B) \leq \gamma^*scl(A) \wedge \gamma^*scl(B)$. Hence proved.

IV. CONCLUSION

Fuzzy γ - closed set and fuzzy γ - open set are play major role in fuzzy topology. Since its inception several weak forms of fuzzy γ -closed sets and fuzzy γ - open sets have been introduced in general fuzzy topology. The present paper is investigated in the new weak forms fuzzy γ^* -semi interior and fuzzy γ^* -semi closure in fuzzy topological spaces. Hence the propositions and theorems are justify the results. We hope that the findings in this paper will help researcher enhance and promote the further study on general fuzzy topology to carry out a general framework for their applications in practical life. This paper, not only can form the theoretical basis for further applications of fuzzy topology , but also lead to the development of information systems.

REFERENCES

1. C.L.Chang, Fuzzy Topological Spaces, J.Math. Anal, Appl.24 (1968), 182-190.
2. T.Noiri and O.R.Sayed, Fuzzy γ -open sets and fuzzy γ -continuity in fuzzifying topology, Sci. Math. Jpn. 55(2002).
3. L.A.Zadeh, Fuzzy Sets, Information and control, vol.8, Pp 338-353, 1965.
4. D.Andrijevic,semi open sets ,Mat.Vesnik,38(1)(1998)24-32
5. K. K. Azad "On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity J. Math Annual Appl. 82 (1) (1981) 14-32
6. Luay A.Al.Swidi, Amed S.A.O on, Fuzzy γ -open sets and fuzzy γ -closed sets, Americal Journal of scientific research, 27(2011), 62-67.
7. N. Palaniappan, Fuzzy topology second edition 2005.
8. J.Dontchev, on generalizing semi-preopen sets.Mem.Fac.sci.Kochi Univ.ser.A.Math.



WEAKLY COMPATIBLE MAPS AND COMMON COUPLED FIXED POINT THEOREMS IN GENERALIZED FUZZY METRIC SPACE

Seema Mehra

Deptt. of Mathematics, M.D. University, Rohtak, Haryana, India.
E-mail : sberwal2007@gmail.com

ABSTRACT :

The purpose of this paper is to prove coupled fixed point theorems for six mappings in generalized fuzzy metric spaces.

Keywords and phrases: Fuzzy metric space, Generalized fuzzy metric space, weakly compatible mapping, common coupled fixed point.

2010 Mathematics Subject Classification : 47H10,54H25

1. INTRODUCTION AND PRELIMINARIES

Brouwer(1912) laid the foundation of fixed point theory by proving the result that the continuous mappings of an n-dimensional element into itself has a fixed point. Later on Schauder(1930) extended Brouwer's result for infinite dimensional spaces. The first contractive definition is due to Banach . In 1922, a Polish mathematician Stefan Banach proved the contraction principle, popularly known as Banach Contraction Principle, which states that "If T is a mapping of a complete metric space (X, d) into itself satisfying $d(Tx, Ty) \leq k d(x, y)$ for all $x, y \in X$ and $0 \leq k < 1$, then T admits a unique fixed point." This theorem was generalized by various authors including those of Ciric (1974), Edelstein(1961), Kannan(1968), Gahler(1963-66), Cronic, Ishikawa, Pathak (1995) and Wong(1973) etc. As a generalization of fixed point, the concept of coupled fixed point was introduced by Opoitsev ([19]- [21]) and then by Guo and Lakshmikantham (see [7]) in connection with coupled quasi solutions of an initial value problem for ordinary differential equations. For other results on coupled fixed point theory see([4-5], [12-13], [18], [23-27] and [30-32]). The concept of Fuzzy set was published in 1965 by Zadeh[10], Prof. of Computer Science in Univ. of California. Some of the key contributors to the theory are Wyllis Bandler, Didier Dubois, Brian R. Gaines, Ladislav J. Kohout, Mari Nowakowska, Henri Prade, Ronald R. Yager and H. J. Zimmermann[8]. Using notion of fuzzy sets, Kramosil and Michalek[15] introduced the concept of fuzzy metric spaces. Many authors have studied fixed point theory in fuzzy metric spaces. George and Veeramani[1] modified the concept of fuzzy metric space which was introduced by Kramosil and Michalek[15]. In 1963, Gahler [16] introduced the concept of 2-metric space akin to the metric space (X, d) . But different authors proved that there is no relation between these two functions, for instance, Ha et al. in (1988) showed that 2-metric need not be continuous function, further there is no easy relationship between results obtained in the two settings. Motivated by the measure of nearness, between two or more objects with respect to a specific property or characteristic, called the parameter of nearness, in 1992, Dhage in his Ph.D. thesis introduced a new class of generalized metric space called D-metric space ([2],[3]). He

claimed that D-metrics provide a generalization of ordinary metric functions and went on to present several fixed point results. In 2006, Mustafa and Sims[33] have pointed out that Dhage's notion of a D-metric space is fundamentally flawed and most of the results claimed by Dhage and others are invalid. In 2006, Mustafa and Sims[33] introduced a new structure of generalized metric spaces, which are called G-metric spaces as generalization of metric space (X, d) , to develop and introduce a new fixed point theory for a various mappings in this new structure and demonstrated that most of the claims concerning the fundamental topological structure of D-metric space by Dhage [3] are incorrect. Afterwards, they proved some fixed point results for mappings satisfying various contractive conditions on this newly defined space. Recently, Mustafa, Obiedat and Awawdeh(2008) proved fixed point results for mappings satisfying sufficient conditions on complete G-metric space and also showed that if the G-metric space (X, G) is symmetric, then the existence and uniqueness of these fixed point results follow from well-known theorems in usual metric space (X, dG) , where (X, dG) is the usual metric space which defined from the G-metric space (X, G) . In 2011, Shatanawi[24] proved a coupled coincidence fixed point theorem in the setting of a generalized metric space in the sense of Z. Mustafa and B. Sims[33]. In current time, the existence of common or coupled fixed points of a fuzzy version for multiple mappings has attracted much attention. In 2010, Sedghi et al. [17], proved coupled fixed point theorems for contractions in fuzzy metric spaces.. Choudhury et. al [6] established coupled coincidence point results for compatible mappings in partially ordered fuzzy metric space. After that common coupled fixed point results in fuzzy metric spaces were established by Hu [29]. Gordji, Cho and Baghani[13] established coupled fixed point theorems in intuitionistic fuzzy normed spaces. Sun and Yang[8] introduced the concept of Q-fuzzy metric space and obtained some properties related to this space. In 2012, Hu and Luo[28] introduced the concept of mixed g-monotone mapping and proved coupled coincidence and common coupled fixed point theorems for mappings under ϕ – contractive conditions in partially ordered generalized fuzzy metric spaces. Motivated by [28], we prove common coupled fixed point theorems for six mappings in generalized fuzzy metric spaces.

PRELIMINARIES

Definition1.1.[8] A 3-tuple $(X, G, *)$ is called a generalized fuzzy metric space if X is an arbitrary nonempty set, $*$ is a continuous t-norm and G is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions for each $x, y, z \in X$ and $s, t > 0$

1. $G(x, y, z, t) > 0$ for all $x, y, z \in X$ with $x \neq y$;
2. $G(x, x, y, t) \geq G(x, y, z, t)$ for all $x, y, z \in X$ with $y \neq z$;
3. $G(x, y, z, t) = 1$ for all $x, y, z \in X$ and $t > 0$ if and only if $x = y = z$;
4. $G(x, y, z, t) = G(p(x,y,z), t)$ where p is a permutation function;
5. $G(x, a, a, t) * G(a, y, z, s) \leq G(x, y, z, t + s)$ for all $x, y, z, a \in X$ and $s, t > 0$;
6. for all $x, y, z \in X$, $G(x, y, z, \bullet) : (0, \infty) \rightarrow [0, 1]$ is continuous.

Then $(G, *)$ is called generalized fuzzy metric on X . The function $G(x, y, z, t)$ denote the degree of parameter of nearness among x, y and z with respect to t , respectively.

Definition 1.2[8]. Let $(X, G, *)$ be generalized fuzzy metric space. Then

- (a) a sequence $\{x_n\}$ in X is convergent to $x \in X$ if, for all $t > 0$ $\lim_{n \rightarrow \infty} G(x_n, x_n, x, t) = 1$
- (b) a sequence $\{x_n\}$ in X is a Cauchy if for all $t > 0$ and $p, q > 0$ $\lim_{n \rightarrow \infty} G(x_n, x_n, x_m, t) = 1$
- (c) A generalized fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 1.3[8]. Self mappings A and B of fuzzy metric space $(X, G, *)$ is said to be weakly compatible if $ABx = BAx$ when $Ax = Bx$ for some $x \in X$.

Lemma 1.1[8]. In generalized fuzzy metric space X , $G(x, y, z, \cdot)$ is non- decreasing for all $x, y, z \in X$.

Lemma 1.2(see [8]). Let $(X, G, *)$ be a G -fuzzy metric space. Then, G is a continuous function on $X^3 \times (0, \infty)$.

Definition 1.4. Let X be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.5. Let X be a nonempty set. An element $(x, y) \in X \times X$ is called

- (i) a coupled coincidence point of $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $gx = F(x, y)$ and $gy = F(y, x)$.
- (ii) a common coupled fixed point of $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.6. Self mappings A and B of fuzzy metric space $(X, G, *)$ is said to be weakly compatible if $ABx = BAx$ when $Ax = Bx$ for some $x \in X$.

Define $\emptyset = \{\varphi: R^+ \rightarrow R^+\}$, where

$R^+ = [0, +\infty)$ and each $\varphi \in \emptyset$ satisfies the following conditions :

- ($\emptyset - 1$) φ is strict increasing;
- ($\emptyset - 2$) φ is upper semi – continuous from the right;
- ($\emptyset - 3$) $\sum_{n=0}^{\infty} \varphi^n(t) < +\infty$ for all $t > 0$, where $\varphi^{n+1}(t) = \varphi(\varphi^n(t))$.

Lemma 1.3 [8]. Let $(X, G, *)$ be a generalized fuzzy metric space and $\{y_n\}$ be a sequence in X . If there exists $\varphi \in \emptyset$ such that

$$G(y_n, y_n, y_{n+1}, \varphi(t)) \geq G(y_{n-1}, y_{n-1}, y_n, t) * G(y_n, y_n, y_{n+1}, t)$$

for all $t > 0$ and $n = 1, 2, \dots$, then $\{y_n\}$ is a Cauchy sequence in X .

MAIN RESULTS :

Theorem 2.1 : Let $(X, G, *)$ be a complete Generalized Fuzzy metric space where $*$ is a continuous t-norm. Let $P, Q : X \times X \rightarrow X$ and A, B, S and $T : X \rightarrow X$ be six mappings satisfying the following conditions:

- (i) $P(X \times X) \subseteq ST(X)$ and $Q(X \times X) \subseteq AB(X)$
- (ii) P and AB are continuous
- (iii) $AB = BA, ST = TS$,
- (iv) the pairs (P, AB) and (Q, ST) are weakly compatible. Also suppose
- (v) if there exists $\varphi \in \emptyset$ such that

$$G(P(x, y), P(x, y), Q(u, v), \varphi(t)) \geq G(ABx, ABx, STu, t) * G(ABx, ABx, P(x, y), t) * G(STu, STu, Q(u, v), t) \text{ for all } x, y, u, v \in X, t > 0.$$

Then P, Q, A, B, S and T have a unique common coupled fixed point in X .

Proof Let $x_0, y_0 \in X$. From condition (i) there exists $x_1, x_2 \in X$ such that $P(x_0, y_0) = STx_1 = z_0, Q(x_1, y_1) = ABx_2 = z_1$ and $P(y_0, x_0) = STy_1 = p_0, Q(y_1, x_1) = ABx_2 = p_1$. Inductively we can construct sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{p_n\}$ in X such that $P(x_{2n}, y_{2n}) = STx_{2n+1} = z_{2n}, Q(x_{2n+1}, y_{2n+1}) = ABx_{2n+2} = z_{2n+1}$ and $P(y_{2n}, x_{2n}) = STy_{2n+1} = p_{2n}$

$Q(y_{2n+1}, x_{2n+1}) = ABx_{2n+2} = p_{2n+1}$ for all $n \geq 0$. Now we prove $\{z_n\}$ and $\{p_n\}$ are Cauchy sequence in X .

Step 1. Putting $x = x_{2n}, y = y_{2n}, u = x_{2n+1}, v = y_{2n+1}$ for $x > 0$ in (V) we have

$$\begin{aligned} G(z_{2n}, z_{2n}, z_{2n+1}, \varphi(t)) &= G(P(x_{2n}, y_{2n}), P(x_{2n}, y_{2n}), Q(x_{2n+1}, y_{2n+1}), \varphi(t)) \\ &\geq G(ABx_{2n}, ABx_{2n}, STx_{2n+1}, t) * G(ABx_{2n}, ABx_{2n}, P(x_{2n}, y_{2n}), t) * G(STx_{2n+1}, STx_{2n+1}, Q(x_{2n+1}, y_{2n+1}), t) \\ &= G(z_{2n-1}, z_{2n-1}, z_{2n}, t) * G(z_{2n-1}, z_{2n-1}, z_{2n}, t) * G(z_{2n}, z_{2n}, z_{2n+1}, t) \\ &= G(z_{2n-1}, z_{2n-1}, z_{2n}, t) * G(z_{2n}, z_{2n}, z_{2n+1}, t) \end{aligned}$$

Now, by Lemma 1.3, $\{z_n\}$ is a Cauchy sequence in X which is complete.

Putting $x = y_{2n+1}, y = x_{2n+1}, u = y_{2n}, v = x_{2n}$ for $x > 0$ in (3.5) we have

$$\begin{aligned} G(p_{2n}, p_{2n}, p_{2n+1}, \varphi(t)) &= G(P(y_{2n}, x_{2n}), P(y_{2n}, x_{2n}), Q(y_{2n+1}, x_{2n+1}), \varphi(t)) \\ &\geq G(STy_{2n}, STy_{2n}, ABx_{2n+1}, t) * G(STy_{2n}, STy_{2n}, P(y_{2n}, x_{2n}), t) * G(ABx_{2n+1}, ABx_{2n+1}, Q(y_{2n+1}, x_{2n+1}), t) \\ &= G(p_{2n-1}, p_{2n-1}, p_{2n}, t) * G(p_{2n-1}, p_{2n-1}, p_{2n}, t) * G(p_{2n}, p_{2n}, p_{2n+1}, t) \\ &= G(p_{2n-1}, p_{2n-1}, p_{2n}, t) * G(p_{2n}, p_{2n}, p_{2n+1}, t) \end{aligned}$$

Now, by Lemma 1.3, $\{p_n\}$ is also a Cauchy sequence in X which is complete.

Hence sequences $\{z_n\} \rightarrow \alpha, \{p_n\} \rightarrow \beta$ and the subsequences $\{P(x_{2n}, y_{2n})\}, \{STx_{2n+1}\}, \{Q(x_{2n+1}, y_{2n+1})\}, \{ABx_{2n+2}\}$ of $\{z_n\}$ converge to α and $\{P(y_{2n}, x_{2n})\}, \{STy_{2n+1}\}, \{Q(y_{2n+1}, x_{2n+1})\}, \{ABx_{2n+2}\}$ of $\{p_n\}$ converge to β .

Since the pair (P, AB) is w - compatible, we have $P(\alpha, \beta) = AB \alpha$ and $P(\beta, \alpha) = AB \beta$

Suppose that $AB\alpha \neq \alpha$ or $AB\beta \neq \beta$,

$$\begin{aligned} G(AB\alpha, AB\alpha, \alpha, \varphi(t)) &\geq G(AB\alpha, AB\alpha, z_{2n+1}, \varphi(kt)) * G(z_{2n+1}, z_{2n+1}, \alpha, \varphi(t) - \varphi(kt)) \\ &\geq G(P(\alpha, \beta), P(\alpha, \beta), Q(x_{2n+1}, y_{2n+1}), \varphi(kt)) * 1 \\ &\geq G(P(\alpha, \beta), P(\alpha, \beta), Q(x_{2n+1}, y_{2n+1}), \varphi(t) - \varphi(kt)) \\ &\geq G(AB\alpha, AB\alpha, \alpha, t) * G(AB\alpha, AB\alpha, P(\alpha, \beta), t) * G(STx_{2n+1}, STx_{2n+1}, Q(x_{2n+1}, y_{2n+1}), t) \\ &\geq G(AB\alpha, AB\alpha, \alpha, t) * G(AB\alpha, AB\alpha, AB\alpha, t) * G(\alpha, \alpha, \alpha, t) \\ &\geq G(AB\alpha, AB\alpha, \alpha, t) * 1 * 1 \\ &\geq G(AB\alpha, AB\alpha, \alpha, t) \end{aligned}$$

This implies $AB\alpha = \alpha$

Similarly $AB \beta = \beta$.

Thus $P(\alpha, \beta) = AB \alpha = \alpha$ and $P(\beta, \alpha) = AB \beta = \beta$.

Since $P(X \times X) \subseteq ST(X)$, there exists $a, b \in X$ such that $P(\alpha, \beta) = \alpha = STa$ and $P(\beta, \alpha) = \beta = STb$.

$$\begin{aligned} G(\alpha, \alpha, Q(a, b), \varphi(t)) &= G(P(\alpha, \beta), P(\alpha, \beta), Q(a, b), \varphi(t)) \\ &\geq G(AB\alpha, AB\alpha, STa, t) * G(AB\alpha, AB\alpha, P(\alpha, \beta), t) * G(STa, STa, Q(a, b), t) \\ &\geq G(\alpha, \alpha, \alpha, t) * G(\alpha, \alpha, \alpha, t) * G(\alpha, \alpha, Q(a, b), t) \\ &\geq 1 * 1 * G(\alpha, \alpha, Q(a, b), t) \\ &\geq G(\alpha, \alpha, Q(a, b), t) \end{aligned}$$

This implies $Q(a, b) = \alpha = P(\alpha, \beta) = \alpha = STa = AB \alpha$

similarly $Q(b, a) = P(\beta, \alpha) = \beta = STb = AB \beta$.

Since the pair (Q, ST) is w-compatible, we have $ST \alpha = Q(\alpha, \beta)$ and $ST \beta = Q(\beta, \alpha)$.

Suppose that $ST\alpha \neq \alpha$ or $ST\beta \neq \beta$,

$$\begin{aligned} G(\alpha, \alpha, ST\alpha, \varphi(t)) &= G(P(\alpha, \beta), P(\alpha, \beta), Q(\alpha, \beta), \varphi(t)) \text{ ---} \\ &\geq G(AB\alpha, AB\alpha, ST\alpha, t) * G(AB\alpha, AB\alpha, P(\alpha, \beta), t) * G(ST\alpha, ST\alpha, Q(\alpha, \beta), t) \\ &\geq G(P(\alpha, \beta), P(\alpha, \beta), Q(x_{2n+1}, y_{2n+1}), \varphi(kt)) * 1 \\ &\geq G(\alpha, \alpha, \alpha, t) * G(\alpha, \alpha, \alpha, t) * G(\alpha, \alpha, \alpha, t) \geq 1 * 1 * 1 \end{aligned}$$

This implies $ST\alpha = \alpha$

Similarly $ST \beta = \beta$.

Hence $Q(\alpha, \beta) = \alpha = P(\alpha, \beta) = \alpha = ST \alpha = AB \alpha$

similarly $Q(\alpha, \beta) = P(\beta, \alpha) = \beta = ST\beta = AB \beta$.

Now,

$$\begin{aligned} G(P(B\alpha, B\beta), P(B\alpha, B\beta), Q(\alpha, \beta), \varphi(t)) \\ \geq G(ABB\alpha, ABB\alpha, ST\alpha, t) * G(ABB\alpha, ABB\alpha, P(B\alpha, B\beta), t) * G(ST\alpha, ST\alpha, Q(\alpha, \beta), t) \end{aligned}$$

Since $AB=BA$,

We have $AB(B \alpha) = B(AB\alpha) = B\alpha$ and $P(\alpha, \beta) = \alpha$ this implies $P(B\alpha, B\beta) = B \alpha$

$$\begin{aligned} G(B\alpha, B\alpha, Q(\alpha, \beta), \varphi(t)) &\geq G(B\alpha, B\alpha, \alpha, t) * G(B\alpha, B\alpha, B\alpha, t) * G(\alpha, \alpha, \alpha, t) \\ G(B\alpha, B\alpha, \alpha, \varphi(t)) &\geq G(B\alpha, B\alpha, \alpha, t) * 1 * 1 \end{aligned}$$

This implies $B\alpha = \alpha$, similarly $B\beta = \beta$.

Since $\alpha = AB\alpha$, we have $\alpha = A \alpha$

Similarly, $\beta = AB\beta = A\beta$.

Now,

$$\begin{aligned} G(P(\alpha, \beta), P(\alpha, \beta), Q(T\alpha, T\beta), \varphi(t)) \\ \geq G(AB\alpha, AB\alpha, STT\alpha, t) * G(AB\alpha, AB\alpha, P(\alpha, \beta), t) * G(STT\alpha, STT\alpha, Q(T\alpha, T\beta), t) \end{aligned}$$

Since $ST=TS$,

We have $ST(T \alpha) = T(ST\alpha) = T\alpha$ and $Q(\alpha, \beta) = \alpha$ this implies $Q(T\alpha, T\beta) = T \alpha$

$$\begin{aligned} G(\alpha, \alpha, T\alpha, \varphi(t)) &\geq G(\alpha, \alpha, T\alpha, t) * G(\alpha, \alpha, \alpha, t) * G(T\alpha, T\alpha, T\alpha, t) \\ G(\alpha, \alpha, T\alpha, \varphi(t)) &\geq G(\alpha, \alpha, T\alpha, t) * 1 * 1 \end{aligned}$$

This implies $T\alpha = \alpha$, similarly $T\beta = \beta$.

Since $\alpha = ST\alpha$, we have $\alpha = S \alpha$

Similarly, $\beta = ST\beta = S\beta$.

Hence (α, β) is common coupled fixed point of P, Q, A, B, S and T .

Let (α^*, β^*) be another common coupled fixed point of P, Q, A,B, S and T. We have

$$\begin{aligned} G(\alpha, \alpha, \alpha^*, \varphi(t)) &= G(P(\alpha, \beta), P(\alpha, \beta), Q(\alpha^*, \beta^*), \varphi(t)) \\ &\geq G(AB\alpha, AB\alpha, ST\alpha^*, t) * G(AB\alpha, AB\alpha, P(\alpha, \beta), t) * G(ST\alpha^*, ST\alpha^*, Q(\alpha^*, \beta^*), t) \\ &\geq G(\alpha, \alpha, \alpha^*, t) * G(\alpha, \alpha, \alpha, t) * G(\alpha^*, \alpha^*, \alpha^*, t) \\ &\geq G(\alpha, \alpha, \alpha^*, t) * 1 * 1 \\ &\geq G(\alpha, \alpha, \alpha^*, t) \end{aligned}$$

This implies $\alpha = \alpha^*$.

Similarly $\beta = \beta^*$.

Hence (α, β) is the unique common coupled fixed point of P, Q, A,B, S and T.

Now, we will show that $\alpha = \beta$.

Suppose $\alpha \neq \beta$.

$$\begin{aligned} G(P(\alpha, \beta), P(\alpha, \beta), Q(\beta, \alpha), \varphi(t)) &\geq G(AB\alpha, AB\alpha, ST\beta, t) * G(AB\alpha, AB\alpha, P(\alpha, \beta), t) * G(ST\beta, ST\beta, Q(\beta, \alpha), t) \\ G(\alpha, \alpha, \beta, \varphi(t)) &\geq G(\alpha, \alpha, \beta, t) * G(\alpha, \alpha, \alpha, t) * G(\beta, \beta, \beta, t) \\ G(\alpha, \alpha, \beta, \varphi(t)) &\geq G(\alpha, \alpha, \beta, t) * 1 * 1 \\ G(\alpha, \alpha, \beta, \varphi(t)) &\geq G(\alpha, \alpha, \beta, t) \end{aligned}$$

This implies $\alpha = \beta$.

Thus $\alpha = S\alpha = T\alpha = P(\alpha, \alpha) = A\alpha = B\alpha = Q(\alpha, \alpha)$, that is, the common coupled fixed point of P, Q, A,B, S and T has the form (α, α) .

If we put $B = T = I$ in Theorem 2.1, we have the following :

Corollary 2.1. Let P, Q, A and S be self mappings of X satisfying the following conditions:

Let P, Q : $X \times X \rightarrow X$ and A and S : $X \rightarrow X$ be four mappings satisfying the following conditions:

- (i) $P(X \times X) \subseteq S(X)$ and $Q(X \times X) \subseteq A(X)$
- (ii) P and A are continuous
- (iii) the pairs (P, A) and (Q, S) are weakly compatible. Also suppose
- (v) if there exists $\varphi \in \emptyset$ such that

$$G(P(x, y), P(x, y), Q(u, v), \varphi(t)) \geq G(Ax, Ax, Su, t) * G(Ax, Ax, P(x, y), t) * G(Su, Su, Q(u, v), t) \text{ for all } x, y, u, v \in X, t > 0.$$

Then P,Q, A and S have a unique common coupled fixed point in X.

If we put $P = Q = f$ and $A = S = g$ and $B = T = I$ in Theorem 2.1, we have the following :

Corollary 2.2. Let f, g be self mappings of X satisfying the following conditions:

Let f : $X \times X \rightarrow X$ and A and g : $X \rightarrow X$ be four mappings satisfying the following conditions:

- (i) $f(X \times X) \subseteq g(X)$
- (ii) f and g are continuous
- (iii) the pair (f,g) is weakly compatible. Also suppose

(iv) if there exists $\varphi \in \emptyset$ such that

$$G(f(x, y), f(x, y), g(u, v), \varphi(t)) \geq G(gx, gx, gu, t) * G(gx, gx, f(x, y), t) * G(gu, gu, f(u, v), t)$$

for all $x, y, u, v \in X, t > 0$.

Then f and g have a unique common coupled fixed point in X .

REFERENCES

1. A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy sets and Systems*, 46(1992) 107-113.
2. B. C. Dhage, A study of some fixed point theorems, Ph.D. Thesis(1984), Marathwada Univ., Aurangabad, India.
3. B. C. Dhage, Generalized metric spaces and mappings with fixed point, *Bull. Cal. Math. Soc.* 84(4) (1992) 329-336.
4. B. Samet, C. Vetro, Coupled fixed point, F-invariant set and fixed point of N-order, *Ann. Funct. Anal.*, 1 (2010), 46-56.
5. B. Samet, Coupled fixed point theorems for a generalized Meir-Keller contraction in partially ordered metric spaces, *Nonlinear. Anal.*, 72 (12) (2010), 4508-4517.
6. B.S. Choudhury, K. Das and P.Das, Coupled coincidence point results for compatible mappings in partially ordered fuzzy metric spaces, *Fuzzy Sets Syst.* 222(2013), 84-97.
7. D. Guo and V. Lakshmikantham, Coupled fixed points of nonlinear operators with applications, *Nonlinear Anal.*, 11 (1987), 623-632.
8. G. Sun and K. Yang, Generalized fuzzy metric spaces with properties, *Research journal of Applied Sciences, Engineering and Technology*, 2(7)(2010), 673-678.
9. H.J. Zimmerman, *Fuzzy set theory and its application*, Kluwer Academic Publications, Hingham, 1984.
10. K.S.Ha, Y.J. Cho and A. White, Strictly convex and strictly 2-convex 2-normed spaces, *Mathematica Japonica*, 33(1988), 375-384.
11. L.A. Zadeh, *Fuzzy sets*, *Inform. and Control*, 8 (1965), 338-353.
12. M. Abbas, W. Sintunavarat, P. Kumam, Coupled fixed point in partially ordered G-metric spaces, *Fixed Point Theory Appl.*, (2012), 2012:31.
13. M. Berzig, Solving a class of matrix equations via the Bhaskar-Lakshmikantham coupled fixed point theorem, *Appl. Math. Lett.*, (2012), doi:10.1016/j.aml.2012.01.028.
14. M. E. Gordji, Y. J. Cho, H. Baghani, Coupled fixed point theorems for contractions in intuitionistic fuzzy normed spaces, *Math. Comput. Model.*, 54 (2011), 1897-1906
15. O. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetika* 11 (1975) 326-334.
16. S. Gahler, Zur geometric 2-metriche raume, *Revue Roumaine de Mathematiques Pures et Appliquees*, 40(1966), 664-669.
17. S. Sedghi, I. Altun, and N.Shobe, Coupled fixed point theorems for contractions in fuzzy metric spaces, *Nonlinear Anal.* 72(2010), 1298-1304.
18. S.Gahler, 2-metriche Raume und ihre topologische Struktur, *Mathematische Nachrichten*, 26(1963),115-148.
19. V. Berinde, Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, *Nonlinear Anal.*, 74 (2011), 7347-7355.
20. V. I. Opoitsev, Dynamics of collective behavior. III. Heterogenic systems, *Avtomat. i Telemekh.*, 36 (1975), 124ñ138 (in Russian).

21. V. I. Opoitsev, Heterogenic and combined-concave operators, *Syber. Math. J.*, 16 (1975), 781ñ792 (in Russian).
22. V. I. Opoitsev, T.A. Khurodze, Nonlinear operators in spaces with a cone, *Tbilis. Gos. Univ.*, Tbilisi (1984) 271 (in Russian).
23. V. Lakshmikantham and L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, *Nonlinear Anal.* 70, (2009), 4341-4349.
24. W. Shatanawi, Coupled fixed point theorems in generalized metric spaces, *Hacettepe Journal of Mathematics and Statistics* Volume 40 (3) (2011), 441 – 447
25. W. Sintunavarat, A. Petrusel and P. Kumam, Common coupled fixed point theorems for compatible mappings without mixed monotone property, *Rendiconti del Circolo Matematico di Palermo*, DOI 10.1007/s12215-012-0096-0.
26. W. Sintunavarat, P. Kumam and Y. J. Cho, Coupled fixed point theorems for nonlinear contractions without mixed monotone property, *Fixed Point Theory Appl.*, (2012), 2012:170.
27. W. Sintunavarat, Y. J. Cho and P. Kumam, Coupled fixed-point theorems for contraction mapping induced by cone ball-metric in partially ordered spaces, *Fixed Point Theory Appl.* 2012, 2012:128.
28. Xin-Qi Hu and Qing Lu, Coupled coincidence point theorems for contractions in generalized fuzzy metric spaces, *Fixed Point Theory and Applications*, 2012,2012: 196.
29. Xin-Qi Hu, Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces, *Fixed Point Theory and Applications*, volume 2011, Artical ID 363716, 14 pages, doi: 10.1155/2011/363716.
30. Y. J. Cho, B. E. Rhoades, R. Saadati, B. Samet and W. Shantawi, Nonlinear coupled fixed point theorems in ordered generalized metric spaces with integral type, *Fixed Point Theory Appl.*, 2012, 2012:8.
31. Y. J. Cho, G. He, N. J. Huang, The existence results of coupled quasi-solutions for a class of operator equations, *Bull. Korean Math. Soc.*, 47 (2010), 455-465.
32. Y. J. Cho, M. H. Shah, N. Hussain, Coupled fixed points of weakly F-contractive mappings in topological spaces, *Appl. Math. Lett.* 24 (2011), 1185-1190.
33. Z. Mustafa and B. Sims, A new approach to generalized metric spaces, *Journal of Nonlinear and Convex Analysis*, 7(2006), 289–297.



HYPER-ASYMPTOTIC CURVES ON A KAEHLERIAN HYPER-SURFACE

U. S. Negi

Department of Mathematics, H.N.B. Garhwal (A Central) University, S.R.T. Campus Badshahi Thaul, Tehri Garhwal, Uttarakhand, India.
 E-mail: usnegi7@gmail.com

ABSTRACT :

Mishra (1952) studied hyper-asymptotic curves on a Riemannian hypersurface. Further, Tsagas (1969) has studied special curves of a Hyper-surface of Riemannian space. Also, Negi (Jan-June., 2017) defined and studied Union and Special curves on a Kaehlerian hyper-surface. In this paper, we have defined and studied hyper-asymptotic curves on a Kaehlerian hyper-surface and several theorems have been obtained.

Keywords : Hyper-asymptotic, Special curves, Hyper-surface and Kaehlerian Space.

2010 MSC: 32C15, 46A13, 46M40, 53B35, 53C55.

1. INTRODUCTION

Let us consider a $n(= 2m) > 2$ dimensional complex manifold M_{2n} of differentiability class (C^r) with respect to an allowable coordinate system:

$$(z^i, z^{\bar{i}}) \equiv (z^1, z^2, \dots, z^{n+1}, z^{\bar{1}}, z^{\bar{2}}, \dots, z^{\bar{n+1}}).$$

As we shall use the following variations of the indices:

$$\{ i, j, k, \dots = 1, 2, \dots, n + 1; \bar{i}, \bar{j}, \bar{k}, \dots = \bar{1}, \bar{2}, \dots, \overline{n + 1} \}$$

and $\{ \alpha, \beta, \gamma, \dots = 1, 2, \dots, n; \bar{\alpha}, \bar{\beta}, \bar{\gamma}, \dots = \bar{1}, \bar{2}, \dots, \bar{n}. \}$

If there exists a mixed tensor $F_i^h(z^i, z^{\bar{i}})$ of class C^r , which satisfies

$$(1.1) \quad F_j^i F_i^h = -A_j^h,$$

And with Riemannian metric g_{ij} satisfying:

$$(1.2) \quad ds^2 = g_{i\bar{j}}(z, \bar{z}) dz^i d\bar{z}^{\bar{j}},$$

Which also satisfy the condition,

$$(1.3) \quad \nabla_j F_{i\bar{h}} + \nabla_{\bar{i}} F_{j\bar{h}} = 0,$$

Then, the space is called an almost Kaehlerian space. If the conditions

$$(1.4) \quad \frac{\partial^2 z^h}{\partial z^j \partial z^i} + \frac{\partial z^k}{\partial z^i} g^{hs} \partial_k g_{js} - \frac{\partial z^h}{\partial z^k} g^{ks} \partial_i g_{js} = 0$$

And

$$(1.5) \quad \frac{\partial^2 z^{\bar{h}}}{\partial z^{\bar{j}} \partial z^{\bar{i}}} + \frac{\partial z^{\bar{k}}}{\partial z^{\bar{i}}} g^{\bar{h}s} \partial_{\bar{k}} g_{\bar{j}s} - \frac{\partial z^{\bar{h}}}{\partial z^{\bar{k}}} g^{\bar{k}s} \partial_{\bar{i}} g_{\bar{j}s} = 0$$

are satisfied, then the space is said to be a Kaehlerian space. We shall denote such a space by K_{n+1}^c .

Let an n -dimensional hyperspace K_n^c given by the equation relating, Negi (Jan-June, 2017):

$$z^i = z^i(u^\alpha) (i = 1, 2, \dots, n + 1, \alpha = 1, 2, \dots, n)$$

$$z^{\bar{i}} = z^{\bar{i}}(u^{\bar{\alpha}}) (\bar{i} = \bar{1}, \bar{2}, \dots, \overline{n + 1}, \bar{\alpha} = \bar{1}, \bar{2}, \dots, \bar{n})$$

be immersed in a Kaehlerian space K_{n+1}^c . The first two Frenet's formulae of a curve ($u^\alpha = u^\alpha(s), u^{\bar{\alpha}} = u^{\bar{\alpha}}(s)$) (of the hypersurface) are given by:

$$(1.6a) \quad \frac{\delta \eta_{(0)}^i}{\delta s} = \kappa_{(1)} \eta_{(1)}^i, \quad \frac{\delta \bar{\eta}_{(0)}^{\bar{i}}}{\delta s} = \bar{\kappa}_{(1)} \bar{\eta}_{(1)}^{\bar{i}} \quad \text{and}$$

$$(1.6b) \quad \frac{\delta \eta_{(1)}^i}{\delta s} = -\kappa_{(1)} \eta_{(0)}^i + \kappa_{(2)} \eta_{(2)}^i, \quad \frac{\delta \bar{\eta}_{(1)}^{\bar{i}}}{\delta s} = -\bar{\kappa}_{(1)} \bar{\eta}_{(0)}^{\bar{i}} + \bar{\kappa}_{(2)} \bar{\eta}_{(2)}^{\bar{i}}$$

Where, $\left\{ \eta_{(0)}^i \left(\equiv \frac{dz^i}{ds} \right), \bar{\eta}_{(0)}^{\bar{i}} \left(\equiv \frac{d\bar{z}^{\bar{i}}}{ds} \right), (\eta_{(1)}^i, \bar{\eta}_{(1)}^{\bar{i}}), (\eta_{(2)}^i, \bar{\eta}_{(2)}^{\bar{i}}) \right\}$,

are the components of unit tangent, unit principal normal vector and unit first binormal vector, $(\kappa_{(1)}, \bar{\kappa}_{(1)}, \kappa_{(2)}, \bar{\kappa}_{(2)})$ are the first and second curvatures of the curve.

The components $(q^i, \bar{q}^{\bar{i}})$ and $(p^\alpha, \bar{p}^{\bar{\alpha}})$ of the first curvature vectors with respect to K_{n+1}^c and K_n^c are related by

$$(1.7a) \quad q^i = p^\alpha \beta_\alpha^i + K_n^* N^i, \quad \text{and} \quad (1.7b) \quad \bar{q}^{\bar{i}} = \bar{p}^{\bar{\alpha}} \bar{\beta}_{\bar{\alpha}}^{\bar{i}} + \bar{K}_n^* \bar{N}^{\bar{i}},$$

Where, $\beta_\alpha^i = \frac{\partial z^i}{\partial u^\alpha}, \bar{\beta}_{\bar{\alpha}}^{\bar{i}} = \frac{\partial \bar{z}^{\bar{i}}}{\partial u^{\bar{\alpha}}}$.

$(N^i, \bar{N}^{\bar{i}})$ are the components of unit normal vector and (T_n^*, \bar{T}_n^*) is the normal curvature of the hypersurface in the direction of the curve.

Consider two congruences of the curves given by $(\lambda, \bar{\lambda})$ and $(\mu, \bar{\mu})$, which are such that at the point of K_n^c , we have

$$(1.8a) \quad \lambda^i = t^\alpha \beta_\alpha^i + C N^i, \quad \bar{\lambda}^{\bar{i}} = t^{\bar{\alpha}} \bar{\beta}_{\bar{\alpha}}^{\bar{i}} + \bar{C} \bar{N}^{\bar{i}} \quad \text{and}$$

$$(1.8b) \quad \mu^i = s^\alpha \beta_\alpha^i + D N^i, \quad \bar{\mu}^{\bar{i}} = s^{\bar{\alpha}} \bar{\beta}_{\bar{\alpha}}^{\bar{i}} + \bar{D} \bar{N}^{\bar{i}}.$$

2. HYPER-ASYMPTOTIC CURVE

A hyper asymptotic curve of (order one) relative to a congruence $(\mu, \bar{\mu})$ in Riemannian space is characterized by Mishra (1952). Consequently, we have

$$(2.1a) \quad \mu^i = u_{(1)} \eta_{(0)}^i + z_{(1)} \eta_{(2)}^i,$$

$$(2.1b) \quad \bar{\mu}^{\bar{i}} = u_{(1)} \bar{\eta}_{(0)}^{\bar{i}} + \bar{z}_{(1)} \bar{\eta}_{(2)}^{\bar{i}}.$$

From the first two Frenet's Formulae Refer (1.6a) and (1.6b), we deduce

$$(2.2a) \quad \delta q^i / \delta s = -K_{(1)}^2 \eta_{(0)}^i + \left\{ \frac{d}{ds} (\log K_{(1)}) \right\} q^i + K_{(1)} K_{(2)} \eta_{(2)}^i$$

$$(2.2b) \quad \delta \bar{q}^{\bar{i}} / \delta s = -\bar{K}_{(1)}^2 \bar{\eta}_{(0)}^{\bar{i}} + \left\{ \frac{d}{ds} (\log \bar{K}_{(1)}) \right\} \bar{q}^{\bar{i}} + \bar{K}_{(1)} \bar{K}_{(2)} \bar{\eta}_{(2)}^{\bar{i}}$$

Another expression for $(\delta q^i / \delta s, \delta \bar{q}^{\bar{i}} / \delta s)$ will be obtained by (1.7a) and (1.7b) and the relating Mishra (1952)

$$(2.2c) \quad \frac{\delta \beta_\alpha^i}{\delta s} = \Omega_{\alpha\beta} \left(\frac{du^\beta}{ds} \right) N^i, \quad \frac{\delta \bar{\beta}_{\bar{\alpha}}^{\bar{i}}}{\delta s} = \Omega_{\bar{\alpha}\bar{\beta}} \left(\frac{d\bar{u}^{\bar{\beta}}}{ds} \right) \bar{N}^{\bar{i}} \quad \text{and}$$

$$(2.2d) \quad \frac{\delta N^i}{\delta s} = -\Omega_{\beta\gamma} g^{\beta\alpha} \rho_\alpha^i \frac{du^\gamma}{ds}, \quad \frac{\delta \bar{N}^{\bar{i}}}{\delta s} = -\Omega_{\bar{\beta}\bar{\gamma}} g^{\bar{\beta}\bar{\alpha}} \bar{\rho}_{\bar{\alpha}}^{\bar{i}} \frac{d\bar{u}^{\bar{\gamma}}}{ds}.$$

This later expression for $(\delta q^i / \delta s, \delta \bar{q}^{\bar{i}} / \delta s)$ and equations (2.1a), (2.1b), (2.2a) and (2.2b) may be used in the elimination of $(\eta_{(2)}^i, \bar{\eta}_{(2)}^{\bar{i}})$ with the help of (1.7a), (1.7b), (1.8a) and (1.8b) gives

$$(2.3a) \quad s^\alpha = u_{(1)} \frac{du^\alpha}{ds} + y \left\{ \frac{\delta p^\alpha}{\delta s} - K_n^* \Omega_{\beta\gamma} g^{\beta\alpha} \frac{du^\gamma}{ds} - \frac{p^\alpha d(\log K_{(1)})}{ds} + K_{(1)}^2 \frac{du^\alpha}{ds} \right\},$$

$$(2.3b) \quad s^{\bar{\alpha}} = u_{(1)} \frac{d\bar{u}^{\bar{\alpha}}}{ds} + y \left\{ \frac{\delta \bar{p}^{\bar{\alpha}}}{\delta s} - \bar{K}_n^* \Omega_{\bar{\beta}\bar{\gamma}} g^{\bar{\beta}\bar{\alpha}} \frac{d\bar{u}^{\bar{\gamma}}}{ds} - \frac{\bar{p}^{\bar{\alpha}} d(\log \bar{K}_{(1)})}{ds} + \bar{K}_{(1)}^2 \frac{d\bar{u}^{\bar{\alpha}}}{ds} \right\},$$

$$(2.4a) \quad D = y \left\{ \Omega_{\alpha\beta} \frac{du^\beta}{ds} + \frac{dK_n^*}{ds} - K_n^* \frac{d(\log K_{(1)})}{ds} \right\}, \quad \text{Where } y = \frac{z_{(1)}}{K_{(1)}K_{(2)}}.$$

$$(2.4b) \quad \bar{D} = \bar{y} \left\{ \Omega_{\bar{\alpha}\bar{\beta}} \frac{d\bar{u}^{\bar{\beta}}}{ds} + \frac{d\bar{K}_n^*}{ds} - \bar{K}_n^* \frac{d(\log \bar{K}_{(1)})}{ds} \right\}, \quad \text{Where } \bar{y} = \frac{\bar{z}_{(1)}}{\bar{K}_{(1)}\bar{K}_{(2)}}.$$

Let $(\xi_{(0)}^\alpha, \xi_{(0)}^{\bar{\alpha}})$, $(\xi_{(1)}^\alpha, \xi_{(1)}^{\bar{\alpha}})$ and $(\xi_{(2)}^\alpha, \xi_{(2)}^{\bar{\alpha}})$ be the unit tangent, unit principal normal, unit first binormal vectors and $(\kappa_{(1)}, \bar{\kappa}_{(1)})$, $(\kappa_{(2)}, \bar{\kappa}_{(2)})$ be the first and second curvatures of the curve with respect to the hyper surface.

We obtain from first two Frenet's formulae with respect to T_n^c yield

$$(2.5a) \quad \delta p^\alpha / \delta s = -\kappa_{(1)}^2 \frac{du^\alpha}{ds} + \frac{p^\alpha d(\log K_{(1)})}{ds} + \kappa_{(1)}\kappa_{(2)}\xi_{(2)}^\alpha,$$

$$(2.5b) \quad \delta p^{\bar{\alpha}} / \delta s = -\bar{\kappa}_{(1)}^2 \frac{d\bar{u}^{\bar{\alpha}}}{ds} + \frac{p^{\bar{\alpha}} d(\log \bar{K}_{(1)})}{ds} + \bar{\kappa}_{(1)}\bar{\kappa}_{(2)}\xi_{(2)}^{\bar{\alpha}}.$$

From equations $[(K_{(1)}^2 = \kappa_{(1)}^2 + K_n^{*2} \quad \text{and} \quad \bar{K}_{(1)}^2 = \bar{\kappa}_{(1)}^2 + \bar{K}_n^{*2}]$, relating Negi (Jan-June, 2017), (2.4a), (2.4b), (2.3a), (2.3b), (2.5a), (2.5b) and the definitions

$$\cos \phi = \left(\frac{\left(g_{\alpha\beta} s^\alpha \frac{du^\beta}{ds} \right)}{s}, \frac{\left(g_{\bar{\alpha}\bar{\beta}} s^{\bar{\alpha}} \frac{d\bar{u}^{\bar{\beta}}}{ds} \right)}{\bar{s}} \right),$$

We deduce $u_{(1)} = (S \cos \phi, \bar{S} \cos \phi)$, and

$$(2.6a) \quad \left[\Omega_{\alpha\beta} p^\gamma \frac{du^\beta}{ds} + \frac{dK_n^*}{ds} - K_n^* \frac{d(\log K_{(1)})}{ds} \right] \left(s^\alpha - S \cos \phi \frac{du^\alpha}{ds} \right)$$

$$= D \left[K_n^{*2} \frac{du^\alpha}{ds} + \frac{p^\alpha d(\log \frac{\kappa_{(1)}}{K_{(1)}})}{ds} + \kappa_{(1)}\kappa_{(2)}\xi_{(2)}^* - K_n^* \Omega_{\beta\gamma} g^{\alpha\beta} \frac{du^\gamma}{ds} \right]$$

$$(2.6b) \quad \left[\Omega_{\bar{\alpha}\bar{\beta}} p^{\bar{\gamma}} \frac{d\bar{u}^{\bar{\beta}}}{ds} + \frac{d\bar{K}_n^*}{ds} - \bar{K}_n^* \frac{d(\log \bar{K}_{(1)})}{ds} \right] \left(s^{\bar{\alpha}} - \bar{S} \cos \phi \frac{d\bar{u}^{\bar{\alpha}}}{ds} \right)$$

$$= \bar{D} \left[\bar{K}_n^{*2} \frac{d\bar{u}^{\bar{\alpha}}}{ds} + \frac{p^{\bar{\alpha}} d(\log \frac{\bar{\kappa}_{(1)}}{\bar{K}_{(1)}})}{ds} + \bar{\kappa}_{(1)}\bar{\kappa}_{(2)}\xi_{(2)}^* - \bar{K}_n^* \Omega_{\bar{\beta}\bar{\gamma}} g^{\bar{\alpha}\bar{\beta}} \frac{d\bar{u}^{\bar{\gamma}}}{ds} \right]$$

Theorem (3.1): A hyper-asymptotic curve relative to $(\mu, \bar{\mu})$ is characterized by equations (2.6a), (2.6b).

Proof: Multiplying equation (2.6a) by $g_{\alpha t} p^t$ and equation (2.6b) by $g_{\bar{\alpha} \bar{t}} p^{\bar{t}}$ and simplifying, we obtain

$$\Omega_{\alpha\beta} p^\alpha \left(\frac{du^\beta}{ds} \right) \left(S \cos \psi + \frac{DK_n^*}{\kappa_{(1)}} \right) = \left[\frac{D\kappa_{(1)} d(\log \frac{\kappa_{(1)}}{K_{(1)}})}{ds} - \frac{K_n^* S \cos \psi d(\log \frac{K_n^*}{K_{(1)}})}{ds} \right],$$

Where, we have defined $\cos \psi = \frac{g_{\alpha\beta} p^\alpha s^\beta}{\kappa_{(1)} S}$,

And

$$\Omega_{\bar{\alpha}\bar{\beta}} p^{\bar{\alpha}} \left(\frac{d\bar{u}^{\bar{\beta}}}{ds} \right) \left(\bar{S} \cos \psi + \frac{D\bar{K}_n^*}{\bar{\kappa}_{(1)}} \right) = \left[\frac{D\bar{\kappa}_{(1)} d(\log \frac{\bar{\kappa}_{(1)}}{\bar{K}_{(1)}})}{ds} - \frac{\bar{K}_n^* \bar{S} \cos \psi d(\log \frac{\bar{K}_n^*}{\bar{K}_{(1)}})}{ds} \right],$$

Where, we have defined

$$\cos \bar{\psi} = \frac{g_{\alpha\bar{\beta}} p^{\alpha} \bar{s}^{\beta}}{\bar{\kappa}_{(1)} \bar{s}}.$$

REFERENCES

1. Mishra, R. S. (1952), Hyperasymptotic curves on a Riemannian Hyper-surface, Math. Student, 20, 63-65.
2. Tsagas, G. I. (1969), Special curves of a Hypersurface of Riemannian space, Tensor, N. S., 20, 88-90.
3. Springer, C. I. (1950), Union curves of a Hypersurface, Can. J. Math., 2, 457-460.
4. Mishra, R. S. (1965), A course in Tensor with applications to Riemannian Geometry, Pothishala Private Limited.
5. Mann.Wong. P. (2007): A survey of complex finsler geometry, Advanced studied in pure Math, Vol.48, Math.Soc. of Japan, pp.375- 433.
6. Negi, U.S. and Rawat Aparna (2009): Some theorems on almost Kaehlerian spaces with recurrent and symmetric projective curvature Tensors. ACTA Ciencia Indica, Vol. XXXVM, No 3, pp. 947-951.
7. Negi.U.S. (Jan-June., 2017), Union and Special curves on a Kaehlerian hyper-surface, Aryabhata Journal of Mathematics & Informatics Vol. 9, No.1, pp.01-04,



COMPARATIVE ANALYSIS OF TWO RELIABILITY MODELS WITH REGARD TO PROVISION OF ASSISTANT REPAIRMAN AND DISCUSSION TIME

Anil Kumar Taneja

Former Professor, JRE Group of Institutions, Greater Noida, U.P., India
E-mail : dranil_tan@rediffmail.com

ABSTRACT :

A two-unit cold standby system with regular as well as visiting repairmen is studied. On failure of a unit, the repair is undertaken by the regular repairman who always remains with the system. The regular repairman while repairing the failed unit may get tired after some time and then the nature of the fault is discussed by an outside expert repairman telephonically and accordingly he himself comes or sends his assistant (an ordinary repairman). Various measures of system effectiveness are obtained. Profit is evaluated for a particular case.

Keywords: Cold-standby, Regular and two types of visiting repairmen, Telephonic discussion, Rest period, Profit analysis

INTRODUCTION

Reliability of two-unit systems taking various assumptions has been studied by various researchers in the field of reliability. Most of these studies including [1-9, 11] considered the repairman as perfect i.e. a repairman who repairs tirelessly and flawlessly. Some of the studies such as Kumar et al. [6], Parasher and Taneja [9], Makkadis et al.[3], incorporated the concept of assistant of a repairman wherein the repair is first undertaken by the assistant repairman after getting instructions given by his master. However, there may be situations where the expert repairman may discuss the nature of failure to decide whether he himself should go or to send his assistant repairman. This concept has been discussed by Taneja [10] with patience time. However, it may not be economically advisable to wait up to certain time and hence the expert may be called as soon as the assistant declares himself unable to repair.

Keeping the above in view, the present paper is devoted to study a two-unit cold standby system incorporating the idea of a regular repairman who always remains with the system and two types of visiting repairman i.e. an expert and his assistant. The failed unit is first undertaken by the regular repairman who may get tired while repairing the failed unit or may not be able to do some complex repairs. If the regular repairman gets tired or declares himself unable to repair the unit, an outside expert repairman is contacted who first discusses the nature of the failure telephonically to decide whether he himself should go or to send his assistant repairman. It has also been assumed that the assistant repairman may also not be able to do some complex repairs. If the regular/assistant repairman finds himself unable to complete the repair or if the system becomes inoperable, then expert himself comes to repair the unit and he repairs all the units which fail during his stay at the system. This model is compared with the model there is no provision of assistant repairman.

Various measures of system effectiveness such as mean time to system failure, steady state availability, busy period analysis of the assistant/expert repairman, expected number of visits by the assistant/expert repairman, expected

discussion time and expected profit earned by the system are determined. Profit of this model is also compared graphically with the model wherein there is no provision of assistant repairman.

NOTATIONS

λ	constant failure rate
a	the probability that the regular/assistant repairman is able to repair the failed unit
b	the probability that the regular repairman does not need rest
p	$= ab$
q	$= a(1-b)$
p_1	the probability that the regular repairman is available on the completion of repair by the assistant/expert repairman
q_1	the probability that the regular repairman is not available
p_2	the probability that the expert after discussing the nature of failure himself comes to repair
q_2	the probability that the expert after discussing the nature of the failure sends his assistant to repair
$g(t), G(t)$	p.d.f. and c.d.f. of the repair time of the regular repairman
$g_a(t), G_a(t)$	p.d.f. and c.d.f. of the repair time of the assistant repairman
$g_e(t), G_e(t)$	p.d.f. and c.d.f. of the repair time of the expert repairman
$h_1(t), H_1(t)$	p.d.f. and c.d.f. of the discussion time

SYMBOLS FOR THE STATES OF THE SYSTEM :

o	operative unit and regular repairman is available
o_n	operative unit (suffix n represents that regular repairman not available)
cs	cold standby
Fr	failed unit under repair of the regular repairman
F_{wd}	failed unit waiting for repair while discussions are going on
Fra	failed unit under repair of the assistant repairman
Fre	failed unit under repair of the expert repairman
FRe	repair of the failed unit is continuing by the expert repairman from the previous state
F_w	failed unit waiting for repair of the expert.

TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES

The state transition diagram is shown as in Fig.1. The epochs of entry into states 0, 1, 2, 3, 4, 5 and 6 are regeneration points and hence these states are regenerative states. States 4 and 7 are down states.

The transition probabilities are :-

$$\begin{aligned}
 q_{01}(t) &= \lambda e^{-\lambda t} ; & q_{10}(t) &= p e^{-\lambda t} g(t) \\
 q_{12}(t) &= q e^{-\lambda t} g(t) ; & q_{13}(t) &= (1-a) e^{-\lambda t} g(t) \\
 q_{14}(t) &= \lambda e^{-\lambda t} \bar{G}(t) ; & q_{23}(t) &= p_2 e^{-\lambda t} h_1(t) \\
 q_{24}(t) &= \lambda e^{-\lambda t} \bar{H}_1(t) ; & q_{25}(t) &= q_2 e^{-\lambda t} h_1(t) \\
 q_{30}(t) &= p_1 e^{-\lambda t} g_e(t) ; & q_{36}(t) &= q_1 e^{-\lambda t} g_e(t) \\
 q_{37}(t) &= \lambda e^{-\lambda t} \bar{G}_e(t) ; & q_{33}^{(7)}(t) &= [\lambda e^{-\lambda t} \odot 1] g_e(t) = [1-e^{-\lambda t}] g_e(t) \\
 q_{43}(t) &= g_e(t) ; & q_{50}(t) &= p_1 a e^{-\lambda t} g_a(t) \\
 q_{53}(t) &= (1-a) e^{-\lambda t} g_a(t) ; & q_{54}(t) &= \lambda e^{-\lambda t} \bar{G}_a(t)
 \end{aligned}$$

$$\begin{aligned}
 q_{56}(t) &= q_1 a e^{-\lambda t} g_a(t); & q_{61}(t) &= p_1 \lambda e^{-\lambda t} \\
 q_{62}(t) &= q_1 \lambda e^{-\lambda t} & &
 \end{aligned}
 \tag{1 - 19}$$

The non-zero elements p_{ij} are given by

$$\begin{aligned}
 p_{01} &= 1; & p_{10} &= p g^*(\lambda) \\
 p_{12} &= q g^*(\lambda); & p_{13} &= (1-a)g^*(\lambda) \\
 p_{14} &= 1-g^*(\lambda); & p_{23} &= p_2 h_1^*(\lambda) \\
 p_{24} &= 1-h_1^*(\lambda); & p_{25} &= q_2 h_1^*(\lambda) \\
 p_{30} &= p_1 g_e^*(\lambda); & p_{36} &= q_1 g_e^*(\lambda) \\
 p_{37} &= 1-g_e^*(\lambda); & p_{33}^{(7)} &= 1-g_e^*(\lambda) \\
 p_{43} &= 1; & p_{50} &= p_1 a g_a^*(\lambda) \\
 p_{53} &= (1-a)g_a^*(\lambda); & p_{54} &= 1-g_a^*(\lambda) \\
 p_{56} &= q_1 a g_a^*(\lambda); & p_{61} &= p_1 \\
 p_{62} &= q_1. & &
 \end{aligned}
 \tag{20 - 38}$$

By these transition probabilities, it can be verified that

$$p_{01} = 1; \quad p_{10} + p_{12} + p_{13} + p_{14} =$$

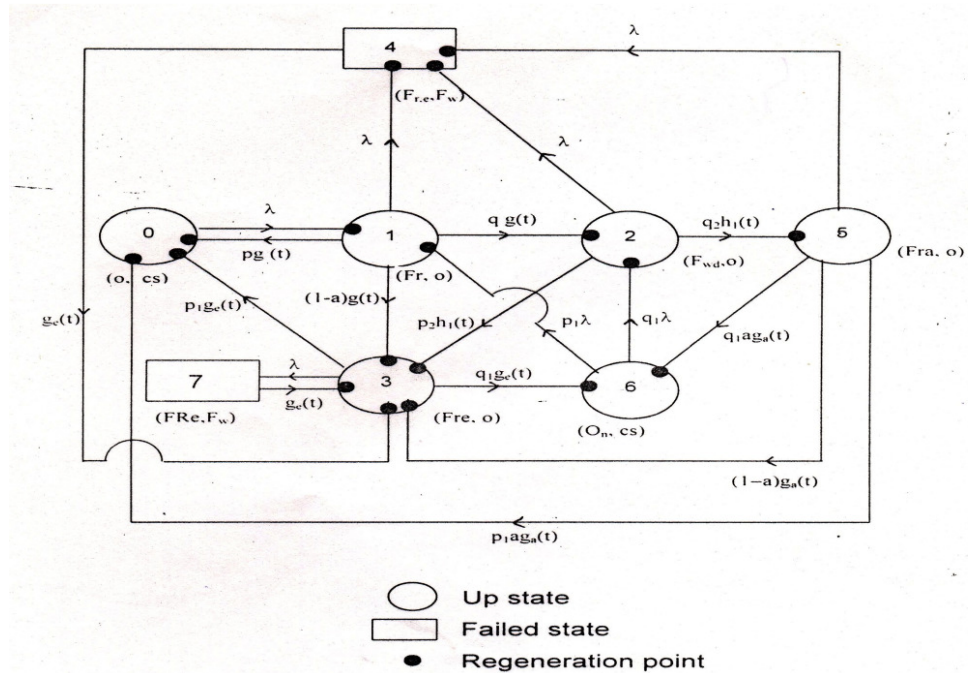


Fig. 1

$$\begin{aligned}
 p_{23} + p_{24} + p_{25} &= 1; & p_{30} + p_{36} + p_{37} &= 1 \\
 p_{30} + p_{36} + p_{33}^{(7)} &= 1; & p_{43} &= 1 \\
 p_{50} + p_{53} + p_{54} + p_{56} &= 1; & p_{61} + p_{62} &= 1
 \end{aligned}
 \tag{39 - 46}$$

The mean sojourn times (μ_i) are

$$\begin{aligned}
 \mu_0 &= \frac{1}{\lambda}; & \mu_1 &= \frac{1 - g^*(\lambda)}{\lambda} \\
 \mu_2 &= \frac{1 - h_1^*(\lambda)}{\lambda}; & \mu_3 &= \frac{1 - g_e^*(\lambda)}{\lambda}
 \end{aligned}$$

$$\begin{aligned} \mu_4 &= -g_e^* '(0) ; & \mu_5 &= \frac{1 - g_a^* (\lambda)}{\lambda} \\ \mu_6 &= \frac{1}{\lambda} = \mu_0 \end{aligned} \tag{47 - 53}$$

The unconditional mean time taken by the system to transit for any state j when it is counted from epoch of entrance into state i is mathematically stated as :

$$m_{ij} = \int_0^\infty t q_{ij}(t)dt = -q_{ij}^* '(0) \tag{54}$$

Thus,

$$\begin{aligned} m_{01} &= \mu_0 ; & m_{10} + m_{12} + m_{13} + m_{14} &= \mu_1 \\ m_{23} + m_{24} + m_{25} &= \mu_2 ; & m_{30} + m_{36} + m_{37} &= \mu_3 \\ m_{30} + m_{36} + m_{33}^{(7)} &= \mu_4 ; & m_{43} &= \mu_4 ; \\ m_{50} + m_{53} + m_{54} + m_{56} &= \mu_5 ; & m_{61} + m_{62} &= \mu_6 \end{aligned} \tag{55 - 62}$$

Mean Time to System Failure

By probabilistic arguments, we obtain the following recursive relations for $\phi_i(t)$:

$$\begin{aligned} \phi_0(t) &= Q_{01}(t)(s) \phi_1(t) \\ \phi_1(t) &= Q_{10}(t)(s)\phi_0(t) + Q_{12}(t)(s)\phi_2(t) + Q_{13}(t)(s)\phi_3(t) + Q_{14}(t) \\ \phi_2(t) &= Q_{23}(t)(s)\phi_3(t) + Q_{24}(t) + Q_{25}(t)(s)\phi_5(t) \\ \phi_3(t) &= Q_{30}(t)(s)\phi_0(t) + Q_{36}(t)(s)\phi_6(t) + Q_{37}(t) \\ \phi_5(t) &= Q_{50}(t)(s)\phi_0(t) + Q_{53}(t)(s)\phi_3(t) + Q_{54}(t) + Q_{56}(t)(s)\phi_6(t) \\ \phi_6(t) &= Q_{61}(t)(s)\phi_1(t) + Q_{62}(t)(s)\phi_2(t) \end{aligned} \tag{63 - 68}$$

Taking Laplace-Stieltjes Transforms of these relations and solving them for $\phi_0^{**}(s)$, the mean time to system failure (MTSF) when the system starts from the state '0' is

$$T_0 = \lim_{s \rightarrow 0} \frac{1 - \phi_0^{**}(s)}{s} = N/D \tag{69}$$

where

$$\begin{aligned} N &= [(p_{10} + p_{14}) (1 - p_{23}p_{36}p_{62}) - p_{62}p_{25}(p_{56} + p_{36}p_{53}) (p_{10} + p_{14}) \\ &+ p_{25}p_{12}(p_{50} + p_{53}p_{30}) + p_{30}(p_{13} + p_{12}p_{23}) + p_{62}p_{13}p_{24}p_{36} \\ &+ p_{62}p_{25}p_{13} \{ p_{56}(p_{30} - p_{37}) + p_{36}(p_{54} - p_{50}) \} \\ &+ p_{13}p_{36} + p_{12} \{ p_{23}p_{36} + p_{25}(p_{56} + p_{36}p_{53}) \}] \mu_0 \\ &+ [1 - p_{62} \{ p_{23}p_{36} + p_{25}(p_{56} + p_{36}p_{53}) \}] \mu_1 \\ &+ (p_{12} + p_{62}p_{13}p_{36}) \mu_2 + [p_{13}(1 - p_{25}p_{56}p_{62}) + p_{12}(p_{23} + p_{25}p_{53})] \mu_3 \\ &+ p_{25}(p_{12} + p_{13}p_{36}p_{62}) \mu_5 \\ D &= p_{12} [p_{37}(p_{23} + p_{25}p_{53}) + (p_{24} + p_{25}p_{54})] + p_{14} + p_{13}p_{37} \\ &+ p_{62} [p_{13}(p_{24}p_{36} + p_{25}p_{36}p_{54} - p_{13}p_{25}p_{37}p_{56}) - p_{14}(p_{23}p_{36} \\ &+ p_{25}p_{56} + p_{25}p_{36}p_{53})] \end{aligned} \tag{70-71}$$

Availability Analysis

The availability $A_i(t)$ is seen to satisfy the following recursive relations:–

$$\begin{aligned}
 A_0(t) &= M_0(t) + q_{01}(t) \odot A_1(t) \\
 A_1(t) &= M_1(t) + q_{10}(t) \odot A_0(t) + A_{12}(t) \odot A_2(t) + q_{13}(t) \odot A_3(t) + q_{14}(t) \odot A_4(t) \\
 A_2(t) &= M_2(t) + q_{23}(t) \odot A_3(t) + q_{24}(t) \odot A_4(t) + q_{25}(t) \odot A_5(t) \\
 A_3(t) &= M_3(t) + q_{30}(t) \odot A_0(t) + q_{36}(t) \odot A_6(t) + q_{33}^{(7)}(t) \odot A_3(t) \\
 A_4(t) &= q_{43}(t) \odot A_3(t) \\
 A_5(t) &= M_5(t) + q_{50}(t) \odot A_0(t) + q_{53}(t) \odot A_3(t) + q_{54}(t) \odot A_4(t) \\
 &\quad + q_{56}(t) \odot A_6(t) \\
 A_6(t) &= M_6(t) + q_{61}(t) \odot A_1(t) + q_{62}(t) \odot A_2(t)
 \end{aligned} \tag{72 – 78}$$

where

$$\begin{aligned}
 M_0(t) &= e^{-\lambda t}, M_1(t) = e^{-\lambda t} \bar{G}(t), M_2(t) = e^{-\lambda t} \bar{H}_1(t), M_3(t) = e^{-\lambda t} \bar{G}_e(t) \\
 M_5(t) &= e^{-\lambda t} \bar{G}_a(t), M_6(t) = e^{-\lambda t}
 \end{aligned} \tag{79 – 84}$$

Taking the Laplace transforms of the above equations and solving them for $A_0^*(s)$, In steady state availability of the system is given by

$$A_0 = N_1/D_1 \tag{85}$$

where

$$\begin{aligned}
 N_1 &= (p_{30} + p_{36})[(1 - p_{25}p_{56}p_{62})(\mu_0 + \mu_1) + \mu_0 p_{12}p_{25}(p_{61} - p_{56}) + p_{12}\mu_2] \\
 &\quad - \mu_0 p_{36} \{ [p_{25}(p_{53} + p_{54}) + (p_{23} + p_{24})(p_{62} + p_{61}p_{12}) + p_{61}(p_{13} + p_{14})] \\
 &\quad - (p_{24} + p_{25}p_{54})[p_{12}\mu_3 + p_{36}(p_{62}\mu_1 - p_{12}\mu_0)] + (p_{13} + p_{12}p_{23})(\mu_3 + p_{36}\mu_0) + \\
 &\quad p_{14}[\mu_3(1 - p_{25}p_{56}p_{62}) + p_{36}\{\mu_0 + p_{62}(\mu_2 + p_{25}\mu_5)\}] + p_{25}[p_{62}p_{13}(\mu_5 p_{36} - \mu_3 p_{56}) \\
 &\quad - p_{53}p_{36}(\mu_1 p_{62} + \mu_0) - p_{12}\{\mu_3 p_{53} + \mu_5(p_{30} + p_{36})\}] + p_{36}p_{62}(\mu_2 p_{13} - \mu_1 p_{23}) \\
 D_1 &= \mu_0[(1 - p_{33}^{(7)})\{p_{10}(1 - p_{56}p_{62}p_{25}) + p_{12}p_{25}p_{50}\} + (p_{23} + p_{24}) \\
 &\quad \{p_{12}p_{30} - p_{10}p_{36}p_{62}\} + (p_{53} + p_{54})p_{25}\{p_{12}p_{30} - p_{10}p_{36}p_{62}\} \\
 &\quad + (p_{13} + p_{14})\{p_{30} + p_{62}p_{25}(p_{50}p_{36} - p_{56}p_{30})\} + (1 - p_{10})p_{36} \\
 &\quad + p_{25}\{p_{12}(p_{56}p_{30} - p_{36}p_{50}) + (p_{13} + p_{14})p_{36}(p_{53} + p_{54})\}] \\
 &\quad + \mu_1[p_{30} + p_{36}p_{61} + p_{36}p_{62}p_{25}(p_{50} + p_{56}) - p_{56}p_{62}p_{25}(p_{30} + p_{36})] \\
 &\quad + \mu_2[p_{12}(p_{30} + p_{36}) + p_{36}p_{62}(p_{13} + p_{14})] + \mu_3[p_{12}(p_{23} + p_{24}) + (p_{13} + p_{14})(1 \\
 &\quad - p_{56}p_{62}p_{25}) + p_{25}p_{12}(p_{53} + p_{54}) + p_{24}\{p_{36}p_{62}(1 - p_{10}) + p_{12}p_{30}\} \\
 &\quad + p_{14}\{p_{62}p_{25}(p_{50}p_{36} - p_{56}p_{30}) + p_{30}\} + p_{25}p_{54}(1 + p_{61}p_{12}p_{36})] \\
 &\quad + \mu_5[p_{12}p_{25}(p_{30} + p_{36}) + p_{62}p_{25}p_{36}(p_{13} + p_{14})]
 \end{aligned} \tag{86–87}$$

Busy Period Analysis of the Expert Repairman (Repair Time only)

Letting $B_i^e(t)$ be the probability that the repairman

$$\begin{aligned}
 B_0^e(t) &= q_{01}(t) \odot B_1^e(t) \\
 B_1^e(t) &= q_{10}(t) \odot B_0^e(t) + q_{12}(t) \odot B_2^e(t) + q_{13}(t) \odot B_3^e(t) + q_{14}(t) \odot B_4^e(t) \\
 B_2^e(t) &= q_{23}(t) \odot B_3^e(t) + q_{24}(t) \odot B_4^e(t) + q_{25}(t) \odot B_5^e(t) \\
 B_3^e(t) &= W_3(t) + q_{30}(t) \odot B_0^e(t) + q_{36}(t) \odot B_6^e(t) + q_{33}^{(7)}(t) \odot B_3^e(t) \\
 B_4^e(t) &= W_4(t) + q_{43}(t) \odot B_3^e(t) \\
 B_5^e(t) &= q_{50}(t) \odot B_0^e(t) + q_{53}(t) \odot B_3^e(t) + q_{54}(t) \odot B_4^e(t) + q_{56}(t) \odot B_6^e(t) \\
 B_6^e(t) &= q_{61}(t) \odot B_1^e(t) + q_{62}(t) \odot B_2^e(t)
 \end{aligned} \tag{88 – 94}$$

where $W_3(t) = W_4(t) = \bar{G}_e(t)$ (95)

Taking L.T. of the above equations and solving them for $B_0^{e*}(s)$,

In steady state, the total fraction of time for which the system is under repair of the expert repairman is given by

$$B_0^e = N_2/D_1 \tag{96}$$

where

$$N_2 = \mu_3[(p_{13}+p_{14})(1-p_{25}p_{56}p_{62}) + p_{12}\{p_{25}(p_{53}+p_{54}) + (p_{23}+p_{24})\} + p_{36}p_{62}\{p_{13}(p_{24}+p_{25})-p_{14}p_{23}\} + (1-p_{33}^{(7)})\{p_{12}(p_{24}+p_{25}p_{54}) + p_{14}(1-p_{25}p_{62})\} - p_{14}p_{25}p_{53}p_{36}p_{62}] \tag{97}$$

Busy Period Analysis of the Assistant Repairman

By probabilistic arguments, we have the following recursive relations for $B_i^a(t)$:

$$\begin{aligned} B_0^a(t) &= q_{01}(t) \odot B_1^a(t) \\ B_1^a(t) &= q_{10}(t) B_0^a(t) + q_{12}(t) \odot B_2^a(t) + q_{13}(t) \odot B_3^a(t) + q_{14}(t) \odot B_4^a(t) \\ B_2^a(t) &= q_{23}(t) \odot B_3^a(t) + q_{24}(t) \odot B_4^a(t) + q_{25}(t) \odot B_5^a(t) \\ B_3^a(t) &= q_{30}(t) \odot B_0^a(t) + q_{36}(t) \odot B_6^a(t) + q_{33}^{(7)}(t) \odot B_3^a(t) \\ B_4^a(t) &= q_{43}(t) \odot B_3^a(t) \\ B_5^a(t) &= W_5(t) + q_{50}(t) \odot B_0^a(t) + q_{53}(t) \odot B_3^a(t) + q_{54}(t) \odot B_4^a(t) \\ &\quad + q_{56}(t) \odot B_6^a(t) \\ B_6^a(t) &= q_{61}(t) \odot B_1^a(t) + q_{62}(t) \odot B_2^a(t) \end{aligned} \tag{98-104}$$

where

$$W_5(t) = e^{-\lambda t} \bar{G}_a(t) \tag{105}$$

Taking L.T. of the above equations and solving them for $B_0^{a*}(s)$,

In steady state, the total fraction of time for which the system is under repair of the assistant repairman is given by

$$B_0^a = N_3/D_1 \tag{106}$$

where

$$N_3 = p_{25}[p_{36}p_{62}(p_{13}+p_{14}) + p_{12}(p_{30} + p_{36})]\mu_5 \tag{107}$$

Expected Number of Visits by the Expert Repairman

By probabilistic arguments, we have the following recursive relations for $V_i^e(t)$:-

$$\begin{aligned} V_0^e(t) &= Q_{01}(t)(s)V_1^e(t) \\ V_1^e(t) &= Q_{10}(t)(s)V_0^e(t) + Q_{12}(t)(s)V_2^e(t) + Q_{13}(t)(s)[1+V_3^e(t)] \\ &\quad + Q_{14}(t)(s)[1+V_4^e(t)] \\ V_2^e(t) &= Q_{23}(t)(s)[1+V_3^e(t)] + Q_{24}(t)(s)[1+V_4^e(t)] + Q_{25}(t)(s)V_5^e(t) \\ V_3^e(t) &= Q_{30}(t)(s)V_0^e(t) + Q_{36}(t)(s)V_6^e(t) + Q_{33}^{(7)}(t)(s)V_3^e(t) \\ V_4^e(t) &= Q_{43}(t)(s)V_3^e(t) \\ V_5^e(t) &= Q_{50}(t)(s)V_0^e(t) + Q_{53}(t)(s)[1+V_3^e(t)] + Q_{54}(t)(s)[1+V_4^e(t)] \\ &\quad + Q_{56}(t)(s)V_6^e(t) \\ V_6^e(t) &= Q_{61}(t)(s)V_1^e(t) + Q_{62}(t)(s)V_2^e(t) \end{aligned} \tag{108-114}$$

Taking L.S.T. of the above equations and in steady state, the number of visits per unit time by the expert is given by

$$V_0^e = N_4/D_1 \tag{115}$$

where

$$N_4 = (p_{30}+p_{36}) [(p_{13}+p_{14})(1-p_{25}p_{56}p_{62}) + p_{12}(p_{23}+p_{24})$$

$$+ p_{12}p_{25}(p_{53}+p_{54})] \tag{116}$$

Expected Number of Visits by the Assistant Repairman

The following recursive relations for $V_i^a(t)$ are obtained :-

$$\begin{aligned} V_0^a(t) &= Q_{01}(t)(s) V_1^a(t) \\ V_1^a(t) &= Q_{10}(t)(s)V_0^a(t) + Q_{12}(t)(s) V_2^a(t) + Q_{13}(t)(s)V_3^a(t) \\ &\quad + Q_{14}(t)(s)V_4^a(t) \\ V_2^a(t) &= Q_{23}(t)(s)V_3^a(t) + Q_{24}(t)(s)V_4^a(t) + Q_{25}(t)(s)[1+V_5^a(t)] \\ V_3^a(t) &= Q_{30}(t)(s)V_0^a(t) + Q_{36}(t)(s)V_6^a(t) + Q_{33}^{(7)}(t)(s)V_3^a(t) \\ V_4^a(t) &= Q_{43}(t)(s)V_3^a(t) \\ V_5^a(t) &= Q_{50}(t)(s)V_0^a(t) + Q_{53}(t)(s)V_3^a(t) + Q_{54}(t)(s)V_4^a(t) \\ &\quad + Q_{56}(t)(s)V_6^a(t) \\ V_6^a(t) &= Q_{61}(t)(s)V_1^a(t) + Q_{62}(t)(s)V_2^a(t) \end{aligned} \tag{117-123}$$

Taking L.S.T. of the above equations and solving them for $V_0^{a**}(s)$,

In steady state, the number of visits per unit time by the expert is given by

$$V_0^a = N_5/D_1 \tag{124}$$

where

$$N_5 = p_{25}[p_{12}(p_{30}+p_{36}) + p_{36}p_{62}(p_{13}+ p_{14})] \tag{125}$$

Expected Discussion Time

The following recursive relations for $DT_i(t)$ are obtained :-

$$\begin{aligned} DT_0(t) &= q_{01}(t) \odot DT_1(t) \\ DT_1(t) &= q_{10}(t) \odot DT_0(t) + q_{12}(t) \odot DT_2(t) + q_{13}(t) \odot DT_3(t) \\ &\quad + q_{14}(t) \odot DT_4(t) \\ DT_2(t) &= W_2(t) + q_{23}(t) \odot DT_3(t) + q_{24}(t) \odot DT_4(t) + q_{25}(t) \odot DT_5(t) \\ DT_3(t) &= q_{30}(t) \odot DT_0(t) + q_{36}(t) \odot DT_6(t) + q_{33}^{(7)}(t) \odot DT_3(t) \\ DT_4(t) &= q_{43}(t) \odot DT_3(t) \\ DT_5(t) &= q_{50}(t) \odot DT_0(t) + q_{53}(t) \odot DT_3(t) + q_{54}(t) \odot DT_4(t) \\ &\quad + q_{56}(t) \odot DT_6(t) \\ DT_6(t) &= q_{61}(t) \odot DT_1(t) + q_{62}(t) \odot DT_2(t) \end{aligned} \tag{126-132}$$

where

$$W_2(t) = \bar{e}^{-\lambda t} \bar{H}_1(t) \tag{133}$$

Taking L.T. of the above equations and solving them for $DT_0^*(s)$, In steady state, the total fraction of time for which the discussion is going on is given by

$$DT_0 = N_6/D_1 \tag{134}$$

where

$$N_6 = \mu_2[p_{12}(p_{30} + p_{36}) + p_{36}p_{62}(p_{13} + p_{14})] \tag{135}$$

Profit Analysis

The expected total profit incurred to the system in steady state is given by

$$P = C_0A_0 - C_1B_0^e - C_2B_0^a - C_3V_0^e - C_4V_0^a - C_5 - C_6(DT_0) \tag{136}$$

where

- C_0 = revenue per unit up time of the system
- C_1 = cost per unit time for which the expert repairman is busy
- C_2 = cost per unit time for which the assistant repairman is busy
- C_3 = cost per visit of the expert repairman
- C_4 = cost per visit of the assistant repairman
- C_5 = cost per unit time for the regular repairman
- C_6 = cost per unit time for which the expert is discussing the nature of failure

NUMERICAL RESULTS AND DISCUSSION

Let us consider the following particular case for obtaining various findings:

$$g(t) = \alpha e^{-\alpha t}, \quad h_1(t) = \beta_1 e^{-\beta_1 t}, \quad g_e(t) = \alpha_1 e^{-\alpha_1 t}, \quad g_a(t) = \alpha_2 e^{-\alpha_2 t}$$

It has been noticed that:

The MTSF decreases as failure rate increases. However, the values of MTSF become higher if we increase the discussion rate (β_1). On increasing the values of λ , the values of the profit have decreasing trend. Also, the values of the profit become lower if we increase the cost for discussion (C_6). The MTSF as well as the Profit have the higher values for the higher values of probability (p_2).

Comparison of the Model with that having No Provision of the Assistant Repairman

The transition diagram in case when there is no provision of the assistant repairman will take the form as shown in Fig. 2. Proceeding in the similar manner as above, the transition probabilities and the expressions for the following measures of system effectiveness have been obtained:

MTSF, Availability analysis, Busy period analysis of the expert repairman, expected number of visits by the expert repairman.

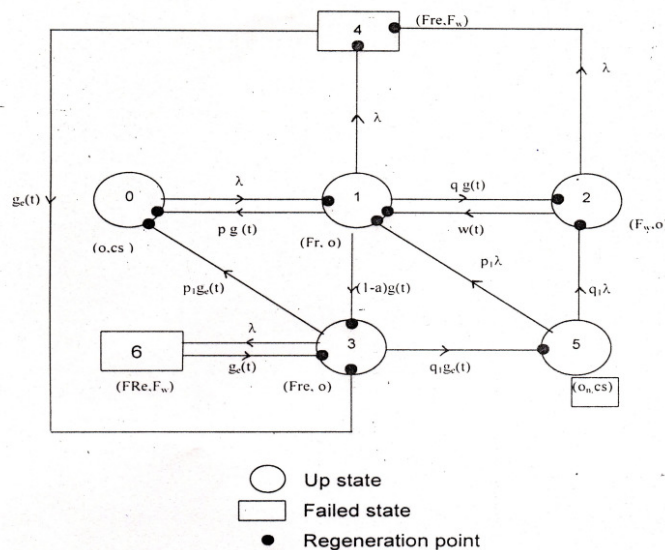


Fig. 2

The expression for the profit, in steady state, is obtained as

$$P_1 = C_0A_0 - C_1B_0^e - C_3V_0^e - C_5 \tag{137}$$

where C_0 , C_1 , C_3 and C_5 are same as already defined in Model 1.

The comparative study done graphically as shown in Fig. 3 revealed that the difference of profits ($P_1 - P$) increases as we increase the waiting rate(β) when the regular repairman is taking rest. But this difference has the lower value if we increase the value of discussion rate. Further,

- (i) If $\beta_1 = 6$, then $P_1 - P < 0$ or $= 0$ or > 0 according as $\beta < 1.21$ or $= 1.21$ or > 1.21 . This implies that there should be provision or no provision of the assistant repairman according as $\beta < 1.21$ or $\beta > 1.21$. The two models are equally good if $\beta = 1.21$.
- (ii) If $\beta_1 = 3$, then $P_1 - P < 0$ or $= 0$ or > 0 according as $\beta < 0.12$ or $= 0.12$ or > 0.12 . This implies that there should be provision or no provision of the assistant repairman according as $\beta < 0.12$ or $\beta > 0.12$. The two models are equally good if $\beta = 0.12$.

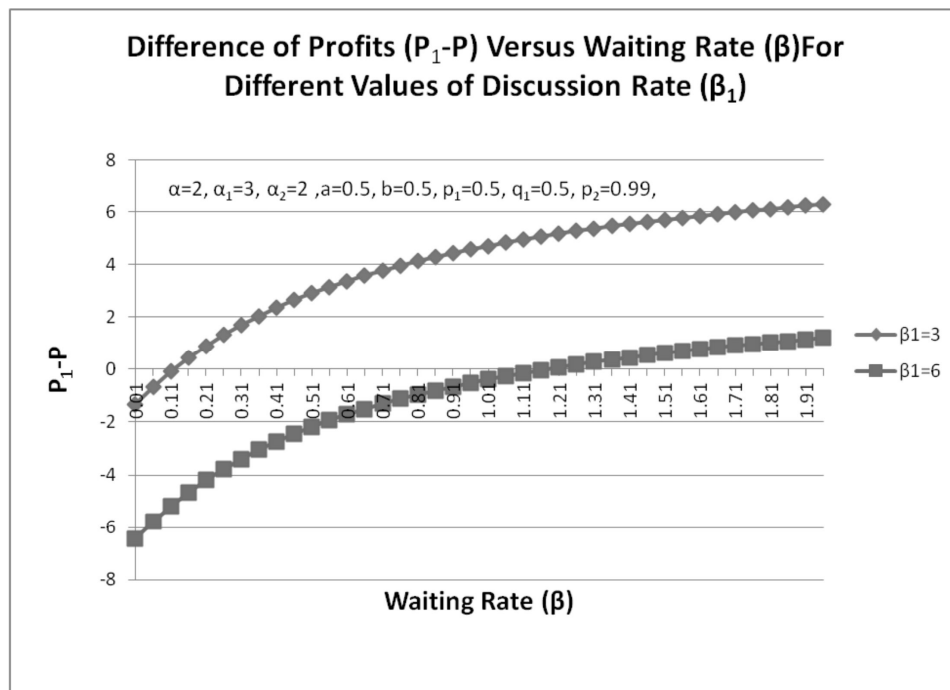


Fig. 3

REFERENCES

1. Adlakha N., Taneja G. and Shilpi. (2017). Reliability and cost benefit analysis of a two-unit standby communication system with assembling of one unit at a time. Aryabhata Journal of Mathematics and Informatics, 9(2), 281-288.
2. El-Said K. and El-Sherbany M.(2005). Profit Analysis of a Two Unit Cold Standby System with Preventive Maintenance and Random Change in the Units. Journal of Mathematics and Statistics,71-77.
3. G. S. Mokaddis, Y. M. Ayed and H. S. AL-Hajeri (2013). Probabilistic Analysis Of Two Warm Standby System With Instructions At Need. African Journal of Mathematics and Computer Science Research, 6(2), 20-25.
4. Goyal A., Taneja G., Singh D.V. (2009). Analysis of a two unit cold standby system working in a sugar mill with operating and rest period. Caledonian Journal of Engineering, 5, 1-5.

5. Kumar, A, Gupta. S. K. and Taneja. G.(1995). Probabilistic analysis of a two-unit cold standby system with instructions at need. *Microelectron Reliab.* 35(5), 829-832.
6. Kumar R. and Bhatia P. (2011). Reliability and cost analysis of one unit centrifuge system with single repairman and inspection. *Pure and Applied Matematika Sciences*, 74 (1-2), 113-121.
7. Malhotra R. and Taneja G.(2015). Comparative Study between a Sinle Unit System and a Two-Unit Cold Standby System with Varying Demand. Springer Plus.
8. Manocha A. and Taneja G.(2014). Stochastic Analysis of Two-Unit Cold Standby System with Arbitrary Distribution for Life, Repair and Waiting Times. *International Journal Of Performability Engineering*, 11(3),293-299.
9. Parasher B. and Taneja G.(2007). Reliability and Profit Evaluation of a PLC Hot Standby System based on a Master Slave Concept and Two Types of Repair Facilities. *IEEE Transcations on Reliability*, 56(3), 534-539.
10. Taneja A.(2016). Stochastic Analysis of a System with Patience Time and Telephonic Discussion Regarding Nature of Failure to Decide Type of Visiting Repairman. *Aryabhata Journal of Mathematics and Informatics*, 8(2), 277-284.
11. Tuteja. R. K and Taneja G.(1991). Proper utilization of a system with two identical repairmen. *International Journal of Management and Systems*. 7,273-281



ON A NEW SUMMATION FORMULA FOR THE H-FUNCTION

Vivek Rohira*, Shantha Kumari. K.** & Arjun K. Rathie***

*Research Scholar, Department of Mathematics, Career Point University, Kota-325003, Rajasthan, INDIA

**Department of Mathematics, A J Institute of Engineering and Technology, Mangaluru- 575006, Karnataka, INDIA

***Department of Mathematics, Vedanta College of Engg. & Tech., Rajasthan Technical University Bundi – 323021, Rajasthan, INDIA
 Email : vivekrohira@yahoo.com, shanthakk99@gmail.com, arjunkumarrathie@gmail.com

ABSTRACT :

In 2010, Kim et al., obtained an interesting extension of the classical Watson summation theorem. The aim of this research paper is to establish an interesting summation formula for the H- function by employing extended classical Watson's summation formula.

A few interesting special cases have also been given.

Keywords : H- function, Watson's Summation Theorem.

1. INTRODUCTION

We recall here by beginning the definition of the well-known, interesting and useful H-function introduced by Fox [2] and studied in detail by Braaksma[1] will be defined and represented in the following manner:

$$H_{p,q}^{m,n} \left[z \left| \begin{matrix} 1(a_j, e_j)_p \\ 1(b_j, f_j)_q \end{matrix} \right. \right] = \frac{1}{2\pi i} \int_L \theta(s) z^s ds \quad (1.1)$$

where

$$\theta(s) = \frac{\prod_{j=1}^m \Gamma(b_j - f_j s) \prod_{j=1}^n \Gamma(1 - a_j + e_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + f_j s) \prod_{j=n+1}^p \Gamma(a_j - e_j s)} \quad (1.2)$$

Also,

(i) $i = \sqrt{-1}$

(ii) $z \neq 0$

(iii) An empty product is to be interpreted as unity.

(iv) m,n,p,q are integers satisfying $0 \leq m \leq q, 0 \leq n \leq p$ (not both zero simultaneously)

(v) $a_j, j = 1, \dots, p; b_j, j = 1, \dots, q$ are complex numbers.

(vi) $e_j, j = 1, \dots, p; f_j, j = 1, \dots, q$ are real positive numbers for standardization purposes.

(vii) L is a contour going from $\sigma - i\infty$ to $\sigma + i\infty$ (σ real) so that all the poles of $\Gamma(b_j - f_j s)$, $j = 1, 2, \dots, m$ lie to the right of L and all the poles of $\Gamma(1 - a_j + e_j s)$, $j = 1, 2, \dots, n$ lie to the left of L.

Braaksma[1] has shown that the integral (1.1) converges absolutely if

$$\theta > 0, |argz| < \frac{\theta\pi}{2},$$

where θ is given by

$$\theta = \sum_{j=1}^m f_j - \sum_{j=m+1}^q f_j + \sum_{j=1}^n e_j + \sum_{j=n+1}^p e_j \tag{1.3}$$

Also from Braaksma [1]

$$H_{p,q}^{m,n}[z] \sim O[z^\alpha]$$

For small values of z , where

$$\alpha = \min_{1 \leq j \leq m} Re \left(\frac{b_j}{f_j} \right)$$

and

$$H_{p,q}^{m,n}[z] \sim O[z^\beta]$$

For large value of z , where

$$\beta = \max_{1 \leq j \leq n} Re \left(\frac{a_j - 1}{e_j} \right)$$

For more detail about H-function, we refer the standard text[4], and a paper by Braaksma [1].

2. RESULT REQUIRED

The following extension of classical Watson’s summation theorem recently obtained by Kim, et al. [3] will be required in our present investigation.

$$\begin{aligned} & {}_4F_3 \left[\begin{matrix} a, b, c, d+1 \\ \frac{1}{2}(a+b+1), 2c+1, d \end{matrix}; 1 \right] \\ &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)\Gamma\left(c + \frac{1}{2}\right)\Gamma\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)\Gamma\left(c - \frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(c - \frac{1}{2}b + \frac{1}{2}\right)} \\ &+ \left(\frac{2c-d}{d}\right) \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)\Gamma\left(c + \frac{1}{2}\right)\Gamma\left(c - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c - \frac{1}{2}a + 1\right)\Gamma\left(c - \frac{1}{2}b + 1\right)} \end{aligned} \tag{2.1}$$

provided $Re(2c - a - b) > -1$ and $d \in \mathbb{C}/\mathbb{Z}_0^-$

3. MAIN SUMMATION FORMULA

In this section, the following interesting summation formula for the H-function will be established.

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (d+1)_r}{\left(\frac{1}{2}(a+b+1)\right)_r (d)_r r!} H_{p+2, q+1}^{m, n+2} \left[z \left| \begin{matrix} (1-c-r, \lambda), (-c, \lambda), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, (-2c-r, 2\lambda) \end{matrix} \right. \right] \\
 &= \frac{\pi \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{2^{2c} \Gamma\left(\frac{1}{2}a + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} H_{p+2, q+2}^{m, n+2} \left[\frac{z}{2^{2\lambda}} \left| \begin{matrix} (1-c, \lambda), \left(\frac{1}{2}-c + \frac{1}{2}a + \frac{1}{2}b, \lambda\right), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2}-c + \frac{1}{2}a, \lambda\right), \left(\frac{1}{2}-c + \frac{1}{2}b, \lambda\right) \end{matrix} \right. \right] \\
 &+ \frac{\pi \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{d 2^{2c} \Gamma\left(\frac{1}{2}a\right) \Gamma\left(\frac{1}{2}b\right)} \\
 &\times H_{p+3, q+3}^{m, n+3} \left[\frac{z}{2^{2\lambda}} \left| \begin{matrix} (1-c, \lambda), \left(\frac{1}{2}-c + \frac{1}{2}a + \frac{1}{2}b, \lambda\right), (d-2c, 2\lambda), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2}a - c, \lambda\right), \left(\frac{1}{2}b - c, \lambda\right), (1+d-2c, 2\lambda) \end{matrix} \right. \right] \tag{3.1}
 \end{aligned}$$

provided $\lambda > 0$, $\text{Re}(c) > 0$, $\min_{1 \leq j \leq m} \text{Re} \left[c + \lambda \left(\frac{b_j}{f_j} \right) \right] > 0$, $d \in \mathbb{C}/\mathbb{Z}_0^-$, $\theta > 0$, $|\text{arg} z| < \frac{\theta\pi}{2}$ where θ is the same as given in (1.3).

4. PROOF

In order to derive the main summation formula (3.1), we proceed as follows. Denoting the left hand side of (3.1) by S , expressing the H-function by its definition (1.1), we have

$$S = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (d+1)_r}{\left(\frac{1}{2}(a+b+1)\right)_r (d)_r r!} \cdot \frac{1}{2\pi i} \int_L \theta(s) z^s \frac{\Gamma(c + \lambda s + r) \Gamma(c + \lambda s + 1)}{\Gamma(2c + 2\lambda s + r + 1)} ds$$

Interchanging the order of summation and integration, which is easily seen to be justified due to the uniform convergence of the series and absolute convergence of the integral and using the result,

$$(a)_r = \frac{\Gamma(a+r)}{\Gamma(a)}$$

we have after some simplification

$$S = \frac{1}{2\pi i} \int_L \theta(s) z^s \frac{\Gamma(c + \lambda s)\Gamma(c + 1 + \lambda s)}{\Gamma(2c + 2\lambda s + 1)} \left\{ \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (d + 1)_r (c + \lambda s)_r}{\left(\frac{1}{2}(a + b + 1)\right)_r (2c + 2\lambda s + 1)_r (d)_r r!} \right\} ds$$

on summing up the inner series, we have

$$S = \frac{1}{2\pi i} \int_L \theta(s) z^s ds \frac{\Gamma(c + \lambda s)\Gamma(c + 1 + \lambda s)}{\Gamma(2c + 2\lambda s + 1)} \cdot {}_4F_3 \left[\begin{matrix} a, b, c + \lambda s, d + 1 \\ \frac{1}{2}(a + b + 1), 2c + 2\lambda s + 1, d \end{matrix}; 1 \right]$$

Now if we use the extended Watson’s summation theorem (2.1), separating into two parts and writing

$$2c + 2\lambda s - d = \frac{\Gamma(2c + 2\lambda s - d + 1)}{\Gamma(2c + 2\lambda s - d)}$$

and using the duplication formula for the Gamma function

$$\Gamma(2z) = \frac{2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)}{\sqrt{\pi}}$$

and finally interpreting the result thus obtained with the help of definition of H-function using (1.1), we easily arrive at the right-hand side of (3.1). This completes the proof of our interesting summation formula (3.1).

5. APPLICATION

It is interesting to mention here that if for the factor $2c - d + 2\lambda s$, we write

$$2c - d + 2\lambda s = 2c - d + 2\lambda \frac{\Gamma(s + 1)}{\Gamma(s)}$$

and proceed on similar lines, we get

$$\begin{aligned} & \sum_{r=0}^{\infty} \frac{(a)_r (b)_r (d + 1)_r}{\left(\frac{1}{2}(a + b + 1)\right)_r (d)_r r!} H_{p+2, q+1}^{m, n+2} \left[z \left| \begin{matrix} (1 - c - r, \lambda), (-c, \lambda), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, (-2c - r, 2\lambda) \end{matrix} \right. \right] \\ &= \frac{\pi \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{2^{2c} \Gamma\left(\frac{1}{2}a + \frac{1}{2}\right) \Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} H_{p+2, q+2}^{m, n+2} \left[\frac{z}{2^{2\lambda}} \left| \begin{matrix} (1 - c, \lambda), \left(\frac{1}{2} - c + \frac{1}{2}a + \frac{1}{2}b, \lambda\right), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2} - c + \frac{1}{2}a, \lambda\right), \left(\frac{1}{2} - c + \frac{1}{2}b, \lambda\right) \end{matrix} \right. \right] \\ &+ \left(\frac{2c - d}{d}\right) \frac{\pi \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right) \Gamma\left(\frac{1}{2}b\right)} \frac{1}{2^{2c}} H_{p+2, q+2}^{m, n+2} \left[\frac{z}{2^{2\lambda}} \left| \begin{matrix} (-c, \lambda), \left(\frac{1}{2} - c + \frac{1}{2}a + \frac{1}{2}b, \lambda\right), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2}a - c, \lambda\right), \left(\frac{1}{2}b - c, \lambda\right) \end{matrix} \right. \right] \\ &+ \frac{2\lambda}{d} \frac{1}{2^{2c}} \frac{\pi \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right) \Gamma\left(\frac{1}{2}b\right)} H_{p+3, q+3}^{m, n+3} \left[\frac{z}{2^{2\lambda}} \left| \begin{matrix} (0, 1), (-c, \lambda), \left(\frac{1}{2} - c + \frac{1}{2}a + \frac{1}{2}b, \lambda\right), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, (1, 1), \left(\frac{1}{2}a - c, \lambda\right), \left(\frac{1}{2}b - c, \lambda\right) \end{matrix} \right. \right] \end{aligned} \tag{5.1}$$

Thus on comparison of (3.1) with (5.1), we get, after some simplification, the following interesting recursive relation for the H-function.

$$\begin{aligned}
 & H_{p+3,q+3}^{m,n+3} \left[\begin{matrix} z \\ 2^{2\lambda} \end{matrix} \middle| \begin{matrix} (1-c, \lambda), \left(\frac{1}{2}-c+\frac{1}{2}a+\frac{1}{2}b, \lambda\right), (d-2c, 2\lambda), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2}a-c, \lambda\right), \left(\frac{1}{2}b-c, \lambda\right), (1+d-2c, 2\lambda) \end{matrix} \right] \\
 = & (2c-d)H_{p+2,q+2}^{m,n+2} \left[\begin{matrix} z \\ 2^{2\lambda} \end{matrix} \middle| \begin{matrix} (-c, \lambda), \left(\frac{1}{2}-c+\frac{1}{2}a+\frac{1}{2}b, \lambda\right), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2}a-c, \lambda\right), \left(\frac{1}{2}b-c, \lambda\right) \end{matrix} \right] \\
 + & 2\lambda H_{p+3,q+3}^{m,n+3} \left[\begin{matrix} z \\ 2^{2\lambda} \end{matrix} \middle| \begin{matrix} (0,1), (-c, \lambda), \left(\frac{1}{2}-c+\frac{1}{2}a+\frac{1}{2}b, \lambda\right), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, (1,1), \left(\frac{1}{2}a-c, \lambda\right), \left(\frac{1}{2}b-c, \lambda\right) \end{matrix} \right] \tag{5.2}
 \end{aligned}$$

6. SPECIAL CASES

In this section, we shall mention two very interesting special cases of our main summation formula.

(i) Let $b = -2\ell$ and replace a by $a + 2\ell$, where ℓ is zero or a positive integer. In such case, one of the two terms on the right hand side of (3.1) will vanish and we get the following interesting result

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \frac{(-2\ell)_r (a+2\ell)_r (d+1)_r}{\left(\frac{1}{2}(a+1)\right)_r (d)_r r!} H_{p,q}^{m,n} \left[\begin{matrix} z \\ \end{matrix} \middle| \begin{matrix} (1-c-r, \lambda), (-c, \lambda), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, (-2c-r, 2\lambda) \end{matrix} \right] \\
 = & \frac{(-1)^\ell \Gamma\left(\frac{1}{2}\right) \left(\frac{1}{2}\right)_\ell}{2^{2c} \left(\frac{1}{2}a + \frac{1}{2}\right)_\ell} H_{p+2,q+2}^{m,n+2} \left[\begin{matrix} z \\ 2^{2\ell} \end{matrix} \middle| \begin{matrix} (1-c, \lambda), \left(\frac{1}{2} + \frac{1}{2}a - c, \lambda\right), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2} + \frac{1}{2}a - c + \ell + \lambda\right), \left(\frac{1}{2} - c - \ell, \lambda\right) \end{matrix} \right] \tag{6.1}
 \end{aligned}$$

(ii) Let $b = -2\ell - 1$ and replace a by $a + 2\ell + 1$, where ℓ is zero or a positive integer. In such case, one of the two terms on the right-hand side of (3.1) will vanish and we get the following interesting result

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \frac{(-2\ell-1)_r (a+2\ell+1)_r (d+1)_r}{\left(\frac{1}{2}(a+1)\right)_r (d)_r r!} H_{p+2,q+1}^{m,n+2} \left[\begin{matrix} z \\ \end{matrix} \middle| \begin{matrix} (1-c-r, \lambda), (-c, \lambda), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, (-2c-r, 2\lambda) \end{matrix} \right] \\
 = & \frac{(-1)^{r-1} \Gamma\left(\frac{1}{2}\right) \left(\frac{3}{2}\right)_\ell}{d 2^{2c+1} \left(\frac{1}{2}a + \frac{1}{2}\right)_\ell} \\
 & \times H_{p+3,q+3}^{m,n+3} \left[\begin{matrix} z \\ 2^{2\lambda} \end{matrix} \middle| \begin{matrix} (1-c, \lambda), \left(\frac{1}{2}a + \frac{1}{2} - c, \lambda\right), (d-2c, 2\lambda), {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q, \left(\frac{1}{2} + \frac{1}{2}a - c + \ell, \lambda\right), \left(-\ell - \frac{1}{2} - c, \lambda\right), (1+d-2c, \lambda) \end{matrix} \right] \tag{6.2}
 \end{aligned}$$

Since H-function is one of the most general function of one variable studied so far which includes as special cases, Meijer's G-function, MacRobert's E-function, Wright's generalized hypergeometric function, Generalized hypergeometric function ${}_pF_q$, Whittaker function, Mittag-Leffler function and almost all elementary functions, so from our generalized summation formulas, a large number of interesting special cases can we obtained. But we shall not record them due to the lack of space.

REFERENCES

1. Braaksma, B.L.J. Asymptotic expansions and analytic continuation for a class of Barnes integrals, *comp.Math.*, 15, 239-341, (1964).
2. Fox, C. The G and H-functions as symmetrical Fourier Kernels, *Trans. Amer. Math. Soc.*, 98, 395-429, (1961).
3. Kim, Y. S., Rakha, M. A. and Rathie, A.K., Extension of certain classical summations theorems for the series ${}_2F_1$, ${}_3F_2$ and ${}_4F_3$ with applications in Ramanujan's summations, *Int.J.Math.Sci.*, Article ID 309503, 26 pages, (2010)
4. Mathai, A.M., Saxena, R.K. and Haubold, H.J., *The H- function: Theory and Applications*, Springer, New York (2010).



CHARACTERISTICS OF RIEMANNIAN MANIFOLDS AND PSEUDO- RIEMANNIAN METRIC

K.C. Petwal and Kulveer Singh Rana

Department of mathematics, HNB Garhwal Central University, SRT Campus Badshahi Thaul,
Tehri Garhwal, Pin- 249 199, Uttarakhand, India
Email: kcpetwal@gmail.com

ABSTRACT :

In the present paper we characterize the Riemannian manifolds, pseudo- Riemannian metric and several applications of Riemannian manifolds have been investigated. We discuss linear connections, connectors, torsion and space of all covariant derivatives on Riemannian manifolds and also geometry of geodesics structure of Riemann manifold. In a pseudo-Riemann manifold there is a torsion-free covariant derivative which is compatible with the Riemann metric. Further characterize the geodesic distance conformal metrics, sectional curvature relations to vector analysis in three dimensions of Riemannian and pseudo-Riemannian manifold and several theorems are investigated.

Keywords: Riemannian, Manifold, curvature, covariant, conformal, connection, tensor, geodesic, sectional, metric.

MSC (2010): 53A30, 53B15, 53B20, 53C22.

1. INTRODUCTION

Riemannian manifold: In differential geometry, a (smooth) Riemannian manifold or (smooth) Riemannian space (M, g) is a real smooth manifold M equipped with an inner product on the tangent space at each point that varies smoothly from point to point in the sense that if X and Y are vector fields on M , then $g(X, Y)$ is a smooth function. The family of inner products is called a Riemannian metric (tensor). These terms are named after the German mathematician Bernhard Riemann. The study of Riemannian manifolds constitutes the subject called Riemannian geometry.

Riemann metrics: Let M be n – dimensional smooth manifold. A Riemann metric g on M is a symmetric $(0,2)$ –tensor field such that $g_x: T_x M \times T_x M \rightarrow \mathbb{R}$ is a positive definite inner product for each $x \in M$. A pseudo-Riemann metric g on M is a symmetric $(0,2)$ –tensor field such that g_x is non-degenerate, i.e. $g_x: T_x M \rightarrow T_x^* M$ is one-one onto $\forall x \in M$

If (U, u) is a chart on M , then we have

$$g|_U = \sum_{i,j=0}^m g \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) du^i \otimes du^j =: \sum_{i,j} g_{ij} du^i \otimes du^j$$

Here $(g_{ij}(x))$ is a symmetric invertible $(n \times n)$ –matrix for each $x \in M$, positive definite in the case of a Riemann metric; thus $(g_{ij}): U \rightarrow Mat_{sym}(n \times n)$. In the case of a pseudo-Riemann metric, the matrix (g_{ij}) has p positive

eigenvalues and q negative ones; (p, q) is called the signature of the metric and $q = m - p$ is called the index of the metric; both are locally constant on M and we shall always assume that it is constant on M .

Lemma 1.1: On each manifold M there exist many Riemann metrics. But there need not to exist a pseudo-Riemann metric of some given signature.

Proof: Let (U_α, u_α) be an atlas on M with a subordinated partition of unity (f_α) . Choose smooth mappings g_{ij}^α from U_α to the convex cone of all positive definite symmetric $(n \times n)$ – matrices for each α and put $g = \sum_\alpha f_\alpha \sum_{ij} g_{ij}^\alpha du_\alpha^i \otimes du_\alpha^j$.

For example, on any even-dimensional sphere S^{2n} there does not exist a pseudo-Riemann metric g of signature $(1, 2n - 1)$: Otherwise there would exist a line subbundle $L \subset TS^{2n}$ with $g(v, v) > 0$ for $0 \neq v \in L$. But since the Euler characteristic $\chi(S^{2n}) = 2$, such a line subbundle of the tangent bundle cannot exist.

Length and energy of a curve: Let $c: [a, b] \rightarrow M$ be a smooth curve. In the Riemann case the length of the curve c is then given by

$$L_a^b(c) := \int_a^b g(c'(t), c'(t))^{1/2} dt = \int_a^b |c'(t)|_g dt.$$

In both cases the energy of the curve c is given by

$$E_a^b(c) := \frac{1}{2} \int_a^b (c'(t), c'(t)) dt.$$

In the Riemann case we have by the Cauchy-Schwarz inequality

$$L_a^b(c)^2 = \left(\int_a^b |c'|_g \cdot 1 dt \right)^2 \leq \int_a^b |c'|_g^2 dt \cdot (b - a) = 2(b - a)E_a^b(c)$$

For piecewise smooth curves the length and the energy are defined by taking it for the smooth pieces and then by summing up over all the pieces. In the pseudo-Riemann case for the length one has to distinguish different classes of curves according to the sign of $g(c'(t), c'(t))$ (the sign then should be assumed constant) and by taking an appropriate sign before taking the root. These leads to the concept of ‘time-like’ curves (with speed less than the speed of light) and ‘space-like’ curves (travelling faster than light).

The length is invariant under reparameterizations of the curve:

$$\begin{aligned} L_a^b(c \circ f) &= \int_a^b g((c \circ f)'(t), (c \circ f)'(t))^{1/2} dt = \int_a^b g(f'(t)c'(f(t)), f'(t)c'(f(t)))^{1/2} dt \\ &= \int_a^b g(c'(f(t)), c'(f(t)))^{1/2} |f'(t)| dt = \int_a^b g(c'(t), c'(t))^{1/2} dt = L_a^b(c) \end{aligned}$$

The energy is not invariant under reparameterizations.

2. COVARIANT DERIVATIVES

Let (M, g) be a pseudo-Riemann manifold. A covariant derivative on M is a mapping $\nabla: \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow \mathfrak{X}(M)$, denoted by $(X, Y) \mapsto \nabla_X Y$, which satisfies the following conditions:

i). $\nabla_X Y$ is $C^\infty(N)$ –linear in $X \in \mathfrak{X}(M)$, i.e., $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$. So for a tangent vector $X_x \in T_x M$ the mapping $\nabla_{X_x}: \mathfrak{X}(M) \rightarrow T_x M$ makes sense and we have $(\nabla_X s)(x) = \nabla_{X(x)} s$.

ii). $\nabla_X Y$ is \mathbb{R} -linear in $Y \in \mathfrak{X}(M)$.

iii). $\nabla_X(f \cdot Y) = df(X) \cdot Y + f \cdot \nabla_X Y$ for $f \in C^\infty(M)$ $f \in C^\infty(M)$, the derivation property of ∇_X .

The covariant derivative ∇ is called symmetric or torsion-free if moreover the following holds:

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

The covariant derivative ∇ is called compatible with the pseudo-Riemann metric if we have:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \quad \forall X, Y, Z \in \mathfrak{X}(M).$$

Theorem 2.1.: On any pseudo-Riemann manifold (M, g) there exists a unique torsion-free covariant derivative $\nabla = \nabla^g$ which is compatible with the metric g . In a chart (U, u) we have

$$\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} = - \sum_k \Gamma_{ij}^k \frac{\partial}{\partial u^k},$$

where the Γ_{ij}^k are the Christoffel symbols.

This unique covariant derivative is called the Levi-Civita covariant derivative.

Proof: We write the cyclic permutations of property (V) equipped with the signs $+, +, -$:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

$$Y(g(Z, X)) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - Z(g(X, Y)) = -g(\nabla_Z X, Y) - g(X, \nabla_Z Y)$$

We add these three equations and use the torsion-free property (IV) to get

$$\begin{aligned} & X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + g(\nabla_Y Z, X) + g(Z, \nabla_Y X) - g(\nabla_Z X, Y) - g(X, \nabla_Z Y) \\ &= g(2\nabla_X Y - [X, Y], Z) - g([Z, X], Y) + g([Y, Z], X) \end{aligned}$$

which we rewrite as an implicit defining equation for $\nabla_X Y$:

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]).$$

Thus by (VII) uniquely determined bilinear mapping $(X, Y) \mapsto (\nabla_X Y)$ indeed satisfies (I) – (VI), which is tedious but easy to check. The final assertion of the theorem follows by using (VII) once more

$$2g\left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^l}\right) = \frac{\partial}{\partial u^i} \left(g\left(\frac{\partial}{\partial u^j}, \frac{\partial}{\partial u^l}\right) \right) + \frac{\partial}{\partial u^j} \left(g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^l}\right) \right) - \frac{\partial}{\partial u^l} \left(g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) \right) = -2 \sum_k \Gamma_{ij}^k g_{kl}$$

Linear connections and connectors: Let M be a smooth manifold. A smooth mapping $C: TM \times_M M$ is called a linear connection or horizontal lift on M if it has the following properties:

i). $(T(\pi_M), \pi_{TM}) \circ C = Id_{TM \times_M TM}$.

ii). $C(\cdot, X_x): T_x M \rightarrow T_{X_x}(TM)$ is linear; this is the first vector bundle structure on $T^2 M$.

iii). $C(X_x, \cdot): T_x M \rightarrow T(\pi_M)^{-1}(X_x)$ is linear; this is the second vector bundle structure on $T^2 M$.

The connection C is called symmetric or torsion-free if moreover the following property holds:

i). $K_M \circ C = C \circ flip: TM \times_M TM \rightarrow T^2 M$, where $K_M: T^2 M \rightarrow T^2 M$ is the canonical flip mapping [1].

From the properties (i) – (iii) it follows that for a chart (U_α, u_α) on M the mapping C is given by

ii). $T^2(u_\alpha) \circ C \circ (T(u_\alpha)^{-1}) \times_M T(u_\alpha)^{-1}((x, y), (x, z)) = (x, z; y, \Gamma_x^\alpha(y, z))$,

Where the Christoffel symbol $\Gamma_x^\alpha(y, z) \in \mathbb{R}^n$ ($n = \dim(M)$) is smooth in $x \in u_\alpha(U_\alpha)$ and is bilinear in $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. For the sake of completeness let us also note the transformation rule of the Christoffel symbols which follows now directly from the chart change of the second tangent bundle in [1]. The chart change on M

$$u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1}: u_\beta(U_\alpha \cap U_\beta) \rightarrow u_\alpha(U_\alpha \cap U_\beta)$$

induces the following transformation of the Christoffel symbols:

$$\Gamma_{u_{\alpha\beta}(x)}^\alpha(d(u_{\alpha\beta})(x)y, d(u_{\alpha\beta})(x)z) = d(u_{\alpha\beta})(x)\Gamma_x^\beta(y, z) + d^2(u_{\alpha\beta})(x)(y, z)$$

Since a spray S on a manifold determines symmetric Christoffel symbols and thus a symmetric connection C . If the spray S is induced by a pseudo-Riemann metric g on M , then the Christoffel symbols are the same as the singular curves of the energy. The promised geometric description of the Christoffel symbols is (5), which also explains their transformation behavior under chart changes: They belong to the vertical part of the second tangent bundle.

Consider now a linear connection $C: TM \times_M TM \rightarrow T^2M$. for $\xi \in T^2M$

we have, $\xi - C(T(\pi_M). \xi, \pi_{TM}(\xi)) \in V(TM) = T(\pi_M)^{-1}(0)$

which is an element of the vertical bundle, since

$$T(\pi_M) \left(\xi - C(T(\pi_M). \xi, \pi_{TM}(\xi)) \right) = T(\pi_M). \xi - T(\pi_M). \xi = 0$$

by (1). Thus we may define the connector $K: T^2M \rightarrow TM$ by

$$K(\xi) = vpr_{TM} \left(\xi - C(T(\pi_M). \xi, \pi_{TM}(\xi)) \right),$$

where the vertical projection vpr_{TM} was defined in[1]. In coordinates induced by a chart on M we have

$$K(x, y; a, b) = vpr(x, y; 0, b - \Gamma_x(a, y)) = (x, b - \Gamma_x(a, y))$$

Obviously the connector K has the following three properties:

We have

$$K \circ vl_{TM} = pr_2: TM \times_M TM \rightarrow TM$$

where $vl_{TM}(X_x, Y_x) = \partial|_0(X_x, tY_x)$ is the vertical lift introduced in (8.12) [1].

A. The mapping $K: TTM \rightarrow TM$ is linear for the (first) vector bundle structure on $\pi_{TM}: TTM \rightarrow TM$.

B. The mapping $K: TTM \rightarrow TM$ is linear for the (second) vector bundle structure on $T(\pi_M): TTM \rightarrow TM$.

A connector, defined as a mapping satisfying (IX)– (XI), is equivalent to a connection, since one can reconstruct it (which is most easily checked in a chart) by

$$C(\cdot, X_x) = \left(T(\pi_M)|_{ker(K: T_{X_x}(TM) \rightarrow T_xM)} \right)^{-1}$$

The connector K is associated to a symmetric connection if and only if $K \circ \kappa_M = K$.

a) Torsion: Let ∇ be a general covariant derivative on a manifold M .

Then the torsion is given by

$$Tor(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y], X, Y \in \mathfrak{X}(M) \tag{1}$$

It is skew-symmetric and $C^\infty(M)$ –linear in $X, Y \in \mathfrak{X}(M)$ and is thus a 2–form with values in $TM: Tor \in \Omega^2(M; TM) = \Gamma(\wedge^2 T^*M \otimes TM)$, since we have

$$\begin{aligned} Tor(f.X, Y) &= \nabla_{f.X} Y - \nabla_Y (f.X) - [f.X, Y] \\ &= f.\nabla_X Y - Y(f).X - f.\nabla_Y(X) - f[X, Y] + Y(f).X = f.Tor(X, Y). \end{aligned}$$

Locally on a chart (U, u) we have

$$\begin{aligned}
 \text{Tor}|U &= \sum_{i,j} \text{Tor} \left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) \otimes du^i \otimes du^j \\
 &= \sum_{i,j} \left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^j} - \nabla_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i} - \left[\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right] \right) \otimes du^i \otimes du^j \\
 &= \sum_{i,j} (-\Gamma_{ij}^k + \Gamma_{ji}^k) du^i \otimes du^j \otimes \frac{\partial}{\partial u^k} \\
 &= -\sum_{i,j} \Gamma_{ij}^k du^i \wedge du^j \otimes \frac{\partial}{\partial u^k} = -2 \sum_{i < j} \Gamma_{ij}^k du^i \wedge du^j \otimes \frac{\partial}{\partial u^k} \quad (2)
 \end{aligned}$$

We may add an arbitrary form $T \in \Omega^2(M; TM)$ to a given covariant derivative and we get a new covariant derivative with the same spray and geodesic structure, since the summarization of the Christoffel symbols stays the same.

3. THE SPACE OF ALL COVARIANT DERIVATIVES

If ∇^0 and ∇^1 are two covariant derivatives on a manifold M , then $\nabla_X^1 Y - \nabla_X^0 Y$ turns out to be $C^\infty(M)$ -linear in $X, Y \in \mathfrak{X}(M)$ and is thus a $\binom{1}{2}$ -tensor on M ; see [1]. Conversely, one may add an arbitrary $\binom{1}{2}$ -tensor field A to a given covariant derivative and get a new covariant derivative. Thus the space of all covariant derivatives is an affine space with modeling vector space $\Gamma(T^*M \otimes T^*M \otimes TM)$.

3.1. The covariant derivative of tensor fields: Let ∇ be covariant derivative on a manifold M , and let $X \in \mathfrak{X}(M)$. Then the ∇_X can be extended uniquely to an operator ∇_X on the space of all tensor fields on M with the following properties:

- A. For $f \in C^\infty(M)$ we have $\nabla_X f = X(f) = df(X)$.
- B. ∇_X respects the spaces of $\binom{p}{q}$ -tensor fields.
- C. $\nabla_X(A \otimes B) = (\nabla_X A) \otimes B + A \otimes (\nabla_X B)$, a derivation with respect to the tensor product.
- D. ∇_X commutes with any kind of contraction C (i.e., any trace): So for $\omega \in \Omega^1(M)$ and $Y \in \mathfrak{X}(M)$ we have $\nabla_X(\omega(Y)) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y)$

The correct way to understand this is to use the concepts of (19.9)–(19.12): in [1]

Recognize the linear connection as induced from a principal connection on the linear frame bundle $GL(R^n, TM)$ and induce it then to all vector bundles associated to the representations of the structure group $GL(n, \mathbb{R})$ in all tensor spaces. Contractions are then equivariant mappings and thus intertwine the induced covariant derivatives, which is most clearly seen from (19.15): in [1]

Nevertheless, we discuss here the traditional proof, since it helps in actual computations. For $\omega \in \Omega^1(M)$ and $Y \in \mathfrak{X}(M)$ and the total contraction C we have

$$\begin{aligned}
 \nabla_X(\omega(Y)) &= \nabla_X(C(\omega \otimes Y)) = C(\nabla_X \omega \otimes Y + \omega \otimes \nabla_X Y) = (\nabla_X \omega)(Y) + \omega(\nabla_X Y), \\
 (\nabla_X \omega)(Y) &= \nabla_X(\omega(Y)) - \omega(\nabla_X Y),
 \end{aligned}$$

which is easily seen (as in (22.10)): [1] to be $C^\infty(M)$ –linear in Y . Thus $\nabla_X \omega$ is again a 1 –form.

For a $\binom{p}{q}$ – tensor -tensor field A we choose $X_i \in \mathfrak{X}(M)$ and $\omega^j \in \Omega^1(M)$ and arrive (similarly using again the total contraction) at

$$\begin{aligned} (\nabla_X A)(X_1, \dots, X_q, \omega^1, \dots, \omega^p) &= X \left(A(X_1, \dots, X_q, \omega^1, \dots, \omega^p) \right) \\ &- A(\nabla_X X_1, \dots, X_q, \omega^1, \dots, \omega^p) - \dots - A(X_1, \dots, \nabla_X X_q, \omega^1, \dots, \omega^p) \\ &- A(X_1, \dots, X_q, \nabla_X \omega^1, \dots, \omega^p) - \dots - A(X_1, \dots, X_q, \omega^1, \dots, \nabla_X \omega^p) \end{aligned}$$

This expression is again $C^\infty(M)$ –linear in each entry X_i or ω^j and defines thus the $\binom{p}{q}$ – tensor field $\nabla_X A$.

Obviously ∇_X is a derivation with respect to the tensor product of fields and commutes with all contractions.

For the sake of completeness we also list the local expression

$$\nabla_{\frac{\partial}{\partial u^i}} du^j = \sum_k \left(\nabla_{\frac{\partial}{\partial u^i}} du^j \right) \left(\frac{\partial}{\partial u^k} \right) du^k = \sum_k \left(\frac{\partial}{\partial u^i} \delta_j^k - du^j \left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^k} \right) \right) du^k = \sum_k \Gamma_{ik}^j du^k$$

from which one can easily derive the expression for an arbitrary tensor field:

$$\begin{aligned} \nabla_{\frac{\partial}{\partial u^i}} A &= \sum \left(\nabla_{\frac{\partial}{\partial u^i}} A \right) \left(\frac{\partial}{\partial u^{i_1}}, \dots, \frac{\partial}{\partial u^{i_q}}, du^{j_1}, \dots, du^{j_p} \right) du^{i_1} \otimes \dots \otimes \frac{\partial}{\partial u^{j_q}} \\ &\sum \left(\frac{\partial}{\partial u^i} \left(A \left(\frac{\partial}{\partial u^{i_1}}, \dots, du^{j_p} \right) \right) - A \left(\nabla_{\frac{\partial}{\partial u^i}} \frac{\partial}{\partial u^{i_1}}, \dots, du^{j_p} \right) - \dots - A \left(\frac{\partial}{\partial u^{i_1}}, \dots, \nabla_{\frac{\partial}{\partial x}} du^{j_p} \right) \right) du^{i_1} \otimes \dots \otimes \frac{\partial}{\partial u^{j_q}} \\ &= \sum \left(\frac{\partial}{\partial u^i} A_{i_1, \dots, i_q}^{j_1, \dots, j_p} + A_{k, i_2, \dots, i_q}^{j_1, \dots, j_p} \Gamma_{i, i_1}^k + \dots + A_{i_1, \dots, i_{q-1}, k}^{j_1, \dots, j_p} \Gamma_{i, i_q}^k - A_{i_1, \dots, i_q}^{k, j_2, \dots, j_p} \Gamma_{i, k}^{j_1} - \dots - A_{i_1, \dots, i_q}^{j_1, \dots, j_{p-1}, k} \Gamma_{i, k}^{j_p} \right) du^{i_1} \otimes \dots \otimes \frac{\partial}{\partial u^{j_q}} \end{aligned}$$

4. GEOMETRY OF GEODESICS STRUCTURE OF RIEMANN MANIFOLD

On a pseudo-Riemann manifold (M, g) we have a geodesic structure which is described by the flow of the geodesic spray on TM .

The geodesic with initial value $X_x \in T_x M$ is denoted by $t \mapsto \exp(t.X_x)$ in terms of the pseudo-Riemann exponential mapping \exp and $\exp_x = \exp|_{T_x M}$. We recall the properties of the geodesics which we will use.

A. $\exp_x: T_x M \supset U_x \rightarrow M$ is defined on a maximal ‘radial’ open zero neighborhood U_x in $T_x M$. Here radial means that for $X_x \in V_x$ we also have $[0, 1].X_x \subset V_x$. This follows from the flow properties since $\exp_x = \pi_M(FI_1^S|_{T_x M})$.

B. $T_{0_x}(\exp|_{T_x M}) = Id_{T_x M}$; thus $\partial|_0 \exp_x(t.X_x) = X_x$.

C. $\exp \left(s. \left(\frac{\partial}{\partial t} \exp(t.X) \right) \right) = \exp((t + s)X)$.

D. $t \mapsto g \left(\frac{\partial}{\partial t} \exp(t.X), \frac{\partial}{\partial t} \exp(t.X) \right)$ is constant in t : for $c(t) = \exp(t.X)$ we have

$\partial_t g(c', c') = 2g(\nabla_{\partial_t} c', c') = 0$. Thus in the Riemann case the length

$\left| \frac{\partial}{\partial t} \exp(t.X) \right|_g = \sqrt{g \left(\frac{\partial}{\partial t} \exp(t.X), \frac{\partial}{\partial t} \exp(t.X) \right)}$ is also constant.

If for a geodesic c the (by (IV)) constant $|c'(t)|_g$ is 1, we say that c is parameterized by arc-length.

Lemma 4.1: Let (M, g) be a Riemann manifold. For $x \in M$ let $\varepsilon > 0$ be so small that

$exp_x: D_x(\varepsilon) := \{X \in T_x M: |X|_g < \varepsilon\} \rightarrow M$ is a diffeomorphism on its image. Then in $exp_x(D_x(\varepsilon))$ the geodesic rays starting from x are all orthogonal to the ‘geodesic spheres’ $\{exp_x(X): |X|_g = k\} = exp_x(k.S(T_x M, g))$ for $k < \varepsilon$.

On pseudo-Riemann manifolds this result holds too, with the following adaptation: Since the unit spheres in $(T_x M, g_x)$ are hyperboloids, they are not small and may not lie in the domain of definition of the geodesic exponential mapping; the result only holds in this domain.

Proof: $exp_x(k.S(T_x M, g))$ is a submanifold of M since exp_x is a Diffeomorphisms on $D_x(\varepsilon)$. Let $s \mapsto v(s)$ be a smooth curve in $kS(T_x M, g) \subset T_x M$,

and let $\gamma(t, 0) = exp_x(t.v(0))$ Then γ is a variation of the geodesic $\gamma(t, 0) = exp_x(t.v(0)) =: c(t)$. In the energy of the geodesic $t \mapsto \gamma(t, s)$ the integrand is constant by (4 of 4):

$$E_0^1(\gamma(\cdot, s)) = \frac{1}{2} \int_0^1 g \left(\frac{\partial}{\partial t} \gamma(t, s), \frac{\partial}{\partial t} \gamma(t, s) \right) dt = \frac{1}{2} g(\partial|_0 \gamma(t, s), \partial|_0 \gamma(t, s)) dt = \frac{1}{2} k^2$$

Comparing this with the first variational formula (22.3): [1] i.e.,

$$\frac{\partial}{\partial s} \Big|_0 (E_0^1(\gamma(\cdot, s))) = \int_0^1 0 dt - g(c(0))(c'(0), 0) + g(c(1)) \left(c'(1), \frac{\partial}{\partial s} \Big|_0 \gamma(1, s) \right)$$

We get $0 = g(c(1)) \left(c'(1), \frac{\partial}{\partial s} \Big|_0 \gamma(1, s) \right)$, where $\frac{\partial}{\partial s} \Big|_0 \gamma(1, s)$ an arbitrary tangent vector of $exp_x(kS(T_x M, g))$.

Corollary 4.1.: Let (M, g) be a Riemann manifold, $x \in M$, and $\varepsilon > 0$ be such that $exp_x: D_x(\varepsilon) := \{X \in T_x M: |X|_g < \varepsilon\} \rightarrow M$ is a diffeomorphism on its image. Let $c: [a, b] \rightarrow exp_x(D_x(\varepsilon)) \setminus \{x\}$ be a piecewise smooth curve, so that $c(t) = exp_x(u(t), v(t))$ where $0 < u(t) < \varepsilon$ and $|v(t)|_{g_x} = 1$.

Then for the length we have $L_a^b(c) \geq |u(b) - u(a)|$ with equality if and only if u is monotone and v is constant, so that c is a radial geodesic, reparameterized by u .

On pseudo-Riemann manifolds this result holds only for in the domain of definition of the geodesic exponential mapping and only for curves with positive velocity vectors (time-like curves).

Proof: We may assume that c is smooth by treating each smooth piece of c separately. Let $\alpha(u, t) := exp_x(u, v(t))$. Then

$$\begin{aligned} c(t) &= \alpha(u(t), t), \\ \frac{\partial}{\partial t} c(t) &= \frac{\partial \alpha}{\partial u}(u(t), t) \cdot u'(t) + \frac{\partial \alpha}{\partial t}(u(t), t), \\ \left| \frac{\partial \alpha}{\partial u} \right|_{g_x} &= |v(t)|_{g_x} = 1, 0 = g \left(\frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial t} \right), \quad \text{by Lemma 4.1.} \end{aligned}$$

Putting this together, we get

$$\begin{aligned} |c'|_g^2 &= g(c', c') = g \left(\frac{\partial \alpha}{\partial u} \cdot u' + \frac{\partial \alpha}{\partial t}, \frac{\partial \alpha}{\partial u} \cdot u' + \frac{\partial \alpha}{\partial t} \right) \\ |u'|_g^2 \left| \frac{\partial \alpha}{\partial u} \right|_g^2 + \left| \frac{\partial \alpha}{\partial t} \right|_g^2 &= |u'|^2 + \left| \frac{\partial \alpha}{\partial t} \right|_g^2 \geq |u'|^2 \end{aligned}$$

with equality if and only if $\left| \frac{\partial \alpha}{\partial t} \right|_g = 0$; thus $\frac{\partial \alpha}{\partial t} = 0$ and $v(t) = \text{constant}$. So finally:

$$L_a^b(c) = \int_a^b |c'(t)|_g dt \geq \int_a^b |u'(t)| dt \geq \left| \int_a^b u'(t) dt \right| = |u(b) - u(a)|$$

with equality if and only if u is monotone and v is constant.

Corollary 4.2.: Let (M, g) be a Riemann manifold. Let $\varepsilon: M \rightarrow \mathbb{R} > 0$ be a continuous function such that for $\bar{V} = \{X_x \in T_x M : |X_x| < \varepsilon(x) \forall X \in M\}$ the mapping $(\pi_m \cdot \exp): TM \supseteq \bar{V} \rightarrow W \subseteq M \times M$ is a diffeomorphism from the open neighborhood \bar{V} of the zero section in TM onto an open neighborhood W of the diagonal in $\times M$.

Then for each $(x, y) \in W$ there exists a unique geodesic c in M which connects x and y and has minimal length: For each piecewise smooth curve γ from x to y we have $L(\gamma) \geq L(c)$ with equality if and only if γ is a reparameterization of c .

Proof: The set $\bar{V} \cap T_x M = D_x(\varepsilon(x))$ satisfies the condition of corollary 4.1. For $X_x = \exp_x^{-1}(y) = (\pi_M \exp|_{\bar{V}})^{-1}(x, y)$ the geodesic $t \rightarrow c(t) = \exp_x(t \cdot X_x)$ leads from x to y . Let $\delta > 0$ be small. Then c contains a segment which connects the geodesic spheres $\exp_x(\delta \cdot S(T_x M, g))$ and $\exp_x(|X_x|_{g_x} \cdot S(T_x M, g))$. By corollary (4.1) the length of this segment is $\geq |X_x|_g - \delta$ with equality if and only if this segment is radial, thus a reparameterization of c . Since this holds for all $\delta > 0$, the result follows.

4.3. The geodesic distance: On a Riemann manifold (M, g) there is a natural topological metric defined by

$$dist^g(x, y) := \inf\{L_0^1(c) : c: [0,1] \rightarrow M \text{ piecewise smooth}, c(0) = x, c(1) = y\}$$

which we call the geodesic distance (since ‘metric’ is heavily used). We either assume that M is connected or we take the distance of points in different connected components as ∞ .

Lemma 4.4: On a Riemann manifold (M, g) the geodesic distance is a topological metric which generates the topology of M . For $\varepsilon_x > 0$ small enough the open ball

$$B_x(\varepsilon_x) = \{y \in M : dist^g(x, y) < \varepsilon_x\}$$

has the property that any two points in it can be connected by a geodesic of minimal length.

Proof: This follows by corollary (4.1) and (4.2). The triangle inequality is easy to check since we admit piecewise smooth curves.

5. CONFORMAL METRICS

Two Riemann metrics g_1 and g_2 on a manifold M are called conformal if there exists a smooth nowhere vanishing function f with $g_2 = f^2 \cdot g_1$. Then g_1 and g_2 have the same angles, but not the same lengths. A local diffeomorphism $\varphi: (M_1, g_1) \rightarrow (M_2, g_2)$ is called conformal if $\varphi^* g_2$ is conformal to g_1 .

As an example, which also explains the name, we mention that any holomorphic mapping with nonvanishing derivative between open domains in \mathbb{C} is conformal for the Euclidean inner product. This is clear from the polar decomposition $\varphi'(z) = |\varphi'(z)| e^{i \arg(\varphi'(z))}$ of the derivative.

As another, not unrelated, example we note that the stereographic projection [1] is a conformal mapping:

$$u_+ : (S^n \setminus \{a\}, g^{S^n}) \rightarrow \{a\}^\perp \rightarrow (\mathbb{R}^n, \langle \cdot, \cdot \rangle), \quad u_+(x) = \frac{x - \langle x, a \rangle a}{1 - \langle x, a \rangle}.$$

To see this, take $X \in T_x S^n \subset T_x \mathbb{R}^{n+1}$, so that $\langle X, x \rangle = 0$. Then we get:

$$\begin{aligned}
 du_+(x)X &= \frac{(1 - \langle x, a \rangle)(X - \langle X, a \rangle a) + \langle X, a \rangle(x - \langle x, a \rangle a)}{(1 - \langle x, a \rangle)^2} \\
 &= \frac{1}{(1 - \langle x, a \rangle)^2} ((1 - \langle x, a \rangle)X + \langle X, a \rangle x - \langle x, a \rangle a), \langle du_+(x)X, du_+(x)Y \rangle \\
 &= \frac{1}{(1 - \langle x, a \rangle)^2} \langle X, Y \rangle = \frac{1}{(1 - \langle x, a \rangle)^2} (g^{S^n})_x(X, Y)
 \end{aligned}$$

Theorem 5.1: Let (M, g) be a connected Riemann manifold. Then, we have:

(1) There exist complete Riemann metrics on M which are conformal to g and are equal to g on any given compact subset of M .

(2) There also exist Riemann metrics on M such that M has finite diameter which are conformal to g and are equal to g on any given compact subset of M . If M is not compact, then a Riemann metric for which M has finite diameter is not complete.

Thus the sets of all complete Riemann metrics and of all Riemann metrics with bounded diameter are both dense in the compact C^∞ -topology on the space of all Riemann metrics.

Proof: (1). For $x \in M$ let

$r(x) := \sup\{r: B_x(r) = \{y \in M: \text{dist}^g(x, y) \leq r\}$ is compact in $M\}$.

If $r(x) = \infty$ for one x , then g is a complete metric. Since \exp_x is a diffeomorphism near 0_x , $r(x) > 0 \forall x$. We assume that $r(x) < \infty \forall x$.

Claim: $|r(x) - r(y)| \leq \text{dist}^g(x, y)$; thus $r: M \rightarrow \mathbb{R}$ is continuous, since: For small $\varepsilon > 0$ the set $B_x(r(x) - \varepsilon)$ is compact, $\text{dist}^g(z, x) \leq \text{dist}^g(z, y) + \text{dist}^g(y, x)$ implies that $B_y(r(x) - \varepsilon - \text{dist}^g(x, y)) \subseteq B_x(r(x) - \varepsilon)$ is compact, and thus $r(y) \geq r(x) - \text{dist}^g(x, y) - \varepsilon$ and $r(x) - r(y) \leq \text{dist}^g(x, y)$. Now interchange x and y .

By a partition of unity argument we now construct a smooth function $f \in C^\infty(M, \mathbb{R} > 0)$ with $f(x) > \frac{1}{r(x)}$.

Consider the Riemann metric $\bar{g} = f^2 g$.

Claim: $\bar{B}_x\left(\frac{1}{4}\right) := \{y \in M: \text{dist}^{\bar{g}}(x, y) \leq \frac{1}{4}\} \subset B_x\left(\frac{1}{2}r(x)\right)$; thus it is compact.

Suppose $y \notin B_x\left(\frac{1}{2}r(x)\right)$ For any piecewise smooth curve c from x to y we have

$$L^g(c) = \int_0^1 |c'(t)|_g dt > \frac{r(x)}{2},$$

$$L^{\bar{g}}(c) = \int f(c(t)) \cdot |c'(t)|_g dt = f(c(t_0)) \int_0^1 |c'(t)|_g dt > \frac{L^g(c)}{r(c(t_0))},$$

for some $t_0 \in [0, 1]$, by the mean value theorem of integral calculus. Moreover,

$$\begin{aligned}
 |r(c(t_0)) - r(x)| &\leq \text{dist}^g(c(t_0), x) \leq L^g(c) =: L, r(c(t_0)) \leq r(x) + L \\
 L^{\bar{g}}(c) &\geq \frac{L}{r(x)+L} \geq \frac{L}{3L} = \frac{1}{3} \text{ so } y \notin \bar{B}_x\left(\frac{1}{4}\right) \text{ either.}
 \end{aligned}$$

Claim: (M, \bar{g}) is a complete Riemann manifold.

Let $X \in T_x M$ with $|X|_{\bar{g}} = 1$. Then $\exp^{\bar{g}}(t.X)$ is defined for $|t| \leq \frac{1}{5} < \frac{1}{4}$ But also $\exp^{\bar{g}}\left(s \cdot \frac{\partial}{\partial t} \Big|_{t=\pm 1/5} \exp^{\bar{g}}(t.X)\right)$ is

defined for $|s| < \frac{1}{4}$ which equals $\exp^{\bar{g}}\left(\left(\pm \frac{1}{5} + S\right)X\right)$, and so on. Thus $\exp^{\bar{g}}(t.X)$ is defined $\forall t \in \mathbb{R}$, and the

metric \bar{g} is complete.

Claim: We may choose f in such a way that $f = 1$ on a neighborhood of any given compact set $K \subset M$.

Let $C = \max \left\{ \frac{1}{r(x)} : x \in K \right\} + 1$. By a partition of unity argument we construct a smooth function f with $f = 1$ on a neighborhood of K and $Cf(x) > \frac{1}{r(x)} \forall x$. By the arguments above, $C^2 f^2 g$ is then a complete metric; thus so is $f^2 g$.

Proof: (2). Let g be a complete Riemann metric on M . We choose $x \in M$, a smooth function h with $h(y) > \text{dist}^g(x, y)$, and we consider the Riemann metric $\tilde{g}_y = e^{-2h(y)} g_y$. For any $y \in M \exists$ a minimal g -geodesic c from x to y , parameterized by arc-length. Then

$$h(c(s)) > \text{dist}^g(x, c(s)) = s \forall s \leq \text{dist}^g(x, y) =: L$$

But

$$L^{\tilde{g}}(c) = \int_0^L e^{-h(c(s))} |c'(s)|_g ds < \int_0^L e^{-s} ds < \int_0^\infty e^{-s} ds = 1,$$

so that M has diameter 1 for the Riemann metric \tilde{g} . We may also obtain that $\tilde{g} = g$ on a compact set as above.

Proposition: Let (M, g) be a complete Riemann manifold. Let $X \in \mathfrak{X}(M)$ be a vector field which is bounded with respect to g , $|X|_g \leq C$. Then X is a complete vector field; it admits a global flow.

Proof: The flow of X is given by the differential equation $\frac{\partial}{\partial t} Fl_t^X(x) = X(Fl_t^X(x))$ with initial value $Fl_0^X(x) = x$. suppose that $c(t) = Fl_t^X(x)$ is defined on (a, b) and that $b < \infty$, say. Then

$$\begin{aligned} \text{dist}^g \left(c(b - 1/n), c(b - 1/m) \right) &\leq L_{b-1/n}^{b-1/m}(c) = \int_{b-1/n}^{b-1/m} |c'(t)|_g dt \\ &= \int_{b-1/n}^{b-1/m} |X(c(t))|_g dt \leq \int_{b-1/n}^{b-1/m} C dt = C \cdot \left(\frac{1}{m} - \frac{1}{n} \right) \rightarrow 0 \end{aligned}$$

so that $c(b - 1/n)$ is a Cauchy sequence in the complete metrical space M and the limit $c(b) = \lim_{n \rightarrow \infty} c(b - 1/n)$ exists. But then we may continue the flow beyond b by $Fl_s^X(Fl_b^X(x)) = Fl_{b+s}^X(x)$.

Proposition 5.1: Let X be a complete vector field on a connected manifold M . Then there exists a complete Riemann metric g on the manifold $M \times \mathbb{R}$ such that the vector field $X \times \partial_t \in \mathfrak{X}(M \times \mathbb{R})$ is bounded with respect to g .

Proof: Since $Fl_t^{X \times \partial_t}(x, s) = (Fl_t^X(x), s + t)$ the vector field $X \times \partial_t$ is also complete. It is now here 0.

Choose a smooth proper function f_1 on M ; for example, if a smooth function f_1 satisfies $f_1(x) > \text{dist}^{\tilde{g}}(x_0, x)$ for a complete Riemann metric \tilde{g} on M , then f_1 is proper.

For a Riemann metric \tilde{g} on M we consider the Riemann metric \bar{g} on the product $M \times \mathbb{R}$ which equals g_x on $T_x M \cong T_x M \times 0_t = T_{(x,t)}(M \times \{t\})$ and satisfies

$$|X \times \partial_t|_{\bar{g}} = 1 \quad \text{and} \quad \bar{g}_{(x,t)} \left((X \times \partial_t)(x, t), T_{(x,t)}(M \times \{t\}) \right) = 0$$

We will also use the fiberwise \bar{g} -orthogonal projections

$$pr_M: T(M \times \mathbb{R}) \rightarrow TM \times 0 \quad \text{and} \quad pr_X: T(M \times \mathbb{R}) \rightarrow \mathbb{R}. (X \times \partial_t) \cong \mathbb{R}.$$

The smooth function $f_2(x, s) = f_1(Fl_{-s}^X(x)) + s$ satisfies the following and is thus still proper:

$$(\mathcal{L}_{X \times \partial_t} f_2)(x, s) = \partial|_0 f_2 \left(Fl_t^{X \times \partial_t}(x, s) \right) = \partial|_0 f_2(Fl_t^X(x), s + t)$$

$$= \partial|_0 \left(f_1 \left(Fl_{-s-t}^X \left(Fl_t^X(x) \right) \right) + s + t \right) = \partial|_0 f_1 \left(Fl_{-s}^X(x) \right) + 1 = 1$$

By a partition of unity argument we construct another smooth function $f_3: M \times \mathbb{R} \rightarrow \mathbb{R}$ which satisfies

$$f_3(x, s)^2 > \max\{|Y(f_2)|^2: Y \in T_{(x,s)}(M \times \{s\}), |Y|_{\bar{g}} = 1\}.$$

Finally we define a Riemann metric g on $M \times \mathbb{R}$ by

$$g_{(x,t)}(Y, Z) = f_3(x, t)^2 \bar{g}_{(x,t)}(pr_M(Y), pr_M(Z)) + pr_X(Y) \cdot pr_X(Z)$$

for $Y, Z \in T_{(x,t)}(M \times \mathbb{R})$ which satisfies $|X \times \partial_t|_g = 1$.

Claim: g is a complete Riemann metric on $M \times \mathbb{R}$.

Let c be a piecewise smooth curve parameterized by g –arc-length. Then

$$|c'|_g = 1, \text{ also } |pr_M(c')|_g \leq 1, |pr_X(c')| \leq 1$$

$$\frac{\partial}{\partial t} f_2(c(t)) = df_2(c'(t)) = \left(pr_M(c'(t)) \right) (f_2) + pr_X(c'(t)) (f_2),$$

$$\begin{aligned} \left| \frac{\partial}{\partial t} f_2(c(t)) \right| &\leq \left| \frac{pr_M(c'(t))}{|pr_M(c'(t))|_g} (f_2) \right| + \left| \frac{pr_X(c'(t))}{|pr_X(c'(t))|_g} (f_2) \right| < 2 \\ &= \left| \frac{1}{f_3(c(t))} \frac{pr_M(c'(t))}{|pr_M(c'(t))|_g} (f_2) \right| + |\mathcal{L}_{X \times \partial_t} f_2| \end{aligned}$$

by the definition of g and the properties of f_3 and f_3 . Thus

$$|f_2(c(t)) - f_2(c(0))| \leq \int_0^t \left| \frac{\partial}{\partial t} f_2(c(t)) \right| dt \leq 2t$$

Since this holds for every such c , we conclude that

$$|f_2(x) - f_2(y)| \leq 2 \text{dist}^g(x, y)$$

and thus each closed and dist^g –bounded set is contained in some

$$\{y \in M \times \mathbb{R}: \text{dist}^g(x, y) \leq R\} \subset f_2^{-1} \left(\left[f_2(x) - \frac{R}{2}, f_2(x) + \frac{R}{2} \right] \right)$$

which is compact since f_2 is proper. So $(M \times \mathbb{R}, g)$ is a complete Riemann manifold.

Theorem 5.2: Let (M, g) be a pseudo-Riemann manifold with vanishing curvature. Then M is locally isometric to \mathbb{R}^m with the standard inner product of the same signature: For each $x \in M$ there exists a chart (U, u) centered at x such that $g|_U = u^* \langle \cdot, \cdot \rangle$.

Proof: Choose an orthonormal basis $X_1(x), \dots, X_m(x)$ of $(T_x M, g_x)$; this means $g_x(X_i(x), X_j(x)) = \eta_{ii} \delta_{ij}$ where $\eta = \text{diag}(1, \dots, 1, -1, \dots, -1)$ is the standard inner product of signature (p, q) . Since the curvature R vanishes, we may consider the horizontal foliation of curvature and integrability of horizontal bundle. Let H_i denote the horizontal leaf through $X_i(x)$ and define $X_i: U \rightarrow TM$ by $X_i = (\pi_M|_{H_i})^{-1}: U \rightarrow H_i \subset TM$ where U is a suitable (simply connected) neighborhood of x in M . Since $X_i \circ c$ is horizontal in TM for any curve c in U , we have $\nabla_X X_i = 0$ for any $X \in \mathfrak{X}(M)$ for the Levi-Civita covariant derivative of g . Vector fields X_i with this property are called Killing fields. Moreover $X(g(X_i, X_j)) = g(\nabla_X X_i, X_j) + g(X_i, \nabla_X X_j) = 0$ thus $g(X_i, X_j) = \text{constant} = g(X_i(x), X_j(x)) = \eta_{ii} \delta_{ij}$ and X_1, \dots, X_j is an orthonormal frame on U . Since ∇ has no torsion, we have

$$0 = \text{Tor}(X_i, X_j) = \nabla_{X_i} X_j - \nabla_{X_j} X_i - [X_i, X_j] = [X_i, X_j]$$

Since there exists a chart (U, u) on M centered at x such that $X_i = \frac{\partial}{\partial u^i}$, i.e., $Tu.X_i(x) = (u(x), e_i)$ for the standard basis e_i of \mathbb{R}^m .

Thus Tu maps an orthonormal frame on U to an orthonormal frame on $u(U) \in \mathbb{R}^n$ and u is an isometry.

REFERENCES

1. Peter W. Michor, Topics in Differential Geometry Fakultät für Mathematik der Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria Kritzendorf, April 2008.
2. D. Alekseevsky, A. Kriegel, M. Losik, and P.W. Michor, The Riemannian geometry of orbit spaces. The metric, geodesics, and integrable systems, *Publicationes Mathematicae* 62 (2003), 247–276.
3. D. Alekseevsky, P.W. Michor, and W.A.F. Ruppert, Extensions of Lie algebras, *math.DG/0005042*, 2000.
4. Extensions of super Lie algebras, *J. Lie Theory* 15 (2005), 125–134, *math.QA/0101190*.
5. J.A. Azcárraga and J.M. Izquierdo, Lie groups, Lie algebras, cohomology and some applications in physics, Cambridge University Press, Cambridge, UK, 1995, Cambridge Monographs on Mathematical Physics.
6. A. Cap and J. Slovak, Parabolic geometries I: Background and general theory, AMS, Providence, 2009, Surveys and Monographs.
7. D.B.A. Epstein, Natural tensors on Riemannian manifolds, *J. Diff. Geom.* 10, 631–645.
8. D.S. Freed and D. Groisser, The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomorphism group, *Michigan Math. J.* 36 (1989), 323–344.
9. F. Gesztesy, H. Holden, J. Michor, and G. Teschl, Soliton equations and their algebrogeometric solutions, volume II: (1+1)-dimensional discrete models, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2008.
10. O. Gil-Medrano and P.W. Michor, The Riemannian manifold of all Riemannian metrics, *Quarterly J. Math. Oxford* 42 (1991), 183–202.
11. O. Gil-Medrano, P.W. Michor, and M. Neuwirth, Pseudo-Riemannian metrics on spaces of bilinear structures, *Quarterly J. Math. Oxford* 43 (1992), 201–221.
12. P.B. Gilkey, The spectral geometry of a Riemannian manifold, *J. Diff. Geom.* 10 (1975), 601–618.
13. S. Kobayashi, Riemannian manifolds without conjugate points, *Ann. Mat. Pura Appl.* 53 (1961), 149–155.
14. The moment mapping for a unitary representation, *J. Global Analysis and Geometry* 8 (1990), 299–313.
15. J. Morrow, The denseness of complete Riemannian metrics, *J. Diff. Geo.* 4 (1970), 225–226.



COMPARATIVE STUDY OF EFFECTS OF RAMP ANGLE ON THE NON-LINEAR PASSIVE DYNAMIC WALKING OF KNEED AND KNEE-LESS BIPED ROBOT

Mahesh A. Yeolekar

Department of Mathematics, Institute of Technology, Nirma University, Ahmedabad, Gujarat, India
E-mail: mahesh.a.yeolekar@gmail.com

ABSTRACT :

This paper presents the stability of gaits of kneed biped robots in compare to kneeless bipeds while walking on the ramp in absence of external forces except gravity. The kneed biped modeled by adding knees in the kneeless biped with the same dimensions. Consequently, the ramp angle is considered as a parameter for the comparative analysis of stability of models. The fixed-point analysis of Poincare map technique is used for the stability analysis of periodic gaits. The bifurcation diagrams are used for pictorial representation of results. The effects of ramp angle show that the kneed biped is fast, stable with small step size for higher ramp angle compare to kneeless biped. As a result, the knees increased the stability of biped.

Keywords : passive dynamic walking, periodic orbit, Poincare map, bifurcation diagram

1. INTRODUCTION

Biped robotic walking is inspired by walking of living organism like Kangaroo, Ostrich, Pangolins, and Humans. All they have their own walking patterns like Kangaroo is famous for hopping gaits, Ostrich for aerial-running gaits at faster speed, Pangolin for small legs walking, human for purely two legs walking. These gaits mechanisms depend on the body and legs structure of the living organism.

This paper studies the effects of knees on the gait patterns of biped robots. The impacts of knees are examined by analyzing the results of kneed and kneeless bipeds. In section 2, the mathematical models of passive dynamic kneed and kneeless bipeds are described. The kneeless biped built of two links which behaves as the legs and they joined at uppers ends which works as the hip of biped. The gait of kneeless biped is the combined effects of two events: the motion of swing leg and heel-strike [1] [2] [3]. In case of kneed biped, four links are used which connected by three joints, the center joint acts as a hip while others work as knees of legs of biped. The gait of kneed biped is the result of three of events: the motion of swing leg, knee-strike and heel-strike [4] [5]. The motion equations of swing legs are obtained from the Lagrangian motion equations which are analogous to two and four links inverted pendulums [6] [7][8]. The transition equations of knee-strike and heel-strike are developed using conservative laws of momentums which are similar to the physical impact models [9] [10] [11].

The section 3 explains the Poincare maps of the both models. The fixed-point analysis of Poincare map is used for the study of stability of gaits of the bipeds which helps to reduce the degrees of freedom of the biped system [12] [13]. The cell-to-cell mapping method is used for finding fixed point of Poincare map which gives the accurate results but time-consuming method [14] [15]. The section 4 contains the simulation results and its pictorial representation through bifurcation diagrams [16].

2. PASSIVE DYNAMICS MODELS

In the context of biped robots, defining the kinematics is the first step for developing bipedal walking. The kinematics of humanoid modeled the hybrid system that show both continuous and discrete behavior.

Consequently, the mathematical models of bipedal walkers are consisting two components; the continuous component contains the dynamics of swing phase determined by Lagrangian motion equations and the discrete component contains the impact equations for the instantaneous change in the velocity of the system when a knee locks or a heel strikes the ground.

The kneeless biped model [2] and kneed biped model without torso [4] are the two frequently used passive dynamic biped models indicated in Figure 1.

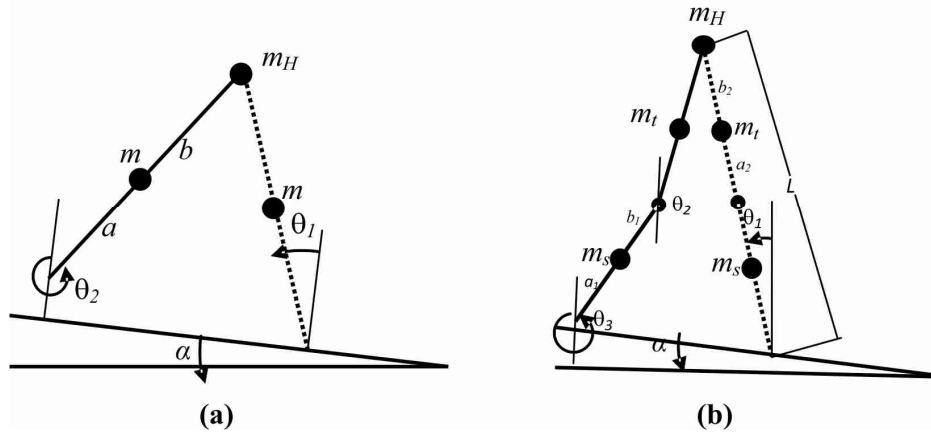


Figure 1 Schematic representation of knee-less (a) and kneed (b) biped walkers on the ramp

2.1 Model for knee-less biped robot

The knee-less biped walker builds by two rigid straight links connected through a joint at the hip. It is a three-point masses system with two degrees of freedom due to two distance constraints. The configuration of the kneeless walker can be described by the four-dimensional state variable $q = [\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]^T$.

The continuous component has the motion equation for the swing phase before the swing leg strikes the ground. Using Lagrange’s method, the motion equation for passive walking is in the form

$$M(\theta)\ddot{\theta} + N(\theta, \dot{\theta})\dot{\theta} + g(\theta) = 0 \tag{I}$$

where $\theta = [\theta_1 \ \theta_2]^T$

$$M(\theta) = \begin{bmatrix} ma^2 + Ml^2 + ml^2 & -mlb \cos(\theta_1 - \theta_2) \\ -mlb \cos(\theta_1 - \theta_2) & mb^2 \end{bmatrix}, \quad N(\theta, \dot{\theta}) = \begin{bmatrix} 0 & -mlb \sin(\theta_1 - \theta_2) \dot{\theta}_2 \\ mlb \sin(\theta_1 - \theta_2) \dot{\theta}_1 & 0 \end{bmatrix},$$

$$G(\theta) = \begin{bmatrix} (-ma - Ml - ml) g \sin \theta_1 \\ mgb \sin \theta_2 \end{bmatrix}$$

The discrete component of the model has impact equation for the instantaneous change in the velocities of swing and stance legs when swing leg strikes the ground and also the equation includes the position change for the subsequent step. Using the conservative law of angular momentums, the impact equation is in the form

$$q^+ = f(q^-) = \begin{pmatrix} J & 0 \\ 0 & (H^+(\varphi))^{-1} H^-(\varphi) \end{pmatrix} q^-$$

where $q = [\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]^T$, $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\theta_2 - \theta_1 = \varphi$, $H^-(\varphi) = \begin{bmatrix} -mab + (2mla + Ml^2) \cos \varphi & -mab \\ -mab & 0 \end{bmatrix}$

$$H^+(\varphi) = \begin{bmatrix} ma^2 + Ml^2 + ml^2 - mlb \cos \varphi & mb^2 - mlb \cos \varphi \\ -mlb \cos \varphi & mb^2 \end{bmatrix}.$$

2.2 Model for kneed biped robot

The kneed walker builds by four rigid links connected thorough three joints: two at the knees and one at the hip. It is a five-point masses system with four degrees of freedom owing to the distance constraints. The configuration of the kneed walker can be described by the six-dimensional state variable $q = [\theta_1 \ \theta_2 \ \theta_3 \ \dot{\theta}_1 \ \dot{\theta}_2 \ \dot{\theta}_3]^T$.

The Lagrangian motion equation of swing phase before knee locks for the passive walking of kneed walker is in the form

$$M(\theta)\ddot{\theta} + N(\theta, \dot{\theta})\dot{\theta} + g(\theta) = 0 \quad (\text{II})$$

where $\theta = [\theta_1 \ \theta_2 \ \theta_3]^T$

$$M(\theta) = \begin{bmatrix} \left(\begin{array}{c} m_s a_1^2 + m_t (a_1 + b_1 + a_2)^2 + \\ (m_H + m_t + m_s) L^2 \end{array} \right) & -(m_t b_2 + m_s (a_2 + b_2)) L \cos(\theta_1 - \theta_2) & -m_s b_1 L \cos(\theta_1 - \theta_3) \\ \left(\begin{array}{c} -(m_t b_2 + m_s (a_2 + b_2)) \\ L \cos(\theta_1 - \theta_2) \end{array} \right) & m_t b_2^2 + m_s (a_2 + b_2)^2 & m_s b_1 (a_2 + b_2) \cos(\theta_2 - \theta_3) \\ -m_s L b_1 \cos(\theta_1 - \theta_3) & m_s b_1 (a_2 + b_2) \cos(\theta_2 - \theta_3) & m_s b_1^2 \end{bmatrix},$$

$$N(\theta, \dot{\theta}) = \begin{bmatrix} 0 & -(m_t b_2 + m_s (a_2 + b_2)) L \sin(\theta_1 - \theta_2) \dot{\theta}_2 & -m_s b_1 L \sin(\theta_1 - \theta_3) \dot{\theta}_3 \\ (m_t b_2 + m_s (a_2 + b_2)) L \sin(\theta_1 - \theta_2) \dot{\theta}_1 & 0 & m_s b_1 (a_2 + b_2) \sin(\theta_2 - \theta_3) \dot{\theta}_3 \\ m_s b_1 L \sin(\theta_1 - \theta_3) \dot{\theta}_1 & -m_s b_1 (a_2 + b_2) \sin(\theta_2 - \theta_3) \dot{\theta}_2 & 0 \end{bmatrix}$$

$$g(\theta) = \begin{bmatrix} -[m_s a_1 + m_t (a_1 + b_1 + a_2) + (m_H + m_t + m_s) L] g \sin(\theta_1) \\ + [m_t b_2 + m_s (a_2 + b_2)] g \sin(\theta_2) \\ + m_s g b_1 \sin(\theta_3) \end{bmatrix}.$$

At the time of knee-strike, $\theta_1^+ = \theta_1^-$, $\theta_2^+ = \theta_2^- = \theta_3^-$ and the impact equation when knee locks is in the form

$$q^+ = g(q^-) = \begin{pmatrix} Q & 0 \\ 0 & (K^+)^{-1} K^- \end{pmatrix} q^-$$

where $q^+ = [\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]^T$, $q^- = [\theta_1 \ \theta_2 \ \theta_3 \ \dot{\theta}_1 \ \dot{\theta}_2 \ \dot{\theta}_3]^T$, $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$K^+ = \begin{bmatrix} \left(\begin{array}{c} m_t (a_1 + b_1 + a_2)^2 + (m_H + m_t + m_s) L^2 + m_s a_1^2 \\ -(m_s (a_2 + b_2 + b_1) + m_t b_2) L \cos(\theta_1^+ - \theta_2^+) \end{array} \right) & \left(\begin{array}{c} m_s (a_2 + b_2 + b_1)^2 + m_t b_2^2 \\ -(m_s (a_2 + b_2 + b_1) + m_t b_2) L \cos(\theta_1^+ - \theta_2^+) \end{array} \right) \\ \left(\begin{array}{c} -(b_2 m_t + m_s (a_2 + b_2 + b_1)) L \cos(\theta_1^+ - \theta_2^+) \\ (m_t b_2^2 + m_s (a_2 + b_2 + b_1)^2) \end{array} \right) & \end{bmatrix}$$

$$K^- = \begin{bmatrix} \left(\begin{array}{c} m_s a_1^2 + m_t (a_1 + b_1 + a_2)^2 + (m_H + m_t + m_s) L^2 \\ -(m_t b_2 + m_s (a_2 + b_2)) L \cos(\theta_1^- - \theta_2^-) \end{array} \right) & \left(\begin{array}{c} m_t b_2^2 + m_s (a_2 + b_2)^2 \\ -(m_t b_2 + m_s (a_2 + b_2)) L \cos(\theta_1^- - \theta_2^-) \end{array} \right) & \left(\begin{array}{c} m_s b_1^2 - m_s L b_1 \cos(\theta_1^- - \theta_3^-) \\ + m_s (a_2 + b_2) b_1 \cos(\theta_2^- - \theta_3^-) \end{array} \right) \\ \left(\begin{array}{c} -m_s L b_1 \cos(\theta_1^- - \theta_3^-) \\ \left(\begin{array}{c} -(m_t b_2 + m_s (a_2 + b_2)) L \cos(\theta_1^- - \theta_2^-) \\ -m_s L b_1 \cos(\theta_1^- - \theta_3^-) \end{array} \right) \end{array} \right) & \left(\begin{array}{c} m_t b_2^2 + m_s (a_2 + b_2)^2 \\ + m_s (a_2 + b_2) b_1 \cos(\theta_2^- - \theta_3^-) \end{array} \right) & \left(\begin{array}{c} m_s b_1^2 + \\ m_s (a_2 + b_2) b_1 \cos(\theta_2^- - \theta_3^-) \end{array} \right) \end{bmatrix}$$

After the knee locks, the system has only two degrees of freedom owing to thigh and shin move forward together which behave like knee-less biped and the motion equation is in the form

$$M(\theta)\ddot{\theta} + N(\theta, \dot{\theta})\dot{\theta} + g(\theta) = 0 \tag{III}$$

where

$$M(\theta) = \begin{bmatrix} \left(m_s a_1^2 + m_t (a_1 + b_1 + a_2)^2 + (m_H + m_t + m_s)L^2 \right) & -(m_t b_2 + m_s (a_2 + b_2 + b_1))L \cos(\theta_1 - \theta_2) \\ -(m_t b_2 + m_s (a_2 + b_2))L \cos(\theta_1 - \theta_2) & m_t b_2^2 + m_s (a_2 + b_2 + b_1)^2 \end{bmatrix}$$

$$N(\theta, \dot{\theta}) = \begin{bmatrix} 0 & -(m_t b_2 + m_s (a_2 + b_2 + b_1))L \sin(\theta_1 - \theta_2) \dot{\theta}_2 \\ -(m_t b_2 + m_s (a_2 + b_2 + b_1))L \sin(\theta_1 - \theta_2) \dot{\theta}_2 & 0 \end{bmatrix}$$

$$g(\theta) = \begin{bmatrix} -[m_s a_1 + m_t (a_1 + b_1 + a_2) + (m_H + m_t + m_s)L]g \sin(\theta_1) \\ +[m_t b_2 + m_s (a_2 + b_2 + b_1)]g \sin(\theta_2) \end{bmatrix}$$

The impact equation at the time of heel strikes is in the form

$$q^+ = f(q^-) = \begin{pmatrix} J & 0 \\ 0 & (H^+(\varphi))^{-1} H^-(\varphi) \end{pmatrix} q^-$$

where $q = [\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]^T$, $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\theta_2 - \theta_1 = \varphi$,

$$H^-(\varphi) = \begin{bmatrix} \begin{pmatrix} -m_t b_2 (a_1 + b_1 + a_2) - m_s a_1 (b_1 + a_2 + b_2) \\ [m_H L + 2m_t (a_2 + a_1 + b_1) + 2m_s a_1]L \cos \varphi \end{pmatrix} & \begin{pmatrix} -m_s a_1 (a_2 + b_2 + b_1) \\ -m_t b_2 (a_2 + a_1 + b_1) \end{pmatrix} \\ \begin{pmatrix} -m_t b_2 (a_1 + b_1 + a_2) \\ -m_s a_1 (b_1 + a_2 + b_2) \end{pmatrix} & 0 \end{bmatrix}$$

$$H^+(\varphi) = \begin{bmatrix} \begin{pmatrix} m_t (a_1 + b_1 + a_2)^2 + (m_H + m_t + m_s)L^2 + m_s a_1^2 \\ -(m_s (a_2 + b_2 + b_1) + m_t b_2)L \cos \varphi \end{pmatrix} & \begin{pmatrix} m_s (a_2 + b_2 + b_1)^2 + m_t b_2^2 \\ -(m_s (a_2 + b_2 + b_1) + m_t b_2)L \cos \varphi \end{pmatrix} \\ \begin{pmatrix} -(m_t b_2 + m_s (a_2 + b_2 + b_1))L \cos \varphi \end{pmatrix} & \begin{pmatrix} m_t b_2^2 + m_s (a_2 + b_2 + b_1)^2 \end{pmatrix} \end{bmatrix}$$

For the subsequent step, the initial state vector can be considered as

$$q = E(q^+) = [\theta_1 \ \theta_2 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2 \ \dot{\theta}_2]^T$$

3. POINCARÉ MAP

In biped robots, the core issues of analysis are uniform gait patterns and its stability. The uniform gait patterns and its stability are associated to periodic orbit and its stability respectively. The Poincaré map is the standard technique to study the stability of periodic orbits. The Poincaré map defined from Poincaré section to Poincaré section which can be considered as the space of Heel-strike. The Poincaré section can be defined as:

forknee-less biped: $S = \left\{ q = [\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]^T \in R^4 \mid \theta_1 - \theta_2 = -2\alpha \right\}$ and

for kneed biped: $S = \left\{ q = [\theta_1 \ \theta_2 \ \theta_3 \ \dot{\theta}_1 \ \dot{\theta}_2 \ \dot{\theta}_3]^T \in R^6 \mid \theta_1 - \theta_2 = -2\alpha, \theta_2 = \theta_3 \right\}$

The Poincaré map $P : S \rightarrow S$ defined by

forknee-less biped: $q_{i+1} = P(q_i) = f(c(\tau_i, q_i)) \tag{i \ge 0}$

where $c(t, q_i)$ is the solution curve of the motion equation (I) with respect the initial condition q_i and τ_i is the time of heel-strike for the i^{th} step.

for kneed biped:
$$q_{i+1} = P(q_i) = E\left(f\left(c_2\left(\tau_{i_2}, g\left(c_1\left(\tau_{i_1}, q_i\right)\right)\right)\right)\right) \quad (i \geq 0)$$

where $c_1(t, q_i)$ and $c_2\left(t, g\left(c_1\left(\tau_{i_1}, q_i\right)\right)\right)$ are the solution curves of the motion equation (II) and (III) with respect the initial conditions q_i and $g\left(c_1\left(\tau_{i_1}, q_i\right)\right)$ respectively and τ_{i_1} and τ_{i_2} are the times of knee-strike and heel-strike for the i^{th} step respectively. The k -periodic orbit O is locally stable if and only if the k -fixed point q^* ($P(q_{i+k}) = q_i$ for the smallest positive integer k) of Poincare map P is locally stable. In this paper, we used the cell-to-cell mapping method to obtain the fixed point of Poincare map [14] [15] [16].

4. COMPARISON OF EFFECTS OF ANGLE OF SLOPE ON KNEELESS AND KNEED ROBOT

The focus of this study is to examine the effects on stability of passive dynamic biped after adding knees in the kneeless biped model. The analysis performed by considering the ramp angle as a parameter of the models. The bifurcation diagrams are used for the comparative study of stability. We observed that the one periodic gaits turn²ⁿ periodic when slope angle is increased and further it directed to chaos for both the models. The bifurcation diagrams of the step lengths, impact time and walking speed of kneeless and kneed biped are showed in Figure 2 by dotted and non-dotted curved lines respectively. The kneeless biped has longer but slower steps while kneed biped is speedy with small steps.

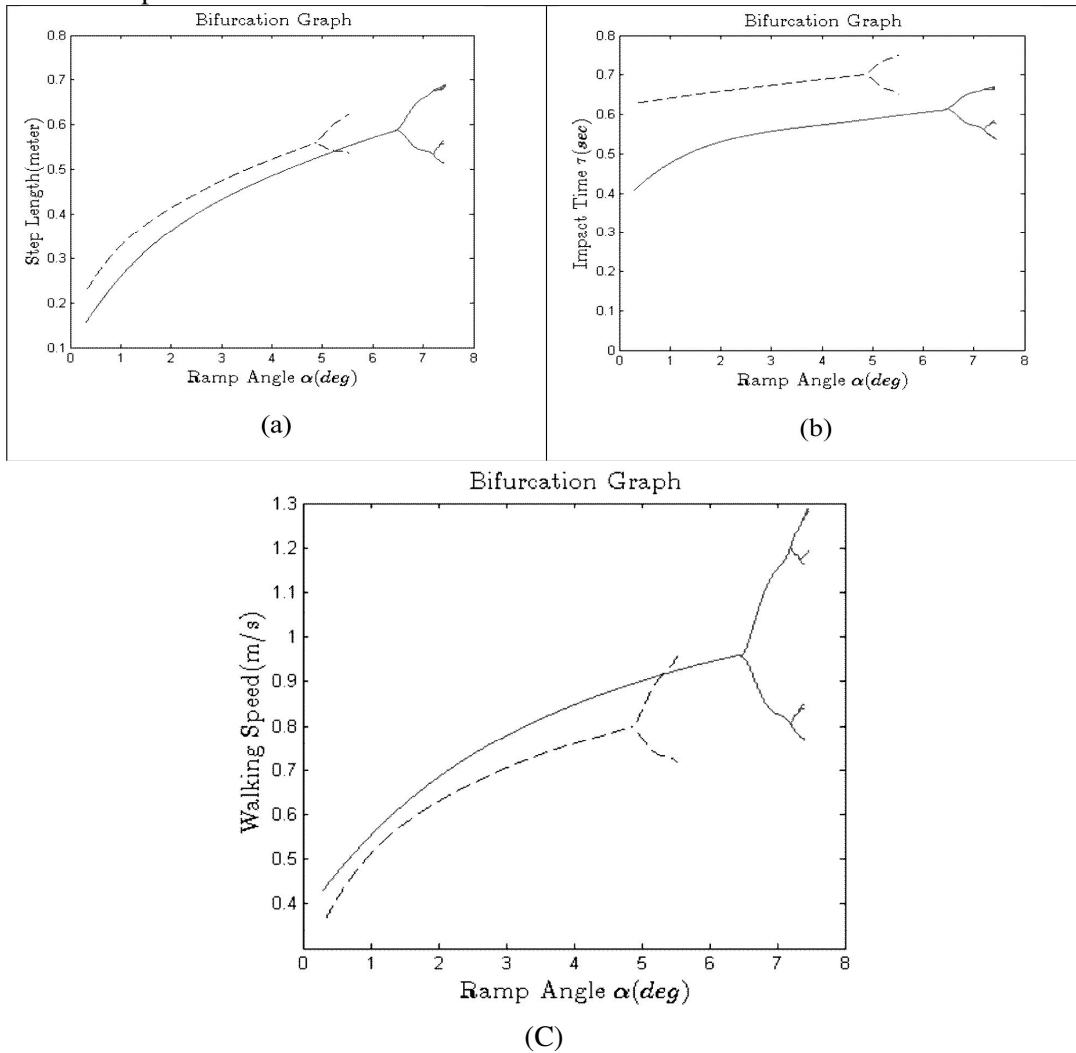


Figure 2 Bifurcation diagrams (a) step length (b) impact time (c) walking speed as the function of ramp angle (----kneeless and -kneed biped)

The bifurcation diagrams of angular velocities of stance and swing legs, kinetic and potential energies of bipeds are displayed in Figure 3. After 1.29 degree ramp angle, the angular velocity of swing leg of kneed biped is increased faster than kneeless biped when the curve of angular velocity of swing leg of kneed biped crossed the curve of kneeless biped. The kneeless biped has low potential energy but the high rate of conversion of kinetic energy as the ramp angle increased which made it unstable earlier while the kneed biped has high potential energy and the low rate of conversion of kinetic energy compare to kneeless made it stable for higher ramp angle.

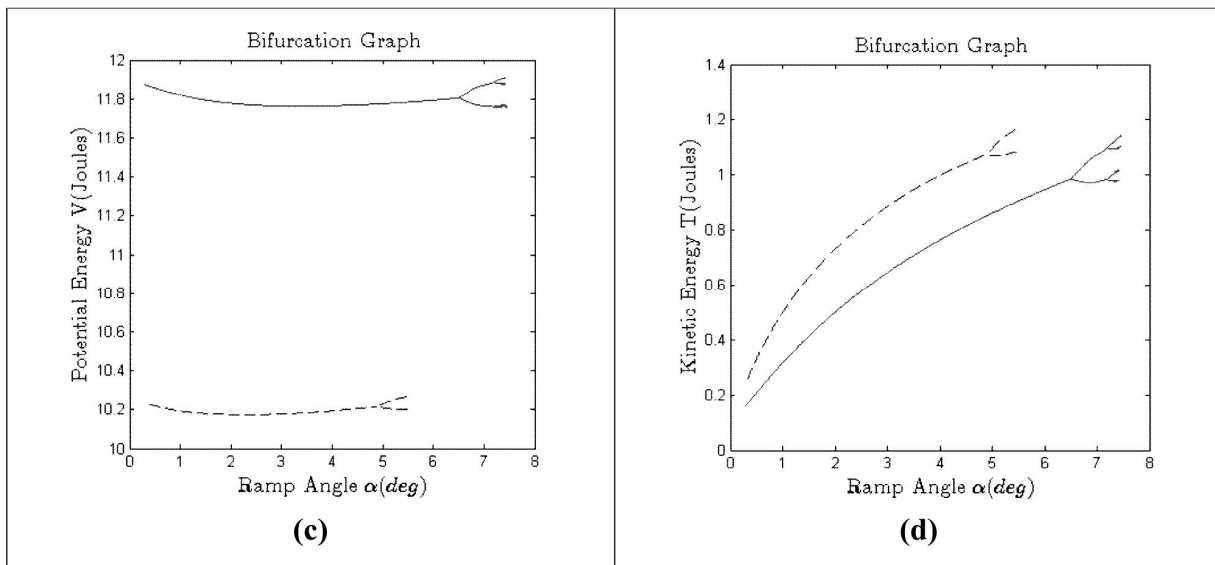
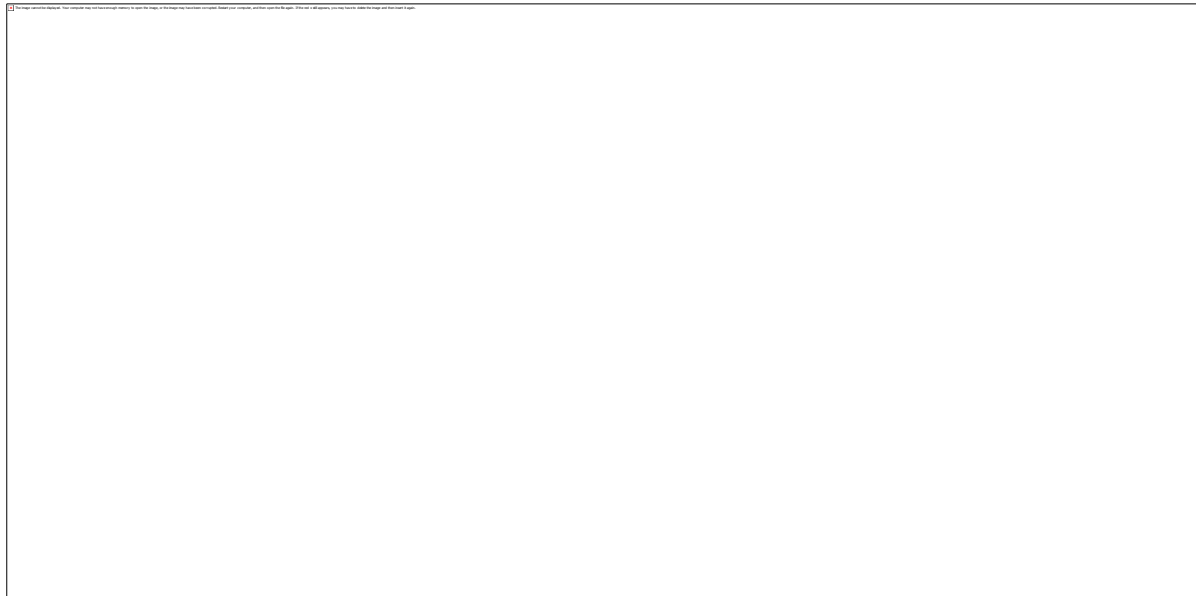


Figure 3 Bifurcation diagrams: (a) angular velocity of stance leg (b) angular velocity of swing leg as the function of ramp angle(---kneeless and -kneed biped)

The phase diagrams display the orbits of gaits of kneeless and kneed biped at the ramp angle 2.5 degree in the Figure 4. The horizontal distance represents step size while the vertical shows the angular velocities of the legs in phase space diagrams. It results that the kneed biped has small step size and high angular velocities compare to kneeless biped.

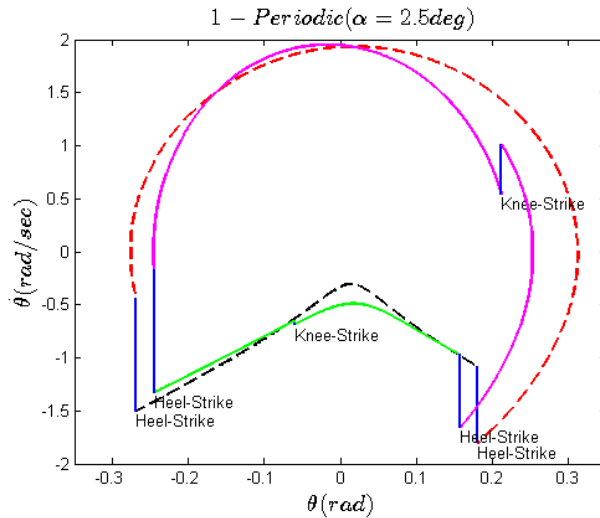


Figure 4 Phase space diagrams:----kneeless and -kneed biped ($\alpha=2$ deg)

Table 1 displayed behavior of periodic gaits of both the models. The kneeless biped has 2 periodic gaits after 4.8 degree while the kneed biped received it after 6.3 degree ramp angle. The 4 and 8-periodic gaits are not available for kneeless biped but the kneed biped gained it in the interval (7.2, 7.3]and (7.3, 7.4] respectively. The chaotic gaits are shown after 5.52 and 7.4 degree for kneeless and kneed bipeds respectively. This analysis showed that the kneed biped is more stable compare to kneeless robot.

n-periodic	Knee-less Robot	Kneed Robot
1-periodic	$0.3 < \alpha \leq 4.8$	$0.3 < \alpha \leq 6.3$
2-periodic	$4.8 < \alpha \leq 5.52$	$6.3 < \alpha \leq 7.2$
4-periodic	Not available	$7.2 < \alpha \leq 7.3$
8-periodic	Not available	$7.3 < \alpha \leq 7.4$
Chaotic	$\alpha > 5.52$	$\alpha > 7.4$

Table 1 Effects of ramp angle on the periodicity of orbit of kneeless and kneed biped

5. CONCLUSION AND FUTURE SCOPE

This paper is the comparative study of passive dynamic models of kneeless and kneed biped. The ramp angle is considered as a parameter of the relative analysis. From the results of bifurcation diagrams of the ramp angle concluded that the kneed biped is fast with the small steps compare to the kneeless biped. The passive model of kneed biped is walked more stably for higher ramp angle than the kneeless biped. This analysis showed that the biped becomes more stable by adding knees in the kneeless passive dynamic model. These results can be used in making of controlled biped and prosthesis legs.

REFERENCES

1. McGeer, T. (1990) Passive dynamic walking, *International Journal of Robotics Research*, 9(2), 62-82.
2. Goswami, A., Thuilot, B., & Espiau, B. (1998). A study of the passive gait of a compass-like biped robot: Symmetry and chaos. *The International Journal of Robotics Research*, 17(12), 1282-1301.
3. Shah, N. H., & Yeolekar, M. A. (2016). Influence of mass and length of biped robot on its passive walking. *International Journal of Applied Nonlinear Science*, 2(4), 235-246.

4. Chen, H. (2007). *Passive dynamic walking with knees: A point foot model* (Doctoral dissertation, Massachusetts Institute of Technology).
5. Yi, Y., Lin, Z., & Yan, G. (2012). Stable running on a kneed biped robot with only hip-joint actuation. In *Intelligent Control and Automation (WCICA), 2012 10th World Congress on* (pp. 3846-3851). IEEE.
6. Lam, S., & Davison, E. J. (2006, June). The real stabilizability radius of the multi-link inverted pendulum. In *American Control Conference, 2006* (pp. 6-pp). IEEE.
7. Shah, N. H., & Yeolekar, M. (2013). Pole placement approach for controlling double inverted pendulum. *Global Journal of Research In Engineering, 13*(2).
8. Liu, Y., & Xin, X. (2016). Controllability and Observability of an n -Link Planar Robot with a Single Actuator Having Different Actuator–Sensor Configurations. *IEEE Transactions on Automatic Control, 61*(4), 1129-1134.
9. Hurmuzlu, Y., & Marghitu, D. B. (1994). Rigid body collisions of planar kinematic chains with multiple contact points. *International Journal of Robotics Research, 13*(1), 82–92.
10. Hurmuzlu, Y., Genot, F. and Brogliato, B. (2004) Modeling, Stability and Control of Biped Robots—A General Framework. *Automatica, 40*, 1647-1664.
11. An, K., & Chen, Q. (2013). A passive dynamic walking model based on knee-bend behaviour: stability and adaptability for walking down steep slopes. *International Journal of Advanced Robotic Systems, 10*(10), 365.
12. Hurmuzlu, Y., & Moskowitz, G. D. (1986). The role of impact in the stability of bipedal locomotion. *Dynamics and Stability of Systems, 1*(3), 217–234.
13. Licer, O., M'Sirdi, N. K., Manamanni, N., & El Alami, N. (2006). Stable periodic gaits of n -link biped robot in three dimensional space. *IFAC Proceedings Volumes, 39*(15), 689-694.
14. Hsu, C. S., & Tongue, B. H. (1988). Cell-to-cell mapping, a method of global analysis for nonlinear systems. *Journal of Applied Mechanics, 55*, 749.
15. Schwab, A. L., & Wisse, M. (2001). Basin of attraction of the simplest walking model. In *Proceedings of the ASME design engineering technical conference* (Vol. 6, pp. 531-539).
16. Shah, N. H., & Yeolekar, M. A. (2015). Influence of slope angle on the walking of passive dynamic biped robot. *Applied Mathematics, 6*(03), 456.



MATHEMATICAL MODEL OF POPULATION DYNAMICS AND GROWTH IN INDIA

Dr. Shankar Lal

Assistant Professor, Department of Mathematics, H.N.B. Garhwal University, S.R.T. Campus, Badshahithaul,
Tehri Garhwal, Uttarakhand (India)
E-mail: shankar_alm@yahoo.com

ABSTRACT :

Generally, Human population dynamics is a track factors related to changes in population such as fertility rate and life expectancy. Ancient India in 300 BC may have a population in the range 100–140 million. It has been estimated that the population was about 100 million in 1600 and remained nearly static until the late 19th century. It reached 255 million according to the first census taken in 1881. Studies of India's population since 1881 have focused on such topics as total population, birth and death rates, growth rates, the rural and urban divide, cities of a million, and the three cities with populations over eight million: Delhi, Greater Mumbai, and Kolkata. Mortality rates fell in the period 1920–45, primarily due to biological. India occupies 2.4% of the world's land area but supports over 17.5% of the world's population. In the present paper we have study to mathematical model of population growth in India by using Logistic model. Population of India has been used ordinary differential equation model known as logistic population model which is parameterized by growth rate along with capacity human population of India. First we test the numerical method for India population data (1981-2013) and we find our population which is very good fit with the population data.

Keywords: Growth rate, Logistic Population Model, vital coefficient, testing of hypothesis etc.

AMS Mathematics Subject Classifications (2010): 35G30, 35G60.

1. INTRODUCTION

In section one, introductory part of human population related problems through some mathematical model. In the section two, we discuss in necessary details of a model for population growth, the logistic model, which is more sophisticated than exponential growth. The logistic model, a slight modification of Malthus's model, is just such a model. In section three, we have to study Analytical solution of logistic population model with accelerated growth. In section four, calculation of vital coefficients, we have the analytical solution of the logistic equation. In section five, we have to introduce the Logistic Model in exponential Population growth governed by differential equations. Again, in section six, the solution of the Logistic Model in exponential Population growth to a maintain carrying capacity. There are many examples in nature that show that when the environment is stable the maximum number of individuals in a population fluctuates near the carrying capacity of the environment. In section seven, we have to use the population in India by the general calculation as a basic population growth depends on Censes. In section eight, we have emphasized the population dynamics includes birth rates, death rates, immigration, and emigration age and sex composition. In section nine, we have show that through graph fertility and mortality trends and Demographic indicators in India. Again, in the end, we have discussed the population problems and showed that

family planning programs have benefited the whole country drastically and even avoided some terrible social or environmental disasters.

Population problem is one of the main problems in India at the current time India is an overpopulated country and the growth in resources has not been keeping pace with the growth in population. So the increasing trend in population is a great threat to the nation. Recognizing the difficulty of feeding the growing population even with considerable increase in food production, they suggested giving priority to population policy for reduction in population.

In this situation, prediction of population is very essential for planning. But he agreed that there was a difficulty in his model in interpreting its parameter unlike those of exponential as

$$(1.1) \quad p_t = p_0 e^{(r+k)t}$$

Where p_0 = current population,

p_t = population after time t ,

r = growth rate,

k = annual migration rate, for prediction of regional population.

In many case, for small population one may use the discrete model as:

$$(1.2) \quad p_t = p_0 (1+r)^t,$$

Where p_0 = current population,

p_t = population after time t ,

r = growth rate.

We see that this discrete model with the modification of the growth rate which is not constant. where the critical point obtained at $t = 2021$, it means that at $t = 2021$, $p'(t)=0$ i.e. the growth rate become zero in the year 2021. We analysis after 2021 population decreases in the parabolic manner of mathematical model.

The continuous analogue of (1.1) is the Malthusian ODE model

$$(1.3) \quad \frac{dp}{dt} = ap,$$

$$(1.4) \quad p(t) = p_0 e^{at}, \text{ where } a \text{ is a constant.}$$

The Malthusian model is very simple and applicable for small population and therefore for large population it is preferable to use logistic ODE population model:

$$(1.5) \quad \frac{dp}{dt} = ap - bp^2, \text{ where } a \text{ and } b \text{ are called vital coefficients.}$$

If p_0 is the population at time t_0 , then $p(t)$, the population at time t , satisfies the initial-value problem (IVP):

$$(1.6) \quad \frac{dp}{dt} = ap - bp^2, \quad p(t_0) = p_0$$

It is easier to calculate the analytical solution for IVP (1.6) when the vital coefficients a and b are considered constants. But it is not so easy to calculate the analytical solution for $a(t)$ and $b(t)$ as functions of t . To compute the vital coefficients $a(t)$ and $b(t)$ as functions of t , it could be more convenient to use numerical methods based on some efficient algorithm. Therefore we are interested to study some well-understood numerical schemes to solve the logistic model where we could also calculate the vital coefficients $a(t)$ and $b(t)$ as functions of t based on some algorithm.

In the population of India based on a non-linear, non-autonomous ordinary differential equation model which is known as generalized Logistic Population Model and parameterized by growth rate along with capacity. In terms of carrying capacity Logistic Differential equation can also be defined as:

$$(1.7) \quad \frac{1}{p} \frac{dp}{dt} = a \left(1 - \frac{p}{k} \right) = R(t)$$

Where (1.7) represents the growth rate and k presents carrying capacity. Here the calculations are based on parameters characterizing growth rate and carrying capacity.

2. THE LOGISTIC POPULATION MODEL

When the population gets extremely large though, Malthus's model cannot be very accurate, since they do not reflect the fact that individual members are now competing with each other for the limited living space, natural resources food available. Thus, we must add a competition term to our linear differential equation. A suitable choice of a competition term is $-bp^2$, where b is a constant, since the statistical average of the number of encounters of two members per unit time is proportional to p^2 . We consider, therefore the modified equation.

$$(2.1) \quad \frac{dp}{dt} = ap - bp^2$$

This equation is known as the logistic law of population growth and the numbers a , b are called the vital coefficients of the population. Now, the constant b , in general, will be very small compared to a , so that if p is not too large then the term $-bp^2$ will be negligible compared to ap and the population will grow exponentially. However, when p is very large, the term $-bp^2$ is no longer negligible, and thus serves to slow down the rapid rate of increase of the population. Needless to say, the more industrialized a nation is, the more living space it has, and the more food it has, the smaller the coefficient b is. Let us now use the logistic equation to predict the future growth of an isolated population. If p is the population at time t , then $p(t)$, the population at time t , satisfies the initial-value problem.

$$(2.2) \quad \frac{dp}{dt} = ap - bp^2 \quad p(t_0) = p_0$$

3. ANALYTICAL SOLUTION OF LOGISTIC POPULATION MODEL

The logistic equation can be solved by separation of variables. From equation (2.2), we have

$$(3.1) \quad \frac{dp}{dt} = ap - bp^2$$

Integrating on both sides of this equation then we get

$$(3.2) \quad \int \frac{dp}{ap - bp^2} = \int dt$$

$$(3.3) \quad \frac{1}{a} [\log p - \log(a - bp)] = t + c$$

Using $t = t_0$ and $p = p_0 \therefore c = \frac{1}{a} [\log p_0 - \log(a - bp_0)] - t$

Now substituting the value of c into equation (3.3) and simplifying, we have

$$(3.4) \quad p(t) = \frac{ap_0}{bp_0 + (a - bp_0)e^{-a(t-t_0)}}$$

This is the required analytical solution of the logistic equation.

Let us now examine Equation (2.5) to see what kind of population it predicts. We observe that as $t \rightarrow \infty$ then

$$p(t) \rightarrow \frac{ap_0}{bp_0} = \frac{a}{b}$$

Thus regardless of its initial value, the population always approaches the limiting value $\frac{a}{b}$. Next,

we observed that $p(t)$ is monotonically increasing function of the time if $0 < p_0 < a/b$.

Moreover, since

$$(3.5) \quad \frac{d^2 p}{dt^2} = a \frac{dp}{dt} - 2bp \frac{dp}{dt} = (a - 2bp) \frac{dp}{dt} = (a - 2bp)p(a - bp)$$

We see that dp/dt is increasing if $p(t) < a/b$, and that dp/dt is decreasing if $p(t) < a/2b$. Hence if $p_0 < a/2b$, the graph of $p(t)$ must have the form given in the figure. Such a curve is called a logistic or S-shaped curve. From its shape we conclude that the time period before the population reaches the half its limiting value is a period of accelerated growth.

4. CALCULATION OF VITAL COEFFICIENTS

We have the analytical solution of the logistic equation as:

$$(4.1) \quad p(t) = \frac{ap_0}{bp_0 + (a - bp_0)e^{-a(t-t_0)}}$$

Let p_0 = current population, p_t = population after time $t = t_0$, and let $t - t_0 = \Delta t$, In this case, calculation of the values of a and b is performed in order to predict the population of India.

In this case, calculation of the values of a and b is performed in order to predict the population of India.

$$\text{Growth rate of population} = \frac{1}{n} \left(\frac{p_1}{p_0} - 1 \right) \times k$$

For this we assume p_0 = population of India in 1991= 891 910000,

p_1 = Population of India in 1996 = 982 553000,

p_2 = Population of India in 2009= 1 207 740000.

$$\text{And natural Growth rate of population} = \left(\frac{B - D}{p} \right) k$$

- Where
- B = total registered birth in a year,
 - D = total registered death in a year,
 - p = total mid-year population, and
 - k = constant 1000 or 100.

For examples, the population of India in1981 and 1982 was 716.289 million and 733.152 million then, percent

$$\text{relative to change per year} = \frac{1}{1} \left(\frac{733.152}{716.493} - 1 \right) \times 1000 = 23.25075, \text{ but this change is not uniform and so simple as}$$

it as it seems to be. The person was a child will become father or mother, through change in population being

uniform in population may produce unequal growth in different years, which may go increasing. Thus, population change behaves is calculated a $\log(1 + r) = \frac{1}{n} \log\left(\frac{P_1}{P_0}\right)$ substituting the above figure, we get

$$\log(1 + r) = \frac{\log\left(\frac{733.152}{716.493}\right)}{1} = \log(1.02325075) = 0.000998207163711371519279845222572$$

After taking anti-log

$$(1 + r) = 2.83$$

$$r = (2.83 - 1) = 1.83 \text{ per year}$$

Thus, growth rate is 1.83 % per year in future population may be possible.
(Population in millions)

Year	Average Population	Birth	Death	Natural change	B.R. (/1000)	D.R. (/ 1000)	N.C. (/1000)	G.R. (/1000)	N.G.R. (/1000)
1981	716.493	24.289	8.956	15.333	33.9	12.5	21.4	-	21.41
1982	733.152	24.781	8.725	16.056	33.8	11.9	21.9	23.25	21.89
1983	750.034	25.276	8.925	16.351	33.7	11.9	21.8	23.02	21.80
1984	767.147	26.006	9.666	16.340	33.9	12.6	21.3	22.81	21.29
1985	784.491	25.810	9.257	16.553	32.9	11.8	21.1	22.60	21.10
1986	802.052	26.147	8.903	17.244	32.6	11.1	21.5	22.38	21.49
1987	819.800	26.316	8.936	17.380	32.1	10.9	21.2	22.12	21.20
1988	837.700	26.388	9.215	17.173	31.5	11.0	20.5	21.83	20.50
1989	855.707	26.185	8.814	17.371	30.6	10.3	20.3	21.49	20.30
1990	873.785	26.388	8.476	17.912	30.2	9.7	20.5	21.12	20.49
1991	891.910	26.133	8.741	17.392	29.3	9.8	19.5	20.74	19.49
1992	910.065	26.392	9.192	17.200	29.0	10.1	18.9	21.39	18.89
1993	928.226	26.640	8.633	18.007	28.7	9.3	19.4	19.95	19.39
1994	946.373	27.161	8.801	18.360	28.7	9.3	19.4	19.55	19.40
1995	964.486	27.295	8.680	18.615	28.3	9.0	19.3	19.13	19.30
1996	982.553	26.824	8.745	18.079	27.3	8.9	18.4	18.73	18.40
1997	1000.558	27.215	8.905	18.310	27.2	8.9	18.3	18.32	18.29
1998	1018.471	26.989	9.166	17.823	26.5	9.0	17.5	17.90	17.49
1999	1036.259	26.943	9.015	17.928	26.0	8.7	17.3	17.46	17.30
2000	1053.898	27.191	8.958	18.233	25.8	8.5	17.3	17.02	17.30
2001	1071.374	27.213	9.000	18.213	25.4	8.4	17.0	16.58	16.99
2002	1088.694	27.217	8.818	18.399	25.0	8.1	16.9	16.16	16.90
2003	1105.886	27.426	8.847	18.579	24.8	8.0	16.8	15.79	16.80
2004	1122.991	27.064	8.422	18.642	24.1	7.5	16.6	15.46	16.60
2005	1140.043	27.133	8.664	18.469	23.8	7.6	16.2	15.18	16.20
2006	1157.039	27.190	8.678	18.512	23.5	7.5	16.0	14.90	15.99
2007	1173.972	27.119	8.687	18.432	23.1	7.4	15.7	14.63	15.70
2008	1190.864	27.152	8.812	18.340	22.8	7.4	15.4	14.38	15.40
2009	1207.740	27.174	8.817	18.357	22.5	7.3	15.2	14.17	15.19
2010	1224.614	27.064	8.817	18.247	22.1	7.2	14.9	13.97	14.90
2011	1242.738	27.092	8.823	18.268	21.8	7.1	14.7	14.79	14.69
2012	1261.006	27.237	8.827	18.410	21.6	7.0	14.6	14.69	14.59
2013	1279.416	27.3795	8.956	18.4236	21.4	7.0	14.4	14.59	14.20

Sources: United Nations Statistics Division website, Demographic Yearbooks, Census of India website, Report of Vital Statistics Sample Registration System in India and calculate some rows and columns by above formulas.

5. LOGISTIC MODEL IN EXPONENTIAL POPULATION GROWTH

Malthus's model is unconstrained growth, i.e. model in which the population increases in size without bound. It is an exponential growth model governed by a differential equation of the form

$$(5.1) \quad \frac{dp}{dt} = ap \Rightarrow \frac{1}{p} \frac{dp}{dt} = a \text{ (Constant)}$$

The equation is exponential growth model

$$(5.2) \quad p = (p_0 e^{-at_0}) e^{at} = ce^{at}$$

Where $c = p_0 e^{-at_0}$ is a constant for the constant a . Therefore, the population number p increases to infinity as time t goes to infinity.

In 1840 a Belgian Mathematician Verhulst modified Malthus's Model to proposed a new model which is,

$$(5.3) \quad \frac{dp}{dt} = ap \left(1 - \frac{p}{m} \right)$$

Where $a > 0$ expresses population growth rate, and $m > 0$ is called the carrying capacity or the maximum supportable population. This equation is also known as a logistic difference equation.

6. THE SOLUTION OF THE LOGISTIC MODEL IN EXPONENTIAL POPULATION GROWTH

We may account for the growth rate declining to 0 by including in the exponential model a factor of a and p which is close to 1 (i.e., has no effect) when p is much smaller than a , and which is close to 0 when p is close to a .

The constant solutions are $p = 0$ and $p = m$, the non-constant solutions may obtained by separating the variables.

$$(6.1) \quad \frac{dp}{p \left(1 - \frac{p}{m} \right)} = adt$$

Taking indefinite integration for the two sides of equation (6.1)

$$(6.2) \quad \int \frac{dp}{p \left(1 - \frac{p}{m} \right)} = \int adt$$

The partial fraction techniques give

$$(6.3) \quad \int \frac{dp}{p \left(1 - \frac{p}{m} \right)} = \int \left(\frac{1}{p} + \frac{\frac{1}{m}}{1 - \frac{p}{m}} \right) dp$$

This gives

$$(6.4) \quad \ln|p| - \ln \left| 1 - \frac{p}{m} \right| = at + c$$

Easy algebraic manipulations give

$$(6.5) \quad \frac{p}{\left(1 - \frac{p}{m} \right)} = ce^{at}$$

Where c is constant and solving for p , we get

$$(6.6) \quad P = \frac{mce^{at}}{m + ce^{at}}$$

If we consider the initial condition $p(0) = p_0$ (assuming that p_0 is not equal to both 0 or m), we get

$$(6.7) \quad c = \frac{p_0 m}{m - p_0}$$

Which, once substituted into the expression for $p(t)$ and simplified, we find

$$(6.8) \quad p(t) = \frac{mp_0}{p_0 + (m - p_0)e^{-at}}$$

It is easy to see that

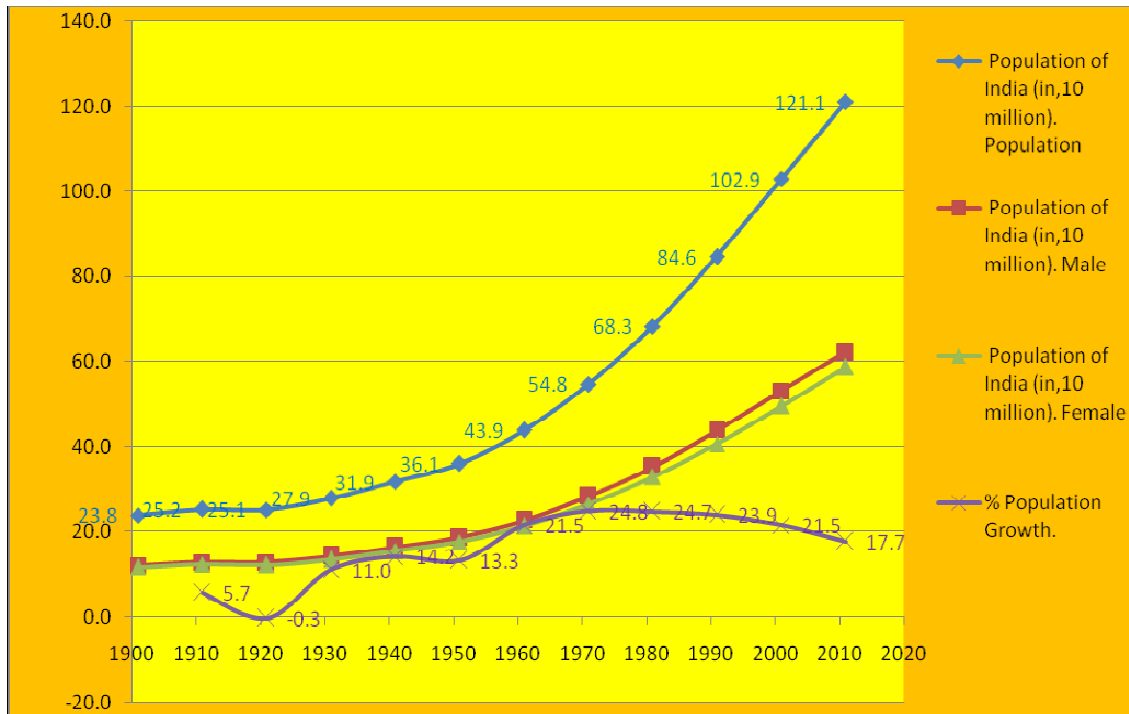
$$(6.9) \quad \lim_{t \rightarrow +\infty} p(t) = m$$

As you can see, when the population starts to grow, it does go through an exponential growth phase, but as it gets closer to the carrying capacity, the growth slows down and it reaches a stable level. This slow down to a carrying capacity is perhaps the result of war, pestilence, and starvation as more and more people contend for the resources that are now at their upper bound. There are many examples in nature that show that when the environment is stable the maximum number of individuals in a population fluctuates near the carrying capacity of the environment. However, if the environment becomes unstable, the population size can have dramatic changes.

7. HUMAN POPULATION DYNAMICS AND GROWTH IN INDIA

As the general solution and we use this to population of India from 1901 to 2021. Basic population growth trend is based on decadal Censuses of India as given below:

Population in India (in 10million)				
Year	Population	Male	Female	% Population Growth
1901	23.8	12.1	11.7	-
1911	25.2	12.8	12.4	5.7
1921	25.1	12.9	12.3	-0.3
1931	27.9	14.3	13.6	11.0
1941	31.9	16.4	15.5	14.2
1951	36.1	18.6	17.6	13.3
1961	43.9	22.6	21.3	21.5
1971	54.8	28.4	26.4	24.8
1981	68.3	35.3	33.0	24.7
1991	84.6	43.9	40.7	23.9
2001	102.9	53.2	49.7	21.5
2011	102.1	62.3	58.8	17.7
2021	133.9	69.4	64.5	23.7



8. THE COMPOSITION OF POPULATION

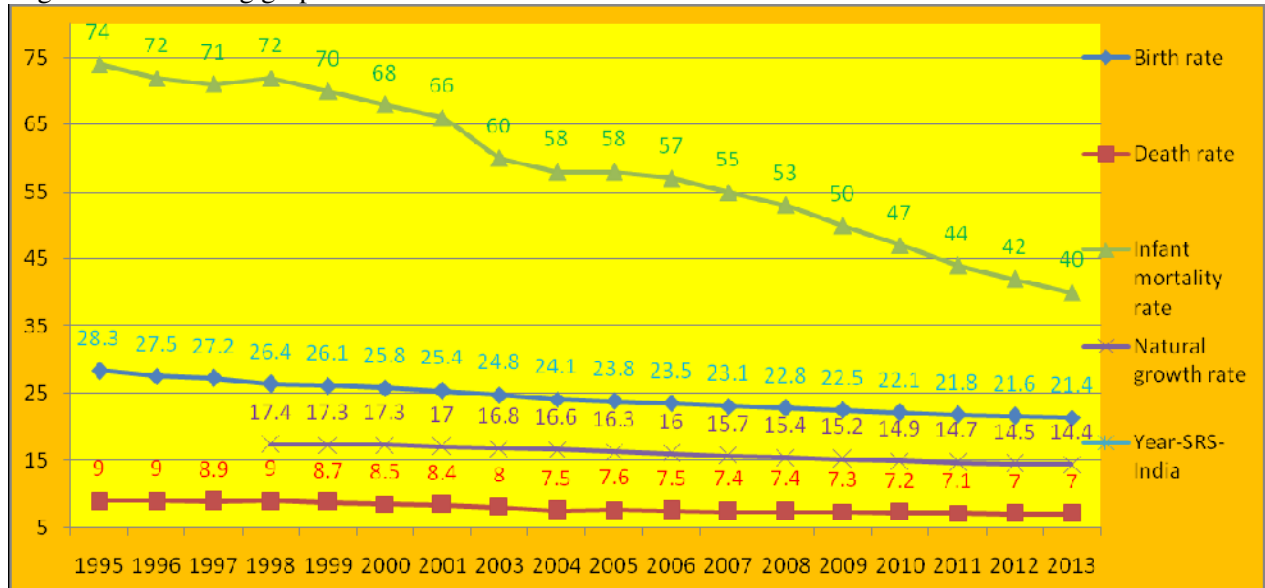
The population dynamics includes birth rates, death rates, immigration, and emigration age and sex composition. Birth and death rates, immigration and emigration are the four primary ecological events that influence the size of a population. This relationship can be expressed in a simple equation:

8.1 Change in population = (Birth + Immigration) - (Death + Emigration)

Birth and death rates are the most important determinants of population growth; in some countries net migration is also important. Until the mid-19th century birth rates were slightly higher than death rates, so the human population grew very slowly. Demographic profiles of the Indian states vary from region to region and state to state. Thus, heterogeneity in demographic profiles of the Indian states affects its population composition. Heterogeneity in demographic profiles of the six southern states (Andhra Pradesh, Karnataka, Kerala, Tamil Nadu, Maharashtra and Goa) and eight empowered action group (EAG) states (viz. Rajasthan, Uttar Pradesh, Uttarakhand, Bihar, Jharkhand, Madhya Pradesh, Chhattisgarh and Orissa) are very significant.

9. FERTILITY AND MORTALITY TRENDS AND DEMOGRAPHIC INDICATORS IN INDIA

Fertility and mortality trend, infant mortality and natural growth rate of India since 1995 to 2013, based on SRS data is given in following graph.



10. DISCUSSION:

In this study a mathematical analysis of the population growth in India is carried out based on an ordinary differential equation model which is called logistic model. Then we establish a non-linear model that gives population for India at the time from 1901 to 2021. To make the non-linear model we use the least square interpolation of growth rate. We examine logistic model, using mathematical techniques of differentiation and integration, we exactly reach the explicit solutions for logistic model. Experiences of a considerable number of countries, especially those, which are less developed, can be speak that over population will but lead to severe problems such as slowing development of economy, instability or even collapse of social systems, vicious circle of poverty and environmental degradation and pollution. Our study, though very limited and shallow, showed that family planning programs have benefited the whole country drastically and even avoided some terrible social or environmental disasters. The success of family planning programs in India to make rigid policies by the Government, then to deserves attention from other developing countries that also face the problem of massive population and its rapid growth and the negative impacts have already affected the whole world. So I suggest a global population program should be planned and launched as an important part of sustainable development which emphasis an ideal relationship among population growth, economic development and environmental protection.

REFERENCES

1. Martin B. (1993): Differential Equation and their Application, Narosa publishing House.
2. Fred Brauer Carols Castillo-Chavez (2001): Mathematical Models in Population Biology and Epidemiology, Springer.

3. Hoque, M. A., Ahmed, F. and Sarker, M. M. (1995): An Estimation of future population of Bangladesh, Journal of Statistical Studies of Jahangir nagar University, vol. 15 pp. 75-88.
4. Basher, M. M. A. (1985): Modern Population Trends in Bangladesh, Rural Demography, vol. XII (1 & 2) pp. 71-83.
5. Beekman, J. A. (1984): Population and Social Security Projection Techniques, Rural Demography, vol. XI (1 & 2) pp. 1-20.
6. Kabir, M. and Chowdhury, A. A. (1982): Population Growth and Food Production in Bangladesh”, Rural Demography, vol. IX (1 & 2) pp. 25-56.
7. Mallick, A. A. (1980): Implausibility of Attaining Zero Population Growth in Banthin Next 100 Years, Rural Demography, VII (1 & 2) pp. 33-39.
8. Obaidullah, M. (1976): Expo-linear Model for Population Growth, Rural Demography, vol. III (1&2) pp. 43-69.
9. Rahman, M. M. (1993): National Growth Rate Method with Varying Internal Migration Rate, The Bangladesh Development Studies, vol. XXI (2) pp. 73-76.
10. Indian Population Census-2009: Bureau of Statistics, Government of India, Delhi.
11. Ghulam Rabbani, A. K. M. and Md. Shahadat Hossain (1981): Population projection of Bangladesh, (1975-2025), B.B.S.
12. Beekman, J. A. (1981): Several Demographic Projection Techniques, Rural Demography, vol. VIII(1) pp. 1-11.
13. Augustus, W. 2012: Mathematical Modeling of Uganda Population Growth, Applied Mathematical Sciences, Vol. 6, , no. 84, pp. 4155 – 4168.
14. Fasagba, I., Adeoye, A.O., Babalola, J. B. and Kuranga, J.L (2014): Modeling of population projection and symptomatic estimated population in Nigeria, IPASJ International Journal of Management, vol. 2, and issue 1.
15. UNDP (1997): Human Development Report, Oxford University Press.
16. Francis A. Lees (1997): China Superpower, Requisites for High Growth, New York: St. Martin’s Press.



COMMON FIXED POINT THEOREMS FOR OCCASIONALLY WEAKLY COMPATIBLE MAPPINGS SATISFYING IMPLICIT RELATION IN MODULAR METRIC SPACES

Prerna Pathak^{*}, Aklesh Pariya^{**}, V. H. Badshah^{***} & Nirmala Gupta^{****}

^{*} & ^{***}School of Studies in Mathematics Vikram University, Ujjain (M.P.), India

^{**}Department of Mathematics, Medi-caps University, Indore (M.P.), India

^{****}Department of Mathematics, Govt. Girls Degree College, Ujjain (M.P.), India

E-mail: prernapathak20@yahoo.com, akleshpariya3@yahoo.co.in, gupta.nirmala70@gmail.com

ABSTRACT :

In this paper, we proved some common fixed point theorems for two pairs of occasionally weakly compatible (owc) mappings satisfying implicit relations in modular metric space. The study is an extension and generalization to the common fixed point theorems for occasionally weakly compatible mappings satisfying implicit relation in modular metric spaces.

MSC: Primary 47H10, Secondary 54H25.

Keywords: Common fixed point, implicit relations, occasionally compatible mapping.

1. INTRODUCTION.

Chistyakov [6, 8] introduced the concept of modular metric spaces. Recently, many authors [7, 12, 13, 17] proved fixed point theorems in modular metric spaces. Jungck [11] generalized the Banach contraction principle by using the notion of commuting mappings. Sessa [18] defined weak commutativity and prove fixed point theorem for weakly commuting maps. Jungck [10] introduced less restrictive concept from weak commutativity which termed as compatibility and discussed few common fixed point theorems in complete metric space. The concept of property E.A. in metric space has been introduced by Aamri et. al. [1]. Al-Thagafi and Shahzad [4] introduced the concept of occasionally weakly compatible (owc) mappings. Abdou & khamsi [2] proved fixed point results for point wise contraction mappings in modular metric spaces. Alfuraidan [3] gave a generalization of the Banach contraction principle on a modular metric space endowed with a graph. Recently, Pathak et al. [16] proved the result of fixed point theorems for contraction type mappings in modular metric spaces.

Pariya et. al. [15] proved the result of some unique common fixed point theorems for generalized contraction type mappings for six self owc mappings in modular metric spaces. In this paper, we proved common fixed point theorems for occasionally weakly compatible mappings satisfying implicit relation using property E.A. and common property E.A. in modular metric spaces.

2. METHODS, MATERIALS, BASIC DEFINITIONS AND PRELIMINARIES :

Let X be a non-empty set, $\lambda \in (0, \infty)$ and due to the disparity of the arguments, function $\omega : (0, \infty) \times X \times X \rightarrow [0, \infty]$ will be written as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $\lambda > 0$ and $x, y \in X$.

Definition 2.1. Let X be a non-empty set. A function $\omega : (0, \infty) \times X \times X \rightarrow [0, \infty]$ is said to be a metric modular on X if it satisfies the following three axioms:

(i) given $x, y \in X, \omega_\lambda(x, y) = 0$ for all $\lambda > 0$ if and only if $x = y$;

- (ii) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0$ and $x, y \in X$;
- (iii) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0$ and $x, y, z \in X$.

If instead of (i), we have only the condition

- (i') $\omega_\lambda(x, x) = 0$ for all $\lambda > 0$ and $x \in X$, then ω is said to be a (metric) pseudo modular on X and if ω satisfies (i') and (i_s) given $x, y \in X$, if there exists a number $\lambda > 0$, possibly depending on x and y , such that $\omega_\lambda(x, y) = 0$, then $x = y$, with this condition ω is called a strict modular on X .

Definition 2.2. [14] Let X_ω be a modular metric space.

- (1) The sequence $(x_n)_{n \in \mathbb{N}}$ in X_ω is said to be convergent to $x \in X_\omega$ if $\omega_\lambda(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$ for all $\lambda > 0$.
- (2) The sequence $(x_n)_{n \in \mathbb{N}}$ in X_ω is said to be Cauchy if $\omega_\lambda(x_m, x_n) \rightarrow 0$ as $m, n \rightarrow \infty$ for all $\lambda > 0$.
- (3) A subset C of X_ω is said to be closed if the limit of the convergent sequence of C always belong to C .
- (4) A subset C of X_ω is said to be complete if any Cauchy sequence in C is a convergent sequence and its limit in C .
- (5) A subset C of X_ω is said to be bounded if for all $\lambda > 0$ $\delta_\omega(C) = \sup\{\omega_\lambda(x, y); x, y \in C\} < \infty$.

We recall the following definitions in metric spaces.

Definition 2.3. Two self mappings S and T of a metric space (X, d) are said to be weakly commuting if $d(STx, TSx) \leq d(Sx, Tx), \forall x \in X$.

It is clear that two commuting mappings are weakly commuting, but the converse is not true.

Definition 2.4 [11]. Let T and S be two self mappings of a metric space (X, d) . S and T are said to be compatible if

$$\lim_{n \rightarrow \infty} d(STx_n, TSx_n) = 0, \text{ whenever } \{x_n\} \text{ is a sequence in } X \text{ such that } \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = t, \text{ for some } t \in X.$$

Definition 2.5. Let X be a set, f, g self maps of X . A point x in X is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$, a point of coincidence of f and g .

Definition 2.6. Two maps S and T are said to be weakly compatible if they commute at coincidence points.

Definition 2.7. Let S and T be two self mappings of a metric space (X, d) . We say that T and S satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n = t ; \text{ for some } t \in X.$$

Definition 2.8. Two pairs of self maps (I, S) and (J, T) of a metric space (X, d) . We say that I, J, S, T satisfy the common property (E.A) if there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Ix_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} Jy_n = t ; \text{ for some } t \in X.$$

Definition 2.9. Two self-maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is a coincidence point of f and g at which f and g commute.

We shall also need the following lemma from Jungck and Rhoades [7].

Lemma 2.1. Let X be a set, f, g owc self-maps of X . If f and g have a unique point of coincidence, $w := fx = gx$, then w is a unique common fixed point of f and g .

Thus we define the above definitions in modular metric spaces as-

Definition 2.10. Let X_ω be a modular metric space. Let f, g self maps of X_ω . A point x in X_ω is called a coincidence point of f and g iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of f and g .

Definition 2.11. Two self mappings S and T of a metric space (X, ω) are said to be weakly commuting if $\omega_\lambda(STx, TSx) \leq \omega_\lambda(Sx, Tx), \forall x \in X_\omega$.

Definition 2.12. Let X_ω be a modular metric space. Two maps f and g of X_ω are said to be weakly compatible if they commute at coincidence points.

Definition 2.13. Let X_ω be a modular metric space. Two self maps f and g of X_ω are occasionally weakly compatible (owc) iff there is a point x in X_ω which is a coincidence point of f and g at which f and g commute.

Definition 2.14. Let X_ω be a modular metric space. Let S and T be two self maps of X_ω , then S and T satisfy the property (E.A.) if there exist a sequence $\{x_n\}$ such that

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n = t ; \text{ for some } t \in X_\omega .$$

Definition 2.15. Let X_ω be a modular metric space. Let the two pairs of self maps (I, S) and (J, T) satisfy the common property (E.A.) if there exist a sequence $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n \rightarrow \infty} Ix_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} Jy_n = z \quad ; \text{ for some } z \in X_\omega .$$

Definition 2.16. [2] Let X_ω be a modular metric space induced by metric modular ω . Two self mapping f, g of X_ω are ω -compatible if $\omega_\lambda(fgx_n, gfx_n) \rightarrow 0$, whenever $\{x_n\}_{n=1}^\infty$ is a sequence in X_ω such that $gx_n \rightarrow z$ and $Tx_n \rightarrow z$ for some point $z \in X_\omega$ and for $\lambda > 0$.

Lemma 2.2. Let X_ω be a modular metric space and f, g owc self-maps of X_ω . If f and g have a unique point of coincidence, $w := fx = gx$, then w is a unique common fixed point of f and g .

Definition 2.17. Two finite families of self maps $\{I_i\}_{i=1}^m$ and $\{J_j\}_{j=1}^n$ on a set X_ω are pairwise commuting

- (i) $I_i I_j = I_j I_i \quad i, j \in \{1, 2, \dots, m\}$
- (ii) $J_i J_j = J_j J_i \quad i, j \in \{1, 2, \dots, n\}$
- (iii) $I_i J_j = J_j I_i \quad i \in \{1, 2, \dots, m\}, j \in \{1, 2, \dots, n\}$

3. IMPLICIT RELATIONS. Let F_6 be the set of all continuous functions satisfying the following conditions:

- (A₁) $\emptyset(u, 0, u, 0, 0, u) \leq 0 \Rightarrow u \leq 0$
- (A₂) $\emptyset(u, 0, 0, u, u, 0) \leq 0 \Rightarrow u \leq 0$
- (A₃) $\emptyset(u, u, 0, 0, u, u) \leq 0 \Rightarrow u \leq 0 ; \text{ for all } 0 < u$.

Example

3.1 Define $\emptyset(t_1, t_2, t_3, t_4, t_5, t_6): \mathcal{R}_+^6 \rightarrow \mathcal{R}$ as

$$\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \frac{t_3 t_5 + t_4 t_6}{t_3 + t_4} - bt_2 \min\{t_2, t_3, t_4\}, \quad \text{where } a, \geq 0.$$

3.2 Define $\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6): \mathcal{R}_+^6 \rightarrow \mathcal{R}$ as

$$\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}, \quad k \in (0, 1).$$

3.3 Define $\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6): \mathcal{R}_+^6 \rightarrow \mathcal{R}$ as

$$\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{1}{2}t_5 + \frac{1}{2}t_6\}, \quad k \in (0, 1).$$

3.4 Define $\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6): \mathcal{R}_+^6 \rightarrow \mathcal{R}$ as

$$\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{t_2, \frac{1}{2}[t_3 + t_4], \frac{1}{2}[t_5, t_6]\}, \quad k \in (0, 1).$$

3.5 Define $\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6): \mathcal{R}_+^6 \rightarrow \mathcal{R}$ as

$$\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \frac{k}{2} \max\{t_2, t_3, t_4, \frac{1}{2}t_5, \frac{1}{2}t_6\}, \quad k \in (0, 1).$$

3.6 Define $\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6): \mathcal{R}_+^6 \rightarrow \mathcal{R}$ as

$$\mathbf{F}(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - t_2 + \frac{t_3 t_4 + t_5 t_6}{t_3 + t_4}, \quad k \in (0, 1)$$

3.7 Define $F(t_1, t_2, t_3, t_4, t_5, t_6): \mathcal{R}_+^6 \rightarrow \mathcal{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min \{t_2, t_3 + t_5, t_4 + t_6\}.$$

3.8 Define $F(t_1, t_2, t_3, t_4, t_5, t_6): \mathcal{R}_+^6 \rightarrow \mathcal{R}$ as

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \min \{t_2, t_3, t_5, t_4, t_6\}.$$

4. OBSERVATIONS, RESULTS AND DISCUSSION

Theorem 4.1. Let (X, ω) be a complete modular metric space and $I, J, S, T: X_\omega \rightarrow X_\omega$ be self-mappings satisfying the conditions:

(4.1.1) Then (I, S) and (J, T) share common property (E.A.);

(4.1.2) for any $x, y \in X_\omega, \emptyset$ in F_6

$$\emptyset(\omega_\lambda(Sx, Ty)\omega_\lambda(Ix, Jy), \omega_\lambda(Sx, Ix), \omega_\lambda(Ty, Jy), \omega_\lambda(Sx, Jy)\omega_{2\lambda}(Ty, Ix)) \leq 0$$

(4.1.3) $I(X_\omega) \subset T(X_\omega)$ and $J(X_\omega) \subset S(X_\omega)$

Then the pairs (I, S) and (J, T) share common (E.A.) property.

Proof. Suppose the pair (I, S) satisfy the property E.A., then there exist a sequence $\{x_n\}$ in X_ω such that

$$\lim_{n \rightarrow \infty} Ix_n = \lim_{n \rightarrow \infty} Sx_n = z \quad \text{for some } z \in X_\omega$$

Since $I(X_\omega) \subset T(X_\omega)$, hence for each $\{x_n\}$ there exist $\{y_n\}$ in X_ω such that $x_n = Ty_n$.

Therefore, $\lim_{n \rightarrow \infty} Ix_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ty_n = z$.

Now we claim that $\lim_{n \rightarrow \infty} Jy_n = z$.

Suppose that $\lim_{n \rightarrow \infty} Jy_n \neq z$, then applying inequality (4.1.2), we obtain

$$\begin{aligned} &\emptyset(\omega_\lambda(Ix_n, Jy_n), \omega_\lambda(Sx_n, Ty_n), \omega_\lambda(Sx_n, Ix_n), \omega_\lambda(Ty_n, Jy_n), \omega_\lambda(Sx_n, Jy_n), \omega_\lambda(Ty_n, Ix_n)) \leq 0 \\ &\emptyset(\omega_\lambda(z, \lim_{n \rightarrow \infty} Jy_n), \omega_\lambda(z, z), \omega_\lambda(z, z), \omega_\lambda(z, \lim_{n \rightarrow \infty} Jy_n), \omega_\lambda(z, \lim_{n \rightarrow \infty} Jy_n), \omega_\lambda(z, z)) \leq 0 \\ &\emptyset(\omega_\lambda(z, \lim_{n \rightarrow \infty} Jy_n), 0, 0, \omega_\lambda(z, \lim_{n \rightarrow \infty} Jy_n), \omega_\lambda(z, \lim_{n \rightarrow \infty} Jy_n), 0) \leq 0, \end{aligned}$$

which is a contradiction using (A_2) ; we get

$$\omega_\lambda(z, \lim_{n \rightarrow \infty} Jy_n) \leq 0$$

and therefore $\lim_{n \rightarrow \infty} Jy_n = z$.

Hence, the pairs (I, S) and (J, T) share the common property (E.A.).

Theorem 4.2. Let (X, ω) be a complete modular metric space and $I, J, S, T: X_\omega \rightarrow X_\omega$ be self-mappings satisfying the conditions (4.1.2) and

(4.2.1) the pair (I, S) and (J, T) share common (E.A.) property,

(4.2.2) $S(X_\omega)$ and $T(X_\omega)$ are closed subsets of X_ω .

Then the pairs (I, S) and (J, T) have a point of coincidence each. Moreover I, S, J, T have a unique common fixed point provided both the pairs (I, S) and (J, T) are weakly compatible.

Proof. Suppose the pair (I, S) and (J, T) satisfies common property (E.A.) there exist sequences $\{x_n\}, \{y_n\}$ in X_ω such that

$$\lim_{n \rightarrow \infty} Ix_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ty_n = \lim_{n \rightarrow \infty} Jy_n = z \quad \text{for some } z \in X_\omega$$

Since $S(X_\omega)$ is a closed subset of X_ω , therefore there exists a point $u \in X_\omega$ such that $z = Su$.

We claim that $Iu = z$.

If $Iu \neq z$, then by condition (4.1.2), take $x = u, y = y_n$

$$\emptyset(\omega_\lambda(Iu, Jy_n), \omega_\lambda(Su, Ty_n), \omega_\lambda(Su, Iu), \omega_\lambda(Ty_n, Jy_n), \omega_\lambda(Su, Jy_n), \omega_\lambda(Ty_n, Iu)) \leq 0$$

taking the limit as $n \rightarrow \infty$, we get

$$\emptyset(\omega_\lambda(Iu, z), \omega_\lambda(z, z), \omega_\lambda(z, Iu), \omega_\lambda(z, z), \omega_\lambda(z, z), \omega_\lambda(z, Iu)) \leq 0$$

$$\emptyset(\omega_\lambda(Iu, z), 0, \omega_\lambda(Iu, z), 0, 0, \omega_\lambda(Iu, z)) \leq 0$$

using (A_1) we get $\omega_\lambda(Iu, z) \leq 0$.

which is a contradiction.

Therefore $Iu = z = Su$, which shows that u is a coincidence point of the pair (I, S) .

Since $T(X_\omega)$ is a closed subset of X_ω , therefore $\lim_{n \rightarrow \infty} Ty_n = z$ in $T(X_\omega)$ and hence there exists a point $v \in X_\omega$ such

that $Tv = z = Iu = Su$.

Now we show that $Jv = z$.

If $Jv \neq z$, then by using (4.1.2), take $x = u, y = v$ we have

$$\emptyset(\omega_\lambda(Iu, Jv), \omega_\lambda(Su, Tv), \omega_\lambda(Su, Iu), \omega_\lambda(Tv, Jv), \omega_\lambda(Su, Jv), \omega_\lambda(Tv, Iu)) \leq 0$$

$$\emptyset(\omega_\lambda(z, Jv), \omega_\lambda(z, z), \omega_\lambda(z, z), \omega_\lambda(z, Jv), \omega_\lambda(z, Jv), \omega_\lambda(z, z)) \leq 0$$

$$\emptyset(\omega_\lambda(z, Jv), 0, 0, \omega_\lambda(z, Jv), \omega_\lambda(z, Jv), 0) \leq 0$$

using (A_2) we get $\omega_\lambda(z, Jv) \leq 0$.

which is a contradiction.

Hence $Jv = z = Tv$, which shows that v is a coincidence point of the pair (J, T) .

Since the pairs (I, S) and (J, T) are weakly compatible and $Iu = Su = Jv = Tv$, therefore

$$Iz = Isu = SJu = Sz, Jz = JTv = TJv = Tz.$$

If $Iz \neq z$, then by using inequality (4.1.2), we have

$$\emptyset(\omega_\lambda(Iz, Jv), \omega_\lambda(Sz, Tv), \omega_\lambda(Sz, Iz), \omega_\lambda(Tv, Jv), \omega_\lambda(Sz, Jv), \omega_\lambda(Tv, Iz)) \leq 0$$

$$\emptyset(\omega_\lambda(Iz, z), \omega_\lambda(Iz, z), \omega_\lambda(Iz, Iz), \omega_\lambda(Jv, Jv), \omega_\lambda(Iz, z), \omega_\lambda(z, Iz)) \leq 0$$

$$\emptyset(\omega_\lambda(Iz, z), \omega_\lambda(Iz, z), 0, 0, \omega_\lambda(Iz, z), \omega_\lambda(Iz, z)) \leq 0$$

using (A_3) we get $\omega_\lambda(Iz, z) \leq 0$.

which is a contradiction.

Hence $Iz = z = Sz$.

Similarly, one can prove that $Jz = Tz = z$. Hence, $Iz = Jz = Sz = Tz$ and z is a common fixed point of I, J, S, T .

Uniqueness. Let z and w be two common fixed point of I, J, S, T .

If $z \neq w$, then by using (4.1.2), we have

$$\emptyset(\omega_\lambda(Iz, Jw), \omega_\lambda(Sz, Tw), \omega_\lambda(Sz, Iz), \omega_\lambda(Tw, Jw), \omega_\lambda(Sz, Jw), \omega_\lambda(Tw, Iz)) \leq 0$$

$$\emptyset(\omega_\lambda(z, w), \omega_\lambda(z, w), \omega_\lambda(z, z), \omega_\lambda(w, w), \omega_\lambda(z, w), \omega_\lambda(w, z)) \leq 0$$

$$\emptyset(\omega_\lambda(z, w), \omega_\lambda(z, w), 0, 0, \omega_\lambda(z, w), \omega_\lambda(z, w)) \leq 0$$

using (A_3) we get $\omega_\lambda(z, w) \leq 0$.

which is a contradiction.

Therefore, $z = w$.

Theorem 4.3. Let (X, ω) be a complete modular metric space and I, J, S, T be self-mappings satisfying the conditions (4.1.2). If the pair (I, S) and (J, T) are owc, then I, J, S, T have a unique common fixed point.

Proof. Since the pair (I, S) and (J, T) are occasionally weakly compatible then there exist $u, v \in X_\omega$.

Such that $Su = Iu$ and $Jv = Tv$

Now we can assert that $Su = Tv$, if not then by (4.1.2)

$$\emptyset(\omega_\lambda(Su, Tv), \omega_\lambda(Iu, Jv), \omega_\lambda(Su, Iu), \omega_\lambda(Tv, Jv), \omega_\lambda(Su, Jv), \omega_\lambda(Tv, Iu)) < 0$$

$$\emptyset(\omega_\lambda(Iu, Tv), \omega_\lambda(Iu, Jv), \omega_\lambda(Iu, Iu), \omega_\lambda(Jv, Jv), \omega_\lambda(Iu, Jv), \omega_\lambda(Jv, Iu)) < 0$$

$$\emptyset(\omega_\lambda(Iu, Tv), \omega_\lambda(Iu, Jv), 0, 0, \omega_\lambda(Iu, Jv), \omega_{2\lambda}(Jv, Iu)) < 0; \text{ a contradiction of } (A_3).$$

Hence $Su = Tv$ and thus $Su = Iu = Tv = Jv$ (4.3.1)

Moreover, if there is another fixed point of coincidence z such that $Sz = Iz$, and using condition (4.1.2).

$$\begin{aligned} \emptyset(\omega_\lambda(Sz, Tv), \omega_\lambda(Iz, Jv), \omega_\lambda(Sz, Iz), \omega_\lambda(Tv, Jv), \omega_\lambda(Sz, Jv), \omega_\lambda(Tv, Iz)) &< 0 \\ \emptyset(\omega_\lambda(Sz, Tv), \omega_\lambda(Sz, Jv), \omega_\lambda(Sz, Sz), \omega_\lambda(Tv, Tv), \omega_\lambda(Sz, Tv), \omega_\lambda(Tv, Sz)) &< 0 \\ \emptyset(\omega_\lambda(Sz, Tv), \omega_\lambda(Sz, Jv), 0, 0, \omega_\lambda(Sz, Tv), \omega_{2\lambda}(Tv, Sz)) &< 0 \end{aligned}$$

Again a contradiction of (A_3) .

Hence we get

$$Sz = Iz = Tv = Jv \tag{4.3.2}$$

Thus from equation (1) and (2) it follows that $Sz = Su$. This implies $z = u$.

Hence $z = Su = Iu$ for some $z \in X_\omega$ is the coincidence point of S and I .

Then by lemma 2.2, z is a unique common fixed point of S and I .

Hence $Sz = Iz = z$.

Similarly, there is a another common fixed point $v \in X_\omega : v = Tv = Jv$

Suppose $v \neq z$, then by (4.1.2) we have

$$\begin{aligned} \emptyset(\omega_\lambda(Sz, Tv), \omega_\lambda(Iz, Jv), \omega_\lambda(Sz, Iz), \omega_\lambda(Tv, Jv), \omega_\lambda(Sz, Jv), \omega_\lambda(Tv, Iz)) &< 0 \\ \emptyset(\omega_\lambda(Sz, Tv), \omega_\lambda(z, v), \omega_\lambda(z, z), \omega_\lambda(v, v), \omega_\lambda(z, v), \omega_\lambda(v, z)) &< 0 \\ \emptyset(\omega_\lambda(Sz, Tv), \omega_\lambda(z, v), 0, 0, \omega_\lambda(z, v), \omega_\lambda(v, z)) &< 0 \end{aligned}$$

Again a contradiction of (A_3) .

Hence z is a unique common fixed point of I, J, S, T .

Theorem 4.4 let $\{I_1, I_2, \dots, I_m\}, \{J_1, J_2, \dots, J_n\}, \{S_1, S_2, \dots, S_p\}, \{T_1, T_2, \dots, T_q\}$ be four finite families of self maps of a modular metric spaces X_ω such that $I = I_1 \cdot I_2 \cdot \dots \cdot I_m, J = J_1 \cdot J_2 \cdot \dots \cdot J_n, S = S_1 \cdot S_2 \cdot \dots \cdot S_p, T = T_1 \cdot T_2 \cdot \dots \cdot T_q$ satisfy the condition (4.1.2). Moreover finite family of self maps I_i, S_k, J_r and T_t have a unique common fixed point provided that the pairs of families $(\{I_i\}, \{S_k\})$ and $(\{J_r\}, \{T_t\})$ are owc for all $i = 1, 2, \dots, m, k = 1, 2, \dots, p, r = 1, 2, \dots, n, t = 1, 2, \dots, q$.

Proof. Since self maps I, S, J, T satisfy all the conditions of theorem 4.3, the pairs (I, S) and (J, T) are owc and have a unique common fixed point. Also the pairs of families $(\{I_i\}, \{S_k\})$ and $(\{J_r\}, \{T_t\})$ are commute pairwise, we first show that $IS = SI$ as

$$\begin{aligned} IS &= (I_1 I_2 \dots I_m)(S_1 S_2 \dots S_p) = (I_1 I_2 \dots I_{m-1})(I_m S_1 S_2 \dots S_p) \\ &= (I_1 I_2 \dots I_{m-1})(S_1 S_2 \dots S_p I_m) = (I_1 I_2 \dots I_{m-2})(I_{m-1} S_1 S_2 \dots S_p I_m) \\ &= (I_1 I_2 \dots I_{m-2})(S_1 S_2 \dots S_p I_{m-1} I_m) = \dots = I_1(S_1 S_2 \dots S_p I_2 \dots I_m) \\ &= (S_1 S_2 \dots S_p)(I_1 I_2 \dots I_m) = SI. \end{aligned}$$

Similarly one can prove that $JT = TJ$ and hence, obviously the pair (I, S) and (J, T) are occasionally weakly compatible.

Now using theorem 4.3, we conclude that I, S, J, T have a unique common fixed point in X_ω , say z .

Now, one needs to prove that z remains the fixed point of all the component maps.

For this consider

$$\begin{aligned} I(I_i z) &= ((I_1 I_2 \dots I_m) I_i) z = (I_1 I_2 \dots I_{m-1})(I_m I_i) z \\ &= (I_1 I_2 \dots I_{m-1})(I_i I_m) z = (I_1 I_2 \dots I_{m-2})(I_{m-1} I_i I_m) z \\ &= (I_1 I_2 \dots I_{m-2})(I_i I_{m-1} I_m) z = I_1(I_i I_2 \dots I_m) z \\ &= (I_1 I_i)(I_2 \dots I_m) z \\ &= (I_i I_1)(I_2 \dots I_m) z = I_i(I_1 I_2 \dots I_m) z = I_i I z = I_i z. \end{aligned}$$

Similarly, one can prove that

$$\begin{aligned} I(S_k z) &= S_k(Iz) = S_k z, S(S_k z) = S_k(Sz) = S_k z, \\ S(I_i z) &= I_i(Sz) = I_i z, J(J_r z) = J_r(Jz) = J_r z, \end{aligned}$$

$$J(T_t z) = T_t(Jz) = T_t z, T(T_t z) = T_t(Tz) = T_t z,$$

and

$$T(J_r z) = J_r(Tz) = J_r z,$$

which shows that (for all i, r, k and t) $I_i z$ and $S_k z$ are other fixed point of the pair (I, S) whereas $J_r z$ and $T_t z$ are other fixed point of the pair (J, T) .

As I, J, S and T have a unique common fixed point, so, we get

$$z = I_i z = S_k z = J_r z = T_t z, \text{ for all } i = 1, 2, \dots, m, \quad k = 1, 2, \dots, p, \\ r = 1, 2, \dots, n, \quad t = 1, 2, \dots, q,$$

which shows that z is a unique common fixed point of $\{I_i\}_{i=1}^m, \{S_k\}_{k=1}^p, \{J_r\}_{r=1}^n$ and $\{T_t\}_{t=1}^q$.

Corollary 4.5 The conclusion of Theorem 4.2 and 4.3 remain true if the inequality (4.1.2) is replaced by the following conditions.

$$(4.5) \omega_\lambda(Ix, Iy) \leq k \max\{\omega_\lambda(Sx, Sy), \omega_\lambda(Sx, Ix), \omega_\lambda(Iy, Sy), \frac{1}{2}[\omega_\lambda(Sx, Iy) + \omega_\lambda(Ix, Sy)]\} \\ , k \in (0, 1).$$

$$(4.6) \omega_\lambda(Ix, Iy) \leq k \max\{\omega_\lambda(Sx, Sy), \omega_\lambda(Sx, Ix), \omega_\lambda(Iy, Sy), \frac{1}{2}\omega_\lambda(Sx, Iy), \frac{1}{2}\omega_\lambda(Ix, Sy)\} \\ , k \in (0, 1).$$

$$(4.7) \omega_\lambda(Ix, Iy) \leq k \max\{\omega_\lambda(Sx, Sy), \frac{1}{2}[\omega_\lambda(Sx, Ix) + \omega_\lambda(Iy, Sy)], \frac{1}{2}[\omega_\lambda(Sx, Iy), \frac{1}{2}\omega_\lambda(Ix, Sy)]\} , k \in (0, 1).$$

$$(4.8) \omega_\lambda(Ix, Iy) \leq \frac{k}{2} \max\{\omega_\lambda(Sx, Sy), \omega_\lambda(Sx, Ix), \omega_\lambda(Iy, Sy), \frac{1}{2}\omega_\lambda(Sx, Iy), \frac{1}{2}\omega_\lambda(Ix, Sy)\} , k \in (0, 1).$$

$$(4.9) \omega_\lambda(Ix, Iy) \leq a_1 \omega_\lambda(Sx, Sy) + \frac{a_2 \omega_\lambda(Sx, Ix) \omega_\lambda(Iy, Sy) + a_3 \omega_\lambda(Sx, Iy) \omega_\lambda(Ix, Sy)}{\omega_\lambda(Sx, Ix) + \omega_\lambda(Iy, Sy)},$$

where $a_1, a_2, a_3 \geq 0$ such that $1 < 2a_1 + a_2 < 2$.

CONCLUSION.

In this paper, we proved some general common fixed point theorems for owc mappings satisfying an implicit function in modular metric spaces which generalizes several results from the literature. In process, our results generalize several fixed point theorems in following respects.

- (i) The class of modular metric spaces is a generalization of a metric spaces.
- (ii) The class of implicit functions is also enriched significantly as it requires merely one condition to satisfy.
- (iii) The condition on completeness/compactness of the space is completely relaxed.
- (iv) The condition of weak compatibility is weakened to owc.

In proving fixed point theorems for four maps, step one is by far the most difficult part of the proof. In theorem 4.3 we have imposed the condition owc, which automatically gives the result of step one. Other author have circumvented this difficulty by hypothesizing a property, known as property (E.A.), which implies owc.

REFERENCES.

1. Aamri, M., El Moutawakil, D., Some new common fixed point theorems under strict contractive conditions. J Math Anal Appl. 270, 181–188 (2002).
2. Abdou, AAN and Khamsi, M. A., Fixed points results of point wise contractions in modular metric spaces, Fixed Point Theory and Applications, 163(2013).
3. Alfuraidan, M. R., The contraction principle for mappings on a modular metric space with a graph, Fixed Point Theory and Applications (2015) 2015:46.
4. Al-Thagafi, M. and A. N. Shahzad, Generalized I-nonexpansive selfmaps and invariant approximations, Acta Math. Sinica 24(2008), 867-876.

5. Azadifar et. al., Integral type contractions in modular metric spaces, *Journal of inequality and applications* 2013, 2013:483.
6. Chistyakov, V. V., Modular metric spaces generated by F -modulars, *Folia Math.* 14, (2008) 3–25.
7. Chistyakov, V.V., Modular metric spaces I basic concepts, *Nonlinear Anal.* 72 (2010) 1–14, doi:10.1016/j.na.2009.04.057.
8. Chistyakov, V.V., A fixed point theorem for contractions in modular metric spaces, arXiv: 1112. 5561 v1 [math.FA] Dec 2011.
9. Jungck, G, and Rhoades, B.E., fixed point theorems for occasionally weakly compatible mappings, *Fixed Point Theory*, Vol.7, No.II, (2006), 287-296.
10. Jungck, G., Commuting mappings and fixed points, *Amer. Math. Monthly*, 83(1976), 261-263.
11. Jungck, G., Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.* 9(4), (1986), 771 – 779.
12. Mongkolkeha, C., Sintunavarat, W., Kumam, P., Fixed point theorems for contraction mappings in modular metric spaces, *Fixed Point Theory and Applications* 2011,2011:93.
13. Mongkolkeha, C., Sintunavarat, W., Kumam, P., Fixed point theorems for contraction mappings in modular metric spaces, *Fixed Point Theory and Applications* 2012,2012:103.
14. Nakano, H., *Modulared : Semi-Ordered Linear Spaces*. In Tokyo Math Book Ser, vol. 1, Maruzen Co., Tokyo (1950).
15. Pariya et. al., Common fixed point theorems for generalized contraction mappings in modular metric spaces, *Adv. Inequal. Appl.* 2017, 2017:9.
16. Pathak et al., fixed point theorems for contraction type mappings in modular metric spaces, *Bulletin of mathematics and statistics research*, Vol.5. Issue.1.2017 (Jan.-Mar), 133-139.
17. Rahimpoor, H., Ebadian, A., Gordji, M. E., Zohri, A., Common fixed point theorems in modular metric spaces, *International Journal of Pure and Applied Mathematics*, vol. 99, No. 3, 2015, 373-383.
18. Sessa, S., On a weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.* 32 (1982), 149-153.



EVOLUTIONARY COMPUTATIONAL ALGORITHM AND ANN SUPERVISED CLASSIFIER FOR MICROARRAY GENE EXPRESSION DATA

Manaswini Pradhan

*Assistant Professor, P.G Department of Information and Communication Technology, Fakir Mohan University, Balasore, Odisha, India
E-mail : mrs.manaswini.pradhan@gmail.com

ABSTRACT :

DNA microarray is an efficient new technology that allows to analyze, at the same time, the expression level of millions of genes.. In DNA microarray technology, gene classification is considered to be difficult because the attributes of the data, are characterized by high dimensionality and small sample size. Classification of tissue samples in such high dimensional problems is a complicated task. Furthermore, there is a high redundancy in microarray data and several genes comprise inappropriate information for accurate classification of diseases or phenotypes. Consequently, an efficient classification technique is necessary to retrieve the gene information from the microarray experimental data. In this paper, a classification technique is proposed that classifies the microarray gene expression data well. In the proposed technique, the dimensionality of the gene expression dataset is reduced by Probabilistic PCA. Then, an Artificial Neural Network (ANN) is selected as the supervised classifier and it is enhanced using Evolutionary programming (EP) technique. The enhancement of the classifier is accomplished by optimizing the dimension of the ANN. The enhanced classifier is trained using the Back Propagation (BP) algorithm and so the BP error gets minimized. The well-trained ANN has the capacity of classifying the gene expression data to the associated classes. The proposed technique is evaluated by classification performance over the cancer classes, Acute Myeloid leukemia (AML) and Acute Lymphoblastic Leukemia (ALL). The classification performance of the enhanced ANN classifier is compared over the existing ANN classifier and SVM classifier.

Keywords: Microarray gene expression data, Probabilistic PCA (PPCA), Artificial Neural Network (ANN), Evolutionary Programming (EP), Back Propagation (BP), Supervised Classifier, Dimensionality reduction

1. INTRODUCTION

Enormous amount of genomic and proteomic data are available in the public domain. The ability to process this information in ways that are useful to humankind is becoming increasingly important [1]. The computational recognition is a basic step in the understanding of a genome and it is one of the challenges in the analysis of newly sequenced genomes. For analyzing genomic sequences and for interpreting genes, precise and fast tools are necessary [2]. In such situation, conventional and modern signal processing methods play a significant role in these fields [1]. A relatively new area in bio-informatics is Genomic signal processing [14] (GSP). It deals with the utilization of traditional digital signal processing (DSP) methods in the representation and analysis of genomic data.

Gene is a segment of DNA, which contains the code for the chemical composition of a particular protein. Genes serve as the pattern for proteins and some additional products, and mRNA is the main intermediary that translates gene information in the production of genetically encoded molecules [4]. The strands of DNA molecules usually

contain the genomic information represented by sequences of nucleotide symbols, symbolic codons (triplets of nucleotides), or symbolic sequences of amino acids in the corresponding polypeptide chains [2]. Simultaneously monitoring of the expression levels of tens of thousands of genes under diverse experimental conditions has been enabled by gene expression microchip, which is perhaps the most rapidly expanding tool of genome analysis. This provides a powerful tool in the study of collective gene reaction to changes in their environments, and provides indications about the structures of the involved gene networks [3].

Today, using microarrays it is possible to simultaneously measure the expression levels of thousands of genes, possibly all genes in an organism, in a single experiment [4]. Microarray technology has become an indispensable tool in the monitoring of genome-wide expression levels of gene [5]. The analysis of the gene expression profiles in various organs using microarray technologies reveal about separate genes, gene ensembles, and the metabolic ways underlying the structurally functional organization of organ and its physiological function [6]. Diagnostic task can be automated and the accuracy of the conventional diagnostic methods can be improved by the application of microarray technology. Microarray technology enables simultaneous examination of thousands of gene expressions [7].

Efficient representation of cell characterization at the molecular level is possible with microarray technology which simultaneously measures the expression levels of tens of thousands of genes [8]. Gene expression analysis [10] [18] that utilizes microarray technology has a wide range of potential for exploring the biology of cells and organisms [9]. Microarray technology assists in the precise prediction and diagnosis of diseases. Three common types of machine learning techniques utilized in microarray data analysis are clustering [11] [15], classification [12] [16], and feature selection [13] [17]: Of these, classification plays a crucial role in the field of microarray technology. However, classification in microarray technology is considered to be very challenging because of the high dimensionality and small sample size of the gene expression data. Numerous works have been carried out for the effective classification of the gene expression data. A few recent works available in the literature are reviewed in the following section.

Katharina J Hoff *et.al.* proposed a new combination of feature selection/extraction approach for Artificial Neural Networks ANNs classification of high-dimensional microarray data, which uses an Independent Component Analysis ICA as an extraction technique and Artificial Bee Colony ABC as an optimisation technique [30].

Microarray gene expression based medical data classification has remained as one of the most challenging research areas in the field of bioinformatics, machine learning and pattern classification. [31]. DNA microarray is an efficient new technology that allows analyzing, at the same time, the expression level of millions of genes [32].

2. RELATED WORKS

Some of the recent related research works are reviewed here. Liu *et al.* [19] have offered an analytical method for categorizing the gene expression data. In the proposed method, dimension reduction has been achieved by utilizing the kernel principal component analysis (KPCA) and categorization has been achieved by utilizing the logistic regression (discrimination). KPCA is a generic nonlinear form of principal component analysis. Five varied gene expression datasets related to human tumor samples has been categorized by utilizing the proposed algorithm. The high potential of the proposed algorithm in categorizing gene expression data has been confirmed by comparing with other well-known classification methods like support vector machines and neural networks.

Roberto Ruiz et al. [20] have proposed a novel heuristic method for selecting appropriate gene subsets which can be utilized in the classification task. Statistical significance of the inclusion of a gene to the final subset from an ordered list is the criteria on which their method is based. Comparison result has proved that the method was more effective and efficient than other such heuristic methods. Their method exhibits outstanding performance both in identification of important genes and in minimization of computational cost.

Peng *et al.* [21] have performed a comparative analysis on different biomarker discovery methods that includes six filter methods and three wrapper methods. After this, they have presented a hybrid approach known as FR-Wrapper for biomarker discovery. The objective of their approach was to achieve an optimum balance between precision and computation cost, by exploiting the efficiency of the filter method and the accuracy of the wrapper method. In their hybrid approach, the majority of the unrelated genes have been filtered out utilizing the Fisher's ratio method, which is simple, easy to understand and implement. Then the redundancy has been minimized utilizing a wrapper method. The performance of the FR-Wrapper approach has been appraised utilizing four widely used microarray datasets. Experimental results have proved that the hybrid approach is capable of achieving maximum relevance with minimum redundancy.

Mramor et al. [22] have proposed a method for the analysis of gene expression data that gives an unfailing classification model and gives useful insight of the data in the form of informative perception. The proposed method is capable of finding simple perceptions of cancer gene expression data sets utilizing a very small subset of genes by projection scoring and ranking however presents a clear visual classification between cancer types. They have proposed in view of data visualization's promising part in penetrative data analysis, short runtimes and interactive interface, that data visualization would enhance other recognized techniques in cancer microarray analysis assisted by efficient projection search methods and become part of the standard analysis toolbox. Wong *et al.* [23] have proposed regulation-level method for symbolizing the microarray data of cancer classification that can be optimized utilizing genetic algorithms (GAs). The proposed symbolization decreases the dimensionality of microarray data to a greater extent compared with the traditional expression-level features. Several statistical machine-learning methods have become usable and efficient in cancer classification because noise and variability can be accommodated in the proposed symbolization. It has been confirmed that the three regulation level representation monotonically converges to a solution by experimental results on real-world microarray datasets. This has confirmed the presence of three regulation levels (up-regulation, down-regulation and non-significant regulation) associated with each particular biological phenotype. In addition to improvement to cancer classification capability, the ternary regulation-level promotes the visualization of microarray data.

Ahmad M. Sarhan [7] has developed an ANN and the Discrete Cosine Transform (DCT) based stomach cancer detection system. Classification features are extracted by the proposed system from stomach microarrays utilizing DCT. ANN does the Classification (tumor or no-tumor) upon application of the features extracted from the DCT coefficients. In his study he has used the microarray images that were obtained from the Stanford Medical Database (SMD). The ability of the proposed system to produce very high success rate has been confirmed by simulation results. Papachristoudis *et al.* [24] have offered SoFoCles, an interactive tool that has made semantic feature filtering a possibility in microarray classification problems by the utilization of external, unambiguous knowledge acquired from the Gene Ontology. By improving an initially created feature set with the help of legacy methods, genes that are associated with the same biological path during the microarray experiment are extracted by the

utilization of the idea of semantic similarity. As one of its many functions, SoFoCles offers a huge repository of semantic similarity methods for deriving feature sets and marker genes. Discussion about the structure and functionality of the tool, and its ability in improving the classification accuracy has been given in detail. By means of experimental evaluation, the improved classification accuracy of the SoFoCles has been demonstrated utilizing different semantic similarity computation methods in two real datasets

Debnath *et al.* [25] have proposed an evolutionary method that is capable of selecting a subset of potentially informative genes that can be used in support vector machine (SVM) classifiers. The proposed evolutionary method estimates the fitness function utilizing SVM and a specified subset of gene features, and new subsets of features were chosen founded on the frequency of occurrence of the features in the evolutionary approach and amount of generalization error in SVMs. Hence, theoretically, the selected genes reflect the generalization performance of SVM classifiers to a certain extent. Comparison with several existing methods has confirmed that better classification accuracy can be achieved by the proposed method with fewer numbers of selected genes. From the review, it can be seen that most of the recent works have performed the classification using selective gene expression data. The selected gene expression sub-dataset has been optimized and classified using traditional classifiers. Though the optimization is effective the ultimate objective is not attained because the effectiveness of classification is inadequate. Hence, the enhancement of classifier becomes an essential pre-requisite for effective classification of microarray gene expression data.

Maxwell W. Libbrecht and William Stanford Noble presented considerations and recurrent challenges in the application of supervised, semi-supervised and unsupervised machine learning methods, as well as of generative and discriminative modeling approaches and they provided general guidelines to assist in the selection of these machine learning methods and their practical application for the analysis of genetic and genomic data sets. [28]

Zena M. Hira and Duncan F. Gillies summarized various ways of performing dimensionality reduction on high-dimensional microarray data. Many different feature selection and feature extraction methods exist and they are being widely used. All these methods aim to remove redundant and irrelevant features so that classification of new instances will be more accurate. A popular source of data is microarrays, a biological platform for gathering gene expressions. Analysing microarrays can be difficult due to the size of the data they provide. In addition the complicated relations among the different genes make analysis more difficult and removing excess features can improve the quality of the results. We present some of the most popular methods for selecting significant features and provide a comparison between them. Their advantages and disadvantages are outlined in order to provide a clearer idea of when to use each one of them for saving computational time and resources. [29]

In this paper, proposes an effective classification technique that uses an enhanced supervised classifier. It is well known that microarray gene expression datasets are characterized by high dimension and small sample size. The dimension of the gene expression dataset is reduced using PPCA. With the aid of the dimensionality reduced gene expression dataset, the ANN, which is selected as supervised classifier in our work, is enhanced using EP technique. The enhanced classifier is utilized for classification and so it is trained using BP algorithm. The well-trained classifier is then subjected to the classification of microarray gene expression dataset. The rest of the paper is organized as follows. Section 3 details the proposed classification technique with required mathematical formulations and illustrations. Section 4 discusses about the implementation results and Section 5 concludes the paper.

3. CLASSIFICATION TECHNIQUE FOR MICROARRAY GENE EXPRESSION DATA

Here, an efficient technique to classify microarray gene expression data is proposed. The proposed technique is comprised of three fundamental processes, namely, dimensionality reduction, development of supervised classifier and gene classification. The development of enhanced supervised classifier is illustrated in the Fig. 1 and the training process is depicted in Fig. 2.

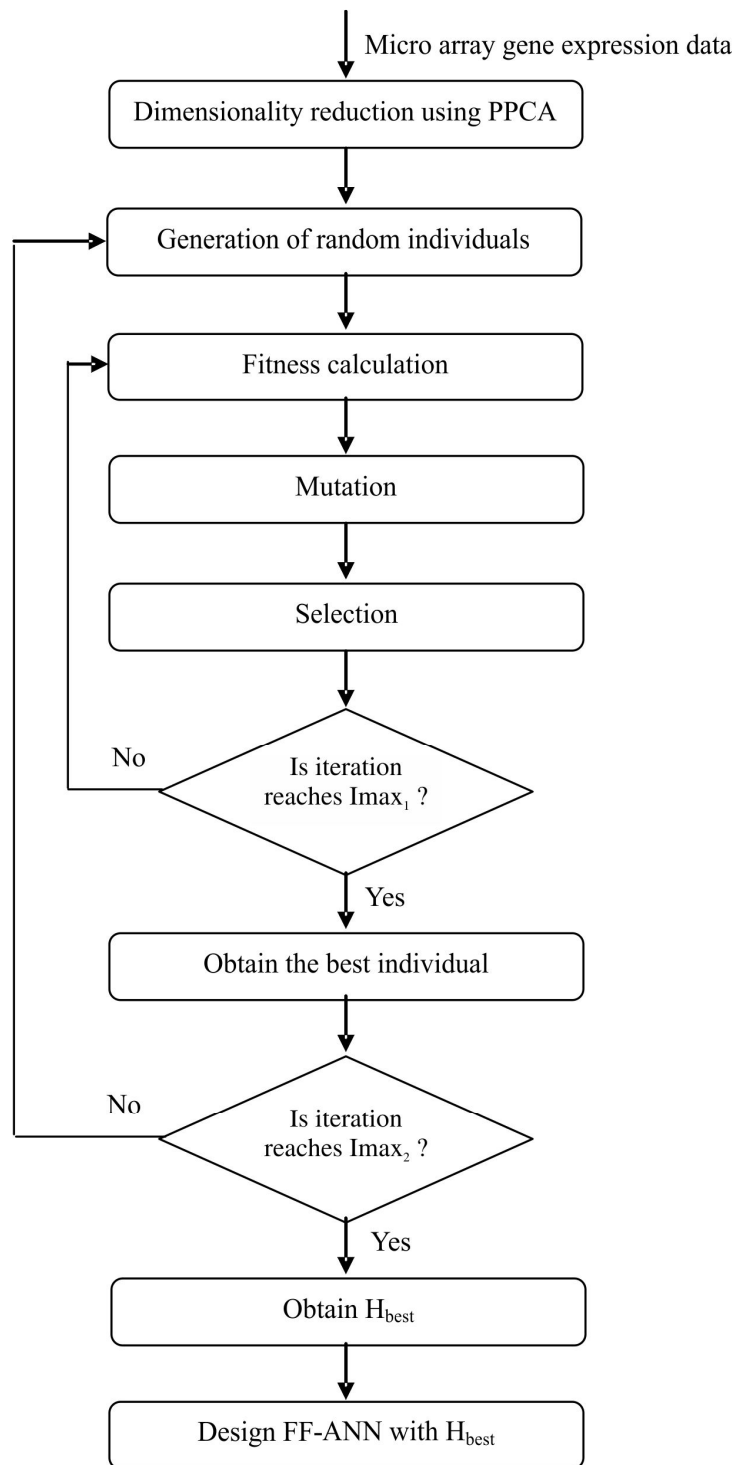


Figure 1: Enhancement of Feed Forward ANN using EP technique

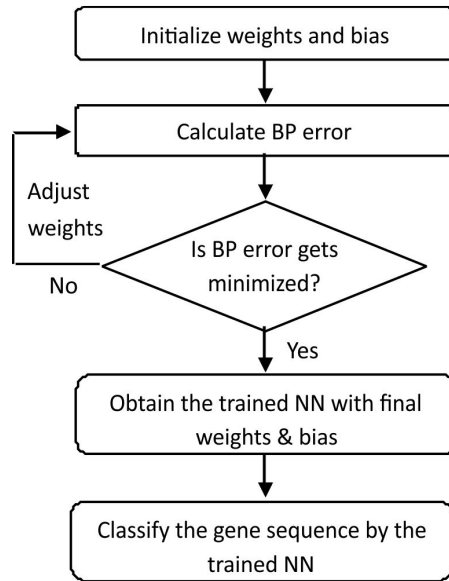


Figure 2: Training process of enhanced supervised classifier using BP algorithm

The dimensionality reduction involves the process of reducing the dimension of the microarray gene expression data using PPCA. In the second process, a supervised classifier is developed using feed forward ANN, which is enhanced using EP technique. In the gene classification, the enhanced classifier is trained using the gene expression data and then the testing process is conducted. So, given a microarray gene expression data, the classifier effectively classifies the data by representing the class to which the data belongs.

3.1. Dimensionality Reduction using PPCA

Let, the microarray gene expression data be M_{jk} ; $0 \leq j \leq N_s - 1$, $0 \leq k \leq N_g - 1$, where, N_s represents the number of samples and N_g represents the number of genes. The dimension of gene data is higher and so it is subjected to dimensionality reduction. In dimensionality reduction, the high dimensional gene data M_{jk} is converted to a low dimensional gene data. To reduce the dimensionality, we use PPCA, which is a PCA with the presence of probabilistic model for the data. The PPCA algorithm composed by Tipping and Bishop [26] is capable of calculating a low dimensional representation utilizing a rightly formed probability distribution of the higher dimensional data.

The instinctive attraction of the probabilistic representation is because of the fact that the definition of the probabilistic measure allows comparison with other probabilistic techniques, at the same time making statistical testing easier and permitting the utilization of Bayesian methods. Dimensionality reduction can be achieved by making use of PPCA as a generic Gaussian density model. Dimensionality reduction facilitates efficient computation of the maximum-likelihood estimates for the parameters connected with the covariance matrix from the data principal components. By performing the dimensionality reduction using PPCA, microarray gene expression data of dimension $N_s \times N_g$ is reduced to $N'_s \times N'_g$. The dimensionality reduced matrix is given as \hat{M} .

Other than dimensionality reduction, the PPCA finds more practical advantages such as finding missing data, classification and novelty detection [26].

3.2. Enhancement of Feed Forward ANNs

Here, an enhanced supervised classifier using multi-layer feed forward ANNs is developed. The enhancement of the neural network is accomplished by optimizing the dimension of the hidden layer using EP technique. EP is a stochastic optimization strategy primarily formulated by Lawrence J. Fogel in 1960, which is similar to genetic algorithm, but it stresses on the behavioral linkage between parents and their offspring instead of attempting to imitate specific genetic operators as seen in nature. The EP technique is comprised of (1) population initialization, (2) fitness calculation (3) selection and (4) mutation. The EP technique used to enhance the classifier is discussed below.

Step 1: A population set X_a ; $0 \leq a \leq N_p - 1$ is initialized, where, X_a is an arbitrary integer generated within the interval $(0, N_H + 1)$ and N_p is the population size.

Step 2: N_p neural networks, each with an input layer, a hidden layer and an output layer are designed. In every a^{th} neural network, N'_s (dimensionality reduced) input neurons and a bias neuron, X_a hidden neurons and a bias neuron and an output neuron are present.

Step 3: The designed NN is weighted and biased randomly. The developed NN is shown in Fig. 3.

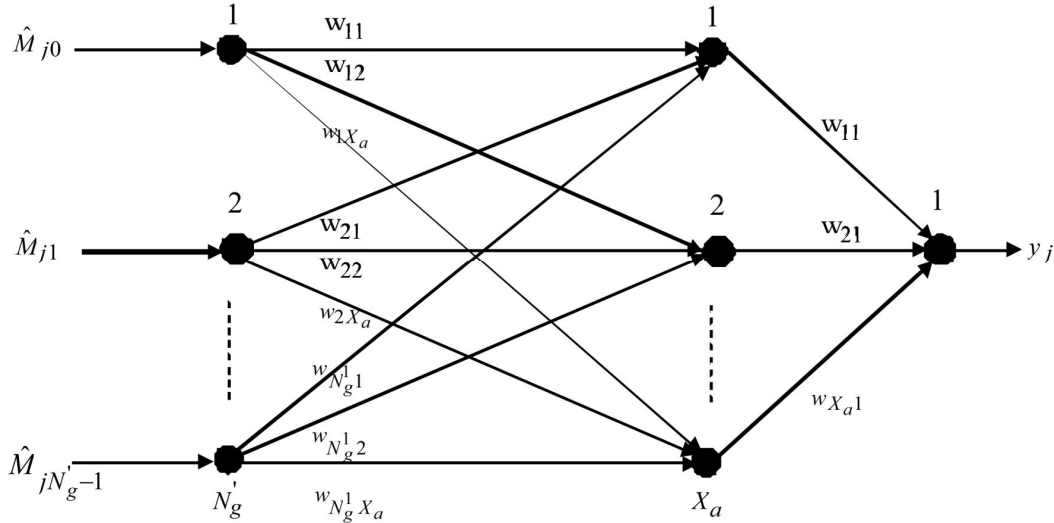


Figure 3: The ANN developed with hidden neurons that are recommended by EP individuals

Step 4: The basis function and activation function are selected for the designed NN as follows

$$y_j = \alpha + \sum_{k=0}^{N'_g-1} w_{jk} \hat{M}_{jk}, \quad 0 \leq j \leq N'_s - 1 \quad (1)$$

$$g(y) = \frac{1}{1 + e^{-y}} \quad (2)$$

$$g(y) = y \quad (3)$$

Eq. (1) is the basis function (given only for input layer), Eq. (2) and Eq. (3) represents the sigmoid and identity activation function, which is selected for hidden layer and output layer respectively. In Eq. (1), \hat{M} is the dimensionality reduced microarray gene data, w_{jk} is the weight of the neurons and α is the bias. The basis function given in Eq. (1) is commonly used in all the remaining layers (hidden and output layer, but with the number of hidden and output neurons, respectively). The \hat{M} is given to the input layer of the N_p ANNs and the output from the all those ANNs are determined.

Step 5: The learning error is determined for all the N_p networks as follows

$$E_a = \frac{1}{N_s'} \sum_{b=0}^{N_s'-1} D - Y_{ab} \tag{4}$$

where, E_a is the error in the a^{th} NN, D is the desired output and Y_{ab} is the actual output.

Step 6: Fitness is determined for every individual, which is present in the population pool, using the fitness function as follows

$$F_a = 1 - \frac{E_a}{\sum_{a=0}^{N_p-1} E_a} \tag{5}$$

Step 7: The individuals which have maximum fitness are selected for the evolutionary process, mutation. So, $N_p/2$ individuals are selected from the population pool and subjected to mutation.

Step 8: In mutation, new $N_p/2$ individuals X^{new} are generated to fill the population pool and the generation is given as follows

$$X^{new} = \begin{cases} M_d & ; \text{if } M_d = N_p/2 \\ M_{d_1} & ; \text{if } M_d < N_p/2 \\ M_{d_2} & ; \text{otherwise} \end{cases} \tag{6}$$

In Eq. (6), the mutation set M_d is determined as $M_d = M_{in} - N^{lbest}$, where, $M_{in} = \{1,2,3,\dots,\mu\}$; μ is the median of N^{lbest} and N^{lbest} is a set of best individuals that has maximum fitness M_{d_1} is determined as

$M_d \cup M'_d$, where, M'_d is a set of random integers that are generated within the interval $(\mu, N_H + 1)$. The set M'_d is generated in such a way that it satisfies the following conditions

$$(i) \quad |M'_d| = N_P/2 - |M_d| \quad (7)$$

$$(ii) \quad M'_d \cap N^{l_{best}} = \phi \quad (8)$$

In Eq. (6), M_{d_2} is the set of random elements which are taken from the set M_d such that $|M_{d_2}| = N_P/2$ and $M_{d_2} \subset M_{in}$.

Step 9: The newly obtained individuals X^{new} occupy the population pool and so the pool retains its size N_P . Then, NNs are developed as per the individuals present in the new population pool and the process is iteratively repeated until it reaches the maximum number of iteration I_{max_1} . Once, the process is completed, the best individual is obtained from the population pool based on the fitness value.

Step 10: The obtained best individual is stored and the process is again repeated from step1 for I_{max_2} iterations. In each iteration, a best individual is obtained and so I_{max_2} best individuals (the best individual represents number of hidden units, which is termed as H_{best}) are obtained after completion of all the iterations. Among the I_{max_2} iterations, the best individual which has maximum frequency i.e. the individual, which is selected as best for the most number of times is selected as the final best individual. Thus obtained best individual is selected as the dimension of the hidden layer and so the NN is designed. Hence, an enhanced NN is developed by optimizing the dimension of the hidden layer using the EP technique.

3.3. Classification of Microarray Gene Expression using the Enhanced Classifier

In the classification of microarray gene expression data, two phases of operation are performed that include training phase and testing phase. In the training phase, the enhanced supervised classifier is trained using the BP algorithm. The dimensionality reduced microarray gene expression dataset is utilized to train the NN.

3.3.1. Training Phase: Minimization of Error by BP algorithm

The training phase of the NN using BP algorithm is discussed below.

(1) The weights are randomly generated within the interval $[0,1]$ and assigned to the hidden layer as well as output layer. For input layer, the weights maintain a constant value of unity.

(2) The training gene data sequence is given to the NN so that the BP error is determined using the Eq. (4). The basis function and transfer function are similar to that used in the optimization (given in Eq. (1), Eq. (2) and Eq. (3)).

(3) When the BP error is calculated, the weights of all the neurons are adjusted as follows

$$w_{jk} = w_{jk} + \Delta w_{jk} \quad (9)$$

In Eq. (3), Δw_{jk} is the change in weight which can be determined as $\Delta w_{jk} = \gamma \cdot y_{jk} \cdot E$, where, E is the BP error and γ is the learning rate, usually it ranges from 0.2 to 0.5.

(4) Once the weights are adjusted as per the Eq. (9), the process is repeated from step 2 until the BP error gets minimized to a least value. Practically, the criterion to be satisfied is $E < 0.1$.

The BP algorithm is terminated when the error gets minimized to a minimum value, which construes that the designed ANN is well trained for its further testing phase.

3.3.2. Testing Phase: Classification of given Microarray Gene Sequence

In the training phase, the ANN learns well about the training gene sequence and the class under which it is present. The well-trained ANN can classify the microarray gene sequence in an effective manner. Given a test sequence, the dimensionality reduction is performed using the PPCA. The dimensionality reduced gene sequence is given as input to the well-trained enhanced supervised classifier. The classifier effectively classifies the gene sequence by determining the class to which it belongs. The supervised classifier is designed with the intention of classifying the microarray gene sequence and hereby, it is accomplished well.

4. RESULTS AND DISCUSSION

The proposed classification technique is implemented in the MATLAB platform (version 7.8) and it is evaluated using the microarray gene expression data of human acute leukemias. The standard leukemia dataset for training and testing is obtained from [27]. The training leukemia dataset is of dimension $N_g = 7192$ and $N_s = 38$. This high dimensional training dataset is subjected to dimensionality reduction using PPCA and so a dataset of dimension $N_g = 30$ and $N_s = 38$ is obtained. In developing the enhanced supervised classifier, the dimensionality reduced microarray gene dataset is used to find the optimal dimension of the hidden layer. The enhancement of the ANN is performed with the parametric values given in the Table I. In each iteration, the error gets minimized and the optimal value for the dimension of the hidden layer is found. While enhancing ANN, the error, which is determined for different iterations and the calculated fitness, while enhancing ANN, are depicted in the Fig. 4. The training of enhanced ANN classifier is implemented using the Neural Network Toolbox in MATLAB. The error versus epochs, which is obtained in the training of ANN using BP, is illustrated in Fig. 5.

Table I: EP parameters used in the enhancement of ANN

S.No	EP Parameters	Values
1	N_p	10
2	N_H	20
3	D	0.25 (for ALL) 0.75 (for AML)
4	$I_{\max 1}$	50
5	$I_{\max 2}$	100

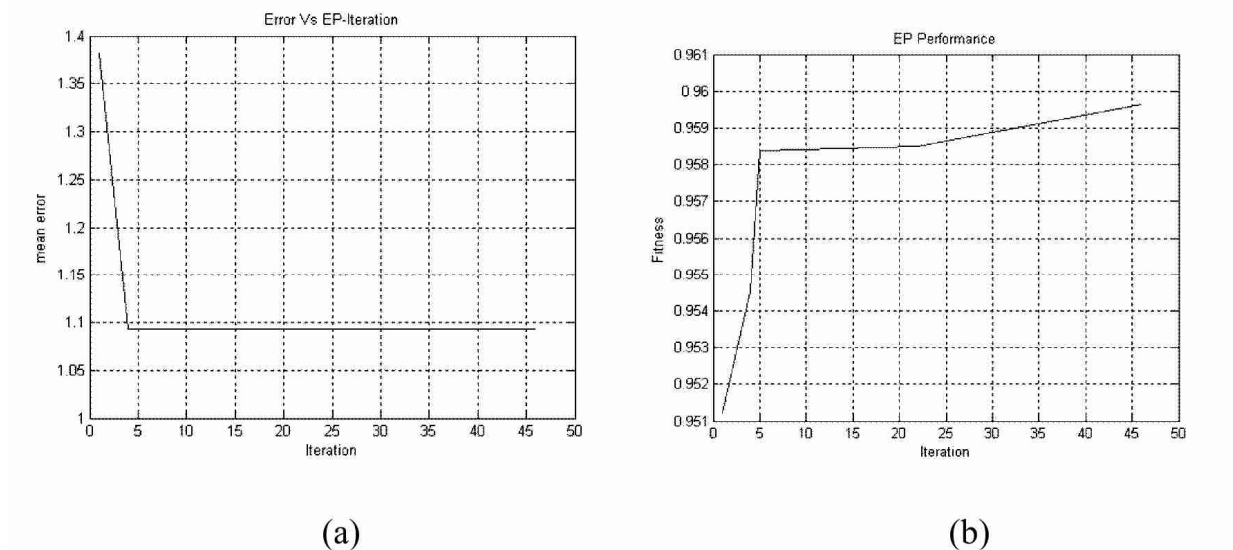


Figure 4: EP performance in enhancement of ANN classifier: (a) Error versus Number of iterations and (b) Fitness versus Number of iterations.

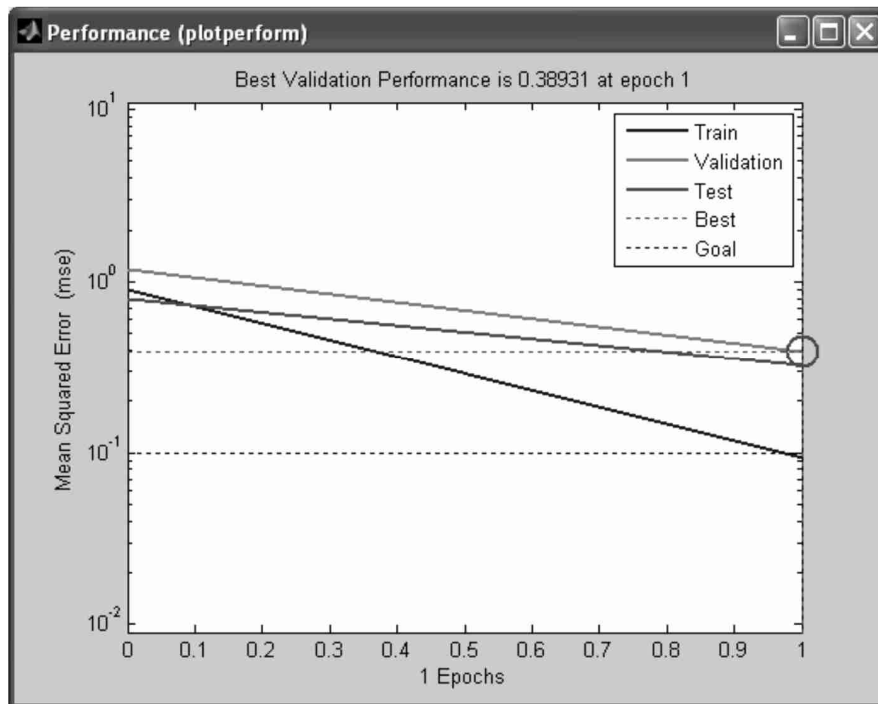


Figure 5: Performance of BP in training the enhanced ANN

Once the enhanced supervised classifier is developed and trained well, the classification is performed by providing the microarray gene expression test dataset. The classifier detects the type of cancer from the dataset with a good accuracy. The significance of the enhanced ANN classifier is demonstrated by comparing its classification

performance with that of the existing ANN classifier. The comparison results are provided in the Table II. Moreover, the performance of the classifier is also compared with the existing SVM classifier and the results are given in the Table III.

Table II: Comparison between enhanced ANN classifier and existing ANN classifier

Cancer class	Enhanced ANN classifier $H_{best}=4$		Existing ANN classifier					
	Classification accuracy (in %)	Error rate (in %)	H=2		H=6		H=8	
			Classification accuracy (in %)	Error rate (in %)	Classification accuracy (in %)	Error rate (in %)	Classification accuracy (in %)	Error rate (in %)
ALL	92.5926	7.4074	70.3704	29.6296	55.5556	44.4444	88.8889	11.1111
AML	90.9091	9.0909	83.1845	16.8155	63.7314	36.2686	81.9983	18.0017

Table III: Comparison between enhanced ANN classifier and existing SVM classifier

Cancer class	Enhanced ANN classifier		Existing SVM classifier	
	Classification accuracy (in %)	Error rate (in %)	Classification accuracy (in %)	Error rate (in %)
ALL	92.5926	7.4074	70.3704	29.6296
AML	90.9091	9.0909	72.7273	27.2727

The comparison results given in the Table demonstrate that the classification accuracy of the enhanced classifier with optimized hidden layer dimension is good, and 90% more than that of the ANN classifier with arbitrary hidden layer dimension. From Table II and Table III results, it can be seen that the proposed technique has good classification accuracy and less error rate when compared with the SVM classifier. The results show that the enhanced supervised classifier performs well in classifying the microarray gene expression dataset.

5. CONCLUSION

In this paper, we propose an efficient classification technique with an enhanced supervised classifier using ANN. The proposed technique has been demonstrated by performing the classification of AML and ALL cancers. The implementation results have shown that the classification of the cancer is performed with good classification rate. The better classification performance is achieved mainly because of the enhancement of the ANN. The enhancement is performed with the intention of finding the dimension of the hidden layer such that the error is minimized. Using the EP, an optimal dimension for hidden layer has been identified. The training of ANN using BP has reduced the BP error to a considerable amount. The comparison results for existing ANN classifier and SVM classifier has demonstrated that the classification accuracy is more in the enhanced ANN classifier rather than the other classifier. Hence, it can be concluded that the proposed classification technique is more effective in classifying the microarray gene expression data for cancers with remarkable classification accuracy.

REFERENCES

1. Vaidyanathan and Byung-Jun Yoon, "The role of signal processing concepts in genomics and proteomics", Journal of the Franklin Institute, Vol.341, No.2, pp.111-135, March 2004
2. Anibal Rodriguez Fuentes, Juan V. Lorenzo Ginori and Ricardo Grau Abalo, "A New Predictor of Coding Regions in Genomic Sequences using a Combination of Different Approaches", International Journal of Biological and Life Sciences, Vol. 3, No.2, pp.106-110, 2007

3. Ying Xu, Victor Olman and Dong Xu, "Minimum Spanning Trees for Gene Expression Data Clustering", *Genome Informatics*, Vol. 12, pp.24–33, 2001
4. Anandhavalli Gauthaman, "Analysis of DNA Microarray Data using Association Rules: A Selective Study", *World Academy of Science, Engineering and Technology*, Vol.42, pp.12-16, 2008
5. Chintanu K. Sarmah, Sandhya Samarasinghe, Don Kulasiri and Daniel Catchpoole, "A Simple Affymetrix Ratio-transformation Method Yields Comparable Expression Level Quantifications with cDNA Data", *World Academy of Science, Engineering and Technology*, Vol. 61, pp.78-83, 2010
6. Khlopova, Glazko and Glazko, "Differentiation of Gene Expression Profiles Data for Liver and Kidney of Pigs", *World Academy of Science, Engineering and Technology*, Vol. 55, pp.267-270, 2009
7. Ahmad m. Sarhan, "cancer classification based on microarray gene expression data using DCT and ANN", *Journal of Theoretical and Applied Information Technology*, Vol.6, No.2, pp.207-216, 2009
8. Huilin Xiong, Ya Zhang and Xue-Wen Chen, "Data-Dependent Kernel Machines for Microarray Data Classification", *IEEE/ACM Transactions on Computational Biology and Bioinformatics (TCBB)*, Vol.4, No.4, pp.583-595, October 2007
9. Javier Herrero, Juan M. Vaquerizas, Fatima Al-Shahrour, Lucia Conde, Alvaro Mateos, Javier Santoyo Ramon Diaz-Uriarte and Joaquin Dopazo, "New challenges in gene expression data analysis and the extended GEPAS", *Nucleic Acids Research*, Vol. 32, pp.485–491, 2004
10. Sveta Kabanova, Petra Kleinbongard, Jens Volkmer, Birgit Andrée, Malte Kelm and Thomas W. Jax, "Gene expression analysis of human red blood cells", *International Journal of Medical Sciences*, Vol.6, No.4, pp.156-159, 2009
11. Sungwoo Kwon and Chonghun Han, "Hybrid Clustering Method for DNA Microarray Data Analysis", *Genome Informatics*, Vol.13, pp, 258-259, 2002
12. Sungwoo Kwon, young-Hwan Chu, Heui-Seok yi and Chonghun Han, "DNA Microarray Data Analysis for Cancer Classification Based on Stepwise Discriminant Analysis and Bayesian Decision Theory", *Genome Informatics*, Vol.12, pp.252-254, 2001
13. Supoj Hengpraprom and Prabhas Chongstitvatana, "Feature Selection By Weighted-SNR for Cancer Microarray Data Classification", *International Journal of Innovative Computing, Information and Control*, Vol.4, No.12(A), pp.4627-4635, December 2009
14. Anastassiou, "Genomic Signal Processing," *IEEE Signal Processing Magazine*, Vol. 18, PP. 8-20, 2001
15. Ho Sun Shon, Sunshin Kim, Chung Sei Rhee and Keum Ho Ryu, "Clustering Approach Using MCL Algorithm for Analyzing Microarray Data", *International Journal of Bioelectromagnetism*, Vol.9, No.2, Pp.65-66, 2007
16. Hieu Trung Huynh, Jung-Ja Kim and Yonggwon Won, "Classification Study on DNA Microarray with Feedforward Neural Network Trained by Singular Value Decomposition", *International Journal of Bio-Science and bio-Technology*, Vol.1, no.1, pp.17-24, December 2009
17. Changjing Shang and Qiang Shen, "Aiding Classification of Gene Expression Data with Feature Selection: A Comparative Study", *International Journal of computational Intelligence Research*, Vol.1, No.1, pp.68-76, 2005
18. Chen-Hsin Chen, Henry Horng-Shing Lu, Chen-Tuo Liao, Chun-houh Chen, Ueng-Cheng Yang and Yun-Shien Lee, "Gene Expression Analysis Refining System (GEARS) via Statistical Approach: A Preliminary Report", *Genome Informatics*, Vol.14, pp.316-317, 2003.
19. Zhenqiu Liu, Dechang Chen and Halima Bensmail, "Gene Expression Data Classification with Kernel Principal Component Analysis", *J Biomed Biotechnol*, Vol.2005, No.2, pp. 155–159, 2005

20. Roberto Ruiz, Jose C. Riquelme and Jesus S. Aguilar-Ruiz, "Incremental wrapper-based gene selection from microarray data for cancer classification," *Pattern Recognition*, Vol. 39 , No. 12, pp. 2383-2392, 2006.
21. Yanxiong Peng, Wenyuan Li and Ying Liu, "A Hybrid Approach for Biomarker Discovery from Microarray Gene Expression Data for Cancer Classification, *Cancer Inform.*, Vol.2, pp.301-311, February 2007.
22. Minca Mramor, Gregor Leban, Janez Demar and Bla Zupan, "Visualization-based cancer microarray data classification analysis", *Bioinformatics*, Vol. 23, No.16, pp.2147-2154, June 2007
23. Hau-San Wong and Hong-Qiang Wang, "Constructing the gene regulation-level representation of microarray data for cancer classification", *Journal of Biomedical Informatics*, Vol.41, No.1, pp.95-105, February 2008.
24. Georgios Papachristoudis, Sotiris Diplaris and Pericles A. Mitkas, "SoFoCles: Feature filtering for microarray classification based on Gene Ontology", *Journal of Biomedical Informatics*, Vol.43, No.1, pp.1-14, February 2010
25. Rameswar Debnath and Takio Kurita, "An evolutionary approach for gene selection and classification of microarray data based on SVM error-bound theories", *BioSystems*, Vol.100, No.1, pp.39-46, April 2010
26. M. E. Tipping and C. M. Bishop, "Probabilistic principal component analysis", *Journal of the Royal Statistical Society, Series B*, Vol. 21, No. 3, p.p. 611–622, 1999
27. ALL/AML datasets from <http://www.broadinstitute.org/cancer/software/genepattern/datasets/>
28. Maxwell W. Libbrecht and William Stanfford Noble, "Machine learning applications in genetics and genomics" , *Nature Reviews Genetics*, Vol. 16, pp. 321-32, May 2015
29. Zena M. Hira and Duncan F. Gillies, "A Review of Feature Selection and Feature Extraction Methods Applied on Microarray Data", *Advances in Bioinformatics*, Vol. May 2015
30. Beatriz A Garro, Katva Rodriguez, Roberto A Vazquez," Classification of DNA microarrays using artificial neural networksand ABC algorithm" , in the *Applied Soft Computing*, Volume 38, pages 548-60, January 2016.
31. Artificial neural network classification of microarray data using new hybrid gene selection method, *International Journal of Data Mining and Bioinformatics*, Volume 17 No. 1, p.42-65, January 2017
32. P. Mohapatra, S. Chakravarty, P.K. Dash," Microarray medical data classification using kernel ridge regression and modified cat swarm optimization based gene selection system", in the *Swarm and Evolutionary Computing*, Volume 28, Pages-144-60, June 2016.



CORRIGENDUM

We want to bring authors & scholars attention that the paper entitled. "A Laplacian Matrix in Algebraic Graph Theory." authors with A. Ramesh Kumar, R. Palani Kumar & S. Deepa published in Aryabhata J. of Mathematics & Informatics Vol. 6 (1) Jan.-July 2014 has been substantially plagiarised from the paper entitled. "Pattern vectors from Algebraic Graph Theory." IEEE Trans Pattern Anal. Mach. Intell. (PAMI) 27 (7) 1112-1124 (2005) authors with Richard C. Wilson, Edwin R. Hancock & Binluo. The authors neither cite the work of reference authors in Introduction nor mention the name and title journal in references. The plagiarism includes verbatim copying of large section of the said paper.

Aryabhata J. of Mathematics & Info. discourages such type of act and believe in maintaining ethical norms fully and dignity of the journal. Further the paper is to be treated remove from the list of publication of volume 6 (1), 2014. The electronic version of paper will be removed from the journal website.

Chief Editor

FORM IV

Statement of relevant particulars regarding "**ARYABHATTA JOURNAL OF MATHEMATICS & INFORMATICS**".

Published in accordance in Form -1, Rule-3 of the Press and Registration of News Paper/Journals/Books Act, 1867.

1. Place of Publication : 2255, Sector-17, Urban Estate, HUDA,
JAGADHRI (Yamuna Nagar) - 135 003
Haryana - INDIA

2. Periodicity of Publication : Twice a year

3. Printer's Name : Dr. T.P. Singh
Nationality : Indian
Address : 2255, Sector-17, HUDA,
JAGADHRI (Yamuna Nagar) - 135 003
Haryana - INDIA

4. Publisher's Name : Dr. T.P. Singh
Nationality : Indian
Address : 2255, Sector-17, HUDA,
JAGADHRI (Yamuna Nagar) - 135 003
Haryana - INDIA

5. Editor's Name : Dr. T.P. Singh
Nationality : Indian
Address : 2255, Sector-17, HUDA,
JAGADHRI (Yamuna Nagar) - 135 003
Haryana - INDIA

6. Name and Address of the owner : Aryans Research & Educational Trust
(Regd. No. 2655)

I, Dr. T. P. Singh, hereby declare that the particulars given above are true to the best of my knowledge and belief.

Dated :
June 3, 2018



Dr. T. P. Singh

Editor-in-Chief / Printer & Publisher

SUBSCRIPTION FORM

I/We wish to subscribe **Aryabhatta Journal of Mathematics & Informatics**. The Filled performas is :

NAME & ADDRESS OF SUBSCRIBER

Name _____ Organisation _____

Designation _____ E-mail : _____

Address of Communication _____

Subscription : Annual/2 years/Life subscription

PAYMENT

I/We wish to make the payment through (Please tick ✓)

Crossed Cheque

Demand Draft

enclosed crossed cheque/demand draft No. _____ Dated _____

Drawn on _____ in favour of **ARYANS RESEARCH &**

EDUCATIONAL TRUST for Rs. _____ payable at

Jagadhri/Yamuna Nagar

Signature of Subscriber

Address to be sent :

Dr. T.P. Singh

Chief Editor

2255, Sec-17, Huda,

Jagadhri - 135003 (Haryana) India

Subscription Information

	India	Abroad
Subscription for individuals/research scholars	Rs. 500/- per issue	US\$ 30/-
Life subscription for individuals/research scholars	Rs. 5,000/-	US\$ 300/-
Annual subscription for Libraries / Institutions	Rs. 1500/-	US\$ 40/-
Subscription for two years	Rs. 2500/-	US\$ 60/-
Life subscription for Libraries / Institutions	Rs. 7,000/-	US\$ 400/-

Including mailing expenses.

Payments : Payments should be made in favour of "**ARYANS RESEARCH & EDUCATIONAL TRUST**" through **Demand Draft** of any Nationalized Bank payable at **JAGADHRI/YAMUNANAGAR** (Haryana) India.