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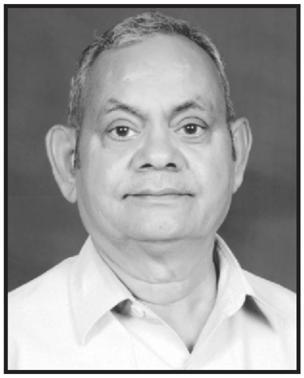


Dr. T.P. Singh
Professor in Maths & O.R.
Chief Editor

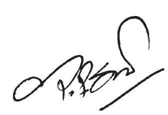
Life Journey of Maths Wizard :
Dr. Vashishtha Narayan Singh

The Journey of Prof. Vashishtha, from a small village **Basantpur** (born on April 2, 1946) in Bihar to **Patna Science College** to **California University** USA and then to **NASA** is truly remarkable & inspiring. In this editorial I would like to record the academic brilliance and contribution of him.

From childhood, Vashishtha (differ from other children) was in absolute love in maths to the extent of often, not getting restless till some tricky problems remained unsolved. I would like to recall one extra ordinary event which was happened with him while studying at Patna Science College. Dr. Nagendra Nath, the then Principal of College, also a maths teacher got many complaints about Vashishtha often disturbing maths classes by posing questions. The Principal called him to his office and gave some hard questions, much beyond to the class he was student of, to solve. Not only did he solve them promptly right in front of Principal but also showed his skill in solving each problem more than one way. His remarkable recognition as a students to amend university rules allowed by Patna University straight away to appear B.Sc. (Maths Hons.) and Later M.Sc. final exam in next year. He topped both classes with distinction. Whenever he was student of Patna Science College, the visiting Professor. John F. Kelley, a famous mathematician specialist in field of Topology & Functional analysis recognized his talent and called him to do research work in USA.

In 1966 he started Ph. D work in **University of California** Berkley USA under the supervision Prof. John L. Kelley and in early 1969 he was awarded Ph. D. degree. His work on **Cycle Vector Space Theory** was appreciated. In 1969 he got a prestigious assignment with **NASA** as associate Scientist. One event at **NASA** is also remarkable. During Launching of **Apollo Space Mission** in NASA, the main computer software suddenly crashed, 30 computer closed working. Dr. Vashishtha counted all geometric orientation calculations on hand & by pen. When the computer start working, these calculations were found correct. Dr. Vashishtha even raised some question mark on famous Gauss Theory in mathematics. From 1969 to earlier 1972 he remained in NASA. In mid 1972 he returned to India and joined as assitt. Prof. in **IIT Kanpur**, later in **TIFR Bombay** for 8 months and then in 1976-77 at Indian statistical Institute **(ISI)** Kolkatta. A research paper titled, **"Reproducing Kernels and operators with a cycle vector"** published in international Journal **'Pacific Journal of Mathematics'** Vol. 52 (2) 1974 pp 565-584 got recognition in research community.

Things were going fine with him till 1977 and what happened one day changed all that. He was referred to doctor a psychiatrist. He was diagnosed as one suffering from schizophrenia. He land up in a mental hospital Central Institute of Psychiatry (CIP) Kanke Ranchi. Though in 2014, Dr. Vashistha was assigned as a visiting prof. at Bhupendra Narayan Mandal University Madhepur but he could not overcome from his prolonged illness. At last, on Nov. 14, 2019 this eminent mathematician passed away between us. We lost a brilliant mathematician. He was an important asset to the nation. In fact, he was such a mathematician who ignited minds.



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DEFINITION OF DERIVATIVE FUNCTION: LOGICAL ERROR IN MATHEMATICS

Temur Z. Kalanov
Home of Physical Problems, Yozuvchilar (Pisatelskaya) 6a, 100128 Tashkent, Uzbekistan
E-mail : tzk_uz@yahoo.com, t.z.kalanov@mail.ru, t.z.kalanov@rambler.ru

ABSTRACT :
The critical analysis of the foundations of the differential calculus is proposed. Methodological basis of the analysis is the unity of formal logic and of rational dialectics. It is shown that differential calculus is fictitious mathematical theory because the concept of the limiting process is the starting point for definition of the derivative function. The passage to the limit "zero" in the definition of the derivative function signifies that the variable quantity takes the only essential value "zero". This fact leads to the following errors. (1) The definition of the derivative function is based on the violation of the necessary and sufficient condition for the validity of the relationship between the increment of the function argument and the increment of the function because the increment of the function is divided by the zero increment of the argument in the case of the limiting process. (2) The definition of the derivative function is based on the contradiction which is that the increment of the argument is both zero and not zero in the same relationship. This contradiction represents a violation of the formal-logical law of identity and of the formal-logical law of the lack of contradiction. (3) The definition of the differential of function is based on two contradictory (mutually exclusive) features: the differential of the argument is not zero while the increment of the argument in the definition of the derivative function is zero.
Keywords: general mathematics, foundations of mathematics, differential calculus, integral calculus, methodology of mathematics, philosophy of mathematics, philosophy and mathematics, mathematics education, logic, physics.
MSC: 00A30, 00A35, 03A05, 00A05, 00A69, 00A79, 97I40, 97I50, 97E20, 97E30, 97A99.

INTRODUCTION

As is known, the formalism of differential and integral calculus is widely and successfully used in natural sciences. However, this does not mean that the problem of substantiation of differential and integral calculus is completely solved in 20-21 centuries, and the foundations of differential and integral calculus are not in need of formal-logical analysis now. Recently, necessity of the critical analysis of the foundations of differential and integral calculus within the framework of the correct methodological basis – unity of formal logic and of rational dialectics – has arisen [1-14]. It is shown in [1-14], that errors in science (for example, in physics) often arise because of the existence of methodological errors in mathematics and because of the "thoughtless application of mathematics" (L. Boltzmann).

The purpose of this work is to propose the critical analysis of the foundations of differential calculus within the framework of methodological basis – the unity of formal logic and of rational dialectics. The critical analysis is based on the dialectical principle of functional connection and of movement.

1. THE PRINCIPLE OF FUNCTIONAL CONNECTION AND OF MOVEMENT

Movement is change in general. In other words, movement is a change in state.

The principle of movement (change) is a theoretical generalization of practice and represents a concretization of the laws of dialectics and formal logic. The principle of functional connection and of movement in mathematics is formulated as follows.

a) If the continuous function y of one argument x is given, then the function

$$y = f(x)$$

is a mathematical (quantitative) representation of the dialectical principle of functional connection. The principle of quantitative change in the functional connection reads as follows: a change in the values of the argument x leads to a change in the values of the function y; a change in the values of the function y characterizes a change in the values of the argument x. (In other words, a change in the argument determines a change in the function; a change in the function characterizes a change in the argument).

b) The change in the values of the argument x is characterized by the increment Δx of the argument. The quantity Δx takes certain numerical values. The definition of the argument increment is the following: the increment of the argument is the difference of the two numerical values of the argument. Therefore, the dimension of the quantity Δx is identical to the dimension of the quantity x.

c) The change in the numerical values of the function y is characterized by the increment Δy of the function. The quantity Δy takes certain numerical values. The definition of the function increment is the following: the function increment is the difference of two numerical values of the function. Therefore, the dimension of the quantity Δy is identical to the dimension of the quantity y.

d) The relationships $\Delta x \neq 0$ and $\Delta y \neq 0$ represent a necessary and sufficient condition for movement (change). The relationships $\Delta x = 0$ and $\Delta y = 0$ represent a necessary and sufficient condition for the lack of movement (i.e., the condition for the lack of change).

e) The coefficient k of the relative increment (i.e., the ratio of the quantity of the increment of the function to the quantity of the increment of the argument) is defined by the following relationship: $k \equiv \Delta y / \Delta x$ where the permitted values of the increments are in the regions $\Delta x \neq 0$ and $\Delta y \neq 0$. If the movement (change) is lack (that is, if $\Delta x = 0$ and $\Delta y = 0$), then the coefficient of relative increment loses its meaning: $k \equiv \Delta y / \Delta x = 0/0$. In other words, the values $\Delta x = 0$ and $\Delta y = 0$ are inadmissible values in the calculation of $k \equiv \Delta y / \Delta x$.

f) The expression $\lim_{\Delta x \to q} \Delta x$ (where $q \neq 0$ is a given number in the region of existence of Δx) signifies that the values $\Delta x \neq q$ are nonessential values, and the value $\Delta x = q$ is essential one. Therefore, the values $\Delta x \neq q$ do not appear (do not show itself) in the expression (symbol) $\lim_{\Delta x \to q} \Delta x$. The value $\Delta x = q$ expresses the true meaning of the expression (symbol) $\lim_{\Delta x \to q} \Delta x$.

g) The expression $\lim_{\Delta x \to q} \Delta y$ (where $q \neq 0$ is a given number in the region of existence of Δx) signifies that the values $\Delta x \neq q$ are nonessential values, and the value $\Delta x = q$ is essential one. Therefore, the values $\Delta x \neq q$ do not appear (do not show itself) in the expression (symbol) $\lim_{\Delta x \to q} \Delta y$. The value $\lim_{\Delta x \to q} \Delta y = p$

(where p is a number in the region of existence of Δy) expresses the true meaning of the expression (symbol) $\lim_{\Delta x \to q} \Delta y$.

h) The true meaning of the expression (symbol) $\lim_{\Delta x \to q} \frac{\Delta y}{\Delta x}$ is the following:

$$\lim_{\Delta x \to q} \frac{\Delta y}{\Delta x} \equiv \frac{\lim_{\Delta x \to q} \Delta y}{\lim_{\Delta x \to q} \Delta x} = \frac{p}{q}.$$

i) In the point of view of formal logic, the right and left sides of the mathematical relationship for $\Delta y / \Delta x$ must satisfy the formal-logical law of identity:

$$\left(\frac{(concept\ of\ \Delta y)}{(concept\ of\ \Delta x)} \right) = \left(\frac{(concept\ of\ \Delta y)}{(concept\ of\ \Delta x)} \right),$$

$$\left(\frac{(definition\ of\ concept\ of\ \Delta y)}{(definition\ of\ concept\ of\ \Delta x)} \right) = \left(\frac{(definition\ of\ concept\ of\ \lim_{\Delta x \to q} \Delta y)}{(definition\ of\ concept\ of\ \lim_{\Delta x \to q} \Delta x)} \right)$$

j) In order to clarify the meaning of the above quantities and designations, one must concretize (specify) the quantities and designations. The quantities x and y take numerical values as a result of measurements (observations). The result of measurements (observations) of the quantity x represents the following values: x_n, $n = 0, 1, 2, \ldots$. These values correspond to the following result of measurements (observations) of quantity y: y_n, $n = 0, 1, 2, \ldots$. In this case, the increments are designated as follows: $x_{n+1} - x_n \equiv \Delta x_{n+1,\ n}$, $y_{n+1} - y_n \equiv \Delta y_{n+1, n}$. Increments $\Delta x_{n+1,\ n}$ and $\Delta y_{n+1, n}$ represent the results of mathematical operations. In other words, increments $\Delta x_{n+1,\ n}$ and $\Delta y_{n+1, n}$ are numbers. If $x_{n+1} - x_n = h$, $h \neq 0$, then

$$x_n = nh,$$

$$k_{n+1,\ n} = \frac{\Delta y_{n+1,\ n}}{\Delta x_{n+1,\ n}} = \frac{\Delta y_{n+1,\ n}}{h},$$

$$\Delta y_{n+!,\ n} = k_{n+1,\ n}\, h, \quad y_{n+1} = y_n + k_{n+1,\ n}\, h.$$

These algebraic relationships express arithmetic relationships between numbers.

k) Proportion is the only correct relationship between changes in the values of the argument and of the function:

$$\left(\frac{y(x) - y_1(x_1)}{y_1(x_1)} \right) = \left(\frac{x - x_1}{x_1} \right), \quad \text{under } x - x_1 \neq 0,$$

$$y(x) - y_1(x_1) = \frac{y_1(x_1)}{x_1}(x - x_1),$$

$$y(x) = \frac{y_1(x_1)}{x_1}\, x;$$

$$\left(\frac{y(x+\Delta x)\ -\ y(x)}{y(x)}\right) = \left(\frac{x+\Delta x - x}{x}\right),$$

$$y(x+\Delta x)\ -\ y(x) = \frac{y(x)}{x}\Delta x,$$

$$y(x+0)\ -\ y(x) = \frac{y(x)}{x}0,\quad 0 = \frac{y(x)}{x}0,\quad 0 = 0 \ \text{ under } \ \Delta x = 0;$$

$$\frac{y(x+\Delta x)\ -\ y(x)}{\Delta x} = \frac{y(x)}{x},\ \text{under } \ \Delta x \neq 0,$$

$$y(x+\Delta x) = \frac{y(x)}{x}(x+\Delta x),$$

$$y(x\ ) = \frac{y(x)}{x}x,\quad y(x) = y(x) \ \text{ under } \ \Delta x = 0.$$

2. DEFINITION OF THE DERIVATIVE FUNCTION

As is known [15, 16], if $y = f(x)$ represents the continuous function y of one argument x, then the derivative function is defined as follows:

$$y'(x) = f'(x);$$

$$f'(x) \equiv f'(x;\ \Delta x = 0) \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \lim_{q \to 0}\left[\frac{\lim\limits_{\Delta x \to q \neq 0} \Delta y}{\lim\limits_{\Delta x \to q \neq 0} \Delta x}\right]$$

where

$$y'(x) \neq \frac{\lim\limits_{\Delta x \to 0} \Delta y}{\lim\limits_{\Delta x \to 0} \Delta x} = \frac{0}{0},\qquad \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x},$$

Δx and $\Delta y = f(x+\Delta x) - f(x)$ are the increments of the argument and of the function, respectively.
As is known [15, 16],

$$\frac{dy}{dx} = f'(x),\ dy = f'(x)\,dx,\ dy = d f(x),$$

where $dx = \Delta x \neq 0$ and $dy = \Delta y \neq 0$ are the differentials of the argument and of the function, respectively. The differential dy is a function of two variable quantities x and dx which are independent of each other.

The essence of the concept of the derivative function becomes the apparent
(obvious, evident, certain) fact in the following example.

Example.

If $y = x^2$, *then*

$$y' = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\,\Delta x + (\Delta x)^2}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x + 0 = 2x,$$

where the symbol $\lim\limits_{\Delta x \to 0} \Delta x$ *signifies that the quantity* Δx *takes the only essential value* $\Delta x = 0$.

The result $y' = 2x$ is not free of the following objections.

(a) If one divides both sides of the relationship

$$\Delta y = (x + \Delta \mathrm{x})^2 - x^2$$

by $\Delta x \neq 0$, then one obtains the following equality:

$$\frac{\Delta y}{\Delta x} = \frac{2x\,\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x, \quad \Delta x \neq 0.$$

The condition $\Delta x \neq 0$ represents the necessary and sufficient condition of validity of this equality. In other words, the left and right sides of true equality must satisfy the condition $\Delta x \neq 0$. Therefore, the equality is not valid if $\Delta x = 0$.

In this point of view, the relationship

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\,\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

contradicts to the condition $\Delta x \neq 0$. Consequently, the result $y' = 2x$ is erroneous.

(b) The result $y' = 2x$ is a consequence of the contradiction which is that $\Delta x \neq 0$ and $\Delta x = 0$ in the same relationship.

Thus, the above example discovers (ascertains, reveals, detects) a formal-logical error in differential calculus.

3. LOGICAL ERRORS IN THE DEFINITION OF THE DERIVATIVE FUNCTION

To understand the essence (nature) of the error in the definition of the derivative function, one must know how the computer performs the calculations. As is known, a programmer and a computer perform the concretization of mathematical (quantitative) relationships expressed in terms of letters and symbols of operations. The computer cannot perform, for example, the operation of addition $x + \Delta x$ if the programmer does not set (specify) numerical values to the quantities x and Δx. If the programmer sets (gives, specifies) numerical values to the quantities x and Δx, then the computer can calculate the result of the mathematical operation. The computer does not distinguish between the quantities $\frac{\Delta y}{\Delta x}$ and $\lim\limits_{\Delta x \to q} \frac{\Delta y}{\Delta x}$ because the computer operates only with numerical values that the programmer sets (gives, specifies) to mathematical quantities. The symbol $\lim\limits_{\Delta x \to q} \frac{\Delta y}{\Delta x}$ signifies that the quantity Δx takes the only essential value $\Delta x = q$. Therefore, the computer divides the number $\Delta y \neq 0$ by the number $\Delta x \neq 0$ and gives the result in the form of the numerical fraction $\frac{p}{q}$. If $\Delta x = q = 0$, then the computer gives the information that the specified division operation is the inadmissible operation $\frac{0}{0}$. Therefore, the condition $\Delta x \neq 0$ represents the necessary and sufficient condition of validity of the quantity $\frac{\Delta y}{\Delta x}$.

But in order to analyze and to understand why this condition is not satisfied in the definition of the derivative function, one must know the formal logic. The formal-logical analysis is not accessible to a computer because a computer cannot operate with concepts. Formal logic (as the science of the laws of correct thinking) operates with concepts and is accessible only to man.

The formal-logical errors in the definition of the derivative function are as follows.

1) In accordance with the law of identity, the object Δx of thought must be identical with itself in the process of reasoning: $\Delta x \equiv \Delta x$. But the definition of the derivative function $f'(x)$ contains the contradiction which is that $\Delta x \neq 0$ and $\Delta x = 0$ in the same relationship. This is a violation of the law of identity and the law of lack of contradiction.

2) In accordance with the law of lack of contradiction, it is not permitted that the same object of thought contains two contradictory features at the same time, in the same sense or in the same relation. But $dy = f'(x)\,dx$ contains two contradictory features: $dx \neq 0$ and $\Delta x = 0$ (in the definition of $f'(x)$). The feature included in the content of the concept $dx \neq 0$ negates the feature included in the concept $\Delta x = 0$ (in the definition of $f'(x)$). One concept excludes another concept. But both features cannot belong to the same relationship. Therefore, one of two contradictory (inconsistent) features (or both) is a lie. Just because the feature $\Delta x = 0$ is a lie.

Thus, differential calculus is a false theory because it contains formal-logical errors.

DISCUSSION

1. The idea of mechanical movement played a “disgusting joke” with Isaac Newton. Newton entered (introduced) the concept of movement (change) into the mathematical expression of the function $y = f(x)$ by means of the increment Δx of the argument x. He obtained the movement (change) Δy of the function y. The “disgusting joke” is that Newton canceled (deleted) the change in the argument (i.e., he putted $\Delta x = 0$), but, contrary to logic, he didn’t canceled (he didn’t deleted) the change in the function (i.e., he didn’t put $\Delta y = 0$). Newton was unable to detect the logical error because he could not understand the essence of the limiting process. (The essence of the limiting process $\lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ is that the quantity Δx takes the only essential value 0). As a result, Newton obtained $\Delta y \neq 0$, $f'(x) \neq 0$ under $\Delta x = 0$. This signifies that movement (or the cause of movement) does not exist (i.e., $\Delta x = 0$), but such a feature (property) of movement as movement speed (i.e., derivative) exists.

This is physical absurdity. Newton probably did not understand that the properties (speed, acceleration) of motion do not exist if motion does not exist. Thus, the absurdity in the form of differential and integral calculus entered in mathematics.

The absurdity took an elegant form (shape) thanks to the canon of differential calculus which was created by logician G. Leibniz. (For the first time, Leibniz's canon was published in the journal Acta Eruditorum, Leipzig, 1684). But Leibniz could not find, understand, and detect Newton's logical errors.

2. Today, mathematicians and physicists all over the world use differential and integral calculus. They believe in the correctness, firmness, and inviolability of this theory. Therefore, scientists do not work for mastery of the correct methodological basis of science: the unity of formal logic and of rational dialectics. The unity of formal logic and of rational dialectics is also a criterion of truth. But errors in science (for example, physics) often arise because of the existence of methodological errors in mathematics and the "mindless, thoughtless application of mathematics" (L. Boltzmann).

Is there "problem of existence of science for science" today? As the history of science shows, scientists are in no hurry to cast doubts on old theories within the framework of the correct criterion of truth because they are afraid to loss prestige and well-being.

CONCLUSION

Thus, the critical analysis of the foundations of differential calculus, carried out within the framework of the correct methodological basis, leads to the following statements:

1) If the continuous function of one argument is given, then this function is a mathematical (quantitative) representation of the dialectical principle of the functional connection. The dialectical principle of the quantitative change in the functional connection is that a change (increment) in the argument leads to a change (increment) in the function.

2) The necessary and sufficient condition for the validity of the relation between the increment of the function argument and the increment of the function is that the increment of the argument must be non-zero in all cases. But this condition is violated in determination of the derivative function: in the case of the passage to limit "zero", the increment of the function is divided by the zero increment of the argument.

3) The definition of the derivative function contains the contradiction which is that the increment of the argument is both zero and non-zero in the same relationship. This contradiction represents a violation of the formal-logical law of identity and the formal-logical law of the lack of contradiction.

4) In accordance with the formally-logical law of the lack of contradiction, one and the same object of thought should not contain two contradictory features at the same time, in the same sense or in the same relation. But the definition of the differential of function contains two contradictory (mutually exclusive) features that cannot belong to the same relationship: the differential of the argument is not zero, but the increment of the argument is zero in the definition of the derivative function.

Thus, differential calculus is a fallacious mathematical theory because it contains formal-logical errors.

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MATHEMATICAL MODEL ON STATIONARY INFORMATION SYSTEM IN A DISTRIBUTED SERVICE NETWORK

Arun Pratap Singh
Assistant Professor, Deptt. of Mathematics, Jubail Industrial College, Jubail, Saudi Arabia
E-mail : arungeit@yahoo.co.in

ABSTRACT :
In Distributed Service Network, the main concern of the user is mostly focused on the immediate response time of a call for service. The user evaluates the efficiency of a service network in terms of how long he or she has been waiting until a service unit arrives at the scene of a call. Short-term policies such as dispatching, repositioning or routing depend critically on the information available and the communications system operated by the dispatcher at the moment a service request arises. Thus, information has great value in distributed service network. In this paper the mathematical model is proposed for the expected response time under stationary information system.
Keywords: Distributed Service Network, Expected Response time, Stationary Information system.

INTRODUCTION:

A distributed service network is a concept related to distribution and traveling: distribution of resources among facilities located at various locations and traveling of resources along a distributed network. There are many types of businesses and organizations that can fit into models of distributed service networks. In theory, almost every service provider can be modeled by means of a network even when one wanders through the long corridors of a mammoth bureaucratic organization while being transferred from one clerk to another; services are in fact, being received from a network.

The information system is a combination of two components: knowledge about the network status (units location and availability to communicate instructions) to the service units. Each component affects the viability of various dispatching decisions. The first component- knowledge about the network status can be obtained from two sources of information (i) direct report upon departing from a node and a computer can estimate its location at any later time, or, alternatively, the unit can contribute to report its location every now and then.

The second component- communication to various network facilities depends on the technology acquired by management. Telephone lines connected only to stations (home nodes) provide limited communication.

The least informative system from the dispatching point of view is the case where assignments are scheduled ahead: at beginning of the working period (day, shift) each service unit is provided with a list of tasks to be accomplished during that period. Once a unit has left the home node, modifications in dispatching instructions cannot be relayed until the unit is back at the home node. This system is labelled the periodical information system.

A periodical information system is acceptable only when the required response time is longer than the duration of a single operational shift because requests cannot be responded to before a new shift begins.

MATHEMATICAL FORMULATION:

Let G (N, L) be adirected network, where N is the set of n nodes and L is the set of links. The fraction of service calls (demands) associated to each node i is h_i such that$\sum_{i=1}^{n} h_i = 1$.

The inter-arrival time of demand requests is a random variable with cumulative distribution F(t), independent of the location of the servers, where t denotes travel time. There are $k > 1$ service units in the system. Of these $k-1$ service units are stationary, say at nodes $i_1, i_2, \dots\dots, i_{k-1}$and k^{th}service unit starts travelling from node V to node W. We consider that travel is at a constant speed and that U-turns are permitted and instantaneous.

Let node $i^{(1)} \in \{i_1, i_2, \dots\dots, i_{k-1}\}, 1 \in N$ be a node such that $d\big(i^{(1)}, 1\big) = \min\{d(i_1, e), \dots\dots, d(i_{k-1}, 1)$; in other words node $i^{(1)}$is the home node of the closest stationary server to node 1. Let X be the random variable describing the time of the incident and let g(X) be the location of the mobile server at the time of the incident. Let Y be the location of the incident. The expected response time to a random incident can be expressed as

$$E_{XY}\big[\min\big\{d\big(i^{(y)}, y\big); d(g(x), y)\big\}\big] = E_X\Big[\sum_{1\in N} P(Y = 1)\min\big\{d\big(i^{(1)}, 1\big); d(g(x), 1)\big\}$$

$$= \textstyle\sum_{1\in N} h_1\, E_X\,\big[\min\big\{d\big(i^{(1)}, 1\big); d(g(x), 1)\big\}\big]$$

We take into account that an incident can occur at any node 1 with probability h_1 and at any time $X = x$ while moving server is located at the point g(X) in the instant of the incident.

EXPECTED RESPONSE TIME:

We consider that the dispatching center maintains communication only with the stationary servers located at nodes $i_1, i_2, \dots\dots, i_{k-1}$ for any node $1 \in N$at the time of service call the dispatcher may assign a stationary server at node $i^{(1)}$ or the moving server if it has arrived at W. Therefore,

$\min\big\{d\big(i^{(1)}, 1\big); d(g(x), 1)\big\}$

$= d\big(i^{(1)}, 1\big)$if the moving server is still in motion at time .

$\min\big\{d\big(i^{(1)}, 1\big); d(g(x), 1)\big\}$if the moving server is at node W at time x or later.

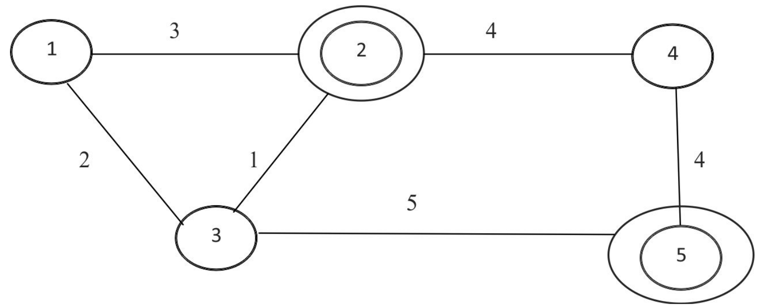
The probability that the moving server is still in motion at time x if$F(X), X < d(V, W)$, while the probability that the moving server is at node W at the time of the incident is $1 - F(X), X \geq d(V, W)$ therefore, the expected response time under stationary information system can be given as

$$ERT_{SIS} = \sum_{1\in N} h_1\big[\, d\big(i^{(1)}, 1\big)P(X < d(V, W) + \min\big\{d\big(i^{(1)}, 1\big); d(W, 1)\big\} + P(X \geq d(V, W)\big]$$

$$= \textstyle\sum_{1\in N} h_1\big[\, F\big\{d(V, W), d\big(i^{(1)}, 1\big)\big\} + \{1 - F(d(V, W)\}\min\big\{d\big(i^{(1)}, 1\big); d(W, 1)\big\}$$

APPLICATION:

Suppose the dispatcher in a network has to assign an unit to serve a call issued at node 1. Suppose in addition to the two stationary servers located at node 2 and 5, there is a moving server currently travelling from node 1 to node 4 via nodes 3 and 2; the server has already left node 1, but at the moment it is not being dispatched to any specific call for service.



What are the dispatching options available to the dispatcher? The answer to this question certainly depends on the information the dispatcher possesses and the ability to communicate instructions to the moving server. For instance, the dispatcher may have no contact with the traveling server from the time it starts moving until it reaches the node of destinations; in this cases, the stationary unit from node 2 will be assigned to node 1. Another extreme case should be when the dispatcher is continuously informed of the server's location (real time information) and can transmit instructions at all times; in this case, the moving server will be assigned to node 1 as long as it has not reached node 2 (assuming that U-turns are permissible and sufficiently short), because the distance for the moving server to return to node 1 is shorter than for the stationary server to travel from node 2. In between these two extreme cases, there are variety of intermediate possibilities. For example, the dispatcher may have communication facilities only with nodes, in such a case, the moving server will be assigned to node 1 only if it has passed halfway on link (1, 3), but it has not reached node 3, where it can be contacted. Once it has left node 3, the stationary unit at node 2 is to be assigned.

Let us assume that travel times are deterministic and that the system parameters are the following:

Arrival of request is at a mean rate of 0.1 per unit time. The inter-arrival time distribution is negative exponential i.e. $F(t) = 1 - e^{-(0.1)t}$. All nodes share the same demand for services i.e. $h_i = \frac{1}{5}$ for every node$i$, $i = 1,2,3,4,5$. The distances on the network are timewise.

Now let us follow the server moving from node 1 to node 4. It takes 7 units of time to travel that way. Meanwhile if there is a call at node i, it will be served by either station 2 or 5, whichever is closer to the calling node. The ERT (expected response time) in this case (given that the third unit is in motion) is

$\sum_{i=1}^{5} h_i \min\,[d(2,i), d(5,i)]$ ....(1)

When we insert figure in equation (1), we get

$\frac{1}{5}(3 + 0 + 1 + 4 + 0) = \frac{8}{5}$ ....(2)

However once the moving server has arrived at node 4, it joins service force and can be assigned to calls. Hence,

ERT=$\sum_{i=1}^{5} h_i \min[d(2,i), d(4,i), d(5,i)] = \frac{1}{5}(3 + 0 + 1 + 0 + 0) = \frac{4}{5}$ ....(3)

We can see that conditional ERT in equation (3) is half that of equation (2) because there is one more service unit available for dispatcher. The question is how often the network's status is described by equation (1) and how often by equation (3). This depends on the probability of a call arriving during the travel period and during the stationary period of the third service unit.

We know that it takes 7 units of time to go from 1 to 4,thus the probability of receiving a call during this period is

$F(7) = 1 - e^{-(0.1)7} = 0.503$ ....(4)

The complementary probability is of course

$1 - F(7) = e^{-(0.1)7} = 0.497$ ....(5)

In order to calculate the overall ERT for this specific case, we have to multiply equation (2) and equation (3) by their relative weights, i.e. the probability in equation (4) and equation (5) respectively. This will yield
ERT= $\left(1-e^{-(0.1)7}\right)\times\frac{8}{5}+e^{-(0.1)7}\times\frac{4}{5}=1.2303$ units of time.

CONCLUSION:

Distributed service network is a term which is widely used in our daily life such as ambulance, fire, police, courier, tax etc. The short term policies depend on the information available. The information plays a major role therefore we have formulated a mathematical model on stationary information system.

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PERFORMANCE MEASURE OF 2 - STAGE FUZZY SCHEDULING WITH SINGLE TRASPORT AGENT USING ROBUST RANKING TECHNIQUE

Kusum*, T.P. Singh**
*Lecture in Mathematics faculty of Science B.M.N University Asthal Bohar Rohtak.
** Professor Dept. of Maths B.M.N. University Asthal Bohar Rohtak.
E-mail: kusumsansanwal123@gmail.com, tpsingh78@yahoo.com

ABSTRACT :

Scheduling problems are concerned in searching an optimal or near optimal sequence in which a number of tasks can be performed subject to the number of constraints. In most of these problems, the performance measures are a function of the order or sequence of jobs. Though a number of heuristic techniques have been developed to solve the scheduling problems, yet many of these are often not practical in dynamic real world scenario due to complex constraints & unexpected disruptions.

This paper put forward an algorithm to find the performance measure for fuzzy scheduling in which the transport agent returns on first machine after delivering the task on second machine. The processing time has been considered with triangular fuzzy parameters. The basic idea is to convert the fuzzy processing time in the crisp value by applying Robust Ranking technique. Robust ranking index is easier to apply and helpful tool in fuzzy decision making environment. The algorithm is based on a theorem which has been verified. Numerical illustration is also given to justify the study.

Keywords: Transport agent, fuzzy scheduling, transportation time, practical situation, triangular fuzzy number, Robust Ranking technique.

1. INTRODUCTION

Motivated by the work of Johnson's (1954) and developing further study by Ignall (1965), Coompbell, Dudek and smith (1970), Maggu and Dass (1980), Singh T.P. (1985,1986) and further T.P. Singh and Deepak Gupta (2005, 2008) up to the present decade, a lot of work has been done in flow shop scheduling mainly in deterministic situations. Mac carthy and Liu (1993) addressed the nature of the gap between the scheduling theory and practice. The shortcomings and limitations of classical scheduling theory responds to the need of practical environments. It has practically seen that most manufacturing plants operate in fuzzy situation where unpredictable real time events may cause a change in the schedule plans or optimal schedule. The estimated time or cost data may become irrelevant when they are released to shop floor. It may be due to machine failures, arrivals of priority jobs, due date change etc. Sunita and Singh T.P. (2008, 2009) extended the work of earlier researchers tracing different performance measures as satisfaction level of demand maker, due date on flow shop and parallel machine under fuzzy environment and obtained encouraging results. Singh T.P. & Sunita (2010) applied α – cut approach to determine average higher ranking of fuzzy processing time.

The present paper differs with the study made by Sunita and T.P. Singh (2009, 2010) in the sense that Robust Ranking technique has been applied rather than the average high Ranking technique or Yager's formula. This technique is comparatively easier and can be applied to solve large number of fuzzy scheduling problems. The study explores heuristic algorithm to find the optimal sequence for fuzzy scheduling in which there is a transporting agent which returns to first machine after delivering the task on second machine. The algorithm is based on a theorem which is justified for the application of fuzzy ranking robust index for optimality of sequence. A numerical example has also been presented to clear the algorithm.

2. PRACTICAL SITUATION

Literature on fuzzy scheduling has considered a significant number of real-time events and their effects considering various manufacturing system including single and parallel machine system, flow shops, job shops and flexible manufacturing system. In sugar manufacturing industry the device named as trolley used to deliver sugarcane for crushing and then returning back acts as transporting agent. Similar many times, truck loaded the material sand & bazri from mine and then deliver on either construction of road side or multistory building by a colonizer or contractor is under progressive work, acts as a transport agent.

3. PRELIMINARIES

3.1 Fuzzy Set

Zadeh in 1965 first introduced fuzzy set as a mathematical way of representing impreciseness or uncertainty in day to day life. A fuzzy set is a generalization of a crisp set. It is defined on a domain X by its membership function from X to [0, 1].

Mathematically, it is defined by,

$$\mu_A(x) : X \rightarrow [0,1]$$

3.2 Triangular fuzzy number

For a triangular number A(x), it can be represented by three parameters (a,b,c) with membership function $\mu(x)$ given by

$$\mu(x) = \left\{ \begin{array}{ll} \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x = b \\ \dfrac{c-x}{c-b}, & b \le x \le c \\ 0, & otherwise. \end{array} \right\}$$

3.3 $\alpha - cut\ of\ a\ fuzzy\ number$

The $\alpha - cut$ of a fuzzy number A(x) is defined as:

A (α) = $\{X: \mu(x) \geq \alpha, \alpha \epsilon [0,1]\}$

Addition of two triangular fuzzy number can be performed as:

$(a_1,b_1, c_1) + (a_2,b_2, c_2) = (a_1 +a_2, b_1+b_2, c_1+c_2)$

3.4 Robust Ranking Technique

To find the performance measure in term of crisp value we defuzzify the fuzzy numbers into crisp ones by a fuzzy robust ranking index method. Robust Ranking technique which satisfy compensation, linearity and additive properties and provides result which are consistent with perceptions. Given a convex fuzzy number $\tilde{a}$, the Robust Ranking index is defined by:-

$$R(\tilde{a}) = \int_0^1 0.5 \ (a_\alpha^L + a_\alpha^U) d\alpha$$

Where (a_α^L, a_α^U) is the α −level cut of the fuzzy number $\tilde{a}$.

4. FLOW SHOP SCHEDULING WITH TRANPORTATION TIME BETWEEN MACHINES

Let us consider n jobs (I_1, I_2,.........I_{i-1}, I_i, I_{i+1}.......... I_n) being processed through two machines (A and B) in the order AB with an agent who transport a job processed at machine A to machine B and then returns back empty to A again to transport next job. Let A_i and B_i be the service time on A and B respectively. Let t_i be the transportation for i^{th} job to carry it from A to B and r_i is the returning time from machine B to A after delivering i^{th} job. The Objective of problem is to find the optimal schedule of jobs which minimizes the total production time for completing all the jobs.

Now we state a theorem which provides a procedure to get an optimal schedule:

5. THEOREM

The optimal Schedule of jobs where the processing time of jobs are under fuzzy parameters is given by sequencing the jobs $i-1, i, i+1$ such that:

$$\min\,(R_i\tilde{A}) + t_i + K_{i-1}, R_{i+1}(\tilde{B}) + t_{i+1} + K_i) < \min(R_{i+1}(\tilde{A}) + t_{i+1} + K_i, R_i(B) + t_i + K_{i-1})$$

$$K_{i-1} = \begin{cases} (t_{i-1} + r_{i-1} - A_i), & \textit{if it is positive} \\ 0 \quad , & \textit{otherwise} \end{cases}$$

where, $t_{i-1\rightarrow}$ transportation time of $(i-1)^{th}$ job

r_{i-1} →time of transporting Agent for $(i-1)^{th}$job

Where, $R_i(\tilde{A})$ and $R_i(\tilde{B})$ = Robust Ranking fuzzy index for machines A & B as defined in section 3.4 by:-

$$R_i(\tilde{A}) = \int_0^1 0.5\,(a_\alpha^L + a_\alpha^U)\, d\alpha$$

and

$$R_i(\tilde{B}) = \int_0^1 0.5\,(a_\alpha^L + a_\alpha^U)\, d\alpha$$

where (a_α^L, a_α^U) is the lower and upper bound of α – level cut of fuzzy processing time $\tilde{A}_i$ & $\tilde{B}_i$ respectively.

Proof:- Denoting the sequences of jobs by S and S^1 :-

$$S = [I,\ I_2, \dots\dots\dots\dots I_{i-1}, I_i, I_{i+1} \dots\dots\dots I_n]$$

$$S' = (I_1', I_2', \dots\dots\dots.. I_{i-1}', I_i', I_{i+1}' \dots\dots\dots.. I_n')$$

Consider (X_p, X'_p) and (CX_p, CX'_p) the processing time and completion time of job p on machine X(= A or B) for the sequencs (S,S'). R_p (X), $R_p(X')$ and R_p C(X), R_p C(X') are the Robust Ranking index of the processing time and completion time of job P on machine X (= A or B), for the sequences. Let (t_p, t'_p) be the transportation times of job p form machine A to machine B for the sequences. r_p is the returning time of the transportation agent from machine B to machine A after delivering the p^{th} job at machine B.

The Completion time of p^{th} job on machine B is given by:-

R_P C(B) = max (R_p C (A) + t_p + K_{p-1}, R_{p-1} C (B) + R_p (B) ………………. (1)

Choose the sequences S in such a way, so that :-

R_p C(B) < R_p C' (B) ………………. (2)

i.e. if max (R_n C(A) + t_n + $K_{n-1,}$ R_{n-1} C(B) + R_n (B) < max (R_n C' (A) + t'_n + K'_{n-1}, h_{n-1} C'(B)) + R_n (B'),

As R_n C(A) + t_n + K_{n-1} = $\sum_{i=1}^{n} R_i(A) + t_n + K_{n-1} = R_n C'(A) + t'_n + K'_{n-1}$ and R_n (B) = R_{n+1} (B'),

Inequality (2) will be true if : R_{n-1} C(B) < R_{n-1} C' (B), ………………. (3)

Moving in the same way, we find that inequality (2) is true if:-

R_p C(B) < R_p C' (B) (P = i+1, i+2, ……….. n and i = 1,2, ……… n-1)…. ………………. (4)

Calculating the value R_{i+1} C(B) and R_{i+1} C'(B);

R_{i+1} C(B)= max (R_{i+1} C(A) + t_{i+1} + K_i, R_i C(B)) + R_{i+1} $(\tilde{B})$

= max (R_{i+1} C(A) + t_{i+1} + K_i,

max (R_i C(A) + t_i + K_{i-1}, R_{i-1} C(B)) + R_i $(\tilde{B})$ + R_{i+1} $(\tilde{B})$

= max (R_{i+1} C(A) + t_{i+1-} + K_i+R_{i+1} $(\tilde{B})$, R_i C(A) + t_{i-} + K_{i-1} + R_i $(\tilde{B})$ + R_{i+1} $(\tilde{B})$,

R_{i-1} C(B) + R_i $(\tilde{B})$ + R_{i+1} $(\tilde{B})$)

= max (R_{i-1} C(A) + R_i $(\tilde{A})$ + R_{i+1} $(\tilde{A})$ + t_{i+1-} + K_i + R_{i+1} $(\tilde{B})$, R_{i-1} C(A) + R_i $(\tilde{A})$

+ t_{i-} + K_{i-1} + B_i + R_{i+1} $(\tilde{B})$, R_{i-1} $(\tilde{B})$ + R_i $(\tilde{B})$ + R_{i+1} $(\tilde{B})$ ………………. (5)

On the similar pattern,

R_{i+1} C'(B) = max (R_{i-1} C' (A) + R_i $(\tilde{A}')$ + R_{i+1} $(\tilde{A}')$ + R_{i+1} $(\tilde{A}')$ + t'_{i+1-} + K'_i + R_{i+1} $(\tilde{B}')$,

R_{i-1} C' (A) + R_i $(\tilde{A}')$ + t'_i + K'_{i-1} + R_i $(\tilde{B}')$ + R_{i+1} $(\tilde{B}')$, R_{i-1} $(\tilde{B}')$ + R_i $(\tilde{B}')$ + R_{i+1} $(\tilde{B}')$)

For the sequence S and S' it is obvious that

R_{i-1} C (A) = R_{i-1} C'(A),

R_{i-1} C(B) = R_{i-1} C'(B)

R_i (x) = R_{i-1} (x'); (x = A or B) ; t_i = t'_{i+1}

R_{i+1}(x)= R_i (x'); (x = A or B); t_{i+1} = t'_I ………………. (6)

K_{i-1} = K'_i ; K_i = K'_{i-1}

expressing inequality (4) for p = i+1, using (6) we find max [R_{i-1} C(A) + R_i $(\tilde{A})$ + R_{i+1} $(\tilde{A})$ + t_{i+1} + K_i + R_{i+1} $(\tilde{B})$,

R_{i-1} C(A) + R_i $(\tilde{A})$ + t_i + K_{i-1}+ R_i $(\tilde{B})$ + R_{i-1} $(\tilde{B})$, R_{i-1} C(B) + R_i $(\tilde{B})$ + R_{i+1} $(\tilde{B})$]

< max [R_{i-1} C(A) + R_{i+1} $(\tilde{A})$ + R_i $(\tilde{A})$ + t_i + K_{i-1} + R_i $(\tilde{B})$,

R_{i-1} C(A) + R_{i+1} $(\tilde{A})$ + t_{i+1} + K_i + R_{i+1} $(\tilde{B})$ + R_i $(\tilde{B})$, R_{i-1} C(B) + R_i $(\tilde{B})$ + R_{i+1} $(\tilde{B})$ ………………. (7)

Subtracting (R_{i-1} C(B) + R_i $(\tilde{B})$ + R_{i+1} $(\tilde{B})$ from both sides, the inequality (7) reduces to :-

max [R_{i-1} C(A) + R_i $(\tilde{A})$ + R_{i+1} $(\tilde{A})$ + t_{i+1} + K_i + R_{i+1} $(\tilde{B})$,

R_{i-1} C(A) + $R_i(\tilde{A})$ + t_i + K_{i-1} + R_i $(\tilde{B})$ + R_{i-1} $(\tilde{B})$]

< max [R_{i-1} C(A) + R_{i+1} $(\tilde{A})$ + R_i $(\tilde{A})$ + t_i + K_{i-1} + R_i $(\tilde{B})$,

R_{i-1} C(A) + R_{i+1} $(\tilde{A})$ + t_{i+1} + K_i + R_{i+1} $(\tilde{B})$ + R_i $(\tilde{B})$]

Again subtracting :-

(R_{i-1} C(A) + R_i $(\tilde{A})$ + R_{i+1} $(\tilde{A})$ + t_i + K_{i-1} + K_i + t_{i+1} + R_i $(\tilde{B})$ + R_{i+1} $(\tilde{B})$, from each side we have:-

max (-R_i $(\tilde{B})$ – t_i – $K_{i-1,-}$ R_{i+1} $(\tilde{A})$ – t_{i+1} – K_i)

< max (- R_{i+1} $(\tilde{B})$ – t_{i+1} – $K_{i,}$ – R_i $(\tilde{A})$ – t_i – K_{i-1}) or

min (R_i $(\tilde{A})$ + t_i + K_{i-1}, R_{i+1} $(\tilde{B})$ + t_{i+1} + K_i)

< min (R_{i+1} $(\tilde{A})$ + t_{i+1} + K_i, R_i $(\tilde{B})$ + t_i + K_{i-1})

Which verify the stated theorem for optimality of job sequences.

6. STATEMENT OF PROBLEM

In general, the problem in tabular form can be stated:-

Job (i)	Machine (A) (processing time Ai)	t_i	r_1	Machine (B) (processing time Bi)
1	A_1	t_i	r_1	B_1
2	A_2	t_2	r_2	B_2
3	A_3	t_3	r_3	B_3
.	.	.	.	.
.	.	.	.	.
n	A_n	t_n	r_n	B_n

Where $A_{i,}$ B_i are the processing times on machine A and B in fuzzy parameters. t_i is the transportation times of jobs i from machine A to B and r_i is the returning time of the transport agent from B to A after delivering the i^{th} job.

Our objective is to find the sequence of jobs which minimize total processing time.

Based on the above stated theorem an **algorithm** for optimal sequence can be summarized as follows:-

Step 1 :- First convert fuzzy processing time into crisp one using Robust Ranking Technique.

Step 2 :- Assuming two fictitious machines (G and H) in place of A and B respectively, defining the times G_i & H_i as :-

$$G_i = K_{i-1} + t_i + R_i(\tilde{A}_i)$$

$$H_i = K_{i-1} + t_i + R_i(\tilde{B}_i)$$

Step 3:- Apply Johnson's (1954) technique for 2 stage problem we get optimal sequence of jobs.

Numerical Example:-

Consider a machine tandem scheduling flow shop problem given in the following table form:-

Job(i)	A_i (in fuzzy parameters)	t_i	r_i	B_i (in fuzzy parameters)
1	(6,8,9)	6	3	(6,7,8)
2	(4,5,7)	5	3	(5,9,11)
3	(3,6,7)	2	3	(4,7,9)
4	(5,8,10)	4	3	(6,9,12)
5	(6,7,9)	12	3	(9,11,13)

Where $r_i = 3$ (constant) for all i and A_i, B_i and t_i are defined as already.

SOLUTION PROCEDURE

Setp 1 :- Applying above remark 5, i.e. convert fuzzy processing time into crisp one using Robust Ranking Index.

$(6,7,8) = \frac{x-6}{2} = \alpha, \quad x = 2\alpha + 6$

and $\frac{9-x}{1} = \alpha, \quad x = 9 - \alpha$

$R(6,7,8) = \int_0^1 0.5\,(2\alpha + 6 + 9 - \alpha)d\alpha$

$= 7.75$

Likewise other values :

Job (i)	Machine (A)	$R_i(\tilde{A})$	t_i	r_i	k_{i-1}	machine B	$R_i(\tilde{B})$
1	(6,8,9)	7.75	6	3	--	(6,7,8)	7
2	(4,5,7)	5.25	5	3	3.75	(5,9,11)	7.50
3	(3,6,7)	5.50	2	3	2.50	(4,7,9)	6.75
4	(5,8,10)	7.75	4	3	0	(6,9,12)	9
5	(6,7,9)	7.25	12	3	0	(9,11,13)	11

Step 2 - The processing time for these fictitious machines G and H in place of A, B defined in Algorithm, can be put in tabular form :-

Job	Machine G	Machine H
1	13.75	13
2	14	16.25
3	10	11.25
4	11.75	13
5	19.25	23

Using Johnson's technique to the reduced problem we find optimal or near optimal sequence is 3-4-2-5-1.

Table for calculating total processing time & other performance measure for sequence <3-4-2-5-1>.

Job	Machine(A)	k_A	t_i	r_i	k_B	machine(B)	I_A	I_B	I_k
3	(3,6,7)	(0,0,0)	2	3	(5,8,9)	(9,15,18)	(0,0,0)	(5,8,9)	(3,6,7)
4	(8,14, 17)	(8,11,12)	4	3	(12,15,16)	(18,24,30)	(0,0,0)	(3,3,0)	(0,3,5)
2	(12,19,24)	(15,21,24)	5	3	(20,26,29)	(25,36,44)	(0,0,0)	(2,0,0)	(0,0,0)
5	(18,26,33)	(23,29,32)	12	3	(35,41,44)	(44,52,57)	(0,0,0)	(10,5,0)	(0,0,1)
1	(24,34,42)	(38,44,47)	6	3	(44,50,53)	(50,59,63)	(26,25,21)	(0,0,0)	(0,0,0)

k_A represents the time at which the transport agent returns to machine A to take the next job and k_B represent the time at which the transport agent reaches to machine B. The total processing time of all the jobs in the system (i.e. total productions times) is (50, 59, 63) hours, Idle time for the machine A is (26,25,21) hours and for the machine B is (20,16, 12) and the transport agent is free for (9,8,23) hours.

7. CONCLUSION

In the present study a single constraint of transporting agent has been taken in account. Idea of high inventory cost between two or more machine with more constraints or more transporting agents can also be added. Moreover, the approach can also be extended with more fuzzy parameters. The transporting agent which is free for certain hours can be assigned another work during this vacation period.

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APPLICATIONS OF EULERIAN GRAPH FOR MATHEMATICAL AND REAL SITUATION

Ajay Kumar*, Manju Pruthi**
* & ** Department of Mathematics, Indira Gandhi University, Meerpur-Rewari, Haryana
E-mail : ajay61286@gmail.com, manju.pruthi@yahoo.com

ABSTRACT :

Graph Theory is an important topic of discrete mathematics and computer science having applications in modeling a variety of real life situations in many disciplines. This article is intended for the attention of young researchers and provides an introductory discussion of some well known application problems in which role of eulerian graph and eulerian circuit is significant. In this paper, we have made an attempt to use the eulerian graph to solve the mathematical and real life situations.
Keywords: Graph, eulerian graph, graph coloring, degree of graph.

1. INTRODUCTION

Graph theory is the branch of mathematics, which has been applied to many problems in mathematics, computer science, biochemistry, electrical engineering, operational research and other scientific and not-so-scientific areas. Recently, Ajay & Pruthi M (2018) search out some applications of eulerian graph. In this paper we have extended our study to search out more applications of eulerian graph in real world scenario.

2. SOME DEFINITIONS

2.1 Graph: A graph G= (V,E) is a mathematical structure where V(G) is the vertex set and E(G) is the edge set. called its endpoints.

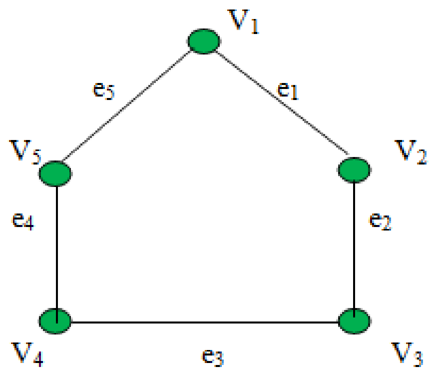
2.2 Degree of a vertex: The number of edges incident on a vertex v with self loops counted twice is called the degree of v and is denoted by deg (v).

2.3 Eulerian path (or Eulerian trail): In graph theory, an eulerian path is a path (in graph), which visits every edge exactly once.

2.4 Eulerian Circuit (or Eulerian cycle): An Eulerian circuit is an Eulerian trail which starts and ends on the same vertex.

2.5 Euler Graphs: In graph theory an Eulerian graph is a graph containing an eulerian cycle. It is assumed that Euler graphs do not have any isolated vertices and are thus connected.

Example : Consider the graph shown in given figure. Clearly, V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_5 e_5 V_1 is an Euler line.



3. THEOREM ON EULERIAN GRAPH

3.1 Theorem: Arrange the 10 vertices in such a manner that 5 of which are of degree 2 and 5 of which are of degree 4 and these vertices follow the eulerian path.

Proof: Suppose that we have 10 vertices in which 5 are blue color vertices and 5 are green color vertices. Now we arrange the 10 vertices in such a manner that 5 of which are of degree 2 and 5 of which are of degree 4 and these vertices follow the eulerian path.

For this we have follow the following steps:

(1) Let we start with blue vertex V_1 and link the vertex V_1 to vertex W_1 throw path A.

(2) Next we link the vertex W_1 to vertex W_2 throw path B.

(3) Next we link the vertex W_2 to vertex V_3 throw path C.

(4) Next we link the vertex V_3 to vertex W_3 throw path D.

(5) Next we link the vertex W_3 to vertex W_4 throw path E.

(6) Next we link the vertex W_4 to vertex V_5 throw path F.

(7) Next we link the vertex V_5 to vertex W_5 throw path G.

(8) Next we link the vertex W_5 to vertex W_1 throw path H.

(9) Next we link the vertex W_1 to vertex V_2 throw path I.

(10) Next we link the vertex V_2 to vertex W_2 throw path J.

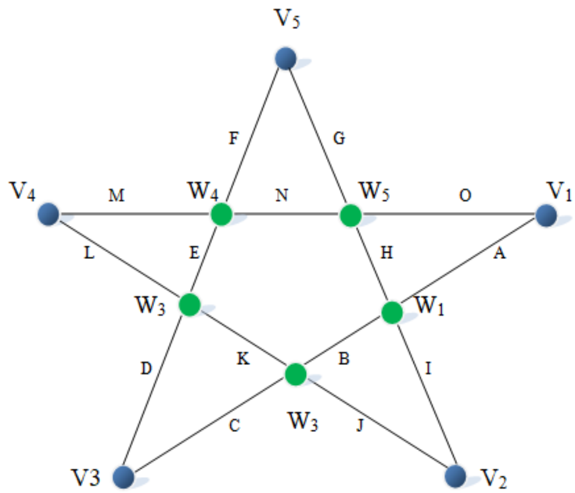
(11) Next we link the vertex W_2 to vertex W_3 throw path K.

(12) Next we link the vertex W_3 to vertex V_4 throw path L.

(13) Next we link the vertex V_4 to vertex W_4 throw path M.

(14) Next we link the vertex W_4 to vertex W_5 throw path N.

(15) Next we link the vertex W_5 to vertex V_1 throw path O.



Eulerian path is, A→B→C→D→E→F→G→H→I→J→K→L→M→N→O .

3.2 Diagram Tracing and Eulerian Graph: A popular old game that entertains children runs as follows:-
Can you trace with a pencil a diagram of points (with the small circles representing the points) and lines as shown in Figures 1(a) and 1(b). The condition is that the diagram is to be traced beginning at a point and on completion end at the same point but the pencil should not be lifted till the diagram is completely traced and a line in the diagram should not be retraced (i.e. can be traced only once). A curious child will certainly try to find the answer by trial and error and will arrive at the conclusion after sometime that it is not possible in diagram in Figure 1(b) but it is possible in the diagram in Figure 1(a). The question of whether it is possible to trace such a diagram, can be quickly answered if the concept of Eulerian graph is known. Indeed the diagram in Figure 1(a) is Eulerian whereas the diagram in Figure 1(b) is not.

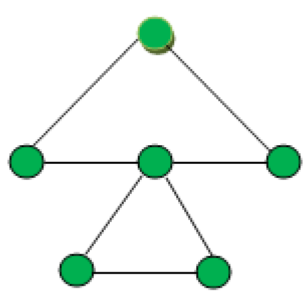


Figure 1 (a)

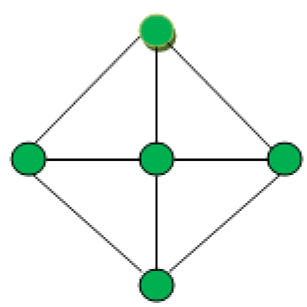


Figure 1(b)

3.3 DNA fragment assembly: DNA (deoxyribonucleic acid) is found in every living organism and is storage medium for genetic information. A DNA strand is composed of bases which are denoted by A (adenine), C (cytosine), G (Guanine) and T (thymine). The familiar DNA double helix arises by the bondage of two separate strands with the Watson-Crick complementary (A and T are complementary; C and G are complementary) leading to the formation of such double strands. DNA sequencing and fragment assembly is the problem of reconstructing full strands of DNA based on the pieces of data recorded. It is of interest to note that ideas from graph theory, especially Eulerian circuits have been used in a recently proposed approach to the problem of DNA fragment assembly. We do not enter into the details but only mention that this brings out the application of graph theory in the field of bioinformatics.

Some Other Applications of Eulerian Graph

Graph theory is the branch of mathematics which is used to solve many real life problems. In this way, eulerian graph plays an important role in mathematical and real life situations. More applications of eulerian graph are as :

(1) With the help of above theorem 3.1 we arrange the 10 vertices in such a way that we have total 5 rows and every row contain 4 vertices.

(2) Chinese post man Problem is solved by eulerian graph.

(3) Application area in which group leader of an organization or company organize training program for new recruited trainee in which some know already each other, some do not know each other.

(4) Surveillance of art gallery with cameras arrangement using eulerian graph.

(5) Eulerian graph is used for networks arrangement to provide best response to users.

CONCLUSION

Main objective of this paper is to study eulerian graph and its various aspects in our real life. Eulerian circuit represents the optimal solution with limited resources. Eulerian circuit provide the desirable result with minimum cost and minimum time. Thus we see that eulerian graph got height of achievement in many situations that occur in a relatively new situation.

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CORRELATION BETWEEN PYRAMIDAL NUMBERS

G. Janaki* & C. Saranya**

* Associate Professor, PG & Research Department of Mathematics, Cauvery College for Women, Trichy-18.
**Assistant Professor, PG & Research Department of Mathematics, Cauvery College for Women, Trichy-18.
E-mail: janakikarun@rediffmail.com* & c.saranyavinoth@gmail.com**

ABSTRACT :

A pyramidal number is a figurate number that represents to a pyramid with a polygonal base and a given number of triangular sides. In this paper, we have made an attempt to obtain various interesting relations among pyramidal numbers and other special numbers.

Keywords: Pyramidal number, Jarasandha number, Polygonal number & Special numbers.

2010 Mathematics Subject Classification: 11D99.

INTRODUCTION

Number theory has always fascinated amateurs as well as professional mathematicians. Rather than different parts of science, a considerable lot of the problems and theorems of number theory can be comprehended by laypersons, although solutions to the problems and proofs of the theorems often require a sophisticated mathematical background.

Until the mid-twentieth century, number theory was viewed as the most perfect part of arithmetic, with no immediate applications to this present reality. The advent of digital computers and digital communications revealed that number theory could provide unexpected answers to real-world problems.

In [1&2], nasty numbers were discussed. In [3,4,6,7&8], theory of numbers were discussed. In [5], icosagonal pyramidal number is observed for its properties. In [9,10], pyramidal numbers and pentatope number were assessed using z-transform and Division Algorithm. Recently in [11-12], pyramidal numbers were utilized for discovering sums of squares and lateral surface area of a cube.

In this paper, we obtain various interesting relations among pyramidal numbers and other special numbers.

NOTATIONS

P_n^m = Pyramidal number of rank 'n' with sides 'm'.

$T_{m,n}$ = Polygonal number of rank 'n' with sides 'm'.

$Star_n$ = Star number of rank 'n'.

CS_n = Centered square number of rank 'n'.

CH_n = Centered hexagonal number of rank 'n'.

Gno_n = Gnomonic number of rank 'n'.

j_n = Jacobsthal Lucas number of rank 'n'.

J_n = Jacobsthal number of rank 'n'.

Tha_n = Thabit-ibn-kurrah number of rank 'n'.

Mer_n = Mersenne number of rank 'n'.

INTERESTING RELATIONS

Relation 1:

The difference of pyramidal numbers of same order m with consecutive ranks n & n + 1 is the polygonal number of order m with rank n + 1.

Proof:

$$P_{n+1}^{m} - P_{n}^{m} = \frac{(n+1)(n+2)}{6}\left[(m-2)(n+1)+(5-m)\right] - \frac{n\,(n+1)}{6}\left[(m-2)\,n+(5-m)\right]$$

$$= (n+1)\left[1+\frac{n\,(m-2)}{2}\right]$$

$$= T_{m,\,n+1}$$

Relation 2:

The triples $(P_n^m, P_n^{m+k}, P_n^{m+2k})$ are in Arithmetic Progression.

Proof:

$$P_n^m + P_n^{m+2k} = \frac{n\;(n+1)}{6}\left[(m+k-2)\,n+(5-m-k)\right]$$

$$\Rightarrow \frac{P_n^m + P_n^{m+2k}}{2} = P_n^{m+k}$$

The triples are in Arithmetic Progression.

Relation 3:

The difference of pyramidal numbers of same rank n with orders m & m +k is k times the triangular pyramidal number.

Proof:

$$P_n^{m+k} - P_n^m = \frac{n\;(n+1)}{6}\left[(m+k-2)\;n+(5-m-k)\right]-\left[(m-2)\,n+(5-m)\right]$$

$$= \frac{n\;(n+1)}{6}(kn-k)$$

$$P_n^{m+k} - P_n^m = k\,P_{n-1}^3$$

Relation 4:

The product of pyramidal numbers of rank n with orders m & m+ 2 is the difference of the squares of pyramidal number of rank n with order m + 1 & triangular pyramidal number of rank n-1.

Proof:

$$\left(P_n^{m+2}\right)^2 + \left(P_n^m\right)^2 = 2\left[\left(P_n^{m+1}\right)^2 + \left(P_{n-1}^3\right)^2\right] \qquad (1)$$

Also, $P_n^{m+2} - P_n^m = 2P_{n-1}^3$

Squaring, $\left(P_n^{m+2}\right)^2+\left(P_n^m\right)^2=2P_n^mP_n^{m+2}+4\left(P_{n-1}^3\right)^2$ (2)

From (1) & (2),

$$P_n^mP_n^{m+2}=\left(P_n^{m+1}\right)^2-\left(P_{n-1}^3\right)^2$$

Relation 5:

For $m=3,4,\ldots 15$ we have $P_n^{m+15}-P_n^m=3(P_n^7-T_{3,n})$

Proof:

$$P_n^{m+15}-P_n^m=\frac{n\,(n+1)}{6}\left[(m+15-2)\,n+(5-m-15)\right]-\left[(m-2)\,n+(5-m)\right]$$

$$=\frac{n\,(n+1)}{6}\left[15\,(n-1)\right]$$

$$P_n^{m+15}-P_n^m=3(P_n^7-T_{3,n})$$

Relation 6:

$$P_{2^n}^m=\begin{cases}\dfrac{J_n}{2}\left[(m-2)2^n\,Mer_n+Tha_n+1\right], & n \text{ is odd}\\ \dfrac{j_n}{6}\left[(m-2)2^n\,Mer_n+Tha_n+1\right], & n \text{ is even}\end{cases}$$

Proof:

(i) $$2P_{2^n}^m=\frac{2^n\,(2^n+1)}{3}\left[(m-2)\,2^n+(5-m)\right]$$

$$=2^n J_n\left[(m-2)\,(2^n-1)+3\right]$$

$$=J_n\left[(m-2)2^n\,Mer_n+Tha_n+1\right]$$

$$\Rightarrow P_{2^n}^m=\frac{J_n}{2}\left[(m-2)2^n\,Mer_n+Tha_n+1\right], \quad \text{when } n \text{ is odd}$$

(ii) Similarly, $P_{2^n}^m=\dfrac{j_n}{6}\left[(m-2)2^n\,Mer_n+Tha_n+1\right]$, when n is even

Relation 7:

$2(P_{n+3}^5+P_{n-1}^5+2P_{n+3}^5P_{n-1}^5-4n^2-30\,n)-11$ is a perfect square.

Proof:

$2(P_{n+1}^5+P_{n-1}^5+2P_{n+3}^5P_{n-1}^5-4n^2-30\,n)-11=(n^3+4n^2-n-5)^2$ is a perfect square.

Relation 8:

$$6\,P_n^m+P_n^5-3P_n^3=2\,(n+1)\,T_{m,n}\,.$$

Proof:

$$3\,P_n^3-P_n^5=2\;T_{3,n}$$

Since, $T_{3,n}=3P_n^m-(n+1)\,T_{m,n}$

so that, $3P_n^3-P_n^5=2\left(3P_n^m-(n+1)\,T_{m,n}\right)$

$$\Rightarrow 6\,P_n^m+P_n^5-3P_n^3=2\,(n+1)\,T_{m,n}$$

Relation 9:

For $a,b=3,4,...30$ & $a>b$ we have $P_n^{m+k}-P_n^{m}=\frac{k}{a-b}(P_n^{a}-P_n^{b})$

Proof:

$$P_n^{m+k}-P_n^{m}=k\,(P_n^{4}-P_n^{3})$$
$$P_n^{m+k}-P_n^{m}=k\,(P_n^{5}-P_n^{4})$$
$$\cdots$$
$$\cdots$$
$$P_n^{m+k}-P_n^{m}=k\,(P_n^{30}-P_n^{29})$$

Adding all the above equations,

$$27\,(P_n^{m+k}-P_n^{m})=\frac{k}{a-b}(P_n^{30}-P_n^{3})$$

In general, $P_n^{m+k}-P_n^{m}=\frac{k}{a-b}(P_n^{a}-P_n^{b})$

Relation 10:

For $a,b=3,4,...30$ & $a>b$ we have $P_n^{m+k}-P_n^{m}=\frac{k\,(n+1)}{3}\left(\frac{T_{a,n}-T_{b,n}}{a-b}\right)$

Proof:

$$P_n^{m+k}-P_n^{m}=\frac{k\,(n+1)}{3}\left[T_{6,n}-T_{4,n}\right]$$
$$P_n^{m+k}-P_n^{m}=\frac{k\,(n+1)}{3}\left[T_{8,n}-T_{6,n}\right]$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$P_n^{m+k}-P_n^{m}=\frac{k\,(n+1)}{3}\left[T_{30,n}-T_{28,n}\right]$$

Adding all the above,

$$13\,(P_n^{m+k}-P_n^{m})=\frac{k\,(n+1)}{6}\left[T_{30,n}-T_{4,n}\right]$$
$$P_n^{m+k}-P_n^{m}=\frac{k\,(n+1)}{3(30-4)}\left[T_{30,n}-T_{4,n}\right]$$

In general, $P_n^{m+k}-P_n^{m}=\frac{k\,(n+1)}{3}\left(\frac{T_{a,n}-T_{b,n}}{a-b}\right)$

Relation 11:

When $n=3$, the expression $2\left(2P_{n+3}^{5}P_{n-1}^{5}+P_{n+3}^{5}P_{n-1}^{5}\right)-(8n^2+60n+11)$ is a Jarasandha number of order 4.

Proof:

$$2\left(2P_{n+3}^{5}P_{n-1}^{5}+P_{n+3}^{5}P_{n-1}^{5}\right)-(8n^2+60n+11)=(n^3+4n^2-n-5)^2$$

When $n=3$, The expression = 55^2 = 3025 which is a Jarasandha number of order 4.

Relation 12:

$$3\,P_n^m = (n+1)\,T_{m,n} + T_{3,n}$$

Proof:

$$3\,P_n^m = \frac{3n\,(n+1)}{6}\left[(m-2)n + (5-m)\right]$$

$$= (n+1)\,n\left[1 + \frac{(n-1)(m-2)}{2}\right] + T_{3,n}$$

$$3\,P_n^m = (n+1)\,T_{m,n} + T_{3,n}$$

Relation 13:

$$P_n^5 = \begin{cases} \text{Nasty number, if } \; n = 12k^2 - 1 \\ \text{Perfect square, if } \; n = 2k^2 - 1 \end{cases} \qquad \text{where } k = 1,2,...$$

Proof:

(i) $P_n^5 = \dfrac{n^2\,(n+1)}{2} = \dfrac{(12k^2-1)^2}{2}(12k^2) = 6\left(k\,(12k^2-1)\right)^2 =$ Nasty number

(ii) $P_n^5 = \dfrac{(2k^2-1)^2}{2}(2k^2) = \left(k\,(2k^2-1)\right)^2 =$ Perfect Square

Relation 14:

$$3(P_n^{m+k} - P_n^m) = k\,(P_n^5 - T_{3,n})$$

Proof:

$$3(P_n^{m+k} - P_n^m) = k\left[\frac{n^2(n+1)}{2} - \frac{n\,(n+1)}{2}\right]$$

$$= k\,(P_n^5 - T_{3,n})$$

$$3(P_n^{m+k} - P_n^m) = k\,(P_n^5 - T_{3,n})$$

Relation 15:

For $a,b,c,d = 3,4,...30$ & $a > b,\ c > d$ & $a - b = c - d = 1$ we have $\dfrac{P_n^a - P_n^b}{P_n^c - P_n^d} = 1$

Proof:

Since $P_n^m + P_n^{m+2k} = 2\,P_n^{m+k}$

Put $k = 1$, $P_n^m + P_n^{m+2} = 2\,P_n^{m+1}$

When $m = 3$, $P_n^3 + P_n^5 = 2\,P_n^4$

When $m = 4$, $P_n^4 + P_n^6 = 2\,P_n^5$

……

……

When $m = 28$, $P_n^{28} + P_n^{30} = 2P_n^{29}$

Adding all the above equations,

$$P_n^3 + P_n^4 + P_n^{29} + P_n^{30} = 2P_n^4 + 2P_n^{29}$$

$$P_n^{30} - P_n^{29} = P_n^4 - P_n^3$$

$$\therefore \quad \frac{P_n^{30} - P_n^{29}}{P_n^4 - P_n^3} = 1$$

In general , $\frac{P_n^a - P_n^b}{P_n^c - P_n^d} = 1$ where $a - b = c - d = 1$

Relation 16:

$$(P_n^m)^2 - (P_{n-1}^m)^2 = \frac{1}{3}\left[(T_{m,n})^2 \mathrm{Gno_n} + 2T_{m,n}\, T_{3,n}\right]$$

Proof:

Since $P_{n+1}^m - T_{m,n+1} = P_n^m$

$$P_n^m - T_{m,n} = P_{n-1}^m$$

Squaring, $\left(P_n^m\right)^2 + \left(T_{m,n}\right)^2 - 2P_n^m T_{m,n} = \left(P_{n-1}^m\right)^2$

$$(P_n^m)^2 - (P_{n-1}^m)^2 = \frac{1}{3}\left[(T_{m,n})^2 \mathrm{Gno_n} + 2T_{m,n}\, T_{3,n}\right]$$

Relation 17:

$$mP_n^m - (m-2)P_n^{m+2} = 2T_{3,n}$$

Proof:

$$mP_n^m - (m-2)P_n^{m+2} = m\left[\frac{n(n+1)}{6}\big((m-2)n + (5-m)\big)\right] - (m-2)\left[\frac{n(n+1)}{6}\big(mn + (3-m)\big)\right]$$

$$= \frac{n(n+1)}{6}(6)$$

$$= 2T_{3,n}$$

Relation 18:

$$P_n^a - P_n^b = \frac{n+1}{3}\left(T_{a,n} - T_{b,n}\right) \text{where } a > b \,\&\, a,b = 3,4,...20$$

Proof:

From the relations (9) and (10)

$$P_n^a - P_n^b = \frac{n+1}{3}\left(T_{a,n} - T_{b,n}\right) \text{where } a > b \,\&\, a,b = 3,4,...20$$

Relation 19:

$$P_n^{m+k} - P_n^m = \frac{k(n+1)}{108}\left[\mathrm{Star}_n + 2CH_n + 3CS_n - 6\right]$$

Proof:

$$P_n^{m+k} - P_n^m = \frac{k\,(n+1)}{6}(n^2 - n)$$

$$= \frac{k\,(n+1)}{12}\left(CS_n - 1\right)$$

$$P_n^{m+k} - P_n^m = \frac{k\,(n+1)}{12}\left(CS_n - 1\right)$$

Similarly, $P_n^{m+k} - P_n^m = \frac{k\,(n+1)}{36}\left(\mathrm{Star}_n - 1\right)$

$$P_n^{m+k} - P_n^m = \frac{k\,(n+1)}{18}\left(CH_n - 1\right)$$

Adding all the above equations,

$$P_n^{m+k} - P_n^m = \frac{k\,(n+1)}{108}\left[\mathrm{Star}_n + 2\,CH_n + 3\,CS_n - 6\right]$$

Relation 20:

$$P_n^5 + 3P_{n-1}^3 = Gno_n\;T_{3,n}$$

Proof:

$$P_n^5 + 3P_{n-1}^3 = \frac{n^2\,(n+1)}{2} + \frac{3(n-1)\,n\,(n+1)}{6}$$

$$= Gno_n\;T_{3,n}$$

Relation 21:

$$P_n^3 P_{n+1}^3 = \frac{2}{9} T_{3,n}\;T_{3,n+1}\;T_{3,n+2}$$

Proof:

$$P_n^3 P_{n+1}^3 = \frac{n(n+1)(n+2)(n+1)(n+2)(n+3)}{36}$$

$$= \frac{2}{9} T_{3,n}\;T_{3,n+1}\;T_{3,n+2}$$

Relation 22:

$$P_{n-1}^3 = \frac{n+1}{6}\left(T_{m+2,n} - T_{m,n}\right)$$

Proof:

Since $\left(P_n^m\right)^2 + \left(P_n^{m+2}\right)^2 = 2\left(P_n^{m+1}\right)^2 + 2\left(P_{n-1}^3\right)^2$ (3)

$$\left(P_n^m\right)^2 + \left(P_n^{m+2}\right)^2 = \frac{(n+1)^2}{9}\left(T_{m+2,\,n} - T_{m,\,n}\right)^2 + 2\left(P_n^{m+1}\right)^2 - 2\left(P_{n-1}^3\right)^2 \quad (4)$$

From (3) & (4) ,

$$P_{n-1}^3 = \frac{n+1}{6}\left(T_{m+2,\,n} - T_{m,\,n}\right)$$

CONCLUSION

Various interesting relations among pyramidal numbers and other special numbers have been search out. To conclude, one may obtain different interesting relations for other numbers also.

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RELIABILITY ANALYSIS OF DIE CASTING MACHINE SYSTEM HAVING TWO TYPES REPAIR FACILITY WITH CONDITION OF REST

Renu*, Pooja Bhatia**
*Research Scholar, Department of Mathematics, Baba Mastnath University, Rohtak, Haryana
**Associate Prof. Department of Mathematics, Baba Mastnath University, Rohtak, Haryana
E-mail : renushiv85@gmail.com, budhirajapooja92@gmail.com

ABSTRACT :

We are taking two high pressure die casting machines case, initially the primary unit is working while the second unit is at cold standby, non priority is standby mode the unit cannot fail in standby mode. All the state possibilities for the system have been discussed. in up states and down states. On failure of the primary unit, standby and secondary unit is operative and failed unit is put under repair immediately. The secondary unit may be working to certain time called maximum allowable operative time. The maximum allowable operative time is over the second unit is put under rest if the primary unit is in progress. In this mode two types of repair facility is considered one ordinary repairman and other expert repairman. The expert repairman is only need based when the ordinary repairman fails to repair the failed primary unit .Secondary unit is always repaired by ordinary repairman. The Failure time of primary as well as secondary unit are exponentially distributed but with different parameters. The repair time distribution of primary unit is arbitrary for both expert repairman and ordinary repairman with different probability distributions. The repair time of standby unit follows exponential distribution .The distribution of rest time, maximum allowable operative time and patience time of ordinary repairman are also assumed to follow exponential distribution distributions with different parameters. Using mean time to system failure, steady state, availabilities, expected no of visit of the system,, system behavior & profit function have been discussed in all possible cases.

Keywords: MTSF, Markov process, regenerative state, reliability, standby system.

INTRODUCTION

Recently, Renu & Pooja B. (2019) made an attempt to find out the reliability of two high pressure die casting machines case, initially the primary unit is working and the secondary unit is at cold standby and calculated the several characteristic of interest such as mean sojourn time, MTSF using Markov processes and regenerative point technique. Further Renu & Pooja (2019) explored the reliability & profitability of the system in different failure mode.

This paper is further an extended work of the system for two high pressure die casting machine system in which all the possible states of the system have been discussed. There are some states which are called up states and some down states. The up states are those states in which at least unit machine is in operative mode either primary or secondary. The down states are in which both machines are not in operative condition. On the failure of the primary unit, standby and secondary unit becomes operative. The failed unit is put under repair immediately. The secondary unit is allowed working to certain time called maximum allowable operative time. When the maximum allowable operative time is over the secondary unit is put under rest if the primary unit is in progress. In this model are two types of repair facility one is ordinary repairman and expert repairman. The expert repairman is only needed when the ordinary repairman fails to repair the failed primary unit and secondary unit is always repaired by the ordinary repairman. It is assumed that the Failure time of primary as well as secondary unit are exponentially distributed but with different parameters. The repair time distributions of primary unit is arbitrary for both expert repairman and ordinary repairman but with different probability distributions. The repair time of standby unit follows exponential distribution .The distribution of rest time, maximum allowable operative time and patience time of ordinary repairman are also assumed to follow exponential distributions with different parameters. Using mean time to system failure, system behavior & profit function have been studied in different cases.

NOTATIONS

X_0	:	Priority unit is operative
Y_0	:	Non priority unit is operative
Y_S	:	Non priority unit is in standby mode
X_r	:	Priority unit is under repair, repaired by ordinary repairman.
X_{er}	:	Priority unit is under repair, repaired by expert repairman.
Y_r	:	Non priority unit is under repair
Y_W	:	Non priority is waiting for repair
Y_{rest}	:	Non priority put under rest after maximum allowable operative time.
$H_1(.)$	:	c d f repair time of priority unit for ordinary repairman
$H_2(.)$	:	c d f repair time of priority unit for expert repairman
μ_1	:	Parameter of repair time distribution for non priority unit
μ_2	:	Parameter of rest time distribution for non priority unit
α_1	:	Parameter of failure time distribution for main unit
α_2	:	Parameter of failure time distribution for secondary unit
p_1	:	Probability of repaired unit in working unit
p_2	:	Probability of repaired unit require post repair
r	:	Repair rate of priority unit
*	:	Symbol for Laplace transformation $F^*(s)=\int_0^\infty e^{-st}\,f(t)\,dt$
□	:	Symbol for Laplace Stieltjes transformation F □(s)= $\int_0^\infty e^{-st}\,dF(t)$
Ψ_i	:	Mean sojourn time in state S_i.
$M_i(t)$	:	Probability that the sojourns in state S_i upto time t.
$\Phi_i(t)$	:	where starting from up state S_0 cdf of time to the system.
©	:	Symbol for Laplace convolution
*	:	Symbol for Laplace transformation $F^*(s)=\int_0^\infty e^{-st}\,f(t)\,dt$
□	:	Symbol for Laplace Stieltjes transformation F □(s)= $\int_0^\infty e^{-st}\,dF(t)$

THE STATES OF THE SYSTEM

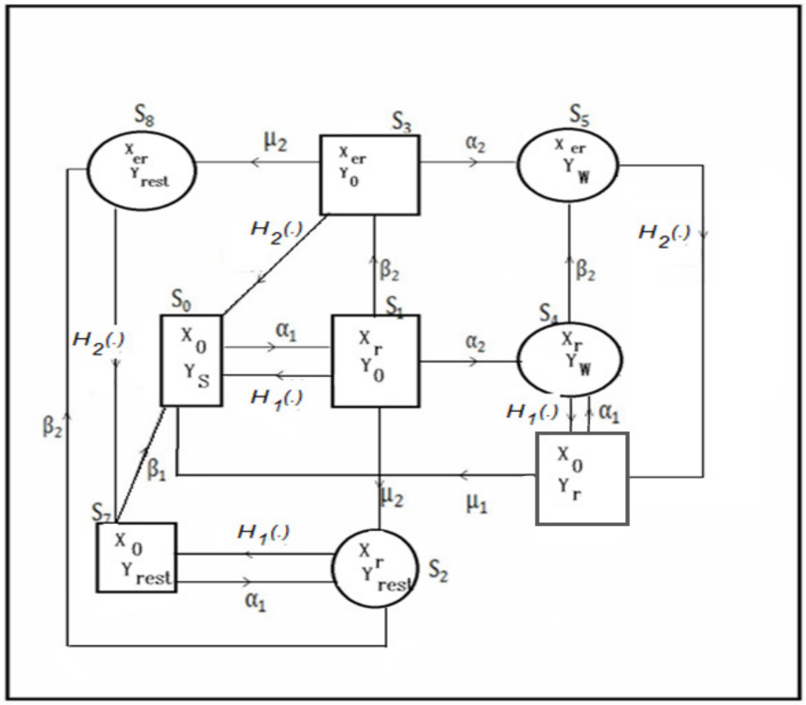
The different states of the system having all the possibilities either main unit is in operative state, failed state, repairing state, inspection state and similarly for second standby unit, the possibilities are in operative state, repairing state, inspection state, failed state, waiting state are taken into account.

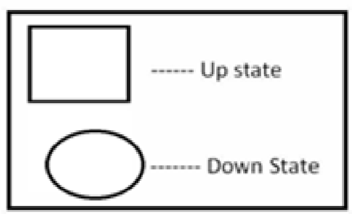
$S_0=[X_0, Y_S]$;	$S_1=[X_r, Y_0]$;	$S_2=[X_r, Y_{rest}]$;
$S_3=[X_{er}, Y_0]$;	$S_4=[X_r, Y_W]$;	$S_5=[X_{er}, Y_W]$;
$S_6=[X_0, Y_r]$;	$S_7=[X_0, Y_{rest}]$;	$S_8=[X_{er}, Y_{rest}]$;

THE MODEL

The figure showing all the possible states of the system, some states are up states and some are down states. The states S_0, S_1, S_2, S_3, S_6 are up states and the states S_4, S_5, S_7 are down states. The all possibilities are shown

TRANSITION DIAGRAM





TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES

Here $Q_{ij}(t)$ denotes the $c\ d\ f$ (cumulative distribution function) of transition time from state Si to Sj in 0 to t .To determine the transition probabilities of states. Let T_0, T_1, T_2 ,....denotes the regenerative epochs. Then{ Xn, Tn} constitute a space E, set of regenerative states and $Q\ ij\ (t) = P[Xn + 1 = \ j, Tn + 1 - Tn\ \leq\ t\ /Xn = i]$ is the semi Markov over E .The various transition probabilities are:

$Q_{01}(t) = \alpha_1 \int_0^t e^{-\alpha_1 u} du$ $\qquad$ $Q_{10}(t) = \int_0^t e^{-(u_2+\alpha_2+\beta_2)u} dH_1(u)$

$Q_{13}(t) = \beta_2 \int_0^t e^{-(u_2+\alpha_2+\beta_2)u}\, \overline{H}_1(u)$

$Q_{15}^{(4)}(t) = \alpha_2 \int_0^t e^{-(u_2+\alpha_2+\beta_2)u}\, \overline{H}_1(u) \int_u^t \frac{\beta_2 e^{-\beta_2(v-u)}\overline{H}_1(v)}{\overline{H}_1(u)} dv$ $\qquad$ $= \frac{\beta_2\alpha_2}{(\alpha_2+u_2)} \int_0^t e^{-\beta_2 v}\overline{H}_1(v)\ \left[1 - e^{-(\alpha_2+u_2)v}\right] dv$

$Q_{16}^{(4)}(t) = \frac{\alpha_2}{(\alpha_2+u_2)} \int_0^t e^{-\beta_2 v}\overline{H}_1(v)\ \left[1 - e^{-(\alpha_2+u_2)v}\right] dv$

$Q_{17}^{(2)}(t) = \frac{\mu_2}{(\alpha_2+u_2)} \int_0^t e^{-\beta_2 v})\ \left[1 - e^{-(\alpha_2+u_2)v}\right] dH_1 v$

$Q_{18}^{(2)}(t) = \frac{\beta_2\mu_2}{(\alpha_2+u_2)} \int_0^t e^{-\beta_2 v}\overline{H}_1(v)\ \left[1 - e^{-(\alpha_2+u_2)v}\right] dv$

$Q_{27}(t) = \int_0^t e^{-\beta_2 u} d\, H_1(u)$ $\quad Q_{28}(t) = \beta_2 \int_0^t e^{-\beta_2 u}\, \overline{H}_1(u) du$

$Q_{30}(t) = \int_0^t e^{-(\alpha_2+u_2)u} dH_2\,(u)$

$Q_{36}^{(5)}(t) = \frac{\alpha_2}{(\alpha_2+u_2)} \int_0^t \left[1 - e^{-(\alpha_2+u_2)v}\right] dH_2(v)$

$Q_{37}^{(8)}(t) = \frac{\mu_2}{(\alpha_2+u_2)} \int_0^t \left[1 - e^{-(\alpha_2+u_2)v}\right] dH_2(v)$ $\quad Q_{45}(t) = \beta_2 \int_0^t e^{-\beta_2 u}\, \overline{H}_1(u) du$

$Q_{46}(t) = \int_0^t e^{-\beta_2 u} d\, H_1(u)$ $\quad Q_{56}(t) = \int_0^t dH_2\,(u)$

$Q_{60}(t) = \mu_1 \int_0^t e^{-(\alpha_1+u_1)u} du$ $\quad Q_{64}(t) = \alpha_1 \int_0^t e^{-(\alpha_1+u_1)u}\, du$

$Q_{70}(t) = \beta_1 \int_0^t e^{-(\alpha_1+\beta_1)u}\, du$

$Q_{72}(t) = \alpha_1 \int_0^t e^{-(\alpha_1+\beta_1)u}\, du$ $\quad Q_{87}(t) = \int_0^t dH_2\,(u)$

STEADY STATE TRANSITION PROBABILITES :

Taking the limit of t tends to infinity, the stated state transition probabilities are calculated as:.

$P_{01} = P_{56} = P_{87} = 1$ $\quad P_{10} = \widetilde{H_1}(u_2 + \alpha_2 + \beta_2)$ $\quad P_{13} = \frac{\beta_2[1-\widetilde{H_1}(u_2+\alpha_2+\beta_2)]}{u_2+\alpha_2+\beta_2}$

$p_{15}^{(4)} = \frac{\alpha_1[1-\widetilde{H_1}(\beta_2)]}{(\alpha_2+u_2)} - \frac{\alpha_1[1-\widetilde{H_1}(u_2+\alpha_2+\beta_2)]}{(\alpha_2+u_2)(u_2+\alpha_2+\beta_2)}$ $\quad p_{16}^{(4)} = \frac{\alpha_1[\widetilde{H_1}(\beta_2)-\widetilde{H_1}(u_2+\alpha_2+\beta_2)]}{(\alpha_2+u_2)}$

$p_{18}^{(2)} = \frac{\mu_1[1-\widetilde{H_1}(\beta_2)]}{(\alpha_2+u_2)} - \frac{\beta_1[1-\widetilde{H_1}(u_2+\alpha_2+\beta_2)]}{(\alpha_2+u_2)(u_2+\alpha_2+\beta_2)}$ $\quad p_{17}^{(2)} = \frac{\mu_1[\widetilde{H_1}(\beta_2)-\widetilde{H_1}(u_2+\alpha_2+\beta_2)]}{(\alpha_2+u_2)}$

$P_{27} = \widetilde{H_1}(\beta_2) = P_{46}$ $\quad P_{28} = 1 - \widetilde{H_1}(\beta_2) = P_{45}$ $\quad P_{30} = \widetilde{H_2}\,(\alpha_2 + u_2)$

$p_{36}^{(5)} = \frac{\alpha_2[1-\widetilde{H_2}(\alpha_2+u_2)\,]}{(\alpha_2+u_2)}$ $\quad p_{37}^{(8)} = \frac{\mu_2[1-\widetilde{H_2}(\alpha_2+u_2)\,]}{(\alpha_2+u_2)}$ $\quad P_{60} = \frac{\mu_1}{\alpha_1+u_1}$

$P_{64} = \frac{\alpha_1}{\alpha_1+u_1}$ $\quad P_{70} = \frac{\beta_1}{\alpha_1+\beta_1}$ $\quad P_{72} = \frac{\alpha_1}{\alpha_1+\beta_1}$

MEAN SOJOURN TIME

The sojourns time Si , denoted by Ψi which is the time spent in a particular state before going to another state. The sojourn times are $\Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4, \Psi_5, \Psi_6, \Psi_7$ and they are calculated as:

$$\Psi i = E[Ti] = \int P(Ti > t)\, dt$$

$\Psi_0 = 1/\alpha_1$ $\quad \Psi_1 = \frac{1-\widetilde{H_1}(u_2+\alpha_2+\beta_2)}{u_2+\alpha_2+\beta_2}$

$\Psi_6 = \frac{1}{u_1+\alpha_1}$ $\quad \Psi_7 = \frac{1}{\beta_1+\alpha_1}$

$M_{ij} = -Q_{ij}(0) = -\frac{d}{ds} \int_0^\infty e^{-st} dQ_{ij}(t)|_{s=0}$

$$\sum_j mij = \Psi i, for\ different\ values\ of\ i\ and\ j$$

m_{ij} is the mean elapsed time of the system in the state S_i to any other regenerative state S_j.

Thus one may obtain the following expressions for different m_{ij}'s

$m_{01} = \alpha_1 \int_0^\infty t e^{-\alpha_1 t} dt$ $\quad m_{10} = \int_0^\infty t e^{-(u_2+\alpha_2+\beta_2)t} dH_1(t)$

$m_{01} = \Psi_0$ $\quad m_{10} + m_{10} + m_{15}^{(4)} + m_{16}^{(4)} + m_{17}^{(2)} + m_{18}^{(2)} = \Psi_1$

$m_{27} + m_{28} = \Psi_2$ $\quad m_{30} + m_{36}^{(5)} + m_{37}^{(8)} = \Psi_3$

$m_{45} + m_{46} = \Psi_4$ $\quad m_{56} = \Psi_5$

$m_{60} + m_{64} = \Psi_6$ $\quad m_{70} + m_{72} = \Psi_7$ $\quad m_{87} = \Psi_8$

PROFIT ANALYSIS

The two profit analysis of the system can be carried out by considering the all the factors in time period (0, t). First for ordinary repairman and second is expert repairman ; $P_1(t)$ and $P_2(t)$ Therefore, the expected profit of system is:

$P_1(t)$ = expected total revenue in (0,t]-expected total expenditure in (0,t]

$P_2(t)$ = expected total revenue in (0,t]-expected total expenditure in (0,t]

In steady state, expected no of profit per unit time

P = $\lim_{t\to\infty}$ [P(t)/t] = $\lim_{s\to 0}$ $s^2 P^*(s)$

$P_1(t) = H_0 A_0 - H_0 B_0 - H_2 B_0^E$

$P_2(t) = H_0 A_0 - H_3 V_0 - H_4 V_0^E$

Where,

H_0=Revenue per unit for up state of system

H_1= is the cost per unit time for which ordinary repair man is busy in repair of the failed unit.

H_2= is the cost per unit time for which expert repair man is busy in repair of the failed unit.

H_3= Cost per unit for ordinary repair.

H_4= Cost per unit for expert repair.

PARTICULAR CASE

The repair time's distributions of primary unit and secondary standby unit different failures were assumed to be arbitrary while analyzing the proposed model. If we assume repair time distribution

$M_i(t)=\lambda_i e^{-\lambda_i(t)}$ for i=1,2,…Then under this assumption the expressions MTSF and steady state transition probabilities.

STEADY STATE TRANSITION PROBABILITIES

$P_{10} = \frac{\lambda_1}{(\mu_2+\beta_2+\alpha_2+\lambda_1)}$ $\quad P_{17}^{(2)} = \frac{\mu_2\lambda_1}{(\beta_2+\lambda_1)(\mu_2+\beta_2+\alpha_2+\lambda_1)}$

$P_{18}^{(2)} = \frac{\beta_2\mu_2}{(\beta_2+\lambda_1)(\mu_2+\beta_2+\alpha_2+\lambda_1)}$ $\quad P_{16}^{(4)} = \frac{\alpha_2\lambda_2}{(\beta_2+\lambda_1)(\mu_2+\beta_2+\alpha_2+\lambda_1)}$

$P_{15}^{(4)} = \frac{\beta_2\alpha_2}{(\beta_2+\lambda_1)(\mu_2+\beta_2+\alpha_2+\lambda_1)}$ $\quad P_{27} = \frac{\lambda_1}{(\beta_2+\lambda_1)}$ $\quad P_{28} = \frac{\beta_2}{(\beta_2+\lambda_1)}$

$P_{30} = \frac{\lambda_1}{(\mu_2+\alpha_2+\lambda_2)}$ $\quad P_{36}^{(5)} = \frac{\alpha_2}{(\mu_2+\alpha_2+\lambda_2)}$ $\quad P_{37}^{(8)} = \frac{\mu_2}{(\mu_2+\alpha_2+\lambda_2)}$

$P_{45} = \frac{\beta_2}{(\beta_2+\lambda_1)}$ $\quad P_{46} = \frac{\lambda_1}{(\beta_2+\lambda_1)}$

MEAN SOJOURN TIMES

$\Psi_1 = \frac{1}{(\mu_2+\beta_2+\alpha_2+\lambda_1)}$ $\quad \Psi_0 = \frac{1}{\alpha_1}$ $\quad \Psi_2 = \frac{1}{(\beta_2+\lambda_1)}$

$\Psi_3 = \frac{1}{(\mu_2+\beta_2+\alpha_2+\lambda_1)}$

$\Psi_4 = \frac{1}{(\beta_2+\lambda_1)}$ $\quad \Psi_5 = \frac{1}{\lambda_1}$

$\Psi_6 = \frac{1}{(\mu_1+\alpha_1)}$ $\quad \Psi_7 = \frac{1}{(\beta_1+\alpha_1)}$ $\quad \Psi_8 = \frac{1}{\lambda_2}$

GRAPHICAL STUDY OF MODEL

In this model we study the system behavior of MTSF, profit function graphically, first we assume that repair time distribution of primary and secondary unit follow exponential with parameter $(\boldsymbol{\lambda_1})$ and$(\boldsymbol{\lambda_2})$. We plot the graph and table of MTSF and, Profit function with respect to failure rate of primary unit (α_1) for different values of repair rate of primary unit($\boldsymbol{\lambda_1}$).

FIGURE 6.2 Mean Time To System Failure of the System for given value of α_1 (0.01 to 0.95) and λ_1 as values $0.35, 0.55, 0.75$, λ_1 =0.35, $\lambda_2 = 0.55$, $\beta_1 = 0.30, \beta_2 = 0.65$, $\alpha_2 = 0.50, \mu_1 = 0.70, \mu_2 = 0.45$. It is observed and plot graph MTSF decrease with failure rate of primary unit (α_1) increase and increase with the repair rate.

FIGURE 6.3 MTSF of the System for given value of λ_1 (0.01 to 0.95) and α_1 as values $0.35, 0.55, 0.75$, λ_1=0.35,$\lambda_2 = 0.55, \beta_1 = 0.30, \beta_2 = 0.65$, $\alpha_2 = 0.50, \mu_1 = 0.70, \mu_2 = 0.45$. It is observed and plot graph MTSF increase with increases in repair rate and decrease with increase the failure rate.

FIGURE 6.4 Profit function of the System for given value of α_1(0.01 to 0.95) and λ_1 as values 0.35,0.55 ,0.75 , λ_1 =0.35, $\lambda_2 = 0.55, \beta_1 = 0.30, \beta_2 = 0.65, \alpha_2 = 0.50, \mu_1 = 0.70, \mu_2 = 0.45, H_0 = 1500, H_1 = 500, H_2 = 250, H_3 = 450, H_4 = 200$. It is observed and plot graph Profit function decrease and failure rate increase and increase with the repair rate. The profit functionP_2 is always greater than P_1.

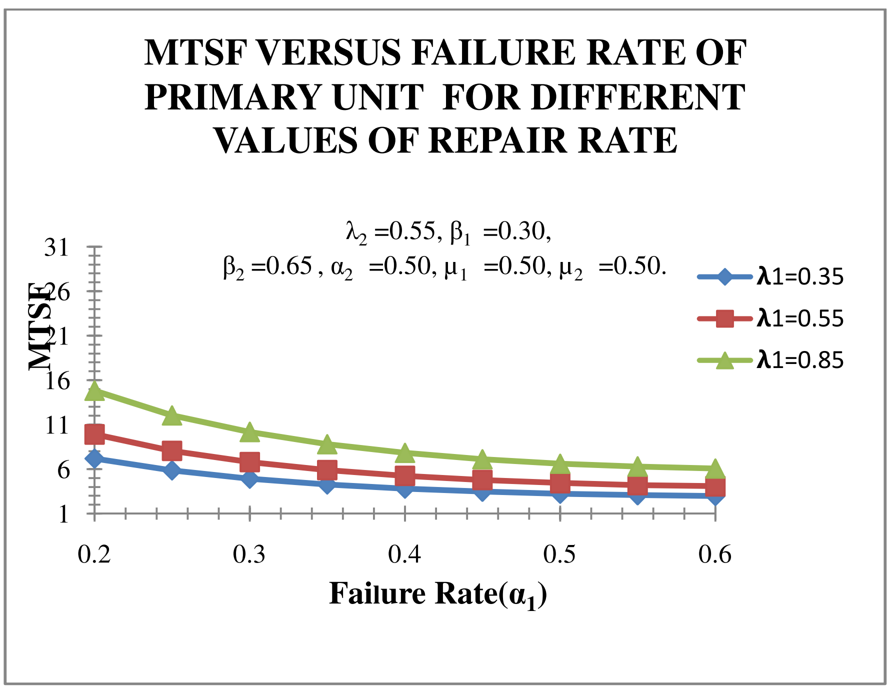


Figure 6.2 MTSF versus Failure Rate (α_1) of Primary Unit for Different values of Repair Rate (λ_1)

Figure 6.2 Mean Time To System Failure of the System for given value of α_1 (0.01 to 0.95) and λ_1 as values $0.35, 0.55, 0.75$, λ_1=0.35,$\lambda_2 = 0.55$,$\beta_1 = 0.30$,$\beta_2 = 0.65$,$\alpha_2 = 0.50$, $\mu_1 = 0.70, \mu_2 = 0.45$. It is observed and plot graph MTSF decreases with failure rate (α_1) increases and increases with the repair rate.

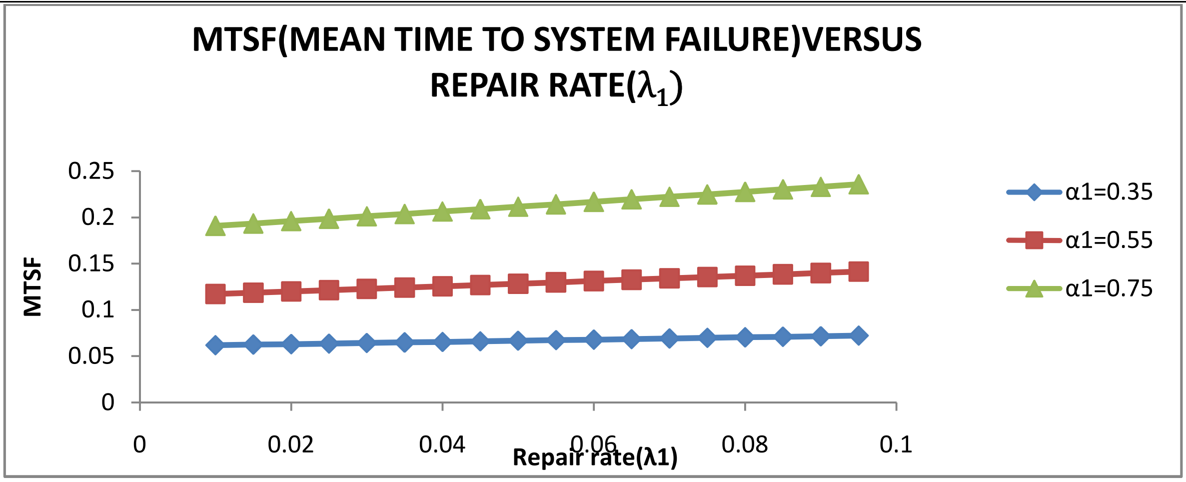
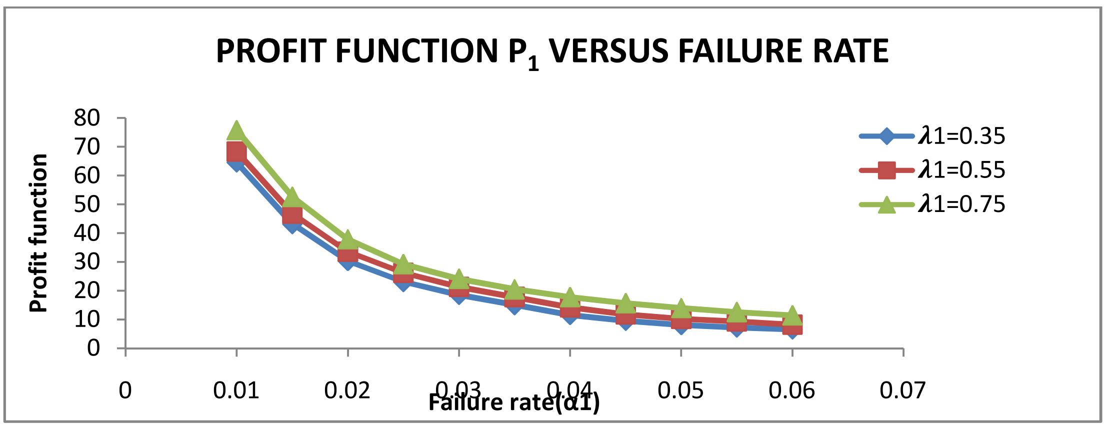


Figure 6.3 MTSF of the System for given value of λ_1 (0.01 to 0.95) and α_1 *as values* 0.35,0.55 ,0.75 , λ_1 =0.35, $\lambda_2 = 0.55$, $\beta_1 = 0.30$, $\beta_2 = 0.65$ $\alpha_2 = 0.50$, $\mu_1 = 0.70$, $\mu_2 = 0.45$,. It is observed and plot graph MTSF increases with increase in repair rate and decreases with increase the failure rate.



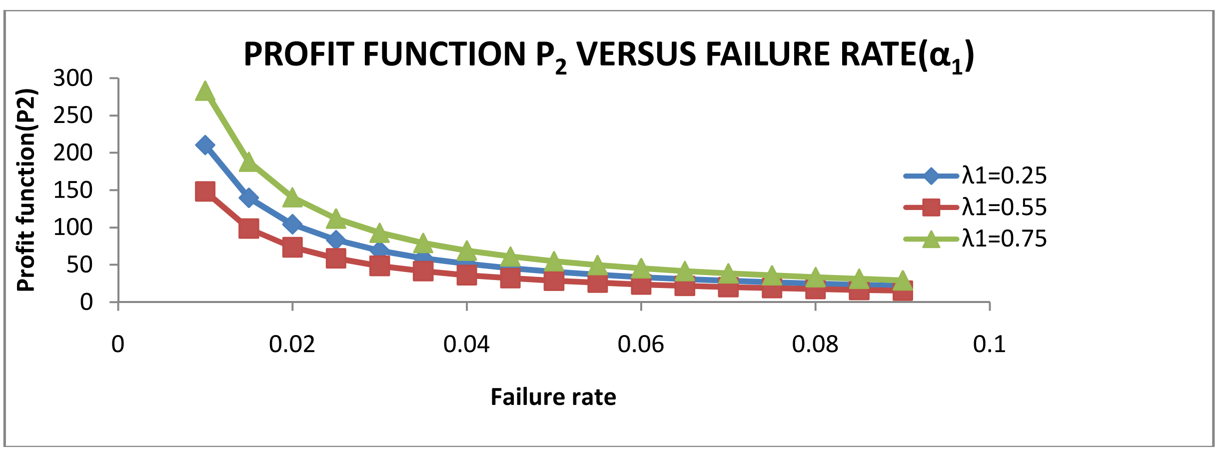


Figure 6.4 Profit function of the System for given value of α_1(0.01 to 0.95) and λ_1 *as values* 0.35,0.55 ,0.75 , λ_1 =0.35, $\lambda_2 = 0.55$, $\beta_1 = 0.30$, $\beta_2 = 0.65$ $\alpha_2 = 0.50$, $\mu_1 = 0.70$, $\mu_2 = 0.45, H_0 = 1500, H_1 = 500, H_2 = 250, H_3 = 450, H_4 = 200$. It is observed and plot graph Profit function decrease and failure rate increase and increase with the repair rate. The profit function P_2 is always greater than P_1.

CONCLUSION:

MTSF is plotted with failure rate of unit. It is observed that with increase in failure rate, MTSF is decreased, implies that the system will go in failure state in less time. So reliability of the system is decreased with increase in failure rate. It is observed through graph that MTSF decrease and failure rate increase and increase with the repair rate. So if failure rate of unit is increased, then the MTSF is decreased, so system is less reliable with more failure rate of unit. MTSF is plotted with repair rate of primary unit, with increase in repair rate, MTSF is increased, the meaning is that the system will go in failure state in more time. So reliability of the system is increased with increase in repair rate of unit.

Profit function is plotted with failure rate of unit, showing increase in failure rate, Profit is decreased, implies that the system will go in failure state in lesser time. So reliability of the system is decreased with increase in failure rate of unit. It is observed through graph Profit function decreases and failure rate increases and increase with the repair rate. The profit function P_2 is always greater than P_1.

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OPTIMAL COMPONENT SELECTION FOR EMBEDDED SYSTEMS WITH MANDATORY REDUNDANCY IN CRITICAL MODULES BASED ON REUSE-BUILD-BUY DECISION

A. Ramandeep Kaur*, B. Stuti Arora, C. P.C.Jha*****
*Institute of Information Technology & Management, D-29, Institutional Area, Janakpuri, New Delhi, India
** Institute of Management Udaipur, Balicha, Udaipur, Rajasthan, India
***Department of Operational Research, University of Delhi, Delhi, India
E-mail: rrdk_07@yahoo.com, stu.aro@gmail.com, jhapc@yahoo.com

ABSTRACT :

Embedded systems are developed in the form of modular, distributed components connected in serial or parallel. The modularized software components are a result of CBSD, wherein the components can be either obtained as COTS components, developed in-house or re-used after modification. This decision is based upon a number of internal and external factors and is known as Reuse-Build-Buy Decision. Software industries are majorly involved in reusing the components in order to save development& testing time and associated cost. Reusability of components depends upon several parameters such as cost and time for modifications & testing, reliability etc. In life-critical embedded systems, rendering the system to run in the external, uncontrolled environment will be unwise. It becomes obligatory to identify the critical modules and incorporate fault tolerance, which further implies additional cost. In this paper, RB/1/1 fault tolerant architecture has been implemented on embedded systems with mandatory redundancy for critical modules to make the system more reliable. The aim of the paper is to select hardware and software components for the modular system to maximize the overall reliability and minimizing the overall cost and SLOC with a constraint of mandatory redundancy for critical modules under RB/1/1 architecture incorporating Reuse-build-or-buy decision.

Keywords : Reusability, CBSD, COTS, Reuse-Build-Buy, RB/1/1 architecture, Critical Modules, Fault Tolerance, Embedded system

NOMENCLATURE

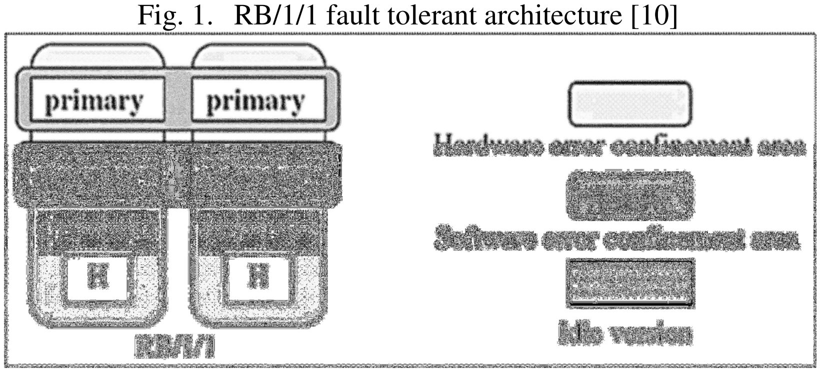
COTS: Commercial Off-The-Shelf ; CBSD: Component Based Software Development; SLOC: Source Lines Of Code; RB: Recovery Block; ; Rel: Reliability; DT: Delivery Time(COTS)/Development Time; ET: Execution Time; FT: Fabrication Time; SS: Subsystem

1. INTRODUCTION

Embedded Systems (formed as a result of inclusion of software on programmable hardware) have intercepted every facet of our daily life, from simple usage of microwave ovens to satellite based TV transmissions. Unlike software development, [1] described "embedded computing as unique because it is a hardware-software co-design problem-the hardware and software must be designed together to ensure that implementation not only functions properly but also meets performance, cost, and reliability goals". Their applicability in the form of safety-critical systems has been popular since ages. Failure of embedded systems can result in severe repercussions of loss of life, drowning of huge capital investments etc., which supports the induction of fault tolerance in these systems. Hence, it becomes obligatory to develop a fault tolerating reliable system[12]. Fault tolerance techniques were initially restricted to tolerating faults only in hardware, while later it became evident to incorporate the same in software systems as well[2]. A number of state-of-art exception handling techniques were proposed in the past literature such as N-Version Programming (NVP) Scheme [3-4], Triple Modular Redundancy (TMR)[5] and Recovery Block Scheme(RBS)[6,7]. The extensions of NVP and RBS are NVP/0/1, NVP/1/1 and RB/1/1 schemes designed for embedded systems[8-9]. Subsequently, fault tolerance in distributed architectures was implemented to optimize overall reliability of the system[10]. On the other hand, incorporating redundancy in embedded system to improve reliability entails additional resources [11].

This paper is based on RB/1/1 architecture with a generalized form of X/i/j, where X represents a fault toleranttechnique with i as the number of hardware faults tolerated & j being the number of software faults tolerated[10]. The RB/1/1 architecture resides on two hardware and two autonomous software versions viz., Primary and Secondary as depicted in Fig.1. The execution of primary version initiates the process of subjecting its output to the Acceptance Test (AT) for adjudication. Under the incident of rejection of output by AT, the process is rolled back and the entire course of executing and testing the output of the secondary version

Fig. 1. RB/1/1 fault tolerant architecture [10]



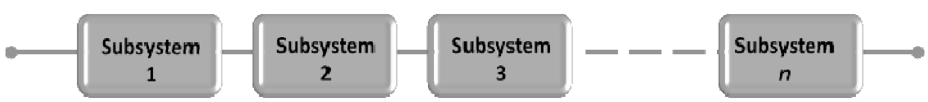


Fig. 2.Framework of subsystem based modularized embedded system

by the AT is repeated. The system shall fail under the following circumstances: existence of related faults amongst both software versions; failure of AT; specification faults; failure of both hardware components; independent failure of both software versions. The ideology behind this scheme is to incorporate redundancy in series-parallel distributed subsystem (or module) arrangement in embedded systems [13]. Allocating redundancy involves the simultaneous selection of components and a system-level design configuration to optimize some objectives satisfying the design constraints [14,20]. A similar architecture has been considered in this paper, as shown in Fig. 2 wherein RB/1/1 architecture may be incorporated in every subsystem (comprising of hardware & software components), with the intent of increasing overall system's reliability. Also, redundancy allocation is a direct way of enhancing system reliability[15]. Further, there are modules which are critical for the functioning of the embedded system i.e. if the most significant module fails then the entire system may fail. Such a module is called a Critical module. If a module is identified as a critical then it is mandatory to incorporate redundancy in that module to prevent system from complete failure. Therefore, identification of critical modules is an important activity in designing a software system.

The hardware components are available from different vendors on competitive prices. Similarly, the software components in the form of Commercial Off-The-Shelf components (COTS)[16-17] may be provided by a third party, which are ready to be integrated in the modular software system. These components promote reusability which is a characteristic feature of Component Based Software Development(CBSD)[17]. Some components can be developed in-house while some previously in-house developed components can be reused with or without fabrication[18-19]. In case of fabrication of a component, effort on additional cost, Source Lines of Code(SLOC), development and testing time are incurred. Certain components for which no COTS or reusable components exist, developing the component in-house becomes compulsory. System developers and designers are always posed with a challenge of optimally selecting the right set of components. This decision is known as Reuse-or-Build-or-Buy decision[21], which depends upon various parameters of cost, expertise availability, specification of requirements, time to develop & market etc. This paper aims at optimal selection of software components(reusable/COTS/ in-house built) and hardware components for each subsystem, as well as selecting the redundancy level for each non-critical modulefor maximizing the reliability of embedded systems, while simultaneously minimizing the overall cost and SLOC(Source Lines Of Code), with a constraint on delivery time (DT)& modular execution time,while imposingmandatory redundancy on critical modules/subsystem under the RB/1/1 architecture.

2. REUSE-OR-BUILD-OR-BUY FRAMEWORK FOR COMPONENT SELECTION

2.1 Notations

N	Number of subsystems within the embedded system
m_i	Number of hardware component choices for subsystem i; $i=1,\ldots,n$
p_i	Number of COTS software alternatives for subsystem i; $i=1,\ldots,n$
I_c	If I ={1,2,…,n} is the set of indices of all subsystems, then $I_c \subseteq I$ is the set of indices of subsystems which are critical.
C^{hw}_{ij}	Cost of hardware component j for subsystem i; $i=1,\ldots,n$; $j=1,\ldots,m_i$
C^{COTS}_{ik}	Cost of kth COTS software component for ith subsystem; $i=1,\ldots n$; $k=1,\ldots p_i$
C^{ihd}_i	Net cost incurred if in-house developed software component is deployed in subsystem i; $i=1,\ldots,n$
C^{fab}_i	Net cost of fabricated reusable component available for subsystem i; $\forall i$
c_i	Unitary development (fabrication) cost of in-house developed (fabricated) software component for subsystem i; $i=1,\ldots,n$
t^{ihd}_i	Estimated development time of in-house developed software instance for subsystem i; $i=1,\ldots,n$
τ^{ihd}_i	Average time required to perform a test case on in-house developed component for ith subsystem;$\forall i$
N^{tot}_i	Total number of tests performed on in-house developed software component for ith subsystem; $\forall i$
N^{suc}_i	Number of successful tests performed on in-house developed component for ith subsystem; $\forall i$
π^{ihd}_i	Testability— probability that single execution of in-house developed software instance fails on a test case from a certain input distribution; $\forall i$
R^{ihd}_i	Probability that the in-house developed software component for ith subsystem is failure free during its execution given that N^{suc}_i test cases have been successfully performed; $\forall i$

$t^{fab}{}_i$	Estimated FT for reusable software component of subsystem i;$\forall i$
$\tau^{fab}{}_i$	Average time required to perform a test case on fabricated reusable software component for subsystem i; $\forall i$
$Nf^{tot}{}_i$	Total number of tests performed on fabricated reusable software component for subsystem i; $\forall i$
$Nf^{suc}{}_i$	Number of successful tests performed on fabricated reusable software component for subsystem i; $\forall i$
$\pi^{fab}{}_i$	Probability that single execution of the fabricated reusable software instance fails on a test case chosen from a certain input distribution; $\forall i$
$R^{fab}{}_i$	Probability that the fabricated software component for i^{th} subsystem is failure free during its execution given that $Nf^{suc}{}_i$test cases have been successfully performed; $\forall i$
$R^{COTS}{}_{ik}$	Reliability of k^{th} COTS software component for i^{th} subsystem; $\forall i, k$
$R^{hw}{}_{ij}$	Reliability of j^{th} hardware component for i^{th} subsystem; $\forall i, j$
R_i	Reliability of subsystem i; $\forall i$
R_{io}	Threshold on rel of critical subsystem $i,i\in I_c$
$l^{COTS}{}_{ik}$	Number of source lines of code for k^{th} COTS software component available for i^{th} subsystem; $\forall i, k$
$l^{ihd}{}_i$	Number of source lines of code for in-house developed software component for subsystem i; $\forall i$
$l^{fab}{}_i$	Number of source lines of code for fabricated software component for subsystem i; $\forall i$
$d^{hw}{}_{ij}$	DT for j^{th} hardware instance of i^{th} subsystem $\forall i, j$
$d^{COTS}{}_{ik}$	DT for k^{th} COTS software instance of i^{th} subsystem; $\forall i, k$
D	Bound set on overall DT of the system
$T^{COTS}{}_{ik}$	ET for k^{th} COTS software instance of i^{th} subsystem; $\forall i, k$
$T^{ihd}{}_i$	ET for in-house developed component of i^{th} subsystem; $\forall i$
$T^{fab}{}_i$	ET for fabricated software instance of i^{th} subsystem; $\forall i$
E_i	Bound set on ET of the i^{th} subsystem
$P^{rv}{}_i$	Probability of failure from the related faults between software versions in subsystem i, which has RB/1/1 redundancy; $i=1,\ldots,n;$ $Q^{rv}{}_i=1-P^{rv}{}_i$
$P^{rv}{}_{i,t,t'}$	Probability of failure of subsystem i from related faults between COTS software versions t and t'; $i=1,\ldots,n;\ \ t,t'=1,\ldots,p_i$
$P^{rv}{}_{i,ihd,t}$	Probability of failure of subsystem i from related faults between in-house developed and COTS software component t; $i=1,\ldots,n;\ \ t=1,\ldots p_i$
$P^{rv}{}_{i,fab,t}$	Probability of failure of subsystem i from related faults between fabricated reusable software & COTS software component t; $i=1,\ldots,n$; $t=1,\ldots p_i$
P_d	Probability of failure of Acceptance Test (AT); $Q_d=1-P_d$
$P^{fs}{}_i$	Probability of failure from faults in specification, that leads to failure of all software versions in subsystem i, which has RB/1/1 redundancy; $i=1,\ldots,n$; $Q^{fs}{}_i=1-P^{fs}{}_i$
$P^{hw}{}_i$	Probability of failure of a hardware in subsystem$i$, which has RB/1/1 redundancy; $i=1,\ldots,n$
$P^{fbv}{}_i$	Probability that both software versions of subsystem i, which has RB/1/1 redundancy, fail independently; $\forall i$
x_{ij}	Decision variable; $i=1,\ldots,n$; $j=1,\ldots,m_i$

$$x_{ij} = \begin{cases} 1; \text{ if } j^{th} \text{ hardware component selected for subsystem } i, \\ \quad \text{without redundancy} \\ 2; \text{ if } j^{th} \text{ hardware component selected for subsystem } i, \\ \quad \text{with redundancy} \\ 0; \text{ if } j^{th} \text{ hardware component not selected for subsystem } i \end{cases}$$

y_{ik} Binary decision variable; $i=1,\ldots,n$; $k=1,\ldots,p_i$

$$y_{ik} = \begin{cases} 1; \text{ if } k^{th} \text{ COTS component of } i\text{th subsystem is selected} \\ 0; \text{ otherwise} \end{cases}$$

v_i Binary decision variable; $i=1,\ldots,n$

$$v_i = \begin{cases} 1; \text{ if in-house developed software component is selected} \\ \quad \text{for } i^{th} \text{ subsystem} \\ 0; \text{ otherwise} \end{cases}$$

w_i Binary decision variable; $i=1,\ldots,n$

$$\mathrm{w}_{\mathrm{i}} = \begin{cases} 1; \text{ if fabricated reusable component is selected} \\ \quad \text{for } i^{th} \text{ subsystem} \\ 0; \text{ otherwise} \end{cases}$$

z_i Binary decision variable; $i=1,\ldots,n$

$$z_i = \begin{cases} 1; \text{ if } i^{th} \text{ subsystem uses RB/1/1 redundancy technique} \\ 0; \text{ otherwise} \end{cases}$$

2.1 Assumptions

1. System is modularized into several but finite number of subsystems connected in series.
2. Each subsystem has finite number of hardware & software components available with known reliability and cost.
3. Each component is non-repairable, and has two states, functional or failed.
4. Software & hardware failures are not identical. A subsystem failure occurs when all software versions or hardware components fail. Failure of even one subsystem leads to system failure.
5. Software components for a subsystem are available as COTS, in-house developed, or fabricated reusable.
6. Information pertaining to Reliability, Cost, SLOC, execution time and delivery time of COTS components is provided by vendor.
7. Cost & reliability for in-house developed components (reusable components) can be specified based on the parameters of development (fabrication) process.
8. COTS software components for a subsystem may be available in abundance but in-house developed component & reusable component for each subsystem can be atmost one.
9. Each subsystem either has RB/1/1 redundancy or has no redundancy.
10. Subsystem without any redundancy has exactly one hardware and software component.
11. Subsystem with RB/1/1 redundancy has two identical hardware components & two distinct software components
12. Subsystem with RB/1/1 redundancy cannot have both in-house developed component and reusable component deployed in it.
13. Failure of individual component are s-independent.
14. Delivery time for hardware component(s) of a subsystem is independent of number of units being purchased.
15. Time taken for integration of components is negligible.

2.2 Model Formulation

The optimization model for component selection can be written as follows:

Problem(P1)

$$Max\ R = \prod_{i=1}^{n} R_i \tag{1}$$

$$Min\ C = \sum_{i=1}^{n}\left[\sum_{j=1}^{m_i} C_{ij}^{hw} x_{ij} + \sum_{k=1}^{p_i} C_{ik}^{\text{cots}} y_{ik} + C_i^{ihd} v_i + C_i^{fab} w_i\right] \tag{2}$$

$$Min\ L = \sum_{i=1}^{n}\left[\sum_{k=1}^{p_i} l_{ik}^{\text{cots}} y_{ik} + l_i^{ihd} v_i + l_i^{fab} w_i\right] \tag{3}$$

subject to

$X \in S = \{x_{ij}, y_{ik}, v_i, w_i, z_i$ are decision variables $\}$

$$C_i^{ihd} = c_i\left(t_i^{ihd} + \tau_i^{ihd} N_i^{tot}\right) \quad ; i = 1,\ldots,n \tag{4}$$

$$C_i^{fab} = c_i\left(t_i^{fab} + \tau_i^{fab} Nf_i^{tot}\right) \quad ; i = 1,\ldots,n \tag{5}$$

$$\left(\sum_{k=1}^{p_i} T_{ik}^{\text{cots}} y_{ik}\right) + T_i^{ihd} v_i + T_i^{fab} w_i \le E_i \quad ; i = 1,\ldots,n \tag{6}$$

$$\sum_{j=1}^{m_i} x_{ij} - z_i = 1 \quad ; i = 1,\ldots,n \tag{7}$$

$$\sum_{k=1}^{p_i} y_{ik} + v_i + w_i - z_i = 1 \quad ; i = 1,\ldots,n \tag{8}$$

$$v_i \times w_i = 0 \quad ; \quad i = 1,\ldots,n; \tag{9}$$

$$x_{ij} \times x_{il} = 0 \quad ; \quad i = 1,\ldots,n; \quad j,l = 1,\ldots,m_i; \; j \ne l \tag{10}$$

$$\underset{\substack{i=1,\ldots,n \\ j=1,\ldots,m_i}}{\text{Max}} \frac{d_{ij}^{hw} \cdot x_{ij}}{(z_i + 1)} \le D \tag{11}$$

$$\underset{\substack{i=1,\ldots,n \\ k=1,\ldots,p_i}}{\text{Max}} d_{ik}^{\text{cots}} \cdot y_{ik} \le D \tag{12}$$

$$\underset{i=1,\ldots,n}{\text{Max}} \left(t_i^{ihd} + \tau_i^{ihd} N_i^{tot}\right) v_i \le D \tag{13}$$

$$\underset{i=1,\ldots,n}{\text{Max}} \left(t_i^{fab} + \tau_i^{fab} Nf_i^{tot}\right) w_i \le D \tag{14}$$

$$R_i^{ihd} = \frac{1 - \pi_i^{ihd}}{(1 - \pi_i^{ihd}) + \pi_i^{ihd}(1 - \pi_i^{ihd})^{N_i^{suc}}} \quad ; i = 1,\ldots,n \tag{15}$$

$$R_i^{fab} = \frac{1 - \pi_i^{fab}}{(1 - \pi_i^{fab}) + \pi_i^{fab}(1 - \pi_i^{fab})^{Nf_i^{suc}}} \quad ; i = 1,\ldots,n \tag{16}$$

$$N_i^{suc} = (1 - \pi_i^{ihd}) N_i^{tot} \quad ; i = 1,\ldots,n \tag{17}$$

$$Nf_i^{suc} = (1 - \pi_i^{fab}) Nf_i^{tot} \quad ; i = 1,\ldots,n \tag{18}$$

$$R_i = (1 - z_i)\left[\left(\sum_{j=1}^{m_i} R_{ij}^{hw} x_{ij}\right) \cdot \left(\sum_{k=1}^{p_i} R_{ik}^{\text{cots}} y_{ik} + R_i^{ihd} v_i + R_i^{fab} w_i\right)\right]$$

$$z_i\left(1 - P_i\right); \forall\ i \tag{19}$$

$$P_i = P_i^{rv} + Q_i^{rv} P_d + Q_i^{rv} Q_d P_i^{fs} + Q_i^{rv} Q_d Q_i^{fs} (P_i^{hw})^2 + Q_i^{rv} Q_d Q_i^{fs}\left(1 - (P_i^{hw})^2\right) P_i^{fbv} \quad ; i = 1,\ldots,n \tag{20}$$

$$P_i^{rv} = \sum_{t=1}^{p_i-1}\sum_{t'=t+1}^{p_i} P_{i,t,t'}^{rv} y_{it} y_{it'} + \sum_{t=1}^{p_i} P_{i,ihd,t}^{rv} y_{it} v_i + \sum_{t=1}^{p_i} P_{i,fab,t}^{rv} y_{it} w_i + P_{i,ihd,fab}^{rv} v_i w_i \; ; i = 1,\dots,n \quad (21)$$

$$P_i^{hw} = \sum_{j=1}^{m_i} \frac{\left(1-R_{ij}^{hw}\right) x_{ij}}{\left(z_i+1\right)} \; ; i = 1,\dots,n \quad (22)$$

$$P_i^{fbv} = \sum_{t=1}^{p_i-1}\sum_{t'=t+1}^{p_i} \left[\left(1-R_{it}^{\text{cots}}\right) y_{it}\right]\cdot\left[\left(1-R_{it'}^{\text{cots}}\right) y_{it'}\right] + \sum_{t=1}^{p_i}\left[\left(1-R_{it}^{\text{cots}}\right) y_{it}\right]\cdot\left[\left(1-R_i^{ihd}\right) v_i\right] + \sum_{t=1}^{p_i}\left[\left(1-R_{it}^{\text{cots}}\right) y_{it}\right]\cdot\left[\left(1-R_i^{fab}\right) w_i\right] + \left[\left(1-R_i^{ihd}\right) v_i\right]\cdot\left[\left(1-R_i^{fab}\right) w_i\right]; i = 1,\dots,n \quad (23)$$

$$z_i = 1 \; ; i \in I\!\!I_C \quad (24)$$

$$R_i \ge R_i^{\text{o}} \quad ; i \in I\!\!I_C \quad (25)$$

$$z_i \in \{0,1\}; \; i \in I\!\!I - I\!\!I_C \quad (26)$$

$$x_{ij} \in \{0,1,2\}; \; i = 1,\dots,n; \; j = 1,\dots,m_i \quad (27)$$

$$y_{ik} \in \{0,1\}; \; i = 1,\dots,n; \; k = 1,\dots,p_i \quad (28)$$

$$v_i \in \{0,1\}; \; i = 1,\dots,n \quad (29)$$

$$w_i \in \{0,1\}; \; i = 1,\dots,n \quad (30)$$

}

Objective function (1) maximizes system reliability, which is the product of reliabilities of its subsystems. Total cost, which is the summation of costs of selected hardware and software(COTS/ In-house developed / Reusable) components, is minimized in objective function (2). Source Lines of Code (SLOC) of the complete system, which is the sum of the SLOC of all selected software components, is minimized in objective function(3).Cost of in-house built components and fabricated components depends upon development time & testing time, and is indicated in (4)& (5) resp. Total execution time of all the components selected for a subsystem cannot be more than the prescribed Execution time of thesubsystem E_i as given by (6). Constraint (7) represents the selection of exactly one hardware component in case of no redundancy, and exactly two components if the subsystem has redundant architecture. Equation (8) denotes the selection of one software component in case of no redundancy, and two software components in case of redundancy for i^{th} subsystem.Eq(9) indicates that for a subsystem, if in-house developed component is selected, then fabricated component will not be selected, and vice versa.Eq. (10) guarantees selection of only one type of hardware component for a subsystem even if the subsystem has redundant architecture. Thus, in case of redundancy in i^{th} subsystem, two hardware components selected will be identical. Constraints (11-14) are indicating the bounds on delivery time(DT) of the system ensuring that all selected hardware & software components will be delivered within D time units.Eq(15) evaluates reliability for i^{th} subsystem's in-house developed components by using the testability and no. of successful test cases as given in (17).Similarly,Eq(16) evaluates reliability for i^{th} subsystem's fabricated/reusable component based on testability and no. of successful test cases as in (18).Eq(19) computes the reliability of i^{th} subsystem, which is product of reliabilities of selected hardware component & software component if subsystem has no redundancy,otherwise it is computed on the basis of the reliability for RB/1/1 architecture. If i^{th} subsystem has RB/1/1 architecture, then its unreliability is given by Eq(20). Auxiliary variables used in Eq(20) are evaluated as given in Eq (21)-(23). Eq(24) imposes redundancy in critical modules. Constraint (25) ensures that the reliability of the critical modules must be greater than the minimum threshold on their reliability. Constraints (26)-(30) represent the decision variables.

3. SOLUTION PROCEDURE

In order to solve this multi-objective optimization problem, weighted sum approach can be used [25]. Here, the upper and lower bound for each of the objective is determined, subject to same set of constraints as in original problem. If S denotes the feasible region, then upper bound for Reliability can be computed by solving Problem R_{max} and lower bound for Reliability can be found by solving Problem R_{min}.

Problem R_{min} :

$$Min\ R = \prod_{i=1}^{n} R_i$$
$$subject\ to$$
$$X \in S$$

Problem R_{max} :

Similarly, following optimization models are solved for upper and lower bounds for Cost and SLOC:

$$Max\ R = \prod_{i=1}^{n} R_i$$
$$subject\ to$$
$$X \in S$$

Problem C_{max}:

$$Max\ C = \sum_{i=1}^{n}\left[\sum_{j=1}^{m_i} C_{ij}^{hw} x_{ij} + \sum_{k=1}^{p_i} C_{ik}^{\text{cots}} y_{ik} + C_i^{ihd} v_i + C_i^{fab} w_i\right]$$
$$subject\ to$$
$$X \in S$$

Problem L_{min}:

$$Min\ L = \sum_{i=1}^{n}\left[\sum_{k=1}^{p_i} l_{ik}^{\text{cots}} y_{ik} + l_i^{ihd} v_i + l_i^{fab} w_i\right]$$
$$subject\ to$$

Problem L_{max}:

$$Max\ L = \sum_{i=1}^{n}\left[\sum_{k=1}^{p_i} l_{ik}^{\text{cots}} y_{ik} + l_i^{ihd} v_i + l_i^{fab} w_i\right]$$
$$subject\ to$$

Problem C_{min}:

$$Min\ C = \sum_{i=1}^{n}\left[\sum_{j=1}^{m_i} C_{ij}^{hw} x_{ij} + \sum_{k=1}^{p_i} C_{ik}^{\text{cots}} y_{ik} + C_i^{ihd} v_i + C_i^{fab} w_i\right]$$
$$subject\ to$$
$$X \in S$$

Now, Problem *P* is converted into Problem *P'*, which has an aggregated objective, with the help of the bounds derived above and the weights associated with each objective. These weights are quantitative measure of the preference of one objective over the other. They can be derived with the help of tools like Analytical Hierarchy process(AHP)[27].

Suppose W_R, W_C and W_L are weights associated with Reliability, Cost and SLOC objective function respectively, then Problem *P'* can be given as:

Problem *P'*

$$Max\ W_R \cdot R' + W_C \cdot C' + W_L \cdot L'$$

subject to

$$X \in S$$

$$W_R + W_C + W_L = 1$$

$$W_R, W_C, W_L \geq 0$$

Problem *P'* is then solved for optimal solution, which is also the pareto-optimal solution for Problem P.

4. ILLUSTRATIVE EXAMPLE

An embedded system with 6 subsystems connected in series has been considered to check the applicability of the model. A number of hardware and software components (COTS/ in-house developed/ reusable) are available for selection. Subsystem 4 and 5 are considered as critical subsystems.

Table I depicts the statistical data set for hardware instance w.r.t Cost, Reliability and Delivery Time. Table II represents the available data set for COTS components w.r.t Cost, SLOC, Reliability, Delivery Time and Execution Time. Table III represents the data set considered for in-house developed components as well as fabricated components.

TABLE I

STATISTICAL DATA SET FOR HARDWARE INSTANCE

SS	**Hardware instance**	**Cost**	**Rel**	**DT**
1	1	10	0.96	8
	2	7	0.93	11
2	1	11	0.91	14
	2	13	0.94	11
3	1	14	0.8	10
	2	9	0.82	14
	3	8	0.81	12
4	1	11	0.83	9
5	1	18	0.91	10
	2	16	0.945	12
6	1	21	0.79	11
	2	16	0.83	19
	3	18	0.81	15

TABLE II

STATISTICAL DATA SET FOR COTS COMPONENTS

SS	COTS instance	Cost	SLOC	Rel	DT	DT
1	1	23	562	0.90	6	0.37
	2	26	582	0.93	8	0.31
2	1	47	917	0.92	7	0.53
	2	39	885	0.90	10	0.50
	3	43	890	0.96	9	0.55
3	1	17	225	0.89	6	0.16
	2	20	207	0.90	4	0.18
4	1	32	714	0.905	9	0.52
	2	29	695	0.92	12	0.56
	3	27	745	0.90	11	0.53
5	1	37	480	0.95	7	0.41
	2	33	473	0.94	9	0.47
6	1	49	570	0.925	9	0.27
	2	58	600	0.94	9	0.24

TABLE III

STATISTICAL DATA SET FOR IN-HOUSE DEVELOPED AND FABRICATED COMPONENTS

SS	Unitary Development/ Fabrication Cost	In-house Developed Components			Fabricated Reusable Components		
		Estimated DT	SLOC	E	Estimated Fabrication Time	SLOC	ET
1	10	-	-	-	2	570	0.35
2	-	-	-	-	-	-	-
3	18	10	230	0.	-	-	-
4	30	11	700	0.	7	755	0.51
5	35	-	-	-	6	887	0.53
6	52	10	608	0.	5	595	0.26

For simplicity, it is assumed that in RB/1/1 architecture, P^{rv}_i= 0.002 $\forall i$,P_d=0.002, P^{fs}_i=0.003 $\forall i$, and the values of τ_i, π_i and πf_i are 0.05, 0.002 & 0.002 respectively for any subsystem. Limit on delivery time of the system is set as D=25, and limit on modular execution time E1, E2, E3, E4, E5, E6 are 0.4, 0.6, 0.35, 1.06, 0.9 and 0.5 resp. The threshold on reliabilities of both the critical subsystems has been set to 0.85.Following the steps for solution approach as described in previous section, this problem of optimal selection of hardware & software components is solved by means of LINGO software [23][26]. Three cases were considered, each with different set of weights W_R (weights of reliability) and W_C(weights of cost) and W_{SLOC} (weights on SLOC), while keeping other parameters same.

Different combinations of selected components were obtained depending upon the prioritization of the customer's non-functional requirements as in Table IV.

TABLE IV

CUSTOMER PRIORITY INDEX

Customer	Reliability	Cost	SLOC	Case
A	Priority 1	Priority 2	Priority 2	1
B	Priority 1	Priority 2	Priority 3	2 a
C	Priority 1	Priority 3	Priority 2	2 b

With respect to the preferences of customer, different weights can be allocated to reliability cost and SLOC. In all 6 cases exist out of which three peculiar cases have been considered in Table V based on the preferences of customer towards objectives.

Case 1:

When Reliability has highest priority and Cost & SLOC holds equal priority.

Here, depending upon the preference of the customer A, reliability has been given maximum weight of W_R=0.7. Rest of the two objectives share same preference, hence, allocated the equal weight of W_C=W_{SLOC}=0.15 each. In this case, the overall reliability comes out to be 0.7405 units with system cost and SLOC of 1041 units and 5532 units respectively.

Case 2:

When reliability is the most preferred alternative for customers and cost & SLOC have different priorities

Case 2 a. Cost has higher priority than SLOC

For customer B, cost was a limiting factor. Hence, cost was given a higher weight of W_C=0.4 than of SLOC with W_{SLOC}=0.1. The reliability of the overall system comes out to be 0.7257 units with system cost and SLOC of 698 units and 5499 units respectively.

Case 2 b. SLOC has higher priority than Cost

For customer C, SLOC was a limiting factor, since its maintenance was impounding him with extra resources. Therefore, it was given a higher weight of W_{SLOC}=0.4 as compared to cost with a weight of W_C=0.1. The reliability of the overall system comes out to be 0.608 units with system cost and SLOC of 919 units and 4865 units respectively.

It can be inferred from the statistical data that depending upon the priority of the customer, overall system's reliability, cost and SLOC varies significantly. Components selected for each case are shown in table V.Weight for reliability in case 2 a. and 2.b was always given valueW_R= 0.5 because it was always the most preferred objective by the customer.

TABLE V Solution for Component Selection

Case	W_R	W_C	W_{SLOC}	SS	z_i	Components Selected	Rel	System Rel	System Cost	System SLOC
1	0.7	0.15	0.15	1	0	$x_{11}=w_1=1$	0.958	**0.7405**	**1041**	**5532**
				2	0	$x_{22}=y_{23}=1$	0.902			
				3	1	$x_{31}=2$, $y_{31}=v_3=1$	0.953			
				4	1	$x_{41}=2$, $y_{41}=w_4=1$	0.964			
				5	1	$x_{51}=2$, $y_{51}=y_{52}=1$	0.982			
				6	1	$x_{61}=2$, $y_{62}=w_6=1$	0.949			
2(a)	0.5	0.4	0.1	1	0	$x_{11}=w_1=1$	0.958	**0.7257**	**698**	**5499**
				2	0	$x_{22}=y_{23}=1$	0.902			
				3	1	$x_{31}=2$, $y_{31}=y_{32}=1$	0.943			
				4	1	$x_{41}=2$, $y_{41}=y_{43}=1$	0.955			
				5	1	$x_{51}=2$, $y_{51}=y_{52}=1$	0.982			
				6	1	$x_{61}=2$, $y_{62}=w_6=1$	0.949			
2(b)	0.5	0.1	0.4	1	0	$x_{11}=w_1=1$	0.958	**0.608**	**919**	**4865**
				2	0	$x_{22}=y_{23}=1$	0.902			
				3	1	$x_{31}=2$, $y_{31}=y_{32}=1$	0.943			
				4	1	$x_{41}=2$, $y_{42}=v_4=1$	0.964			
				5	1	$x_{51}=2$, $y_{51}=y_{52}=1$	0.982			
				6	0	$x_{61}=w_6=1$	0.788			

5. CONCLUSION

Multi-objectives optimization model wasformulated for fault tolerant embedded systems with the objective of maximizing reliability, minimizing cost& SLOC under reuse-or-build-or-buy decision for software components. RB/1/1 fault tolerant architecture was considered for tolerating one hardware & one software fault. Mandatory redundancy was incorporated in critical modules. Depending upon the preferences of different customers, a suitable mix of the components can be obtained giving utmost priority to critical modules.Sum weighted approach was used to demonstrate the variation in Reliability, Cost and SLOC of the components by varying their weights.

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NON-ARCHIMEDEAN STABILITY OF SYSTEM OF AQ RECIPROCAL FUNCTIONAL EQUATIONS

Shalini Tomar*, Nawneet Hooda**
* Department of Mathematics, Kanya Mahavidyalaya, Kharkhoda, Sonepat, Haryana, India
**Department of Mathematics, DCR University of Science & Technology, Sonepat, Haryana
E-mail : s_saroha30@yahoo.com, nawneethooda@gmail.com

ABSTRACT :

In this paper, we will prove generalised Hyers-Ulam-Rassias stability of a system of additive and quadratic reciprocal functional equations in non - Archimedean normed spaces.

Keywords : Generalised Hyers-Ulam-Rassias stability, Additive Reciprocal Functional Equation, Quadratic Reciprocal Functional Equation, non - Archimedean Normed Spaces.

Mathematical subject classification : 39B82, 39B72.

1. INTRODUCTION

The study of stability of functional equations was encouraged by a significant question of Ulam[1], which was to an extent, answered by Hyers[8] in Banach spaces. Latter, Hyers' result was generalised by T.Aoki[2], Th.M.Rassias[10] and Gavruta[7] under various adaptations. Since then, many researchers investigated the result for various functional equations and mappings in various spaces[[3],[6],[9]].

In 2010, Ravi and Senthil Kumar[11] obtained generalised Hyers-Ulam stability for the reciprocal functional equation

$$f(x+y)=\frac{f(x)f(y)}{f(x)+f(y)} \qquad (1)$$

in space of non-zero real numbers. The reciprocal function $f(x)=\frac{c}{x}$ is solution of (1)

For the first time, Bodaghi and Kim[5] introduced and studied the generalised Hyers-Ulam stability for the quadratic reciprocal functional equation

$$f(2x+y)+f(2x-y)=\frac{2f(x)f(y)[4f(y)+f(x)]}{(4f(y)-f(x))^2} \qquad (2)$$

in space of non-zero real numbers. Recently, H. Majani proved the generalised Hyers-Ulam-Rassias stability of the system of additive,quadratic and cubic functional equation in non-archimedean normed spaces(for more information see[4]).

In this paper, we investigate the generalised Hyers-Ulam-Rassias stability for the system of additive quadratic(AQ) reciprocal functional equation of type

$$f(a_1x_1+b_1y_1,x_2,\dots,x_n)=\frac{f(x_1,x_2,\dots,x_n)f(y_1,x_2,\dots,x_n)}{b_1f(x_1,x_2,\dots,x_n)+a_1f(y_1,x_2,\dots,x_n)},$$

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$$f(x_1,..,a_sx_s+b_sy_s,...,x_n)=\frac{f(x_1,...,x_n)f(x_1,..,y_s,..,x_n)}{b_sf(x_1,...,x_n)+a_sf(x_1,..,y_s,..,x_n)},$$

$$f(x_1,...,x_s,(a_{s+1}+1)x_{s+1}+a_{s+1}y_{s+1},.,x_n)+f(x_1,x_2,...,x_s,$$
$$(a_{s+1}+1)x_{s+1}-a_{s+1}y_{s+1},.,x_n)=2f(x_1,...,x_n)f(x_1,..,y_{s+1},...,x_n)$$
$$\frac{[((a_{s+1}+1)^2f(x_1,..,y_{s+1},...,x_n)+a_{s+1}^2f(x_1,...,x_n)]}{((a_{s+1}+1)^2f(x_1,..,y_{s+1},...,x_n)-a_{s+1}^2f(x_1,...,x_n))^2},$$

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$$f(x_1,x_2,..,x_{n-1},(a_n+1)x_n+a_ny_n)+f(x_1,....,x_{n-1},(a_n+1)x_n-a_ny_n)=$$
$$\frac{2f(x_1,...,x_n)f(x_1,x_2,...,y_n)[((a_n+1)^2f(x_1,x_2,....,y_n)+a_n^2f(x_1,...,x_n)]}{((a_n+1)^2f(x_1,...,x_{n-1},y_n)-a_n^2f(x_1,x_2,...,x_{n-1},x_n))^2}. \quad (3)$$

where $s,n\in N$ with $s<n$ and $a_i,b_i\in Z\backslash\{0\}; i=1,2,...,n$ and $a_i\neq -b_i$.

2. PRELIMINARIES

In 1897, Hensel [7] introduced a normed space which does not have the Archimedean property. The most important examples of non-Archimedean spaces are p-adic numbers. By a Non-Archimedean field we mean a field K equipped with a function $|\diamond|: K\to R$ such that for any $a,b\in K$ we have

- $|a|\geq 0$ and equality holds if and only if $a=0$,
- $|ab|=|a||b|$,
- $|a+b|\leq \max\{|a|,|b|\}$.

The third condition is known as strict triangle inequality. By second, we have $|1|=|-1|=1$. From third, with the help of induction, it follows that $|n|\leq 1$ for each integer n. We always assume in addition that $|\diamond|$ is non trivial, i.e. there is an $a_o\in K$ such that $|a_o|\notin\{0,1\}$.

2.1 Definition

Let X be a vector space over a field K with a non-Archimedean valuation $|\circ|$. A function $||\circ||: X\to[0,\infty)$ is called a non-Archimedean norm if the following conditions hold:

- $||x||=0$ if and only if $x=0$ for all $x\in X$;
- $||rx||=|r|||x||$ for all $r\in K$ and $x\in X$;
- (strong triangle inequality) $||x+y||\leq\max\{||x||,||y||\}$for all $x,y\in X$.

Then $(X,||\circ||)$is called a non-Archimedean normed space.

2.2 Definition

Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X.

1. A sequence $\{x_n\}_{n=1}^{\infty}$ in a non-Archimedean space is a Cauchy sequence if and only if, the sequence $\{x_{n+1}-x_n\}_{n=1}^{\infty}$ converges to zero.

2. The sequence $\{x_n\}$ is said to be convergent if, for any $\varepsilon > 0$, there are a positive integer N and $x \in X$ such that $||x_n - x|| \le \varepsilon$ for all $n \ge N$. Then, the point $x \in X$ is called the limit of the sequence $\{x_n\}$, which is denoted by $\lim_{n\to\infty} x_n = x$.
3. If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.

3. STABILITY OF SYSTEM OF AQ RECIPROCAL FUNCTIONAL EQUATIONS (3)

Throughout this paper, suppose that G be a divisible group with the identity element 0 and X be complete non-Archimedean normed space. Also $i, j, m, n, p, s, t \in N \cup \{0\}$ with $1 \le i \le s < j \le n$, unless otherwise explicitly stated.

3.1 Theorem

Let $\Phi_i: G^{2n} \to R^+$ be a function satisfying

$$\lim_{m\to\infty} 2^{sm} a_1^m a_2^m \ldots\ldots a_s^m ((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} \ldots . ((a_n+1)^2 - a_n^2)^{2m}$$
$$\Phi_i(2^m a_1^m x_1, y_1, 2^m a_2^m x_2, y_2, \ldots, 2^m a_{i-1}^m x_{i-1}, y_{i-1}, 2^m a_i^m x_i, 2^m a_i^m y_i,$$
$$2^m a_{i+1}^m x_{i+1}, y_{i+1}, .., 2^m a_s^m x_s, y_s, (2a_{s+1}+1)^m x_{s+1}, y_{s+1}, .., (2a_n+1)^m x_n, y_n) \tag{4}$$

and $$\lim_{m\to\infty} 2^{sm} a_1^m a_2^m \ldots\ldots a_s^m ((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} \ldots . ((a_n+1)^2 - a_n^2)^{2m}$$
$$\Phi_i(2^m a_1^m x_1, y_1 \ldots, 2^m a_s^m x_s, y_s, (2a_{s+1}+1)^m x_{s+1}, y_{s+1} .., (2a_{j-1}+1)^m x_{j-1},$$
$$y_{j-1}, (2a_j+1)^m x_j, (2a_j+1)^m y_j, (2a_{j+1}+1)^m x_{j+1}, y_{j+1}, \ldots, (2a_n+1)^m x_n, y_n) = 0 \tag{5}$$

for all $i = \{1,2,\ldots,s\}$ and $j = \{s+1, s+2, \ldots, n\}$. Suppose that

$$\lim_{m\to\infty} \sigma(x_1, y_1, .., x_n, y_n) = 0 \tag{6}$$

and $$\Phi(x_1, y_1, .., x_n, y_n) = \lim_{p\to\infty} \max\{\sigma(x_1, y_1, .., x_n, y_n); m = 0,1, .., p\} < \infty \tag{7}$$

where

$$\sigma(x_1, y_1, .., x_n, y_n) = \max\{2^s |a_1 \ldots . a_s| \max\{2^{sm} |a_1^m a_2^m \ldots a_s^m| ((a_{s+1}+1)^2$$
$$-a_{s+1}^2)^{2(m+1)} .. ((a_j+1)^2 - a_j^2)^{2(m+1)} ((a_{j+1}+1)^2 - a_{j+1}^2)^{2m} .. ((a_n+1)^2$$
$$-a_n^2)^{2m} \Phi_j(2^m a_1^m x_1, y_1, .., 2^m a_s^m x_s, y_s, (2a_{s+1}+1)^{m+1} x_{s+1}, y_{s+1}, ..,$$
$$(2a_{j-1}+1)^{m+1} x_{j-1}, y_{j-1}, (2a_j+1)^m x_j, (2a_j+1)^m x_j, (2a_{j+1}+1)^m x_{j+1},$$
$$y_{j+1}, .., (2a_n+1)^m x_n, y_n); j = s+1, .., n\}, \max\{2^{sm+i} |a_1^{m+1} a_2^{m+1}$$
$$\ldots a_i^{m+1} a_{i+1}^m \ldots a_s^m| ((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} \ldots . ((a_n+1)^2 - a_n^2)^{2m} \Phi_i(2^{m+1} a_1^{m+1} x_1,$$
$$y_1, .., 2^{m+1} a_{i-1}^{m+1} x_{i-1}, y_{i-1}, 2^m a_i x_i, 2^m a_i x_i, 2^m a_{i+1}^m x_{i+1}, y_{i+1}, .., 2^m a_s^{m+1} x_s,$$
$$(2a_{s+1}+1)^m x_{s+1}, y_{s+1}, .., (2a_n+1)^m x_n, y_n)\}: i = 1, .., s\}$$

for all $x_i, y_i \in G$; $a_i, b_i \in Z\backslash\{0\}$ and $i = 1,2,\ldots,n$. Suppose that $f: G^n \to X$ satisfies following inequalities

$$||f(a_1x_1+b_1y_1,x_2,\ldots,x_n)-\frac{f(x_1,x_2,\ldots,x_n)f(y_1,x_2,\ldots,x_n)}{b_1f(x_1,x_2,\ldots,x_n)+a_1f(y_1,x_2,\ldots,x_n)}||$$
$$\leq \phi_1(x_1,y_1,\ldots,x_n,y_n),$$

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$$||f(x_1,\ldots,a_sx_s+b_sy_s,\ldots,x_n)-\frac{f(x_1,\ldots,x_n)f(x_1,\ldots,y_s,\ldots,x_n)}{b_sf(x_1,\ldots,x_n)+a_sf(x_1,\ldots,y_s,\ldots,x_n)}||$$
$$\leq \phi_s(x_1,y_1,\ldots,x_n,y_n),$$
$$||f(x_1,\ldots,x_s,(a_{s+1}+1)x_{s+1}+a_{s+1}y_{s+1},\ldots,x_n)+f(x_1,\ldots,x_s,$$
$$(a_{s+1}+1)x_{s+1}-a_{s+1}y_{s+1},\ldots,x_n)-2f(x_1,\ldots,x_n)f(x_1,\ldots,y_{s+1},\ldots,x_n)$$
$$\frac{[((a_{s+1}+1)^2f(x_1,\ldots,y_{s+1},\ldots,x_n)+a_{s+1}^2f(x_1,\ldots,x_n)]}{((a_{s+1}+1)^2f(x_1,x_2,\ldots,y_{s+1},\ldots,x_n)-a_{s+1}^2f(x_1,\ldots,x_n))^2}||$$
$$\leq \phi_{s+1}(x_1,y_1,\ldots,x_n,y_n),$$

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$$||f(x_1,\ldots,x_{n-1},(a_n+1)x_n+a_ny_n)+f(x_1,\ldots,x_{n-1},(a_n+1)x_n-a_ny_n)$$
$$-\frac{2f(x_1,\ldots,x_n)f(x_1,\ldots,y_n)[((a_n+1)^2f(x_1,\ldots,y_n)+a_n^2f(x_1,\ldots,x_n)]}{((a_n+1)^2f(x_1,\ldots,y_n)-a_n^2f(x_1,\ldots,x_n))^2}||$$
$$\leq \phi_n(x_1,y_1,\ldots,x_n,y_n), \tag{8}$$

for all $x_i,y_i\in G$; $a_i,b_i\in Z\backslash\{0\}$ and $i=\{1,2,\ldots,n\}$. Also suppose that $f(x_1,x_2,\ldots\ldots x_n)=0$ if $x_j=0$ for some $j=s+1,\ldots,n$. Then there exist a unique reciprocal mapping $S: G^n\to X$ satisfying (3) and the inequality

$$||S(x_1,x_2,\ldots\ldots x_n)-f(x_1,x_2,\ldots\ldots x_n)||\leq \Phi(x_1,y_1,\ldots,x_n,y_n) \tag{9}$$

for all $x_i,y_i\in G$ and $i=\{1,2,\ldots,n\}$.

Proof: Existence- Putting $x_i=y_i$ and $a_i=b_i$ for $i=1,2,\ldots,s$ in (8) and multiplying by $2a_i$, we get

$$||2a_if(x_1,\ldots,2a_ix_i,\ldots,x_n)-f(x_1,\ldots,x_i,\ldots,x_n)||\leq 2|a_i|\phi_i(x_1,y_1,\ldots,x_i,x_i,\ldots,x_n,y_n) \tag{10}$$

for all $x\in X$. Continuing like this, we obtain

$$||2^ia_1a_2\ldots a_if(2a_1x_1,\ldots,2a_ix_i,x_{i+1}\ldots,x_n)-2^{i-1}a_1a_2\ldots a_{i-1}f(2a_1x_1,\ldots,2a_{i-1}x_{i-1},x_i,$$
$$\ldots,x_n)||\leq 2^i|a_1a_2\ldots a_i|\phi_i(2a_1x_1,y_1,\ldots,2a_{i-1}x_{i-1},y_{i-1},x_i,x_i,x_{i+1},y_{i+1}\ldots,x_n,y_n). \tag{11}$$

Thus, it can be easily seen that

$$||2^sa_1a_2\ldots a_sf(2a_1x_1,\ldots,2a_sx_s,x_{s+1},\ldots,x_n)-f(x_1,\ldots,x_n)||\leq \max\{2^i|a_1a_2\ldots a_i|$$
$$\phi_i(2a_1x_1,y_1,\ldots,2a_{i-1}x_{i-1},y_{i-1},x_i,x_i,x_{i+1},y_{i+1},\ldots,x_n,y_n); i=1,2,\ldots,s\}. \tag{12}$$

Putting $y_j=x_j$ for $j=s+1,s+2,\ldots,n$ in (8) and multiplying by $((a_j+1)^2-a_j^2)^2$, we get

$$||((a_j+1)^2-a_j^2)^2f(x_1,x_2,\ldots,(2a_j+1)x_j,\ldots,x_n)-f(x_1,x_2,\ldots,x_j,\ldots,x_n)||$$

$$\leq ((a_j+1)^2-a_j^2)^2\Phi_j(x_1,y_1,\ldots,x_j,x_j,\ldots x_n,y_n) \quad (13)$$

Again, we can write

$$||((a_{s+1}+1)^2-a_{s+1}^2)^2\ldots.((a_j+1)^2-a_j^2)^2f(x_1,..,x_s,(2a_{s+1}+1)x_{s+1},..,$$
$$(2a_j+1)x_j,x_{j+1}\ldots,x_n)-((a_{s+1}+1)^2-a_{s+1}^2)^2\ldots.((a_{j-1}+1)^2-a_{j-1}^2)^2$$
$$f((x_1,..,x_s,(2a_{s+1}+1)x_{s+1},..(2a_{j-1}+1)x_{j-1},x_j,\ldots,x_n)||$$
$$\leq ((a_{s+1}+1)^2-a_{s+1}^2)^2\ldots.((a_j+1)^2-a_j^2)^2\Phi_j(x_1,y_1,..,x_s,y_s,$$
$$(2a_{s+1}+1)x_{s+1},y_{s+1},..,(2a_{j-1}+1)x_{j-1},y_{j-1},x_j,x_j,x_{j+1},y_{j+1}\ldots.,x_n,y_n) \quad (14)$$

Thus, we have

$$||((a_{s+1}+1)^2-a_{s+1}^2)^2\ldots((a_n+1)^2-a_n^2)^2f(x_1,..,x_s,(2a_{s+1}+1)x_{s+1},..,$$
$$(2a_n+1)x_n)-f(x_1,x_2,\ldots\ldots,x_n)||\leq \max\{((a_{s+1}+1)^2-a_{s+1}^2)^2..$$
$$((a_j+1)^2-a_j^2)^2\Phi_j(x_1,y_1,..,x_s,y_s,(2a_{s+1}+1)x_{s+1},y_{s+1},..,$$
$$(2a_{j-1}+1)x_{j-1},y_{j-1},x_j,x_j,x_{j+1},y_{j+1},..,x_n,y_n);j=s+1,s+2,\ldots.,n\} \quad (15)$$

By, (12) and (15), we get

$$||2^sa_1a_2\ldots a_s((a_{s+1}+1)^2-a_{s+1}^2)^2\ldots.((a_n+1)^2-a_n^2)^2f(2a_1x_1,\ldots,2a_sx_s,$$
$$(2a_{s+1}+1)x_{s+1},\ldots,(2a_n+1)x_n)-f(x_1,x_2,\ldots,x_n)||\leq$$
$$\max\{||2^sa_1a_2\ldots a_s((a_{s+1}+1)^2-a_{s+1}^2)^2\ldots((a_n+1)^2-a_n^2)^2f(2a_1x_1,..,$$
$$2a_sx_s,(2a_{s+1}+1)x_{s+1},..,(2a_n+1)x_n)-2^sa_1..a_sf(2a_1x_1,..,2a_sx_s,$$
$$x_{s+1},..,x_n)||,||2^sa_1\ldots.a_sf(2a_1x_1,..,2a_sx_s,x_{s+1}\ldots,x_n)-f(x_1,..,x_n)||\}$$
$$\leq \max\{2^s|a_1\ldots.a_s|\max\{((a_{s+1}+1)^2-a_{s+1}^2)^2\ldots((a_j+1)^2-a_j^2)^2$$
$$\Phi_j(2a_1x_1,y_1,\ldots.,2a_sx_s,y_s,(2a_{s+1}+1)x_{s+1},y_{s+1},\ldots,(2a_{j-1}+1)x_{j-1},$$
$$y_{j-1},x_j,x_j,x_{j+1},y_{j+1}\ldots.,x_n,y_n);j=s+1,s+2,\ldots.,n\},\max\{2^i|a_1a_2$$
$$..a_i|\phi_i(2a_1x_1,y_1,..,2a_{i-1}x_{i-1},y_{i-1},x_i,x_i,x_{i+1},y_{i+1},..,x_n,y_n);i=1,2,..,s\}\}$$

Hence,it follows that

$$||2^{sm}a_1^ma_2^m\ldots a_s^m((a_{s+1}+1)^2-a_{s+1}^2)^{2m}\ldots.((a_n+1)^2-a_n^2)^{2m}f(2^ma_1^mx_1,\ldots,$$
$$2^ma_s^mx_s,(2a_{s+1}+1)^mx_{s+1},.,(2a_n+1)^mx_n)-2^{s(m+1)}a_1^{(m+1)}a_2^{(m+1)}\ldots$$
$$a_s^{(m+1)}((a_{s+1}+1)^2-a_{s+1}^2)^{2(m+1)}..((a_n+1)^2-a_n^2)^{2(m+1)}f(2^{m+1}a_1^{m+1}x_1,$$
$$\ldots,2^{m+1}a_s^{m+1}x_s,(2a_{s+1}+1)^{m+1}x_{s+1},\ldots,(2a_n+1)^{m+1}x_n)||\leq \max\{2^s|a_1\ldots..$$
$$a_s|\max\{2^{sm}|a_1^ma_2^m\ldots a_s^m|((a_{s+1}+1)^2-a_{s+1}^2)^{2(m+1)}..((a_j+1)^2-a_j^2)^{2(m+1)}$$
$$((a_{j+1}+1)^2-a_{j+1}^2)^{2m}..((a_n+1)^2-a_n^2)^{2m}\Phi_j(2^ma_1^mx_1,y_1,..,$$
$$2^ma_s^mx_s,y_s,(2a_{s+1}+1)^{m+1}x_{s+1},y_{s+1},..,(2a_{j-1}+1)^{m+1}x_{j-1},y_{j-1},$$

$(2a_j+1)^m x_j, (2a_j+1)^m x_j, (2a_{j+1}+1)^m x_{j+1}, y_{j+1}, .., (2a_n+1)^m x_n, y_n);$

$j = s+1, .., n\}, \max\{2^{sm+i}|a_1^{m+1}a_2^{m+1} \dots a_i^{m+1}a_{i+1}^m \dots a_s^m|((a_{s+1}+1)^2 -$

$a_{s+1}^2)^{2m} \dots ((a_n+1)^2 - a_n^2)^{2m}\Phi_i(2^{m+1}a_1^{m+1}x_1, y_1, .., 2^{m+1}a_{i-1}^{m+1}x_{i-1}, y_{i-1},$

$2^m a_i x_i, 2^m a_i x_i, 2^m a_{i+1}^m x_{i+1}, y_{i+1}, .., 2^m a_s^{m+1} x_s, (2a_{s+1}+1)^m x_{s+1}, y_{s+1},$

$$.., (2a_n+1)^m x_n, y_n)\}: i = 1, .., s\} \tag{16}$$

for all $m \in N \cup \{0\}$. Next, from (06) and (16), we can say that the sequence

$\{2^{sm}a_1^m a_2^m \dots a_s^m((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} .. ((a_n+1)^2 - a_n^2)^{2m}$

$f(2^m a_1^m x_1, .., 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, \dots (2a_n+1)^m x_n)\}$

is a cauchy sequence. Since X is complete,therefore it is convergent. Hence we can define

$$S(x_1, \dots, x_n) = \lim_{m\to\infty}\{2^{sm}a_1^m a_2^m \dots a_s^m((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} \dots ((a_n+1)^2 - a_n^2)^{2m}$$

$$f(2^m a_1^m x_1, \dots, 2^m a_s^m x_s, (2s_{s+1}+1)^m x_{s+1}, \dots (2a_n+1)^m x_n)\} \tag{17}$$

for all $x_i, y_i \in G, i = \{1,2, \dots, n\}$. With the help of induction ,we can say

$||f(x_1, .., x_n) - 2^{sm}a_1^m a_2^m .. a_s^m((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} .. ((a_n+1)^2 - a_n^2)^{2m} f(2^m a_1^m x_1,$

$.., 2^m a_s^m x_s, (2s_{s+1}+1)^m x_{s+1}, \dots (2a_n+1)^m x_n)|| \le \max\{\max\{2^s|a_1 \dots a_s|$

$\max\{2^{sm}|a_1^m a_2^m \dots a_s^m|((a_{s+1}+1)^2 - a_{s+1}^2)^{2(m+1)} .. ((a_j+1)^2 - a_j^2)^{2(m+1)}$

$((a_{j+1}+1)^2 - a_{j+1}^2)^{2m} .. ((a_n+1)^2 - a_n^2)^{2m}\Phi_j(2^m a_1^m x_1, y_1, .., 2^m a_s^m x_s, y_s,$

$(2a_{s+1}+1)^{m+1}x_{s+1}, y_{s+1}, .., (2a_{j-1}+1)^{m+1}x_{j-1}, y_{j-1}, (2a_j+1)^m x_j, (2a_j+1)^m x_j,$

$(2a_{j+1}+1)^m x_{j+1}, y_{j+1}, .., (2a_n+1)^m x_n, y_n); j = s+1, .., n\}, \max\{2^{sm+i}|a_1^{m+1}a_2^{m+1}$

$\dots a_i^{m+1}a_{i+1}^m \dots a_s^m|((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} \dots ((a_n+1)^2 - a_n^2)^{2m}\Phi_i(2^{m+1}a_1^{m+1}x_1,$

$y_1, .., 2^{m+1}a_{i-1}^{m+1}x_{i-1}, y_{i-1}, 2^m a_i x_i, 2^m a_i x_i, 2^m a_{i+1}^m x_{i+1}, y_{i+1}, .., 2^m a_s^{m+1} x_s,$

$$(2a_{s+1}+1)^m x_{s+1}, y_{s+1}, .., (2a_n+1)^m x_n, y_n)\}: i = 1, .., s\}: m = 0,1,2, \dots, p\} \tag{18}$$

for $p \in N \cup \{0\}$ and for all $x_i, y_i \in G, i = 1,2, \dots, n$. By taking limit $p \to \infty$ in (18) and using (7), we get that inequality (9) is true. By (17), we get

$||S(x_1, , \dots a_i x_i + b_i y_i, \dots, x_n) - \frac{S(x_1, \dots, x_n)S(x_1, x_2, .., y_i, .., x_n)}{b_i S(x_1, \dots, x_n) + a_i S(x_1, x_2, \dots, y_i, \dots, x_n)}|| =$

$\lim_{m\to\infty}|K|||f(2^m a_1^m x_1, ., 2^m a_i^m(a_i x_i + b_i y_i), ., 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, ., (2a_n$

$+1)^m x_n) - \frac{f(2^m a_1^m x_1, .., 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, .., (2a_n+1)^m x_n)F}{f(2^m a_1^m x_1, .., 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, .., (2a_n+1)^m x_n) + F}||$

$\le \lim_{m\to\infty}|K|\Phi_i(2^m a_1^m x_1, y_1, 2^m a_2^m x_2, y_2, \dots, 2^m a_{i-1}^m x_{i-1}, y_{i-1}, 2^m a_i^m x_i, 2^m a_i^m y_i,$

$$2^m a_{i+1}^m x_{i+1}, y_{i+1}, .., 2^m a_s^m x_s, y_s, (2a_{s+1}+1)^m x_{s+1}, y_{s+1}, .., (2a_n+1)^m x_n, y_n) \tag{19}$$

for all i=1,2,...,s; where,

$$K = 2^{sm}a_1^m a_2^m \ldots\ldots a_s^m((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} \ldots.((a_n+1)^2 - a_n^2)^{2m}$$

$$F = f(2^m a_1^m x_1, .., 2^m a_i^m y_i, .., 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, .., (2a_n+1)^m x_n).$$

Also, it follows from (17),

$$\begin{aligned}
&||S(x_1, \ldots., x_s, x_{s+1}, ..((a_j+1)x_j + a_j y_j), \ldots.., x_n) \\
&\quad + S(x_1, \ldots., x_s, x_{s+1}, \ldots\ldots., ((a_j+1)x_j - a_j y_j), \ldots.., x_n) - \\
&\qquad 2S(x_1, \ldots., x_n) S(x_1, \ldots.., y_j, \ldots.., x_n) \\
&\qquad \frac{[((a_j+1)^2 S(x_1,..,y_j,..,x_n) + a_j^2 S(x_1,..,x_n)]}{((a_j+1)^2 S(x_1,x_2,..,y_j,..,x_n) - a_j^2 S(x_1,..,x_n))^2}|| \\
&= \lim_{m\to\infty} |2^{sm}a_1^m a_2^m \ldots\ldots a_s^m((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} \ldots.((a_n+1)^2 - a_n^2)^{2m}| \\
&\quad ||f(2^m a_1^m x_1, .., 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, .., (2a_j+1)^m((a_j+1)x_j + a_j y_j), \\
&\quad .., (2a_n+1)^m x_n) + +f(2^m a_1^m x_1, .., 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, .., (2a_j+1)^m
\end{aligned}$$

$$\begin{aligned}
&((a_j+1)x_j - a_j y_j), \ldots.(2a_n+1)^m x_n) - 2f(2^m a_1^m x_1, \ldots, 2^m a_s^m x_s, (2a_{s+1}+ \\
&\quad 1)^m x_{s+1}, ..(2a_n+1)^m x_n) f(2^m a_1^m x_1, .., 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, .., (a_j+ \\
&\quad 1)^m y_j, ..(2a_n+1)^m x_n) \frac{a_j^* f(2^m a_1^m x_1,..,2^m a_s^m x_s,(2a_{s+1}+1)^m x_{s+1},..,(2a_j+1)^m y_j,..(2a_n+1)^m x_n) + a_j^2 A}{a_j^* f(2^m a_1^m x_1,..,2^m a_s^m x_s,(2a_{s+1}+1)^m x_{s+1},..,(2a_j+1)^m y_j,..(2a_n+1)^m x_n) - a_j^2 A}|| \leq \\
&\lim_{m\to\infty} 2^{sm}|a_1^m a_2^m \ldots\ldots a_s^m|((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} \ldots.((a_n+1)^2 - \\
&\quad a_n^2)^{2m}\Phi_i(2^m a_1^m x_1, y_1, \ldots., 2^m a_s^m x_s, y_s, (2a_{s+1}+1)^m x_{s+1}, y_{s+1}.., \\
&\quad (2a_{j-1}+1)^m x_{j-1}, y_{j-1}, (2a_j+1)^m x_j, (2a_j+1)^m y_j, (2a_{j+1}+ \\
&\quad 1)^m x_{j+1}, y_{j+1}, \ldots., (2a_n+1)^m x_n, y_n) \qquad (20)
\end{aligned}$$

for all j = s+1,.....,n; where,

$$a_j^* = (a_j+1)^2 \text{ and } A = f(2^m a_1^m x_1, \ldots, 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, \ldots.(2a_n+1)^m x_n).$$

Therefore, by (4),(5),(19) and (20) we can say that the function Q satisfies (3).

Uniqueness- To prove the uniqueness of S ,let $S^*: G^n \to X$ be another function which satisfies (3) and (9). Then,

$$\begin{aligned}
&||S(x) - S^*(x)|| \leq 2^{sm}a_1^m a_2^m \ldots\ldots a_s^m((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} \ldots.((a_n+1)^2 - a_n^2)^{2m} \\
&\max\{||S(2^m a_1^m x_1, \ldots, 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, \ldots(2a_n+1)^m x_n) - f(2^m a_1^m x_1, \ldots, \\
&2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, \ldots(2a_n+1)^m x_n)||, ||f(2^m a_1^m x_1, \ldots, 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, \\
&\ldots(2a_n+1)^m x_n) - S^*(2^m a_1^m x_1, \ldots, 2^m a_s^m x_s, (2a_{s+1}+1)^m x_{s+1}, \ldots(2a_n+1)^m x_n)||\} \\
&\leq 2^{sm}|a_1^m a_2^m \ldots\ldots a_s^m|((a_{s+1}+1)^2 - a_{s+1}^2)^{2m} \ldots.((a_n+1)^2 - a_n^2)^{2m}\max\{\Phi(2^m a_1^m x_1, y_1, \ldots, \\
&2^m a_s^m x_s, y_s, (2a_{s+1}+1)^m x_{s+1}, y_{s+1} \ldots(2a_n+1)^m x_n), \Phi(2^m a_1^m x_1, y_1, \ldots, \\
&2^m a_s^m x_s, y_s, (2a_{s+1}+1)^m x_{s+1}, y_{s+1} \ldots(2a_n+1)^m x_n)
\end{aligned}$$

which tends to 0 as $m \to \infty$, therefore $S = S^*$. Hence the proof.

3.2 Theorem

Let $\Phi_i: G^{2n} \to R^+$ be a function satisfying

$$\lim_{m\to\infty} 2^{nm}|a_1^m \dots a_n^m| \Phi_i(2^m a_1^m x_1, y_1, \dots, 2^m a_i^m x_i, 2^m a_i^m y_i, \dots, 2^m a_n^m x_n, y_n) = 0 \qquad (21)$$

for all $i = \{1,2,\dots,n\}$. Suppose that

$$\lim_{m\to\infty} \sigma(x_1, y_1, \dots, x_n, y_n) = 0 \qquad (22)$$

$$\Phi(x_1, y_1, \dots, x_n, y_n) = \lim_{p\to\infty} \max\{\sigma(x_1, y_1, \dots, x_n, y_n); m = 0,1,\dots,p\} < \infty \qquad (23)$$

where

$$\sigma(x_1, y_1, \dots, x_n, y_n) = \max\{2^{mn+i}|a_1^{m+1} a_2^{m+1} \dots a_n^{m+1}| \Phi(2^{m+1} a_1^{m+1} x_1, y_1, \dots,$$
$$2^{m+1} a_{i-1}^{m+1} x_{i-1}, y_{i-1}, 2^m a_i^m x_i, 2^m a_i^m x_i, 2^m a_{i+1}^m x_{i+1}, y_{i+1} \dots 2^m a_n^m x_n, y_n)\}$$

for all $x_i, y_i \in G$; $a_i, b_i \in Z\backslash\{0\}$ and $i = 1,2,\dots,n$. Suppose that $f: G^n \to X$ satisfies following inequalities

$$\left\|f(a_1 x_1 + b_1 y_1, x_2, \dots, x_n) - \frac{f(x_1, x_2, \dots, x_n) f(y_1, x_2, \dots, x_n)}{b_1 f(x_1, x_2, \dots, x_n) + a_1 f(y_1, x_2, \dots, x_n)}\right\|$$
$$\leq \phi_1(x_1, y_1, \dots, x_n, y_n),$$

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$$\left\|f(x_1, \dots, a_s x_s + b_s y_s, \dots, x_n) - \frac{f(x_1, \dots, x_n) f(x_1, \dots, y_s, \dots, x_n)}{b_s f(x_1, \dots, x_n) + a_s f(x_1, \dots, y_s, \dots, x_n)}\right\|$$
$$\leq \phi_s(x_1, y_1, \dots, x_n, y_n),$$

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$$\left\|f(x_1, \dots, a_n x_n + b_n y_n) - \frac{f(x_1, \dots, x_n) f(x_1, \dots, x_{n-1}, y_n)}{b_n f(x_1, \dots, x_n) + a_n f(x_1, \dots, x_{n-1}, y_n)}\right\| \qquad (24)$$

for all $x_i, y_i \in G$; $a_i, b_i \in Z\backslash\{0\}$ and $i = \{1,2,\dots,n\}$. Then there exist a unique reciprocal mapping $S_a: G^n \to X$ satisfying reciprocal additive system and the inequality

$$\|S_a(x_1, x_2, \dots x_n) - f(x_1, x_2, \dots x_n)\| \leq \Phi(x_1, y_1, \dots, x_n, y_n) \qquad (25)$$

for all $x_i, y_i \in G$ and $i = \{1,2,\dots,n\}$.

3.3 Theorem

Let $\Phi_i: G^{2n} \to R^+$ be a function satisfying

$$\lim_{m\to\infty} ((a_1+1)^2 - a_1^2)^{2m} \dots ((a_n+1)^2 - a_n^2)^{2m} \Phi_i((2a_1+1)^m x_1, y_1 \dots,$$
$$(2a_{j-1}+1)^m x_{j-1}, y_{j-1}, (2a_j+1)^m x_j, (2a_j+1)^m y_j,$$
$$(2a_{j+1}+1)^m x_{j+1}, y_{j+1}, \dots, (2a_n+1)^m x_n, y_n) = 0 \qquad (26)$$

for all $i = \{1,2,\dots,n\}$. Suppose that

$$\lim_{m\to\infty} \sigma(x_1, y_1, \dots, x_n, y_n) = 0 \qquad (27)$$

$$\text{and}, \Phi(x_1, y_1, .., x_n, y_n) = \lim_{p\to\infty} \max\{\sigma(x_1, y_1, .., x_n, y_n); m = 0,1, .., p\} < \infty \qquad (28)$$

where

$$\sigma(x_1, y_1, .., x_n, y_n) = \max\{((a_1+1)^2 - a_1^2)^{2(m+1)}, \ldots, ((a_i+1)^2 - a_i^2)^{2(m+1)},$$
$$((a_{i+1}+1)^2 - a_{i+1}^2)^{2m}, \ldots. ((a_n+1)^2 - a_n^2)^{2m}\Phi_j((2a_1+1)^{m+1}x_1, y_1,$$
$$.., (2a_{i-1}+1)^{m+1}x_{i-1}, y_{i-1}, (2a_i+1)^m x_i, (2a_i+1)^m x_i,$$
$$(2a_{i+1}+1)^m x_{i+1}, y_{i+1} .., (2a_n+1)^m x_n, y_n); i = 1, .., n\},$$

for all $x_i, y_i \in G$; $a_i, \in Z\backslash\{0\}$ and $i = 1,2, \ldots, n$. Suppose that $f: G^n \to X$ satisfies following inequalities

$$||f((a_1+1)x_1 + a_1y_1, .., x_s, \ldots, x_n) + f((a_1+1)x_1 - a_1y_1, , .., x_s, \ldots, x_n)$$
$$\frac{-2f(x_1,\ldots,x_n)f(y_1,\ldots,x_n)[((a_1+1)^2 f(y_1,\ldots,x_n) + a_1^2 f(x_1,\ldots,x_n)]}{((a_1+1)^2 f(y_1,x_2,\ldots,x_n) - a_1^2 f(x_1,\ldots,x_n))^2}||$$
$$\leq \phi_1(x_1, y_1, \ldots, x_n, y_n),$$

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$$||f(x_1, .., x_s, (a_s+1)x_s + a_sy_s, .., x_n) + f(x_1, .., x_s, (a_s+1)x_s - a_sy_s, .., x_n)$$
$$\frac{-2f(x_1,\ldots,x_n)f(x_1,\ldots,y_s,\ldots,x_n)[((a_s+1)^2 f(x_1,\ldots,y_s,\ldots,x_n) + a_s^2 f(x_1,\ldots,x_n)]}{((a_s+1)^2 f(x_1,x_2,\ldots,y_s,\ldots,x_n) - a_s^2 f(x_1,\ldots,x_n))^2}||$$
$$\leq \phi_s(x_1, y_1, \ldots, x_n, y_n),$$

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$$||f(x_1, .., x_{n-1}, (a_n+1)x_n + a_ny_n) + f(x_1, \ldots, x_{n-1}, (a_n+1)x_n - a_ny_n)$$
$$-\frac{2f(x_1,\ldots,x_n)f(x_1,\ldots,y_n)[((a_n+1)^2 f(x_1,\ldots,y_n) + a_n^2 f(x_1,\ldots,x_n)]}{((a_n+1)^2 f(x_1,\ldots,y_n) - a_n^2 f(x_1,\ldots,x_n))^2}||$$
$$\leq \phi_n(x_1, y_1, \ldots, x_n, y_n), \qquad (29)$$

for all $x_i, y_i \in G$; $a_i, \in Z\backslash\{0\}$ and $i = \{1, .., n\}$. Also suppose that $f(x_1, x_2, \ldots.., x_n) = 0$ if $x_i = 0$ for some $i = 1, \ldots, n$. Then there exist a unique quadratic reciprocal mapping $S_q: G^n \to X$ satisfying reciprocal quadratic system and the inequality

$$||S_q(x_1, x_2, \ldots.. x_n) - f(x_1, x_2, \ldots.. x_n)|| \leq \Phi(x_1, y_1, \ldots, x_n, y_n) \qquad (30)$$

for all $x_i, y_i \in G$ and $i = \{1,2, \ldots, n\}$.

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DETERMINATION OF MTSF AND AVAILABILITY OF A THREE UNIT STANDBY STOCHASTIC SYSTEM BY USING BASE STATE

Manju Devi*, Dr. V.K. Gupta & Dr. Vinod Kumar*****
*Asstt. Professor in Maths., Govt. College Behrampur (Bapauli), Panipat (Haryana)
**Prof. & HoD (Applied Sciences & Humanities) NC College of Engineering, Israna, Panipat
***Prof. (Maths.) & Registrar Baba Mastnath University Asthal Bohar, Rohtak (Haryana)
E-mail : ghalyan.manju@gmail.com, vijaygupta_1953@yahoo.co.in, kakoriavinod@gmail.com

ABSTRACT :

To evaluate easily and quickly, the key parameters of a stochastic system for its profit/cost analysis, has always remained the need of the hour. While using the Regenerative Point Technique, many state equations are to be written and solved recursively after taking Laplace/Stieltjes transforms of the state equations and then applying the concepts of limits, it takes a lot of time. Also while using Regenerative Point Graphical Technique (RPGT), introduced by Gupta [4], the presence of many circuits make the calculations lengthy. To overcome this difficulty, Gupta et al [3] introduced a new concept of 'base state' for doing reliability analysis to find the key parameters of the system (under steady state conditions) by using modified formulae of RPGT. In the present paper, the path analysis of a three unit stand-by stochastic system, in which initially, two units are in operative mode and a similar unit in cold stand-by mode, has been done to determine the 'base state' of the system, which is further used to determine the MTSF and Availability of the system using modified RPGT (under steady state conditions).

Key Words: Reachable State, Regenerative State, Base State, Primary Circuit, Equivalent Circuits, Path, RPGT.

1. INTRODUCTION

The researchers including Chander & Bansal [1], Kadyan et al [2], Shakeel & Vinod [6], and many others have used the Regenerative Point Technique and other methodologies [9],Gupta [4,5], Manju et al [8] and many others used RPGT introduced by Gupta[4] for doing the reliability and availability analysis of various stochastic systems. With the increase in the number of states in the state-space, to which a system can transit & increase in the number of transitions from any state to the others states, the Regenerative Point Technique and other methodologies, becomes very time consuming and cumbersome for doing the Reliability and Availability analysis of the system. To overcome this difficulty, Gupta et al [3] introduced a new concept known as 'base state' for finding the key parameters of a system (under steady state conditions). The objective of this research paper is to determine the MTSF and Availability of a three unit cold standby stochastic system by finding its 'base-state', in which initially, two units are in operative mode along with the third similar unit in cold stand-by mode and with two types of failures (minor/major) and two types of repairing facility using ordinary/expert server.

2. THE STOCHASTIC SYSTEM:

It is a three unit cold standby system in which initially, two units are in operative mode along with the third similar unit in the cold stand-by mode. There can occur two types of failures (minor/major) and repairing is done by ordinary/expert servers. On the failure of a unit, the cold standby unit becomes operative instantaneously and the inspection is carried out of the failed unit, to detect the type of failure- whether it is minor or major. The unit with minor failure is repaired by the ordinary repairman while that with the major failure, is repaired only by the expert repairman. In case of further failure of any unit, the operating unit is stopped i.e. put in the down state. The system is in up-state/available only if two units are in operative mode.

3. ASSUMPTIONS, NOTATIONS & SYMBOLS:

The various assumptions, notations and symbols used are given as under:

3.1 Assumptions:

i. The system starts from the good state '0' at time t = 0.

ii. All the three identical units and the operating units have the same failure rate.

iii. The inspection/repairs can start only if the server is available and the server cannot leave the system while repairing it.

iv. The expert repairman can do repairs of both types of failures.

v. The unit after repairs works as a new one.

vi. Inspection of a failed unit finishes before the failure of any other unit.

vii. All random variables are independent and un-correlated.

viii. The distributions of the failure times are exponential and that of the inspection times and repair times may have general distributions which are different for minor/major repairs.

3.2 Notations:

*pr/**	:	**Probability/Laplace transformation.**
$\underline{K}$	:	**Non regenerative state '*k*'**
$\{a_0,a_1,...,a_{n-1},a_n\}$	:	**A directed path from the state a_0 to a_n, through the states a_1, …, a_{n-1} to reach the state a_n.**
$q_{i,j}(t)$ / $q_{i,\underline{k},j}(t)$	:	**Probability density function (p.d.f.) of the first passage time from a regenerative state *i* to a regenerative state *j* or to a failed state *j* without visiting any other regenerative state in (0, *t*]/ while visiting $\underline{k}$ only once in (0,**

		t], given that the system entered regenerative state i at $t = 0$.
(i, j) / $p(i, j)$	:	**Steady state transition probability from the regenerative state i to the regenerative state j, without visiting any other states.** $(i, j) = p(i, j) = \lim_{s\to 0} q^{*}_{i,j}(s)$; $(i, j, k) = (i, j)\,(j, k) = p(i, j)\,.\,p(j, k)$
$(i, \underline{k}, j)$ / $p(i, \underline{k}, j)$	:	**Steady state transition probability from the regenerative state i to the regenerative state j, visiting non regenerative state $\underline{k}$.** $(i, \underline{k}, j) = p(i, \underline{k}, j) = \lim_{s\to 0} q^{*}_{i,\underline{k},j}(s)$
$\overline{cycle}$ / k-$\overline{cycle}$		**A circuit formed through un-failed states/ A circuit formed through un-failed states, with terminals at the regenerative state k.**
k-$cycle$	:	**A circuit with terminals at the regenerative state k.**
$V(\overline{k,k})$	:	**Transition probability factor of the reachable state k of the k-$\overline{cycle}$ formed through un-failed states.**
$V(k, k)$	:	**Transition probability factor of the reachable state k of the k-$cycle$.**
$V(i, j)$	:	**Transition probability factor of the reachable state j from the i-state.**
$(i \xrightarrow{sr} j)$	:	**r-th directed simple path from i- state to j- state; r takes positive integral values for different paths from i- state to j- state.**
$(i \xrightarrow{sff} j)$	:	**a directed simple failure free path from i- state to j- state.**
$R_i(t)$	:	**Reliability of the system at time t, given the system is initially in the regenerative state 'i'.**
μ_i / μ_i'	:	**Mean sojourn time of the state 'i'/total un-conditional time spent before transiting to any other regenerative state(s), given that the system entered regenerative state 'i' at $t = 0$.**
f_i	:	**Fuzziness measure of the i-state; $f_i = 0$, if 'i' is a failed state; $f_i = 1$, if 'i' is an up state.**
CS	:	**Unit is in cold standby mode.**
O1/O2/S	:	**Operating Units/ Unit is in the down state.**
λ	:	**Constant failure rate of a unit.**
a/b	:	**Probability of minor/major failure of a unit and a + b =1.**
$g_1(t)/G_1(t)$	:	**p.d.f. /c.d.f. of the repair time for the ordinary repairman.**
$g_2(t)/G_2(t)$	:	**p.d.f. /c.d.f. of the repair time for the expert repairman.**
h(t)/H(t)	:	**p.d.f. /c.d.f. of the inspection time.**
$\overline{G_1}(t)/\overline{G_2}(t)$ / $\overline{H}(t)$	:	$\overline{G_1}(t) = 1 - G_1(t)$ / $\overline{G_2}(t) = 1 - G_2(t)$ / $\overline{H}(t) = 1 - H(t)$
Ui/Ur/Wi	:	**Failed unit is under inspection/ordinary repairs/ waiting for inspection.**

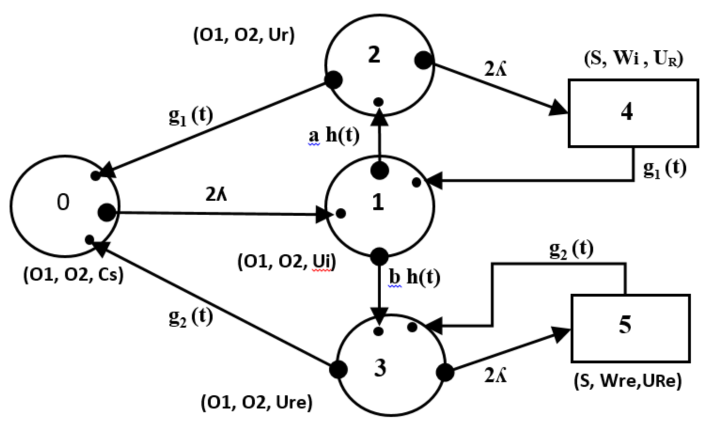
Ure/Wre	**:**	**Failed unit under repair/waiting repairs by an expert repairman.**
UR/Ure	**:**	**Failed unit under repairs from previous state by ordinary repairman /expert repairman.**

4. STATE TRANSITION DIAGRAM:

There are six states belonging to the *state-space* of the stochastic system, to which the system can transit under the given assumptions/conditions (Table-1). The epochs of entry into states 0, 1, 2 & 3 are the regenerative points and hence these states are the regenerative states. The states 4 & 5 are the failed states&non-regenerative states. Accordingly, the transition diagram of the system based upon the above assumptions/conditions is shown in Fig.1.

TABLE -1

TYPE OF STATE	SYMBOL	STATES
Regenerative State	•	**0, 1, 2 & 3**
Up-State	○	**0, 1, 2 & 3**
Failed State	▭	**4 & 5**



(Fig. 1)

5. PATH ANALYSIS OF THE SYSTEM:

There are 6 states (vertices) from 0 to 5 and 9 transitions in the transition diagram of the stochastic system. The terminal states of all the 9 transitions (edges) are shown in Table -2. Firstly, the various directed paths from each state (vertex) to the other reachable states are determined in the Transition Diagram and are shown in Table - 3. Secondly, the primary, secondary and tertiary circuits etc. at all the vertices (corresponding to regenerative states) are determined and shown in Table-4 and the observations are shown in Table- 9. Further all the simple paths along with the corresponding primary, secondary and tertiary circuits etc. w.r.to the different simple paths from the regenerative states 0, 1, 2 & 3 are determined and are shown in Table- 5, 6, 7 & 8 respectively and the summary of observations is given in Table- 10.

TABLE -2

Sr. No.	Transition's Terminal States		Sr. No.	Transition's Terminal States	
	From	To		From	To
1	0	1	6	3	0
2	1	2	7	3	5
3	1	3	8	4	1
4	2	0	9	5	3
5	2	4			

TABLE- 3

Paths from State '*i*' to the Reachable State '*j*': P0						
'*i*'	*j* = 0	*j* =1	*j* =2	*j* =3	*j* =4	*j* =5
0	{0,1,2,0} {0,1,3,0}	{0,1}	{0,1,2}	{0,1,3}	{0,1,2,4}	{0,1,3,5}
1	{1,2,0} {1,3,0}	{1,2,0,1} {1,3,0,1} {1,2,4,1}	{1,2}	{1,3}	{1,2,4}	{1,3,5}
2	{2,0} {2,4,1,3,0}	{2,0,1} {2,4,1}	{2,0,1,2} {2,4,1,2}	{2,0,1,3} {2,4,1,3}	{2,4}	{2,0,1,3,5} {2,4,1,3,5}
3	{3,0}	{3,0,1}	{3,0,1,2}	{3,0,1,3} {3,5,3}	{3,0,1,2,4}	{3,5}
4	{4,1,2,0} {4,1,3,0}	{4,1}	{4,1,2}	{4,1,3}	{4,1,2,4}	{4,1,3,5}
5	{5,3,0}	{5,3,0,1}	{5,3,0,1,2}	{5,3}	{5,3,0,1,2,4}	{5,3,5}

TABLE- 4

Primary, Secondary and Tertiary Circuits at a vertex						
Vertex *i*	(CL1)	(CL2)	(CL3)	N(CL4)	No. of Distinct Circuits	Explanation
0	{0,1,2,0}	{1, 2, 4, 1}	---	---	N (CL1) =2 N (CL2) =2 N (CL3) =0 N(CL) =0	{1,3,0,1}≡{0,1,3,0} {1,2,0,1}≡{0,1,2,0}
	{0, 1, 3, 0}	{1, 2, 4, 1}	---	---		
		{3, 5, 3}	---	---		
1	{1, 2, 0, 1}	-------	---	---	N (CL1) =3 N (CL2) =1 N (CL3) =0 N(CL4) =0	
	{1, 3, 0, 1}	{3, 5, 3}	---	---		
	{1, 2, 4, 1}	-------	---	---		
2	{2, 0, 1, 2}	{0, 1, 3, 0}	{3, 5, 3}	---	N (CL1) =2 N (CL2) =1 N (CL3) =1 N(CL4) =0	
	{2, 4, 1, 2}	{1, 3, 0, 1}	{3, 5, 3}	---		
3	{3, 0, 1, 3}	{0, 1, 2, 0}	{1, 2, 4, 1}	---	N (CL1) =2 N (CL2) =2 N (CL3) =1 N(CL4) =0	
		{1, 2, 4, 1}	---	---		
	{3, 5, 3}	-------	---	---		

TABLE- 5

Vertex 'j'	Paths from initial state '0' to vertex 'j'		Circuits		
	$(0 \xrightarrow{s_r} j)$	P_0	P_1	P_2	Distinct Circuits
0	$(0 \xrightarrow{s_1} 0)$	{0,1,2,0}	{1,2,4,1}	-	$N(P_1) = 2$ $N(P_2) = 0$
	$(0 \xrightarrow{s_2} 0)$	{0,1,3,0}	{1,2,4,1}		
			{3,5,3}		
1	$(0 \xrightarrow{s_1} 1)$	{0,1}	{1,2,4,1}	-	
2	$(0 \xrightarrow{s_1} 2)$	{0,1,2}	{1,2,4,1}	-	
3	$(0 \xrightarrow{s_1} 3)$	{0,1,3}	{1,2,4,1}	-	
			{3,5,3}		

TABLE–6

Vertex 'j'	Paths from initial state '1' to vertex 'j'		Circuits			
	$(1 \xrightarrow{s_r} j)$	P_0	P_1	P_2	P_3	Distinct Circuits
0	$(1 \xrightarrow{s_1} 0)$	{1,2,0}	---	------	------	$N(P_1) = 1$ $N(P_2) = 0$ $N(P_3) = 0$
	$(1 \xrightarrow{s_2} 0)$	{1,3,0}	{3,5,3}	------	------	
1	$(1 \xrightarrow{s_1} 1)$	{1,2,0,1}	---	------	------	
	$(1 \xrightarrow{s_2} 1)$	{1,3,0,1}	{3,5,3}	------	------	
	$(1 \xrightarrow{s_3} 1)$	{1,2,4,1}	---	-----	------	
2	$(1 \xrightarrow{s_1} 2)$	{1,2}	---	------	------	
3	$(1 \xrightarrow{s_1} 3)$	{1,3}	{3,5,3}	------	------	

TABLE- 7

Vertex ‘j’	Paths from initial state ‘2’ to vertex ‘j’		Circuits			
	$(2 \xrightarrow{s_r} j)$	**P_0**	**P_1**	**P_2**	**P_3**	**Distinct Circuits**
0	$(2 \xrightarrow{s_1} 0)$	**{2,0}**	**{0,1,3,0}**	**{1, 2, 4, 1}**	**---**	
				{3,5,3}	**---**	
	$(2 \xrightarrow{s_2} 0)$	**{2,4,1,3,0}**	**{1,3,0,1}**	**{3, 5, 3}**	**---**	
			{3,5,3}	**---**	**---**	
1	$(2 \xrightarrow{s_1} 1)$	**{2,0,1}**	**{0,1,3,0}**	**{3, 5, 3}**	**---**	**$N(P_1)$ =2** **$N(P_2)$ =2** **$N(P_3)$ =0**
	$(2 \xrightarrow{s_2} 1)$	**{2,4,1}**	**{1,3,0,1}**	**{3, 5, 3}**	**---**	
2	$(2 \xrightarrow{s_1} 2)$	**{2,0,1,2}**	**{0,1,3,0}**	**{3, 5, 3}**	**---**	
	$(2 \xrightarrow{s_2} 2)$	**{2,4,1,2}**	**{1,3,0,1}**	**{3, 5, 3}**	**---**	
3	$(2 \xrightarrow{s_1} 3)$	**{2, 0, 1,3}**	**{0, 1, 3, 0}**	**{3, 5, 3}**	**---**	
			{3, 5, 3}	**---**	**---**	
	$(2 \xrightarrow{s_2} 3)$	**{2, 4, 1, 3}**	**{1, 3, 0, 1}**	**{3, 5, 3}**	**---**	

TABLE- 8

Vertex ‘j’	Paths from initial state ‘3’ to vertex ‘j’		Circuits			
	$(3 \xrightarrow{s_r} j)$	**P_0**	**P_1**	**P_2**	**P_3**	**Distinct Circuits**
0	$(3 \xrightarrow{s_1} 0)$	**{3,0}**	**{0,1,2,0}**	**{1,2,4,1}**	**---**	
1	$(3 \xrightarrow{s_1} 1)$	**{3,0,1}**	**{0,1,2,0}**	**{1,2,4,1}**	**---**	**$N(P_1)$ =2** **$N(P_2)$ =1** **$N(P_3)$ = 0**
			{1,2,4,1}	**---**	**---**	
2	$(3 \xrightarrow{s_1} 2)$	**{3,0,1,2}**	**{0,1,2,0}**	**{1,2,4,1}**	**---**	
			{1,2,4,1}	**---**	**---**	
3	$(3 \xrightarrow{s_1} 3)$	**{3,0,1,3}**	**{0,1,2,0}**	**{1,2,4,1}**	**---**	
	$(3 \xrightarrow{s_2} 3)$	**{3,5,3}**	**{1,2,4,1}**	**---**	**---**	

TABLE-9

State	$N_i(CL_1)$	$N_i(CL_2)$	$N_i(CL_3)$	$N_i(CL_4)$	Explanation
0	2	2	0	0	
1	3	1	0	0	$N_i(CL_1) > N_i(CL_1)$
2	2	1	1	0	$N_i(CL_2) \leq N_i(CL_2)$
3	2	2	1	0	$N_i(CL_3) \leq N_i(CL_3)$

TABLE - 10

Regenerative initial state (i)	$N_i(P_1)$	$N_i(P_2)$	$N_i(P_3)$	Explanation
0	2	0	0	$N_i(P_j) \leq N_i(P_j)$
1	1	0	0	j =1,2,3
2	2	1	0	i=0,1,2,3
3	2	1	0	

5.1 Base State of the System:

From Table- 9, it is observed that the vertex '1' is associated with the largest number of primary circuits and minimum number of circuits of higher levels. And from Table-10, it is also concluded that the simple paths from '1' have least number of Primary, Secondary and other level of circuits. From the Table-9 & Table-10, it is concluded that '1' is the base state of the system. And this base state '1' can be used as the initial state (at time t=0) for determining the availability and other global parameters of the system by using modified formulae of RPGT[3].

6. TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES:

The transition probabilities, the mean sojourn times and total un-conditional times are as under:

6.1 Transition Probabilities:

$p(0,1)$=1;$p(1,2) = a$; $p(1,3) = b$;$p(\underline{4},1) = 1$; $p(\underline{5},3)$=1;$p(2,0) = g_1^*(2\lambda)$;$p(2,\underline{4},1) = 1 - g_1^*(2\lambda)$; $p(3,0) = g_2^*(2\lambda)$;$p(3,\underline{5},3) = 1 - g_2^*(2\lambda)$; $p(1,2) + p(1,3) = 1$; $p(2,0) + p(2,\underline{4},1) = 1$; $p(3,0) + p(3,\underline{5},3) = 1$; $p(2,0) + p(2,\underline{4}) = 1$; $p(3,0) + p(3,\underline{5}) = 1$

6.2 Mean Sojourn Times:

$$\mu_0=\frac{1}{2\lambda};\ \mu_1= h^{*/}(0);\ \ \mu_2=\frac{1-g_1^*(2\lambda)}{2\lambda};\ \mu_3=\frac{1-g_2^*(2\lambda)}{2\lambda}$$

6.3 Total Un-Conditional Times:

$\mu_0' = \mu_0$; $\mu_1' = \mu_1$; $\mu_2' = \int_0^\infty t.\,d\{G_1(t)\} = -\,g_1^{*'}(0)$ and $\mu_3' = \int_0^\infty t.d\{G_2(t)\} = -\,g_2^{*'}(0)$.

7. EVALUATION OF MTSF & AVAILABILITY OF THE SYSTEM USING BASE STATE :

The mean time to the system failure (*MTSF*) and availability of the system (under steady state conditions) are evaluated by applying Regenerative Point Graphical Technique (RPGT) and using base state 'ξ'=1 as the initial state (at time t = 0) as under:

7.1 Mean Time to System Failure:

From Fig.1, the regenerative un-failed states to which the system can transit from initial state 'ξ'= 1 (at time't' = 0), before transiting to any failed states are: i = 0, 1, 2 & 3. The simple failure free path from 'ξ'=1 state to un-failed state are shown in Table -11.

TABLE - 11

Vertex '*j*'	**Failure Free Paths from initial state '1' to un-failed state '*j*'**		**Failure Free Circuits**
	$(1 \xrightarrow{s_r(sff)} j)$	**P_0**	**P_1**
0	$(1 \xrightarrow{s_1(sff)} 0)$	**{1,2,0}**	**--**
	$(1 \xrightarrow{s_2(sff)} 0)$	**{1,3,0}**	**--**
1	$(1 \xrightarrow{s_1(sff)} 1)$	**{1,2,0,1}**	**--**
	$(1 \xrightarrow{s_2(sff)} 1)$	**{1,3,0,1}**	**--**
2	$(1 \xrightarrow{s_1(sff)} 2)$	**{1,2}**	**--**
3	$(1 \xrightarrow{s_1(sff)} 3)$	**{1,3}**	**--**

The mean time to system failure (MTSF) is obtained by applying modified formula of RPGT as under:

$$\mathrm{MTSF} = \left[\sum_{i,s_r}\left\{\frac{\{pr(\xi \xrightarrow{s_r(sff)} i)\}.\mu_i}{\prod_{k_1 \neq \xi}\{1-V_{\overline{k_1,k_1}}\}}\right\}\right] \div \left[1-\sum_{s_r}\left\{\frac{\{pr(\xi \xrightarrow{s_r(sff)} \xi)\}}{\prod_{k_2 \neq \xi}\{1-V_{\overline{k_2,k_2}}\}}\right\}\right]$$

$$\Phi_1 = \begin{bmatrix}\{(1,2,0)+(1,3,0)\}.\mu_0+(1,1)\mu_1+\\ (1,2)\mu_2+(1,3)\mu_3\end{bmatrix} \div \left[1-\{(1,2,0,1)+(1,3,0,1)\}\right] = N_1 \div D_1 \text{ where}$$

N_1=[{ $p(1,2)\,p(2,0) + p(1,3)\,p(3,0)$} μ_0+**p(1,1)** μ_1+**p(1,2)** μ_2+**p(1,3)** μ_3] and

D_1= [1 –{ $p(0,1).p(1,2).p(2,0) + p(0,1).p(1,3).p(3,0)$ }].

This MTSF (Φ_1) is different from the value of MTSF (Φ_0) as determined by Manju et al [8] from initial state '0'(at time t = 0) as under:

From Fig.1, the regenerative un-failed states to which the system can transit (if it is assumed that the system starts with initial state '0' at time't' = 0), before transiting to any failed states are: i = 0, 1, 2 & 3. The mean time to system failure (MTSF) is obtained by applying RPGT as under:

$$\textbf{MTSF}=\left[\sum_{i,s_r}\left\{\frac{\{pr(0\xrightarrow{s_r(sff)}i)\}.\mu_i}{\prod_{k_1\neq 0}\{1-V_{\overline{k_1,k_1}}\}}\right\}\right]\div\left[1-\sum_{s_r}\left\{\frac{\{pr(0\xrightarrow{s_r(sff)}0)\}}{\prod_{k_2\neq 0}\{1-V_{\overline{k_2,k_2}}\}}\right\}\right]$$

$$\Phi_0=\begin{matrix}(0,0)\mu_0+(0,1)\mu_1+\\(0,1,2)\mu_2+(0,1,3)\mu_3\end{matrix}\div\left[1-(0,1,2,0)-(0,1,3,0)\right]=N_0\div D_0 \text{ where}$$

N_0 = [$p(0,0)\ \mu_0 + p(0,1)\ \mu_1 + p(0,1).p(1,2)\ \mu_2 + p(0,1).p(1,3)\ \mu_3$] and

D_0 = [1 - $p(0,1).p(1,2).p(2,0)$ - $p(0,1).p(1,3).p(3,0)$]

7.2 Availability of the System Using Base State:

From Fig.1, the regenerative available un-failed/up-states to which the system can transit, are: j = 0, 1, 2, 3. The availability (under steady state conditions) of the system with initial state'ξ'=1 (at time t = 0) is obtained by using the following modified formula RPGT, as under:

$$A_\xi=\left[\sum_{j,s_r}\left\{\frac{\{pr(\xi\xrightarrow{s_r}j)\}f_j.\mu_j}{\prod_{k_1\neq\xi}\{1-V_{k_1,k_1}\}}\right\}\right]\div\left[\sum_{i,s_r}\left\{\frac{\{pr(\xi\xrightarrow{s_r}i)\}.\mu_i^1}{\prod_{k_2\neq\xi}\{1-V_{k_2,k_2}\}}\right\}\right]$$

A_1 = [{(1,2,0)+(1,3,0)/ (1-L_3)}$f_0\mu_0$ + (1,1)$f_1\mu_1$ + (1,2)$f_2\mu_2$ +{(1,3) / (1-L_3)}$f_3\mu_3$]

÷ [{(1,2,0)+(1,3,0) /(1-L_3)} μ_0' +(1,1)μ_1' +(1,2)μ_2' +{(1,3)/ (1-L_3)} μ_3'] =N_1/D_1 where L_3 = (3,$\underline{5}$,3)

N_1= $\left[p(3,0)\left\{\left(1-p(1,2).p(2,\underline{4},1)\right)f_0\mu_0+f_1\mu_1+p(1,2)f_2\mu_2\right\}+p(1,3)f_3\mu_3\right]$

And $D_1=p(3,0)\left\{\left(1-p(1,2).p(2,\underline{4},1)\right)\mu_0+\mu_1+p(1,2)\mu_2'\right\}+p(1,3)\mu_3'$.

This Availability ($A1$) is the same as the value of Availability (A_0)as determined by Manju et al [8] from initial state ‘0’(at time t = 0) by using RPGT:

$$A_0 = \left[\sum_{j,s_r} \left\{ \frac{\{pr(0 \xrightarrow{s_r} j)\} f_j . \mu_j}{\prod_{k_1 \neq 0} \{1 - V_{k_1,k_1}\}} \right\} \right] \div \left[\sum_{i,s_r} \left\{ \frac{\{pr(0 \xrightarrow{s_r} i)\} . \mu_i'}{\prod_{k_2 \neq 0} \{1 - V_{k_2,k_2}\}} \right\} \right]$$

8. CONCLUSION

From the above discussion it is concluded that the MTSF (being a positional measure) depends upon the initial state (at time t=0). And the steady state availability, being a global measure, so it can be evaluated by using ‘base state’ of the system, easily and more quickly and without writing and solving any state equations and thus it saves time & energy and hence is more economical.

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FUZZY INVENTORY MODEL WITH WEIBULL DISTRIBUTED DETERIORATION ALONG VARIABLE DEMAND AND TIME VARYING HOLDING COST

T.P. Singh* & Harish Kumar Yadav**
*Professor, Department of Mathematics, Baba Mast Nath University Asthal Bohar Rohtak
**Research Scholar, Baba Mast Nath University Asthal Bohar Rohtak
E-mail : hrkyadav12@gmail.com, tpsingh78@yahoo.com

ABSTRACT :

The present paper investigates a fuzzy continuous inventory model for deteriorating items with variable demand. The deteriorating rate follows weibull distribution. In Real world situation it has been observed that the cycle time of every supply chain system is uncertain hence, it has described with triangular fuzzy parameters. The Yager's formula and signed distance method is applied to de-fuzzify the cost function. A numerical illustration has been proposed to validate the model. The sensitivity analysis has been carried out to explore the effect of minor changes in parameters with the optimal solution associated to different parameters.

Keywords : Fuzzy Inventory model, Signed distance method, Yagers formula, de-fuzzification.

1. INTRODUCTION

Inventory model plays a significant role in a supply chain system, where the objective of maintaining inventory provides a smooth line between supply and demand for the efficient running of different supply chain operations. The uncertainty is associated with different parameters such as demand, raw material supply, deterioration rate and various relevant cost etc. these parameters may vary with less, moderate or more flexible depending upon the situation and requirement. One has to face such situations while dealing with production inventory system. The fuzzy set given by Bellman and Zadeh in 1970 provides a justified solution in such situation. Zadeh (1973) showed that for new product and seasonal items it will be far better to apply fuzzy approach and compare to probabilistic approach. Park(1987) and Vujosevic etal.(1996) developed the inventory models in which ordering and holding cost are given by Fuzzy parameters. The work was further developed by Yao and Lee (1999), Wang and chen (2001), Kao and Hsu (2002), Maiti (2003), S Jaggi etal. (2012), Mishra etal. (2015) taking major parameters as fuzzy in nature in order to calculate the total inventory cost. But in most of the models the cycle time was considered to be constant though in real situation it is not so. The cycle time generally varies & remains uncertain. The uncertainty is described by triangular fuzzy number.

Following the introductory part rest paper is organized as under. In section 2 preliminary assumptions and notations are presented while in section 3 mathematical modeling has been developed. In section 4 the model has been fuzzified while in section 5 numerical example has been presented to illustrate the result. In section 6 the sensitive analysis of the optimal solution with different parameters of system has been carried out. Finally the conclusion has been carried out in next section and at last the related references have been given.

Section 2

2. Assumptions and Notations:

The model is developed under the following assumptions;

1) Replenishment size is constant and the replenishment rate is infinite.
2) Lead time is zero.
3) T is the length of each production cycle;
4) $\check{T}$ is the fuzzy cycle length.
5) $C_1 = b+ct$ is the inventory holding cost per unit per unit time;
6) C_3 is the cost of each deteriorated unit;
7) C(T) is the total inventory cost ;
8) C($\check{T}$) is fuzzy total cost given by Yager's formula ;
9) C($\check{T}_S$) is fuzzy total cost by Signed distance Method;
10) The deterioration rate function θ(t) represents the on-hand inventory deteriorates per unit time and Moreover in the present study the function assumed of the form
$\theta(t) = \alpha\beta t^{\beta-1}$; $0 < \alpha < 1$, $\beta > 0$, $t > 0$.
When β = 1, θ(t) becomes a constant which is a case of exponential decay. When β < 1, the rate of deterioration is decreasing with t and when β > 1, the rate of deterioration is increasing with t.
11) The demand rate starts from zero and ends at zero during the inventory period. It is assumed of the form D(t) = at(T−t) where T is the cycle period.

Section 3

3. Mathematical model and Analysis

let us assume we get an amount S (S>0) as an initial inventory. Inventory level gradually diminishes due to reasons of market demand and deterioration of the items and ultimately falls to zero at time T . Let I(t) be on hand inventory at any time t. The differential equations which on hand inventory I(t) must satisfy the following :

$$\frac{dI(t)}{dt} + \theta\, I(t) = -D(t) \quad , \qquad 0 \le t \le T \qquad ....(1)$$

By using D(t) = at(T−t) the differential equation (1) can be rewritten as

$$\frac{dI(t)}{dt} + \alpha\beta t^{\beta-1}\, I(t) = -at(T-t) \quad , \qquad 0 \le t \le T \qquad ....(2)$$

Solution of differential equation (2) is obtained on the basis of Yadav H.K. and Singh T.P. (2019)

$$I(t)\, e^{\alpha t^{\beta}} = -\int at(T-t)\, e^{\alpha t^{\beta}}\, dt + C$$

$$I(t)\, e^{\alpha t^{\beta}} = \quad -\int at(T-t)\,(1+\alpha t^{\beta})\, dt + C$$

$$I(t)\, e^{\alpha t^{\beta}} = -a\left(\frac{Tt^2}{2} + \frac{\alpha T t^{\beta+2}}{\beta+2} - \frac{t^3}{3} - \frac{\alpha t^{\beta+3}}{\beta+3}\right) + C \qquad ....(3)$$

Since I(0) =S, we have C = S then Equation (3) gives

$$I(t)\, e^{\alpha t^{\beta}} = -a\left(\frac{Tt^2}{2} + \frac{\alpha T t^{\beta+2}}{\beta+2} - \frac{t^3}{3} - \frac{\alpha t^{\beta+3}}{\beta+3}\right) + S \qquad ....(4)$$

Using the boundary condition I(T)=0 we get

$$S = a\frac{T^3}{6} + a\,\frac{\alpha}{(\beta+2)(\beta+3)}\,T^{\beta+3} \qquad .....(5)$$

Putting the value of S in (4), on simplification we find,

$$I(t)\, e^{\alpha t^{\beta}} = -a\left[\frac{Tt^2}{2}+\frac{\alpha T t^{\beta+2}}{\beta+2}-\frac{t^3}{3}-\frac{\alpha t^{\beta+3}}{\beta+3}-\frac{T^3}{6}-\frac{\alpha}{(\beta+2)(\beta+3)}T^{\beta+3}\right]$$

$$I(t) = -a\left[\frac{Tt^2}{2}+\frac{\alpha T t^{\beta+2}}{\beta+2}-\frac{t^3}{3}-\frac{\alpha t^{\beta+3}}{\beta+3}-\frac{T^3}{6}-\frac{\alpha}{(\beta+2)(\beta+3)}T^{\beta+3}\right] e^{-\alpha t^{\beta}}$$

$$I(t) = -a\left[\frac{Tt^2}{2}+\frac{\alpha T t^{\beta+2}}{\beta+2}-\frac{t^3}{3}-\frac{\alpha t^{\beta+3}}{\beta+3}-\frac{T^3}{6}-\frac{\alpha T^{\beta+3}}{(\beta+2)(\beta+3)}\right](1-\alpha t^{\beta})$$

$$I(t) = -a\left[\frac{Tt^2}{2}+\frac{\alpha T t^{\beta+2}}{\beta+2}-\frac{t^3}{3}-\frac{\alpha t^{\beta+3}}{\beta+3}-\frac{T^3}{6}-\frac{\alpha T^{\beta+3}}{(\beta+2)(\beta+3)}-\frac{\alpha T t^{\beta+2}}{2}-\frac{\alpha^2 T t^{2\beta+2}}{\beta+2}+\frac{\alpha t^{\beta+3}}{3}+\frac{\alpha^2 t^{2\beta+3}}{\beta+3}+\frac{\alpha t^{\beta} T^3}{6}+\frac{\alpha^2 T^{\beta+3} t^{\beta}}{(\beta+2)(\beta+3)}\right] \quad ....(6)$$

Hence total amount of deteriorated units (D)=I(0)−stock loss due to demand

$$= S - \int_0^T at(T-t)\, dt = \frac{a\alpha}{(\beta+2)(\beta+3)} T^{\beta+3} \quad .....(7)$$

Total Inventory held $(I_1) = \int_0^T (b+ct) I(t)\, dt$

$$= \int_0^T \Big\{-ab\left[\frac{Tt^2}{2}+\frac{\alpha T t^{\beta+2}}{\beta+2}-\frac{t^3}{3}-\frac{\alpha t^{\beta+3}}{\beta+3}-\frac{T^3}{6}-\frac{\alpha T^{\beta+3}}{(\beta+2)(\beta+3)}-\frac{\alpha T t^{\beta+2}}{2}-\frac{\alpha^2 T t^{2\beta+2}}{\beta+2}+\frac{\alpha t^{\beta+3}}{3}+\frac{\alpha^2 t^{2\beta+3}}{\beta+3}+\frac{\alpha t^{\beta} T^3}{6}+\frac{\alpha^2 T^{\beta+3} t^{\beta}}{(\beta+2)(\beta+3)}\right] - ac\left[\frac{Tt^3}{2}+\frac{\alpha T t^{\beta+3}}{\beta+2}-\frac{t^4}{3}-\frac{\alpha t^{\beta+4}}{\beta+3}-\frac{T^3 t}{6}-\frac{\alpha T^{\beta+3} t}{(\beta+2)(\beta+3)}-\frac{\alpha T t^{\beta+3}}{2}-\frac{\alpha^2 T t^{2\beta+3}}{\beta+2}+\frac{\alpha t^{\beta+4}}{3}+\frac{\alpha^2 t^{2\beta+4}}{\beta+3}+\frac{\alpha t^{\beta+1} T^3}{6}+\frac{\alpha^2 T^{\beta+3} t^{\beta+1}}{(\beta+2)(\beta+3)}\right]\Big\}\, dt$$

$$I_1 = -a\, c\alpha^2 \frac{1}{(\beta+2)^2(2\beta+5)} T^{2\beta+5} - a\, b\alpha^2 \frac{1}{2(\beta+1)(\beta+2)(2\beta+3)} T^{2\beta+4} + a\, c\alpha \frac{1}{2(\beta+2)(\beta+4)(\beta+5)} T^{\beta+5} + ab\alpha \frac{1}{(\beta+1)(\beta+3)(\beta+4)} T^{\beta+4} + ac\frac{1}{40}T^5 + ab\frac{1}{12}T^4 \quad .....(8)$$

Cost of deteriorated items = C_3 × total amount of deteriorated units

$$= C_3 \frac{a\alpha}{(\beta+2)(\beta+3)} T^{\beta+3} \quad .....(9)$$

Average total cost per unit time C(T)= $\frac{1}{T}$[Total cost per unit time] = $\frac{1}{T}$[Total Inventory held +Cost of deterioration items]

$$= -ac\alpha^2 \frac{1}{(\beta+2)^2(2\beta+5)} T^{2\beta+4} - ab\alpha^2 \frac{1}{2(\beta+1)(\beta+2)(2\beta+3)} T^{2\beta+3} + ac\alpha \frac{1}{2(\beta+2)(\beta+4)(\beta+5)} T^{\beta+4} + ab\alpha \frac{1}{(\beta+1)(\beta+3)(\beta+4)} T^{\beta+3} + ac\frac{1}{40}T^4 + ab\frac{1}{12}T^3 + C_3 \frac{a\alpha}{(\beta+2)(\beta+3)} T^{\beta+2} \quad .....(10)$$

$$\frac{dC(T)}{dT} = -ac\alpha^2 \frac{2}{(\beta+2)(2\beta+5)} T^{2\beta+3} - ab\alpha^2 \frac{1}{2(\beta+1)(\beta+2)} T^{2\beta+2} + ac\alpha \frac{1}{2(\beta+2)(\beta+5)} T^{\beta+3} + ab\alpha \frac{1}{(\beta+1)(\beta+4)} T^{\beta+2} + ac\frac{1}{10}T^3 + ab\frac{1}{4}T^2 + C_3 \frac{a\alpha}{(\beta+3)} T^{\beta+1} \quad ....(11)$$

$$\frac{d^2C}{dT} = -a c\alpha^2 \frac{2(2\beta+3)}{(\beta+2)(2\beta+5)} T^{2\beta+2} - a b\alpha^2 \frac{1}{(\beta+1)} T^{2\beta+1} + a c\alpha \frac{\beta+3}{2(\beta+2)(\beta+5)} T^{\beta+2} + ab\alpha \frac{\beta+2}{(\beta+1)(\beta+4)} T^{\beta+1} + ac\frac{3}{10}T^2 + ab\frac{1}{2}T^1 + C_3 \frac{a\alpha(\beta+1)}{(\beta+3)} T^{\beta} \quad ....(12)$$

For minimum $C(T)$,the necessary condition is

$\frac{dC(T)}{dT} = 0$ After solving we get an equation of odd degree whose last term is negative, then there exists a unique solution $T^* \in (0,T)$ can be solved from equation (11) also clearly $\frac{d^2C}{dT^2} > 0$ at $T = T^*$

$\therefore$ C(T) is minimum at $T = T^*$

So optimum value of T is T^*.

Section 4

FUZZFICATION OF THE MODEL

Let us describe the cycle time as triangular fuzzy parameter, $\check{T} = (T-\mathcal{E}, T, T+\mathcal{E})$.

From equation (10) the total cost function with fuzzy cycle time is –

$$C(\check{T}) = -ac\alpha^2 \frac{1}{(\beta+2)^2(2\beta+5)} \check{T}^{2\beta+4} - ab\alpha^2 \frac{1}{2(\beta+1)(\beta+2)(2\beta+3)} \check{T}^{2\beta+3} + ac\alpha \frac{1}{2(\beta+2)(\beta+4)(\beta+5)} \check{T}^{\beta+4} + ab\alpha \frac{1}{(\beta+1)(\beta+3)(\beta+4)} \check{T}^{\beta+3} + ac\frac{1}{40}\check{T}^4 + ab\frac{1}{12}\check{T}^3 + C_3 \frac{a\alpha}{(\beta+2)(\beta+3)} \check{T}^{\beta+2} \quad \text{........(13)}$$

$$C(\check{T}) = (A,B,C)$$

$$A = -ac\alpha^2 \frac{1}{(\beta+2)^2(2\beta+5)} (T-\mathcal{E})^{2\beta+4} - ab\alpha^2 \frac{1}{2(\beta+1)(\beta+2)(2\beta+3)} (T-\mathcal{E})^{2\beta+3} + ac\alpha \frac{1}{2(\beta+2)(\beta+4)(\beta+5)} (T-\mathcal{E})^{\beta+4} + ab\alpha \frac{1}{(\beta+1)(\beta+3)(\beta+4)} (T-\mathcal{E})^{\beta+3} + ac\frac{1}{40}(T-\mathcal{E})^4 + ab\frac{1}{12}(T-\mathcal{E})^3 + C_3 \frac{a\alpha}{(\beta+2)(\beta+3)} (T-\mathcal{E})^{\beta+2} \quad \text{.........(14)}$$

$$B = -a\,c\alpha^2 \frac{1}{(\beta+2)^2(2\beta+5)} T^{2\beta+4} - a\,b\alpha^2 \frac{1}{2(\beta+1)(\beta+2)(2\beta+3)} T^{2\beta+3} + a\,c\alpha \frac{1}{2(\beta+2)(\beta+4)(\beta+5)} T^{\beta+4} + ab\alpha \frac{1}{(\beta+1)(\beta+3)(\beta+4)} T^{\beta+3} + ac\frac{1}{40}T^4 + ab\frac{1}{12}T^3 + C_3 \frac{a\alpha}{(\beta+2)(\beta+3)} T^{\beta+2} \quad \text{.........(15)}$$

$$C = -a\,c\alpha^2 \frac{1}{(\beta+2)^2(2\beta+5)} (T+\mathcal{E})^{2\beta+4} - a\,b\alpha^2 \frac{1}{2(\beta+1)(\beta+2)(2\beta+3)} (T+\mathcal{E})^{2\beta+3} + a\,c\alpha \frac{1}{2(\beta+2)(\beta+4)(\beta+5)} (T+\mathcal{E})^{\beta+4} + ab\alpha \frac{1}{(\beta+1)(\beta+3)(\beta+4)} (T+\mathcal{E})^{\beta+3} + ac\frac{1}{40}(T+\mathcal{E})^4 + ab\frac{1}{12}(T+\mathcal{E})^3 + C_3 \frac{a\alpha}{(\beta+2)(\beta+3)} (T+\mathcal{E})^{\beta+2} \quad \text{.........(16)}$$

Now from Yager's Formulae $C(\check{T}) = (A,B,C)$

$$= \frac{3B+(C-A)}{3}$$

$$= -a\,c\alpha^2 \frac{1}{3(\beta+2)^2(2\beta+5)} [(T+\mathcal{E})^{2\beta+4} + 3T^{2\beta+4} - (T-\mathcal{E})^{2\beta+4} - a\,b\alpha^2 \frac{1}{6(\beta+1)(\beta+2)(2\beta+3)} [(T+\mathcal{E})^{2\beta+3} + 3T^{2\beta+3} - (T-\mathcal{E})^{2\beta+3}] + a\,c\alpha \frac{1}{6(\beta+2)(\beta+4)(\beta+5)} [(T+\mathcal{E})^{\beta+4} + 3T^{\beta+4} - (T-\mathcal{E})^{\beta+4}] + ab\alpha \frac{1}{3(\beta+1)(\beta+3)(\beta+4)} [(T+\mathcal{E})^{\beta+3} + 3T^{\beta+3} - (T-\mathcal{E})^{\beta+3}] + a\,c\frac{1}{120}[(T+\mathcal{E})^4 + 3T^4 - (T-\mathcal{E})^4] + ba\frac{1}{36}[(T+\mathcal{E})^3 + 3T^3 - (T-\mathcal{E})^3] + C_3 \frac{a\alpha}{3(\beta+2)(\beta+3)} [(T+\mathcal{E})^{\beta+2} + 3T^{\beta+2} - (T-\mathcal{E})^{\beta+2}]$$

........(17)

For minimum $C(\check{T})$,the necessary condition is

$\frac{dC(\check{T})}{dT} = 0$

$$\frac{dC(\check{T})}{dT} = -ac\alpha^2 \frac{2}{(\beta+2)(2\beta+5)}[(T+\mathcal{E})^{2\beta+3} + 3T^{2\beta+3} - (T-\mathcal{E})^{2\beta+3}] - ab\alpha^2 \frac{1}{6(\beta+1)(\beta+2)}[(T+\mathcal{E})^{2\beta+2} + 3T^{2\beta+2} - (T-\mathcal{E})^{2\beta+2}] + ac\alpha \frac{1}{6(\beta+2)(\beta+5)}[(T+\mathcal{E})^{\beta+3} + 3T^{\beta+3} - (T-\mathcal{E})^{\beta+3}] + ab\alpha\frac{1}{3(\beta+1)(\beta+4)}[(T+\mathcal{E})^{\beta+2} + 3T^{\beta+2} - (T-\mathcal{E})^{\beta+2}] + ac\frac{1}{30}[(T+\mathcal{E})^3 + 3T^3 - (T-\mathcal{E})^3] + ba\frac{1}{12}[(T+\mathcal{E})^2 + 3T^2 - (T-\mathcal{E})^2] + C_3\frac{a\alpha}{3(\beta+3)}[(T+\mathcal{E})^{\beta+1} + 3T^{\beta+1} - (T-\mathcal{E})^{\beta+1}]$$

…….(18)

$$\frac{d^2C(\check{T})}{dT^2} = -ac\alpha^2 \frac{2(2\beta+3)}{3(\beta+2)(2\beta+5)}[(T+\mathcal{E})^{2\beta+2} + 3T^{2\beta+2} - (T-\mathcal{E})^{2\beta+2}] + - ab\alpha^2 \frac{1}{3(\beta+2)}[(T+\mathcal{E})^{2\beta+1} + 3T^{2\beta+1} - (T-\mathcal{E})^{2\beta+1}] + ac\alpha\frac{2\beta^2+10\beta+12}{6(\beta+3)(\beta+5)}[(T+\mathcal{E})^{\beta+2} + 3T^{\beta+2} - (T-\mathcal{E})^{\beta+2}] +] + ab\alpha\frac{\beta+2}{3(\beta+1)(\beta+4)}[(T+\mathcal{E})^{\beta+1} + 3T^{\beta+1} - (T-\mathcal{E})^{\beta+1}] + ac\frac{1}{10}[(T+\mathcal{E})^2 + 3T^2 - (T-\mathcal{E})^2] + ba\frac{1}{12}T^1 + C_3\frac{a\alpha(\beta+1)}{3(\beta+3)}[(T+\mathcal{E})^{\beta} + 3T^{\beta} - (T-\mathcal{E})^{\beta}]$$ ……..(19)

For minimum $C(\check{T})$,the necessary condition is

$$\frac{dC(\check{T})}{dT} = 0$$

After solving we get an equation of odd degree whose last term is negative, then there exists a unique solution $\check{T}^* \in (0, \check{T})$ can be solved from equation (18) with the help of Mat Lab also we can show $\frac{d^2C}{dT^2} > 0$ at $T = \check{T}^*$

$\therefore$ $C(\check{T})$ is minimum at $T = \check{T}^*$

So optimum value of $\check{T}$ is $\check{T}^*$.

Now by using signed distance method, we the defuzzified value of C(T).

$$d(\check{T},0) = \frac{1}{2}\int[\check{T}_L(\lambda) + \check{T}_U(\lambda)]\, d\lambda$$

where, $\check{T} = (T-\mathcal{E}, T, T+\mathcal{E})$, $\check{T}_L(\lambda) = (T-\mathcal{E}) + \mathcal{E}\lambda$, $\check{T}_U(\lambda) = (T+\mathcal{E}) - \mathcal{E}\lambda$

Therefore, $d(\check{T},0) = \frac{1}{2}\int[(T-\mathcal{E}) + \mathcal{E}\lambda + (T+\mathcal{E}) - \mathcal{E}\lambda]\, d\lambda = T$

$$C(\check{T}_S) = -ac\alpha^2 \frac{1}{(\beta+2)^2(2\beta+5)}T^{2\beta+4} - ab\alpha^2 \frac{1}{2(\beta+1)(\beta+2)(2\beta+3)}T^{2\beta+3} + ac\alpha \frac{1}{2(\beta+2)(\beta+4)(\beta+5)}T^{\beta+4} + ab\alpha\frac{1}{(\beta+1)(\beta+3)(\beta+4)}T^{\beta+3} + ac\frac{1}{40}T^4 + ab\frac{1}{12}T^3 + C_3\frac{a\alpha}{(\beta+2)(\beta+3)}T^{\beta+2}$$ ....(20)

This equation is same as crisp model, therefore by using Signed Distance Method we get same values of T and C(T).

Section 5

NUMERICAL ILLUSTRATION

To illustrate the model we consider following numerical values of the parameters.

$\alpha = 0.1$,$\beta = 2$, $b=2$, $a=100$, $c=1$, $C_3 = 50$, $\mathcal{E}=.23$

we obtain for the crisp model, Total Cost =337158.9 and cycle time T=.550 years and for the fuzzy model Total Cost = 337159 and cycle time=.5676 years

Section 6

SENSITIVITY ANALYSIS WITH RESPECT TO α,c,a,C_3

Table-1

Change value α		Crisp Model		Fuzzy Model	
		C(T)	T	C(Ť)	Ť
A	0.1	337158.9	6.6601	337159	6.8119
	0.2	221365.7	5.4026	221365.9	5.5544
	0.3	180316.6	4.8116	180316.7	4.9634
	0.4	154751.9	4.4223	154752.1	4.5741
	0.5	139838.4	4.1578	139838.6	4.3096
	0.6	128199	3.9481	128199.1	4.0999
	0.7	130684.8	3.8581	130684.9	4.0099
	0.8	122939.4	3.7175	122939.5	3.8693
	0.9	116491.3	3.5978	116491.4	3.7496
	1.0	100643.8	3.4181	100643.9	3.5699

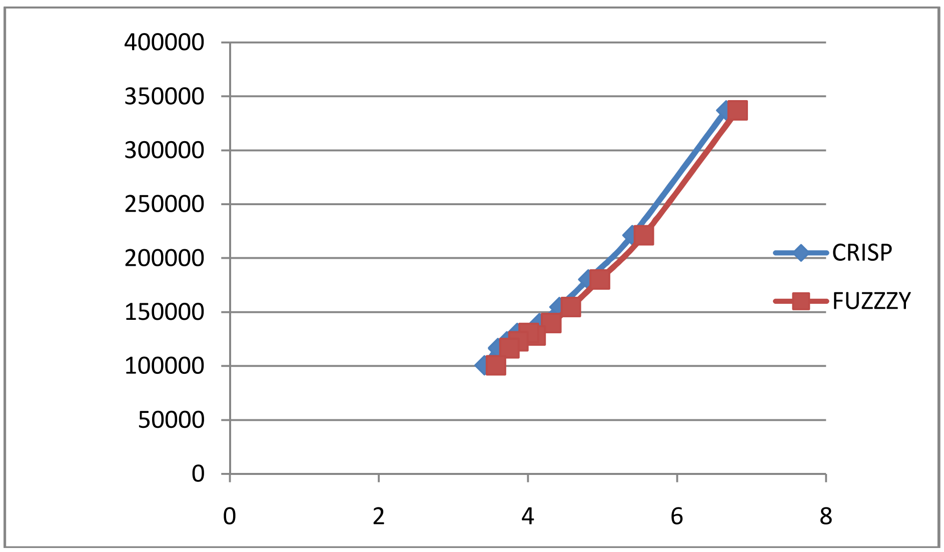


Table-2

Change value		Crisp Model		Fuzzy Model	
		C(T)	T	C(Ť)	Ť
C	1	337158.9	6.6601	337159	6.8119
	2	193935.7	5.8688	193935.9	6.0206
	3	146900.9	5.469	146901.1	5.6208
	4	124497.7	5.2171	124497.9	5.3689
	5	111957.9	5.0406	111958	5.1924
	6	104324.8	4.9087	104324.9	5.0605
	7	99474	4.8057	99474.15	4.9575
	8	96364.25	4.7228	96364.4	4.8746
	9	94401.7	4.6544	94401.85	4.8062
	10	93250.8	4.597	93250.95	4.7488

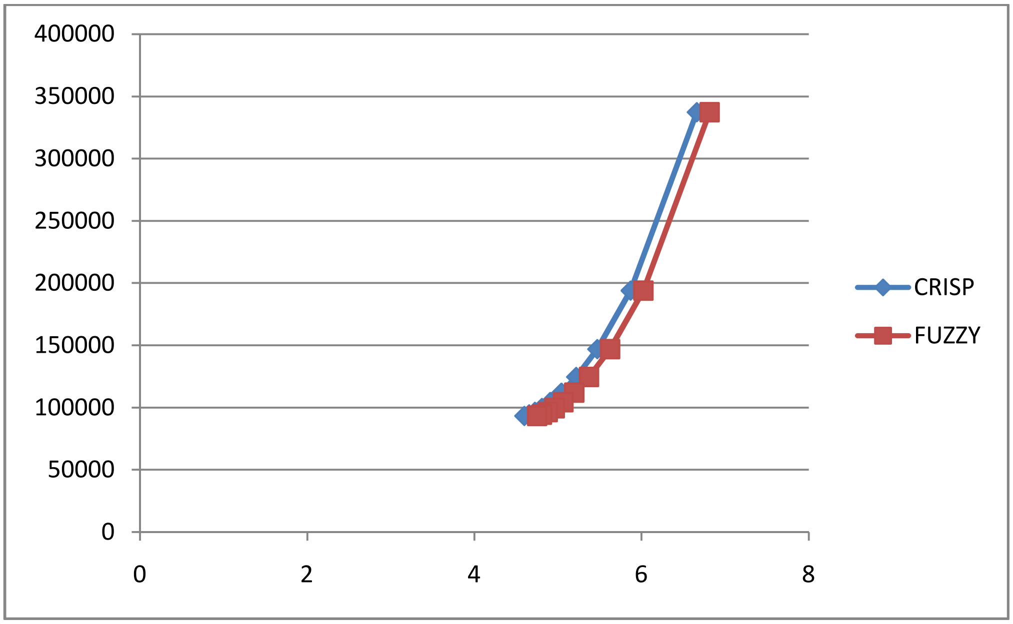


Table-3

Change value		Crisp Model		Fuzzy Model	
		C(T)	T	C(Ť)	Ť
A	100	337158.9	6.6601	337159	6.8119
	200	674889	6.6613	674889.1	6.8131
	300	1011405	6.66	1011405	6.8118
	400	1351588	6.6632	1351588	6.815
	500	1684353	6.6601	1684353	6.8119
	600	2022826	6.66	2022826	6.8118
	700	2359960	6.66	2359960	6.8118
	800	2697102	6.66	2697102	6.8118
	900	3034449	6.6601	3034449	6.8119
	1000	3371615	6.6601	3371615	6.8119

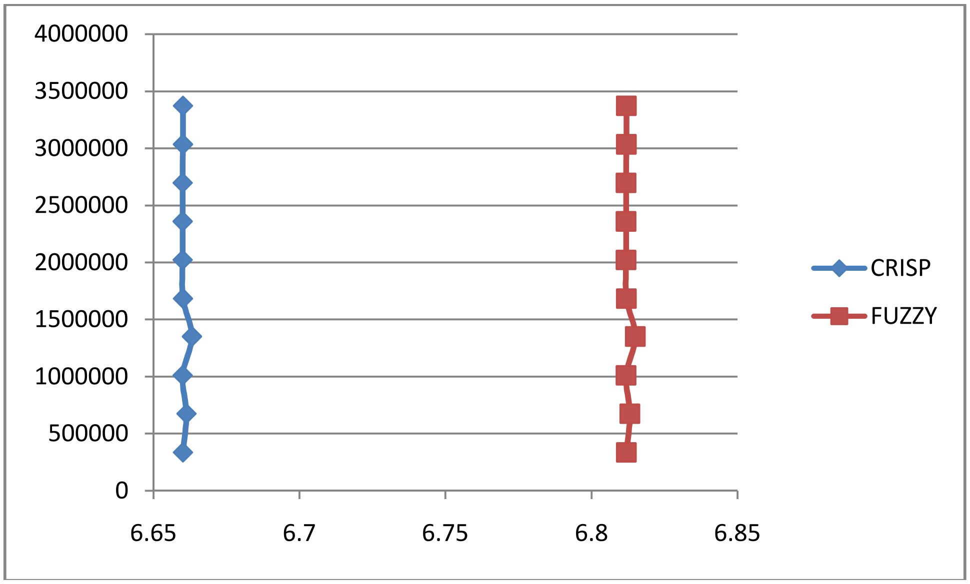
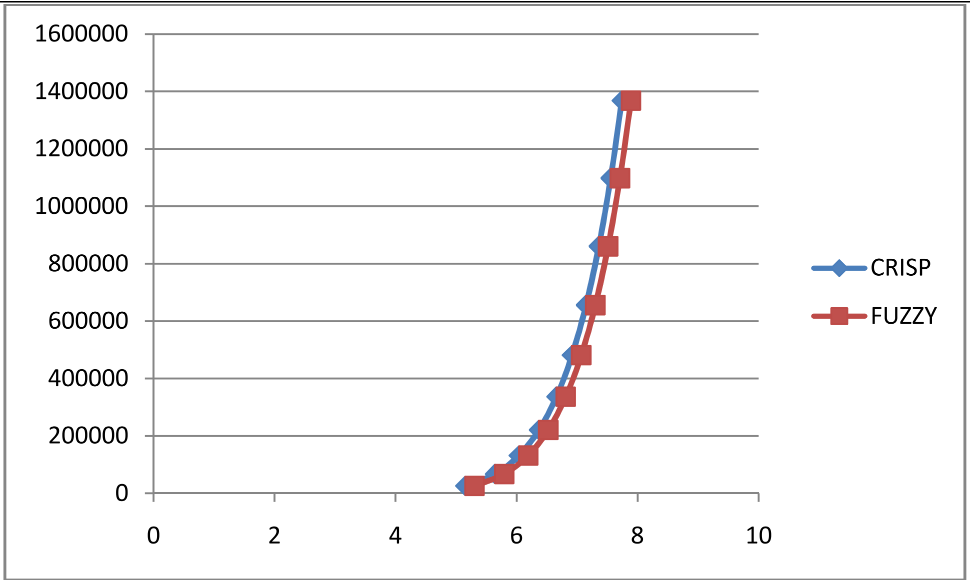


Table-4

Change value		Crisp Model		Fuzzy Model	
		C(T)	T	C(Ť)	Ť
C_3	10	26307.83	5.1527	26307.99	5.3045
	20	67574.09	5.6446	67574.24	5.7964
	30	131764.1	6.039	131764.3	6.1908
	40	221045.5	6.3713	221045.6	6.5231
	50	337158.9	6.6601	337159	6.8119
	60	481652.1	6.9168	481652.2	7.0686
	70	655853.5	7.1486	655853.6	7.3004
	80	860917.7	7.3604	860917.8	7.5122
	90	1097987	7.5559	1097988	7.7077
	100	1368028	7.7377	1368028	7.8895



- Cycle time has been taken in months.

Observations:

1. When α, c increases the difference between points (cycle time, Total cost) in both cases also decreases, also value of cycle time and total cost both decreases.
2. The increase in cost of deterioration items per unit, the total cost also increases.
3. With the increase of the parameter a, the total cost increases while cycle time have some slightly change.

Section 7

CONCLUSION:

A fuzzy Inventory model of deteriorating products mainly arise in supply chain problem has been explored in which demand rate is quadratic function of time while the deterioration rate follows Weibull Deterioration. Cycle time has been fuzzified with the help of triangular fuzzy number and defuzzified with the help of signed distance function and by Yager's formulae. The case of without shortage of items has been discussed. In order to measure the senstivity analysis the total inventory cost, obtained by crisp model and fuzzy model a comparative study has been made. It has been that the total cost of crisp model is slightly lesser than the fuzzy model. However the total cost of both the model increases when the cost associated to the model increases. Also as the cycle time increases the total cost of both models also increases.

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EXTENSION OF LINEAR 2-FUNCTIONALS IN COMPLEX 2-NORMED LINEAR SPACE

C. Sreedhar
Associate Professor, NBKR Institute of Science & Technology "Vidyanagar" 524413
E-mail: csreedhar.nainu@gmail.co

ABSTRACT :

In this paper we extended the theorem of Das in more general way over the field of K where K is the field R of real numbers or field C of complex numbers using norm technique of Hahn Banach theorem in normed linear spaces.

Key words: 2-norms, linear 2-functionals, 2-bounded linear 2-functionals.

AMS subject classification: Primary 46A22 , secondary 46A70

1. INTRODUCTION

Extending the known concept of real 2-normed linear space [1], recently [3] gave the following definition of 2-normed linear space over K where K is the field R of real numbers or the field C of complex numbers.

Definition([1],[3]): Let E be a linear space over K . A mapping $\|.,.\|$ on ExE to the set of real numbers R is called a 2-norm on E if

(1.1) $\|x, y\|$=0 if and only if x and y are linearly dependent in E.

(1.2) $\|x, y\| = \|y, x\|$ if for all x,y∈ E.

(1.3) $\|\alpha x, y\| = |\alpha|\ \|x, y\|$ for all α ∈K and x,y ∈ E.

(1.4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all x,y,z ∈E.

The pair (E, $\|.,.\|$)is a called a 2-normed linear space over K.

2. EXTENSION OF LINEAR 2-FUNCTIONALS

Definition 2.1. f:MxN→K be a linear 2-functional . Then f is called 2-bounded if there exists K>0 such that for all (x,y) ∈MxN (see definition 3.1,3.2 in [3])

(1.5) $|f(x,y)| \leq$ k $\|x,y\|$

For a bounded linear 2-functional f with domain MxN, the norm of f is denoted by $\|f\|$. It has been shown that (see[5],[3])

$$\|f\|=\sup\{\frac{|f(x,y)|}{\|x,y\|} : (x,y) \in MxN, \|x,y\| \neq 0 \}$$

Definition 2.2 Let M and N be two linear subspaces of a linear space E over K where K is the field R of real numbers or field C of complex numbers. f:MxN→ K is a mapping such that

$$f(\alpha_1 x_1 + \beta_1 y_1, \alpha_2 x_2 + \beta_2 y_2) = \alpha_1\alpha_2 f(x_1,x_2) + \alpha_1\beta_2 f(x_1,y_2) + \alpha_2\beta_1 f(y,x_2) + \beta_1\beta_2 f(y_1,y_2)$$

for all $\alpha_{1,}\alpha_{2,}\beta_1,\beta_2 \in K$,$x_1, y_1 \in$ M and $x_2, y_2 \in N$ linear 2 functional with domain M x N

In 1982 S.N.Lal and Das [see Theorem 2.5 in [2]] established the following theorem :

Theorem: Let M be a linear subspace of a real 2-normed linear space E. If for any $x_0, y_0 \in$ E-M, $\delta=$ inf $\{ \|x_0 - x, y_0\|: x \in M \}>0$ and $y_0 \notin \{x + \alpha x_0 : x \in M, \alpha \in R\}$ then there exists a real linear 2-functional on Ex$[y_0]$ such that

(i) $f(x_0,y_0) = \delta$ (ii) $f(x,y_0) = 0$ for all $x \in M$ and (iii) $\|f\|$ =1

3. MAIN RESULT

Theorem 3.1: Let M be a linear subspace of a 2-normed linear space E over the field K where K is the field R of real numbers or field C of complex numbers. For any $x_0 \neq 0 \in E$,

$\|x_0\| = \delta > 0$ and $y_0 \notin \{x + \alpha x_0 : x \in M, \alpha \in K\}$. If f is a bounded linear 2-functional on Mx$[y_0]$ then there exists a bounded linear 2-functional F on Ex$[y_0]$ such that

(i) $F(x_0$,$y_0)$=δ (ii) $F(x$,$y_0)$=0 for all $x \in M$ and (iii) $\|f\| = \|F\|$

Proof: Assume that linear space E is spanned by $[y_0]$ and $\{y_i$:i ∈I (index set)$\}$ be a Hamel basis for E. Then E=$[y_0] \oplus N$, N is the subspace spanned by Hamel basis $\{y_i : i \in I \}$. Since $y_0 \notin$ M∪ $[x_0]$ then M∪ $[x_0]$⊂ E=$[y_0] \oplus N$. Define $\tilde{f}$ on M∪ $[x_0]$ by $\tilde{f}(x + \alpha x_0)$=f($x + \alpha x_0$,$y_0$). Define $\|.\|$ on N by $\|x + \alpha x_0\|$ =$\|x + \alpha x_0, y_0\|$ for every $x + \alpha x_0$ ∈N and $\alpha \in K$. Then $\tilde{f}$ is a linear functional on M∪ $[x_0]$.

Now $\left|f \quad (\widetilde{x + \alpha x_0})\right| = |f(x + \alpha x_0, y_0)| \leq \|f\| \|x + \alpha x_0, y_0\|$

$\left|f \quad (\widetilde{x + \alpha x_0})\right| \leq \|f\| \; \|x + \alpha x_0\| \; \tilde{f}$ is bounded on M∪ $[x_0]$. Now

$\|f\|$ = sup $\{\frac{|f(x+\alpha x_0, \beta y_0)|}{\|x+\alpha x_0, \beta y_0\|}$: $(x + \alpha x_0, \beta y_0)$ ∈ M∪ $[x_0]$ x$[y_0]$, $\|x + \alpha x_0, \beta y_0\| \neq 0\}$

$=\|\tilde{f}\|$

appealing to Hahn Banach theorem we get a bounded linear functional $\tilde{F}$ on N such that

and $\tilde{F}(x + \alpha x_0)$ =$\tilde{f}(x + \alpha x_0)$ for every$(x + \alpha x_0)$ ∈M∪ $[x_0]$.

Define F on Ex$[y_0]$ by F$(x + \alpha x_0, \beta y_0)$={ $\beta\tilde{F}$ $(x + \alpha x_0)$ if $x + \alpha x_0 \in N$

={ $\beta\tilde{F}$ (z)if $x + \alpha x_0 = \beta y_0 + z$ for some $\beta \in K$ and z∈ N

if x+$\alpha x_0 \in N$ *then* |F$(x + \alpha x_0, \beta y_0)$| =$\left|\beta\tilde{F} \; (x + \alpha x_0)\right|$

$\leq |\beta| \|\tilde{F}\| \|x + \alpha x_0\|$ since E=[z] ⊕ N= $|\beta| \|f\| \|x + \alpha x_0\|$

$=|\beta| \|f\| \|x + \alpha x_0, y_0\|$

$=\|f\| \|x + \alpha x_0, \beta y_0\|$

$\|F\| = \|f\|$ if $x + \alpha x_0 \in [y_0]$ ⊕ which implies $x + \alpha x_0$ = z+βy_0 for some $\beta \in K$,z∈N.

Now $|F(x + \alpha x_0, \beta y_0)|$ = $\left|\beta\widetilde{F \; (z)}\right|$ where z∈N $\leq |\beta| \|\tilde{F}\| \|z\|$

$=|\beta| \|f\| \|z, y_0\|$

$=|\beta| \|f\| \|z + \beta y_0, y_0\|$

$=|\beta| \|f\| \|x + \alpha x_0, y_0\|$

Therefore, $\|F\| = \|f\|$

(i) F(x_0, y_0)=F(0+1.x_0,y_0)=$\tilde{F}$(0+1.x_0)= $\tilde{F}(x_0)$=$\|x_0\|$=δ

(ii) F(x,y_0) = F(x+0.x_0,y_0)= $\tilde{F}$(x)= $\|x\|$ =0$\|x_0\|$ =0

(iii) $\|F\| = \|f\|$

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SUPRA SEMI ALPHA OPEN SETS IN SUPRA BITOPOLOGICAL SPACES

R. Gowri*, A.K.R. Rajayal**
*Assistant Professor, Department of Mathematics, Government College for Women(Autonomous), Kumbakonam, India
**Research Scholar, Department of Mathematics, Government College for Women(Autonomous), Kumbakonam, India
E-mail: gowrigck@rediffmail.com, rajayalk@gmail.com

ABSTRACT :

In this paper, we define and study the notion of S_(τ_ij)-Sα-open and closed sets in supra bitopological spaces. Also, we analysed the properties of these sets.

Mathematics Subject Classification: 54D05, 54D10, 54D08, 54D20

Keywords: Supra bitopology, S_(τ_ij)-.Sα-open and S_(τ_ij)-.Sα-closed sets.

1. INTRODUCTION

The concept of bitopological spaces have been established by Kelly[9].The notion of Supra topological spaces was introduced by Mash hour[11] and investigate separation axioms in supra topology. Gowri and Rajayal[6&7] are discussed the concept of supra bitopological spaces and study supra alpha open sets and supra semi open sets in supra bitopological spaces. In 1981, Bose[3] introduced semi open sets in bitopological spaces. Qays Rubaye [1] studied semi α-open sets in bitopological spaces in 2012. In this paper we define the new class of sets called $S_{\tau_{ij}}$-Sα-open and $S_{\tau_{ij}}$-Sα-closed sets and investigate their properties of these sets.

2. PRELIMINARIES

Definition 2.1 [11] (X, S_τ) is said to be a supra topological space if it is satisfying these conditions:

(1) X, $\emptyset \in S_\tau$

(2) The union of any number of sets in S_τ belongs toS_τ.

Definition 2.2 [11] Each element A $\in S_\tau$ is called a supra open set in (X, S_τ) and its complement is called a supra closed set in (X, S_τ).

Definition 2.3 [11] If (X, S_τ) is a supra topological spaces, A $\subseteq$ X, A $\neq \emptyset$, S_{τ_A} is the class of all intersection of A with each element in S_τ, then (A, S_{τ_A}) is called a supra subspace of (X, S_τ).

Definition 2.4 [11] The supra closure of the set A is denoted by S_τ-cl(A) and is defined as S_τ-cl(A) = ∩{B : B is a supra closed and A ⊆ B}.

Definition 2.5[11] The supra interior of the set A is denoted by S_τ-int(A) and is defined as S_τ-int(A) = ∪{B : B is a supra open and B ⊆ A}

Definition 2.6 [6] If S_{τ_1} and S_{τ_2} are two supra topologies on a non-empty set X, then the triplet (X, S_{τ_1}, S_{τ_2}) is said to be a supra bitopological space.

Definition 2.7 [6] Each element of S_{τ_i} is called a supra τ_i-open sets(briefly,S_{τ_i}-open sets) in (X, S_{τ_1}, S_{τ_2}). Then the complement of S_{τ_i}-open sets are called a supra τ_i-closed sets(briefly, S_{τ_i}-closed sets), for i = 1,2.

Definition 2.8 [6] If (X, S_{τ_1}, S_{τ_2}) is a supra bitopological space, Y ⊆ X, Y ≠ ∅ then (Y, $S_{\tau_1^*}$,$S_{\tau_2^*}$) is a supra bitopological subspace of (X, S_{τ_1}, S_{τ_2}) if $S_{\tau_1^*}$= {U ∩ Y ; U is a S_{τ_1}− open in X} and $S_{\tau_2^*}$ = {V ∩ Y ; V is a S_2− open in X}.

Definition 2.9 [6] The S_{τ_i}-closure of the set A is denoted by S_{τ_i}-cl(A) and is defined as S_{τ_i}-cl(A) = ∩{B : B is a S_{τ_i}−closed and A ⊆ B, f or i = 1, 2}.

Definition 2.10 [6] The S_{τ_i}-interior of the set A is denoted by S_{τ_i}-int(A) and is defined as S_{τ_i}-int(A)= ∪{B : B is a S_{τ_i}−open and B ⊆ A, f or i = 1, 2}.

Definition 2.11 [7] Let(X, S_{τ_1}, S_{τ_2}) be a supra bitopological space, A ⊆ X, A is said to be $S_{\tau_{ij}}$-semi-open (briefly, $S_{\tau_{ij}}$-s-open), if A ⊆S_{τ_j}-cl(S_{τ_i}-int(A)), where i≠j, i,j = 1,2.

Definition 2.12 [7] Let(X, S_{τ_1}, S_{τ_2}) be a supra bitopological space, A ⊆ X, A is said to be $S_{\tau_{ij}}$-α-open, if A ⊆S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A))), where i≠j, i,j = 1,2.

Theorem 2.13 [7] Every $S_{\tau_{ij}}$-α-open set is $S_{\tau_{ij}}$-p-open set.

3. SUPRA SEMI ALPHA OPEN SETS IN SUPRA BITOPOLOGICAL SPACES

In this section, we introduced supra semi alpha open sets in supra bitopological spaces and discussed the characterization of these sets.

Definition 3.1 Let (X, S_{τ_1}, S_{τ_2}) be a supra bitopological space, A ⊆ X, A is said to be supra semi α -open set (briefly, $S_{\tau_{ij}}$-Sα-open set) if there exist an $S_{\tau_{ij}}$-α-open set U in X, such that U ⊆ A ⊆S_{τ_j}-cl(U). The family of all $S_{\tau_{ij}}$-Sα-open sets of X is denoted by $S_{\tau_{ij}}$-SαO(X), where i≠j, i,j = 1,2.

Example 3.2

Let X = {a, b, c,d},

S_{τ_1}= {X,∅, {a}, {b}, {a, b}, {a, d}, {b, d}, {a, b, d}, {b, c, d}},

S_{τ_2}= {X,∅, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}},

$S_{\tau_{12}}$-SαO(X) = {X, ∅, {a}, {b}, {a, b}, {a, d}, {b, d}, {a, b, d}, {b, c, d}}.

Remark 3.3

In general $S_{\tau_{12}}$-SαO(X) ≠ $S_{\tau_{21}}$-SαO(X) as shown in the following example.

Example 3.4

In Example 3.2, $S_{\tau_{12}}$-.SαO(X) = {X, ∅, {a}, {b}, {a, b}, {a, d}, {b, d}, {a, b, d}, {b, c, d}},

$S_{\tau_{21}}$-.SαO(X) = {X, ∅, {c}, {a, b}, {a, c}, {a, b, c}, {a, b, d}, {a, c, d}}.

Here $S_{\tau_{12}}$-SαO(X) ≠ $S_{\tau_{21}}$-SαO(X).

The following proposition will give an equivalent definition of $S_{\tau_{ij}}$-Sα-open sets.

Proposition 3.5 Let (X, S_{τ_1}, S_{τ_2}) be a supra bitopological space, A ⊆ X. Then A is an $S_{\tau_{ij}}$-Sα-open set iff A ⊆S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A)))), where i≠j, i,j = 1,2.

Proof .Necessity: Let us assume that A is $S_{\tau_{ij}}$-Sα-open set, then there exists an $S_{\tau_{ij}}$-α-open set U, such that U ⊆ A ⊆S_{τ_j}-cl(U). Since U is an $S_{\tau_{ij}}$-α-open set, then U ⊆S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(U))).

This implies, S_{τ_j}-cl(U)⊆S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(U)))). Since U ⊆ A, Λ

then S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(U)))) ⊆S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A)))).

Hence S_{τ_j}-cl(U) ⊆S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A)))). But A ⊆S_{τ_j}-cl(U).

This implies, A ⊆S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A)))).

Sufficiency part: Let us assume that A ⊆S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A)))).

To prove that A is an $S_{\tau_{ij}}$-Sα-open set. Let us take V = S_{τ_i}-int(A), which implies S_{τ_i}-int(A) ⊆ A, now we show that A ⊆S_{τ_j}-cl(S_{τ_i}-int(A)). Since S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A))) ⊆S_{τ_j}-cl(S_{τ_i}-int(A)). But A ⊆S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A)))) (by hypothesis) implies that A ⊆S_{τ_j}-cl(S_{τ_i}-int(A)). Hence there exist an S_{τ_i}-open set V, such that V ⊆ A ⊆S_{τ_j}-cl(V). Therefore V is an $S_{\tau_{ij}}$-Sα-open set (since V is an S_{τ_i}-open set). Hence A is $S_{\tau_{ij}}$-Sα-open set.

Proposition 3.6 The Union of any family of $S_{\tau_{ij}}$-Sα-open sets is $S_{\tau_{ij}}$-Sα-open set.

Proof. Let $\{A_\lambda$: λ ∈Λ$\}$ be a family of $S_{\tau_{ij}}$-Sα-open subsets of X.

Then A_λ⊆S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A_λ)))), for every λ ∈Λ.

Since $\cup_{\lambda\in\Lambda} int(A_\lambda) \subseteq$ int($\cup_{\lambda\in\Lambda} A_\lambda$) and $\cup_{\lambda\in\Lambda} cl(A_\lambda) \subseteq$ cl($\cup_{\lambda\in\Lambda} A_\lambda$) hold for any topology.

We have $\cup_{\lambda\in\Lambda} A_\lambda \subseteq \cup_{\lambda\in\Lambda}$ S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A_λ)))).

⊆S_{τ_j}-cl($\cup_{\lambda\in\Lambda}$ S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int(A_λ)))).

⊆S_{τ_j}-cl(S_{τ_i}-int($\cup_{\lambda\in\Lambda}$ S_{τ_j}-cl(S_{τ_i}-int(A_λ)))).

⊆S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl($\cup_{\lambda\in\Lambda}$ S_{τ_i}-int(A_λ)))).

⊆S_{τ_j}-cl(S_{τ_i}-int(S_{τ_j}-cl(S_{τ_i}-int($\cup_{\lambda\in\Lambda}$ A_λ)))).

Therefore $\cup_{\lambda\in\Lambda} A_\lambda$ is $S_{\tau_{ij}}$-Sα-open set.

Remark 3.7 The intersection of any two $S_{\tau_{ij}}$-Sα-open sets is need not necessary $S_{\tau_{ij}}$-Sα-open set as seen in the following example.

Example 3.8

Let X = {a, b, c, d},

S_{τ_1}= {X, ∅, {a}, {a, c}, {b, c}, {a, b, c}},

S_{τ_2}= {X, ∅, {c}, {a, d}, {b, d}, {a, b, d}, {a, c, d}},

$S_{\tau_{12}}$-SαO(X) = {X, ∅, {a}, {a, c}, {b, c}, {a, b, c}}.

Here {a, c} ∩ {b, c} = {c} is not $S_{\tau_{12}}$-Sα-open set.

Theorem 3.9 Every S_{τ_i}-open set is $S_{\tau_{ij}}$-Sα-open set.

Proof. Obvious.

Remark 3.10 The converse of the above Theorem is not true as shown in the following example.

Example 3.11

Let X = {a, b, c, d, e},

S_{τ_1}= {X, ∅, {a, b}, {a, d}, {a, b, d}, {a, b, c, d}, {a, c, d, e}},

S_{τ_2}= {X, ∅, {a, d}, {d, e}, {a, d, e}, {b, c, d}},

$S_{\tau_{12}}$-SαO(X) = {X, ∅, {a, b}, {a, d}, {a, b, c}, {a, c, d}, {a, d, e}, {a, b, d}, {b, c, d}, {a, b, c, d}, {a, c, d, e}, {a, b, d, e}}.

Here {a, c, d} is $S_{\tau_{12}}$-Sα-open set but it is not S_{τ_1}-open set.

Theorem 3.12 Every $S_{\tau_{ij}}$-α-open set is $S_{\tau_{ij}}$-Sα-open set.

Proof. The proof of the theorem is similar to the way of the proof of Theorem 2.13.

Remark 3.13 The reverse of the above Theorem is not true as seen in the following example.

Example 3.14 In Example 3.11, we have

$S_{\tau_{12}}$-SαO(X) = {X, ∅, {a, b}, {a, d}, {a, b, c}, {a, c, d}, {a, d, e}, {a, b, d}, {b, c, d}, {a, b, c, d}, {a, c, d, e}, {a, b, d, e}},

$S_{\tau_{12}}$-αO(X) = {X, ∅, {a, b}, {a, d}, {a, c, d}, {a, d, e}, {a, b, d}, {b, c, d}, {a, b, c, d}, {a, c, d, e}, {a, b, d, e}}.

Here {a, b, c} is $S_{\tau_{12}}$-SαO(X) but not $S_{\tau_{12}}$-αO(X).

Definition 3.15 The complement of $S_{\tau_{ij}}$-Sα-open set is called $S_{\tau_{ij}}$-Sα-closed set. Then the family of all $S_{\tau_{ij}}$-Sα-closed sets of X is denoted by $S_{\tau_{ij}}$-SαC(X), where i≠j, i,j = 1,2.

Theorem 3.16 The intersection of any family of $S_{\tau_{ij}}$-Sα-closed sets is $S_{\tau_{ij}}$-Sα-closed set.

Proof. The proof of the theorem is similarly to the proof of Proposition 3.6.

Definition 3.17 Let (X, S_{τ_1}, S_{τ_2}) be a supra bitopological space and A ⊆ X, the intersection of all $S_{\tau_{ij}}$-Sα-closed sets containing A is called $S_{\tau_{ij}}$-Sα-closure of A is denoted by $S_{\tau_{ij}}$-Sα-cl(A) and is defined as $S_{\tau_{ij}}$-Sα-cl(A) = ∩{B⊆ X : B is a $S_{\tau_{ij}}$-Sα-closed set, A ⊆ X, }.

Theorem 3.18 Let (X, S_{τ_1}, S_{τ_2}) be a supra bitopological space and let A ⊆ X, then

(i) $S_{\tau_{ij}}$-Sα-cl(A) is the smallest $S_{\tau_{ij}}$-Sα-closed set containing A.

(ii) A is $S_{\tau_{ij}}$-Sα-closed set iff $S_{\tau_{ij}}$-Sα-cl(A) = A.

Proof.(i) This proof is directly from the Definition 3.17.

(ii) If A is $S_{\tau_{ij}}$-Sα-closed set, then A is itself is the smallest $S_{\tau_{ij}}$-Sα-closed set containing A and hence $S_{\tau_{ij}}$-Sα-cl(A) = A. Conversely, if $S_{\tau_{ij}}$-Sα-cl(A) = A. By (i) $S_{\tau_{ij}}$-Sα-cl(A) is $S_{\tau_{ij}}$-Sα-closed set. Therefore A is also $S_{\tau_{ij}}$-Sα-

closed set.

Theorem 3.19 Let (X, S_{τ_1}, S_{τ_2}) be a supra bitopological space and let A, B be any subsets of X, then (i) A $\subseteq S_{\tau_{ij}}$-$S\alpha$-cl(A).

(ii) If A ⊆ B, then $S_{\tau_{ij}}$-$S\alpha$-cl(A) $\subseteq S_{\tau_{ij}}$-$S\alpha$-cl(B).

(iii) $S_{\tau_{ij}}$-$S\alpha$-cl($S_{\tau_{ij}}$-$S\alpha$-cl(A)) = $S_{\tau_{ij}}$-$S\alpha$-cl(A).

Proof. By Theorem 3.18 part (i). We obtain A $\subseteq S_{\tau_{ij}}$-$S\alpha$-cl(A)

(ii) By part (i) above B $\subseteq S_{\tau_{ij}}$-$S\alpha$-cl(B) is $S_{\tau_{ij}}$-$S\alpha$-closed set containing A. Since $S_{\tau_{ij}}$-$S\alpha$-cl(A) is the smallest $S_{\tau_{ij}}$-$S\alpha$-closed set containing A, hence $S_{\tau_{ij}}$-$S\alpha$-cl(A) $\subseteq S_{\tau_{ij}}$-$S\alpha$-cl(B).

(iii) $S_{\tau_{ij}}$-$S\alpha$-cl(A) is $S_{\tau_{ij}}$-$S\alpha$-closed set, we have by Theorem 3.18 part (ii),

$S_{\tau_{ij}}$-$S\alpha$-cl($S_{\tau_{ij}}$-$S\alpha$-cl(A)) = $S_{\tau_{ij}}$-$S\alpha$-cl(A).

Definition 3.20 Let (X, S_{τ_1}, S_{τ_2}) be a supra bitopological space and let Y be a subset of X. The relative bitopological space for Y is denoted by (Y, $S_{\tau_{1^*}}$, $S_{\tau_{2^*}}$) such that $S_{\tau_{1^*}}$ = {A ∩ Y : A ∈ S_{τ_1}} and $S_{\tau_{2^*}}$ = {B ∩ Y : B ∈ S_{τ_2}} are two supra topologies for Y. Then (Y, $S_{\tau_{1^*}}$, $S_{\tau_{2^*}}$) is called a subspace of a supra bitopological space (X, S_{τ_1}, S_{τ_2}).

The relative bitopological space for Y with respect to $S_{\tau_{ij}}$-$S\alpha$-open sets is the collection $S_{\tau_{ij^*}}$-$S\alpha$O(X) given by $S_{\tau_{ij^*}}$-$S\alpha$O(X) = {A ∩ Y : A ∈ $S_{\tau_{ij}}$-$S\alpha$O(X)}.

Proposition 3.21 Let (Y, $S_{\tau_{1^*}}$, $S_{\tau_{2^*}}$) be a subspace of a supra bitopological space (X, S_{τ_1}, S_{τ_2}). Then (i) A subset A of Y is $S_{\tau_{ij}}$-$S\alpha$-closed in Y iff there exists $S_{\tau_{ij}}$-$S\alpha$-closed F in X such that A = F ∩ Y.

(ii) For every A ⊆ Y, $S_{\tau_{ij^*}}$-$S\alpha$-cl(A) = $S_{\tau_{ij}}$-$S\alpha$-cl(A) ∩ Y.

Proof.(i)Let A is $S_{\tau_{ij}}$-$S\alpha$-closed in Y, iff Y − A = B ∩ Y for some $S_{\tau_{ij}}$-$S\alpha$-open subset B of X iff

A = Y − (B ∩ Y) = (Y − B) ∪ (Y − Y) [De-Margan law] iff A = Y − B iff A = Y ∩ (X − B) iff

A = Y ∩ F(where F = X − B is $S_{\tau_{ij}}$-$S\alpha$-closed in X, since B is $S_{\tau_{ij}}$-$S\alpha$-open in X).

(ii) By definition 3.17, $S_{\tau_{ij^*}}$-$S\alpha$-cl(A) = ∩{G : G is a $S_{\tau_{ij}}$-$S\alpha$-closed in Y, A ⊆ G}

= ∩{F ∩ Y : F is a $S_{\tau_{ij}}$-$S\alpha$-closed in X and A ⊆ F ∩ Y}[by (i) above]

= ∩{F ∩ Y : F is a $S_{\tau_{ij}}$-$S\alpha$-closed in X and A ⊆ F}

= [∩{F : F is a $S_{\tau_{ij}}$-$S\alpha$-closed in X and A ⊆ F] ∩ Y

= $S_{\tau_{ij}}$-$S\alpha$-cl(A) ∩ Y.

Theorem 3.22 Let (Y, $S_{\tau_{1^*}}$, $S_{\tau_{2^*}}$) be a subspace of a supra bitopological space (X, S_{τ_1}, S_{τ_2}). If a subset A of Y is $S_{\tau_{ij}}$-$S\alpha$-open ($S_{\tau_{ij}}$-$S\alpha$-closed) in X, then A is also $S_{\tau_{ij}}$-$S\alpha$-open($S_{\tau_{ij}}$-$S\alpha$-closed) in Y.

Proof. Since A ⊆ Y, then we have A = A ∩ Y so that A is the intersection of Y with a set $S_{\tau_{ij}}$-$S\alpha$-open ($S_{\tau_{ij}}$-$S\alpha$-closed) in X, namely A. Hence by the Definition 3.20 and by Proposition 3.21 part (i), A is $S_{\tau_{ij}}$-$S\alpha$-open ($S_{\tau_{ij}}$-$S\alpha$-closed) in Y.

CONCLUSION

The new class of notion $S_{\tau_{ij}}$-Sα-open and $S_{\tau_{ij}}$-Sα-closed sets are introduced in supra bitopological space. Thereafter, we study the characterization of their sets.

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A GENERAL METHOD OF DEFINING AVERAGE OF FUNCTION OF A SET OF VALUES

Dhritikesh Chakrabarty
Department of Statistics, Handique Girls' College, Guwahati - 781001, Assam, India.
E-mail: dhritikesh.c@rediffmail.com, dhritikeshchakrabarty@gmail.com

ABSTRACT :

A general method has been derived, in this study, for defining the average of a function of a set (or a list) of values which can describe/yield most of the definitions/formulations of average. This paper is based on the derivation of this general method of defining average along with the derivations of various definitions/formulations of average from the method obtained.

Key Words : Average of a function, general defining method, existing averages, derivations

1. INTRODUCTION

Average [2 , 3 , 27] is a concept behind most of the measures based on numerical data. Pythagoras [4 , 5 , 23], one exponent of mathematics, is the pioneer of defining average. He introduced three basic definitions of average. Later on, these three definitions become popularly known as Pythagorean means [4 , 5 , 23 , 25 , 27]. The three Pythagorean means are respectively arithmetic mean, geometric mean and harmonic mean. Currently, there exist a number of definitions/formulations of average. Some of them, which are often used, are Arithmetic Mean , Geometric Mean , Harmonic Mean , Quadratic Mean , Square Root Mean , Cubic Mean , Cube Root Mean , Generalized p Mean & Generalized p^{th} Root Mean which are defined by

Arithmetic Mean $= \frac{1}{n}\sum_{i=1}^{n} x_i$ (1.1)

Geometric Mean $= (\prod_{i=1}^{n} x_i)^{1/n}$ (1.2)

or equivalently

Geometric Mean $= antilog\{\frac{1}{n}\sum_{i=1}^{n} log\, x_i\}$ (1.3)

Harmonic Mean $= (\frac{1}{n}\sum_{i=1}^{n} x_i^{-1})^{-1}$ (1.4)

Absolute Mean $= \frac{1}{n}\sum_{i=1}^{n} |x_i|$ (1.5)

Quadratic Mean $= (\frac{1}{n}\sum_{i=1}^{n} x_i^{2})^{1/2}$ (1.6)

Square Root Mean $= (\frac{1}{n}\sum_{i=1}^{n} x_i^{1/2})^{2}$ (1.7)

Cubic Mean $= (\frac{1}{n}\sum_{i=1}^{n} x_i^{3})^{1/3}$ (1.8)

Cube Root Mean $= (\frac{1}{n}\sum_{i=1}^{n} x_i^{1/3})^{3}$ (1.9)

Generalized p^{th} Mean $= (\frac{1}{n}\sum_{i=1}^{n} x_i^{p})^{1/p}$ (1.10)

Generalized p^{th} Root Mean $= (\frac{1}{n}\sum_{i=1}^{n} x_i^{1/p})^{p}$ (1.11)

respectively

where

$$x_1 , x_2 , \ldots\ldots\ldots , x_n$$

are n values in a list [1 , 17 , 18)].

Recently, there have been a lot of studies on analysis of numerical data based on average in general and on Pythagorean means specially [6 , 7 , 8 , 9 , 10 , 11 , 12 , 13 , 14 , 15 , 20)].

Kolmogorov [24 , 28 , 29 , 30], one great mathematician, generalized the earlier definitions of average, The generalized definition he obtained is known as Generalized f - Mean. The Generalized f - Mean of $x_1 , x_2 , \ldots\ldots\ldots , x_n$, as obtained by him, is

$$\text{Generalized } f\text{ - Mean} = f^{-1}\{\tfrac{1}{n}\textstyle\sum_{i=1}^{n} f(x_i)\} \tag{1.12}$$

where f is an invertible function [18 , 25 , 26].

Recently, two generalized definitions of average have been derived one of them is termed as Generalized f_H – Mean [19)] and the other as Generalized f_G – Mean [16 , 21].

The Generalized f_H – Mean of $x_1 , x_2 , \ldots\ldots\ldots , x_n$ has been found to be

$$\text{Generalized } f_H\text{ - Mean} = f^{-1}\,[\tfrac{1}{n}\textstyle\sum_{i=1}^{n} \{f(x_i)\}^{-1}]^{-1} \tag{1.13}$$

while the Generalized f_G – Mean of $x_1 , x_2 , \ldots\ldots\ldots , x_n$ has been found to be Generalized f_G - Mean =

$$f^{-1}[\,\{\textstyle\prod_{i=1}^{n} f(x_i)\,\}^{1/n}\,] \tag{1.14}$$

or equivalently to be

$$\text{Generalized } f_G\text{ – Mean} = f^{-1}\,[antilog\,\{\tfrac{1}{n}\textstyle\sum_{i=1}^{n} log\, f(x_i)\}] \tag{1.15}$$

where f is an invertible function and $f_i = f(x_i) \neq 0$.

However, none of these three generalized definitions is complete i.e. none of them can describe/yield all types of averages. This leads to the necessity of searching for one general method/definition of average which describe/yield most of the definitions/formulations of average. Accordingly, in another study, an attempt has been made on searching for a generalized method of defining average of a set of values of a variable [22]. In this study, attempt has been made on searching for a generalized method of defining average of a function of a set (or of a list) of values. A general method has been derived for defining the average of a function of a set (or a list) of values which can describe/yield most of the definitions/formulations of average. This paper is based on the derivation of this general method of defining average along with the derivations of various definitions/formulations of average from this method.

2. GENERAL METHOD OF DEFINING AVERAGE OF A FUNCTION

The arithmetic mean

$$A = \tfrac{1}{n}\textstyle\sum_{i=1}^{n} x_i \tag{2.1}$$

satisfies

$x_1 + x_2 + \ldots\ldots\ldots\ldots + x_n$ = A + A + ………. + A

This means, the function f $(x_1 , x_2 , \ldots\ldots\ldots\ldots\ldots\ldots ,x_n)$ of $x_1 , x_2 , \ldots\ldots\ldots\ldots\ldots\ldots ,x_n$

defined by

f $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)\ =\ x_1 + x_2 +\ \ldots\ldots\ldots\ldots\ldots + x_n$

satisfies

f $(A\ ,\ A\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ A)\ =$ f $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)$

Here the function f $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,x_n)$ is

continuous,

strictly increasing in each argument of $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,x_n$

& symmetric (invariant under permutation of the arguments $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,x_n$.

Similarly, the geometric mean

$$G\ =\ (\prod_{i=1}^{n} x_i)1/n \qquad (2.2)$$

satisfies

G.G. ……………….. G $=\ x_1 . x_2 .\ \ldots\ldots\ldots\ldots\ldots . x_n$

This means, the function g $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)$ of $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n$

defined by

g $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)\ =\ x_1 . x_2 .\ \ldots\ldots\ldots\ldots\ldots . x_n$

satisfies

g (G , G , ……………….. , G) $=$ g $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)$

Here also, the function g $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)$ is

continuous,

strictly increasing in each argument of $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,x_n$

& symmetric (invariant under permutation of the arguments $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,x_n$.

Also similarly, the harmonic mean

$$H = (\frac{1}{n}\sum_{i=1}^{n} x_i{}^{-1})^{-1} \qquad (2.3)$$

satisfies

$x_1{}^{-1} + x_2{}^{-1} + \ldots\ldots\ldots + x_n{}^{-1} = H^{-1} + H^{-1} + \ldots\ldots\ldots + H^{-1}$

This means, the function h $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)$ of $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,x_n$

defined by

h $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)\ =\ x_1{}^{-1} + x_2{}^{-1} + \ldots\ldots\ldots + x_n{}^{-1}$

satisfies

h (H , H , ……………….. , H) $=$ h $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)$

In this case also, the function h $(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)$ is

continuous,

strictly increasing in each argument of $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,x_n$

& symmetric (invariant under permutation of the arguments $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,x_n$.

Thus, in general, an average of a list

$x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,x_n$

of numbers can be defined to be a number μ such that

$$\phi\,(\mu\ ,\ \mu\ \ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ \mu)\ =\ \phi\,(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n) \qquad (2.4)$$

where $\phi\,(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)$ is a function of $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n$
which satisfies the following three conditions:

(1) It is continuous.

(2) It is strictly increasing in each argument of $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n$.

(3) it is symmetric (invariant under permutation of the arguments $x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n$.

This definition of average can be regarded as a method of deriving various definitions/formulations of average. Choosing different functions which satisfy the properties (i) , (ii) & (iii), one can obtain different definitions/formulations of average from the equation

$$\phi\,(\mu\ ,\ \mu\ \ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ \mu)\ =\ \phi\,(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n) \qquad (2.5)$$

Now let $y = \xi(.)$ be a function of function so that
$y_1 = \xi(x_1)\ ,\ y_2 = \xi(x_2)\ \ ,\ \ldots\ldots\ldots\ldots\ldots\ldots\ \ ,\ y_n = \xi(x_n)$
are the $\xi(.)$ functional values of
$x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ldots\ ,x_n$
respectively.
Then as per the definition of average, as explained above,
the average of the list
$y_1\ ,\ y_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ y_n$
of numbers can be defined to be a number μ such that

$$\phi\,(\mu\ ,\ \mu\ \ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ \mu)\ =\ \phi\,(y_1\ ,\ y_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ y_n) \qquad (2.6)$$

where $\phi\,(y_1\ ,\ y_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ y_n)$ is a function of $y_1\ ,\ y_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ y_n$
which satisfies the following three conditions:

(1) It is continuous.

(2) It is strictly increasing in each argument of $y_1\ ,\ y_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ y_n$.

(3) it is symmetric (invariant under permutation of the arguments $y_1\ ,\ y_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ y_n$.

This implies that the average of the list
$\xi(x_1\ ) , \xi(x_2)\ ,\ \ldots\ldots\ldots\ldots\ldots\ , \xi(x_n)$
of numbers can be defined to be a number μ such that

$$\phi\,(\mu\ ,\ \mu\ \ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ \mu)\ =\ \phi\,\{\,\xi(x_1\ ) , \xi(x_2)\ ,\ \ldots\ldots\ldots\ldots\ldots\ , \xi(x_n)\} \qquad (2.7)$$

where
$\phi\,\{\,\xi(x_1\ ) , \xi(x_2)\ ,\ \ldots\ldots\ldots\ldots\ldots\ , \xi(x_n)\}$
is a function of
$\xi(x_1\ ) , \xi(x_2)\ ,\ \ldots\ldots\ldots\ldots\ldots\ , \xi(x_n)$
which satisfies the following three conditions:

(1) It is continuous.

(2) It is strictly increasing in each argument of $\xi(x_1\ ) , \xi(x_2)\ ,\ \ldots\ldots\ldots\ldots\ldots\ , \xi(x_n)$.

(3) it is symmetric (invariant under permutation of the arguments

$\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots\ldots, \xi(x_n)$.

3. DERIVATION OF VARIOUS AVERAGES FROM THE METHOD:

Arithmetic Mean

Let the function $\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\}$ be

$\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\} = \xi(x_1) + \xi(x_2) + \ldots\ldots\ldots\ldots + \xi(x_n)$ (3.1)

Then equation (2.5) namely

$\phi(\mu, \mu, \ldots\ldots\ldots\ldots\ldots, \mu) = \phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\}$

implies

$\mu + \mu + \ldots\ldots\ldots\ldots + \mu = \xi(x_1) + \xi(x_2) + \ldots\ldots\ldots\ldots + \xi(x_n)$

which yields,

$$\mu = \frac{1}{n}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \xi(x_i) \quad (3.2)$$

which is nothing but the arithmetic mean of $\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)$.

In particular if $\xi(.)$ is an identity function i.e. if

$$\xi(x_i) = x_i, \quad (i = 1, 2, \ldots\ldots\ldots\ldots, n)$$

then

$$\mu = \frac{1}{n}\sum_{\mathrm{i}=1}^{n} x_i \quad (3.3)$$

which is nothing but the arithmetic mean of $x_1, x_2, \ldots\ldots\ldots\ldots\ldots\ldots, x_n$.

Similarly, if

$\xi(x_i) = x_i 2$, (i = 1 , 2 , ………….. , n)

then

$$\mu = \frac{1}{n}\sum_{\mathrm{i}=1}^{n} x_i 2 \quad (3.4)$$

which is nothing but the arithmetic mean of $x_1 2, x_2 2, \ldots\ldots\ldots\ldots\ldots, x_n 2$.

In general, if

$\xi(x_i) = x_i\,\mathrm{k}$, (i = 1 , 2 , ………….. , n)

then

$$\mu = \frac{1}{n}\sum_{\mathrm{i}=1}^{n} x_i \mathrm{k} \quad (3.5)$$

which is nothing but the arithmetic mean of $x_1\mathrm{k}, x_2\mathrm{k}, \ldots\ldots\ldots\ldots\ldots, x_n\mathrm{k}$.

Geometric Mean

Let the function $\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\}$ be

$\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\} = \xi(x_1).\,\xi(x_2).\,\xi(x_3)\ \ldots\ldots\ldots\ldots\ \xi(x_n)$ (3.6)

Then equation (2.5) implies

$\mu\,.\,\mu.\ \ldots\ldots\ldots\ldots\ \mu = \xi(x_1).\,\xi(x_2).\ \ldots\ldots\ldots\ldots\ \xi(x_n)$

which yields,

$$\mu = \{\prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} \xi(x_i)\}1/\mathrm{n} \quad (3.7)$$

which is nothing but the geometric mean of $\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)$.

In particular if $\xi(.)$ is an identity then

$$\mu = (\prod_{i=1}^{n} x_i)1/n \quad (3.8)$$

which is the geometric mean of x_1 , x_2 , ……………… , x_n .

Similarly, if

$\xi(x_i) = x_i 2$, (i = 1 , 2 , ………….. , n)

then

$$\mu = (\prod_{i=1}^{n} x_n 2)1/n \quad (3.9)$$

This is nothing but the geometric mean of $x_1 2$, $x_2 2$, … … … … ….. , $x_n 2$.

In general, the geometric mean μ of x_1k , x_2k , … … … … ….. , x_nk becomes

$$\mu = (\prod_{i=1}^{n} x_n 2\text{k})1/n \quad (3.10)$$

Harmonic Mean

Let the function ϕ {$\xi(x_1)$, $\xi(x_2)$, … … … ….. , $\xi(x_n)$} be

ϕ {$\xi(x_1)$, $\xi(x_2)$, … … … ….. , $\xi(x_n)$} = $\{\xi(x_1)\}^{-1} + \{\xi(x_2)\}^{-1} + \dots\dots + \{\xi(x_n)\}^{-1}$ (3.11)

Then equation (2.5) implies

$\mu^{-1} + \mu^{-1} + \dots\dots + \mu^{-1} = \{\xi(x_1)\}^{-1} + \{\xi(x_2)\}^{-1} + \dots\dots + \{\xi(x_n)\}^{-1}$

which yields,

$$\mu = [\tfrac{1}{n}\sum_{i=1}^{n}\{\xi(x_i)\}^{-1}]^{-1} \quad (3.12)$$

which is nothing but the harmonic mean of $\xi(x_1)$, $\xi(x_2)$, … … … ….. , $\xi(x_n)$.

In particular if $\xi(.)$ is an identity function then

$$\mu = (\tfrac{1}{n} \sum_{i=1}^{n} x_i^{-1})^{-1} \quad (3.13)$$

This is the harmonic mean of x_1 , x_2 , ……………… , x_n .

Similarly, if

$\xi(x_i) = x_i 2$, (i = 1 , 2 , ………….. , n)

then

$$\mu = (\tfrac{1}{n} \sum_{i=1}^{n} x_i^{-2})^{-1} \quad (3.14)$$

This is the harmonic mean of $x_1 2$, $x_2 2$, … … … … ….. , $x_n 2$.

In general, the harmonic mean μ of x_1k , x_2k , … … … … ….. , x_nk becomes

$$\mu = (\tfrac{1}{n} \sum_{i=1}^{n} x_i^{-2k})^{-1} \quad (3.15)$$

Absolute Mean

Let the function ϕ {$\xi(x_1)$, $\xi(x_2)$, … … … ….. , $\xi(x_n)$} be

ϕ {$\xi(x_1)$, $\xi(x_2)$, … … … ….. , $\xi(x_n)$} = $|\xi(x_1)| + |\xi(x_2)| + \dots\dots + |\xi(x_n|$ (3.16)

Then equation (2.5)

$|\mu| + |\mu| + \dots\dots + |\mu| = |\xi(x_1)| + |\xi(x_2)| + \dots\dots + |\xi(x_n|$

This yields

$$\mu = \tfrac{1}{n}\sum_{i=1}^{n} |\xi(x_i)| \quad (3.17)$$

which is the absolute mean of $\xi(x_1)$, $\xi(x_2)$, … … … ….. , $\xi(x_n)$.

In particular if $\xi(.)$ is an identity function i.e. if

$\xi(x_i) = |x_i|$, (i = 1 , 2 , ………….. , n)

then

$$\mu = \tfrac{1}{n}\sum_{i=1}^{n} |x_i| \quad (3.18)$$

This is nothing but the absolute mean of x_1 , x_2 , ……………… , x_n .
Choosing

$\xi(x_i) = x_i2$, (i = 1 , 2 , ………….. , n) ,

the absolute mean μ of x_1k , x_2k , … … … … ….. , x_nk is obtained as

$$\mu = \frac{1}{n}\sum_{i=1}^{n} | x_i{}^k | \tag{3.19}$$

Quadratic Mean or Root Mean Square

Let the function $\phi \{\xi(x_1), \xi(x_2), \dots\dots, \xi(x_n)\}$ be

$$\phi \{\xi(x_1), \xi(x_2), \dots\dots, \xi(x_n)\} = \{\xi(x_1)\}^2 + \{\xi(x_2)\}^2 + \dots\dots + \{\xi(x_n)\}^2 \tag{3.20}$$

Then equation (2.5) implies

$$\mu^2 + \mu^2 + \dots\dots + \mu^2 = \{\xi(x_1)\}^2 + \{\xi(x_2)\}^2 + \dots\dots + \{\xi(x_n)\}^2$$

which yields,

$$\mu = [\frac{1}{n} \sum_{i=1}^{n} \{\xi(x_i)\}^2]^{1/2} \tag{3.21}$$

which is nothing but the quadratic mean of $\xi(x_1)$, $\xi(x_2)$, … … … ….. , $\xi(x_n)$.

In particular if $\xi(.)$ is an identity function then

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i{}^2)^{1/2} \tag{3.22}$$

which is nothing but the quadratic mean of x_1 , x_2 , ……………… , x_n .

Similarly, if

$\xi(x_i) = x_i2$, (i = 1 , 2 , ………….. , n)

then

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i{}^4)^{1/2} \tag{3.23}$$

This is the quadratic mean of $x_1$2 , $x_2$2 , … … … … ….. , x_n2 .

Choosing

$\xi(x_i) = x_i2$, (i = 1 , 2 , ………….. , n) ,

the quadratic mean μ of x_1k , x_2k , … … … … ….. , x_nk is obtained as

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i{}^{2k})^{1/2} \tag{3.24}$$

Square Root Mean

Let the function $\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\}$ be

$$\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\} = \{\xi(x_1)\}^{1/2} + \{\xi(x_2)\}^{1/2} + \ldots\ldots\ldots\ldots + \{\xi(x_n)\}^{1/2} \quad (3.25)$$

Then equation (2.5) implies

$$\mu^{1/2} + \mu^{1/2} + \ldots\ldots\ldots\ldots + \mu^{1/2} = \{\xi(x_1)\}^{1/2} + \{\xi(x_2)\}^{1/2} + \ldots\ldots\ldots\ldots + \{\xi(x_n)\}^{1/2}$$

which yields,

$$\mu = [\, \tfrac{1}{n} \textstyle\sum_{i=1}^{n} \{\xi(x_i)\}^{1/2} \,]^2 \qquad (3.26)$$

which is nothing but the square root mean of $\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)$.

In particular if $\xi(.)$ is an identity function then

$$\mu = (\, \tfrac{1}{n} \textstyle\sum_{i=1}^{n} {x_i}^{1/2} \,)^2 \qquad (3.27)$$

which is nothing but the square root mean of x_1 , x_2 , ……………… , x_n .

Similarly, if

$\xi(x_i) = x_i 2$, (i = 1 , 2 , ………….. , n)

then

$$\mu = (\, \tfrac{1}{n} \textstyle\sum_{i=1}^{n} x_i \,)^2 \qquad (3.28)$$

This is the square root mean of $x_1 2$, $x_2 2$, …………….. , $x_n 2$.

In general, if

$\xi(x_i) = x_i$ k , (i = 1 , 2 , ………….. , n)

then

$$\mu = (\, \tfrac{1}{n} \textstyle\sum_{i=1}^{n} {x_i}^{k/2} \,)^2 \qquad (3.29)$$

which is nothing but the square root mean of x_1k , x_2k , …………….. , x_nk .

Cubic Mean

Let the function $\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\}$ be

$$\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\} = \{\xi(x_1)\}^3 + \{\xi(x_2)\}^3 + \ldots\ldots\ldots\ldots + \{\xi(x_n)\}^3 \qquad (3.30)$$

Then equation (2.5) implies

$\mu^3 + \mu^3 + \ldots\ldots\ldots\ldots + \mu^3 = \{\xi(x_1)\}^3 + \{\xi(x_2)\}^3 + \ldots\ldots\ldots\ldots + \{\xi(x_n)\}^3$

which yields,

$$\mu = [\ \tfrac{1}{n} \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \{\xi(x_i)\}^3\ ]^{1/3} \qquad (3.31)$$

which is nothing but the cubic mean of $\xi(x_1)$, $\xi(x_2)$, ……………, $\xi(x_n)$.

In particular if $\xi(.)$ is an identity function then

$$\mu = (\ \tfrac{1}{n} \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} {x_i}^3\ )^{1/3} \qquad (3.32)$$

which is nothing but the cubic mean of x_1 , x_2 , ……………… , x_n .

Similarly, if

$\xi(x_i) = x_i 2$, (i = 1 , 2 , ………….. , n)

then

$$\mu = (\ \tfrac{1}{n} \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} {x_i}^6\ )^{1/3} \qquad (3.33)$$

This is the cubic mean of $x_1 2$, $x_2 2$, ……………….. , $x_n 2$.

In general, the cubic mean of x_1k , x_2k , ……………….. , x_nk can be obtained by choosing

$\xi(x_i) = x_i$ k , (i = 1 , 2 , ………….. , n)

which is obtained as

$$\mu = (\ \tfrac{1}{n} \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} {x_i}^{3k}\ )^{1/3} \qquad (3.3\ 4)$$

Cube Root Mean

Let the function φ $\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots, \xi(x_n)\}$ be

φ $\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots, \xi(x_n)\} = \{\xi(x_1)\}^{1/3} + \{\xi(x_2)\}^{1/3} + \ldots\ldots\ldots\ldots + \{\xi(x_n)\}^{1/3}$ (3.35)

Then equation (2.5) implies

$\mu^{1/3} + \mu^{1/3} + \ldots\ldots\ldots\ldots + \mu^{1/3} = \{\xi(x_1)\}^{1/3} + \{\xi(x_2)\}^{1/3} + \ldots\ldots\ldots\ldots + \{\xi(x_n)\}^{1/3}$

which yields,

$$\mu = [\ \tfrac{1}{n} \sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \{\xi(x_i)\}^{1/3}\ ]^3 \qquad (3.36)$$

which is nothing but the cube root mean of $\xi(x_1)$, $\xi(x_2)$, ……………, $\xi(x_n)$.

In particular if $\xi(.)$ is an identity function then

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i^{1/3})^3 \quad (3.37)$$

which is nothing but the cube root mean of x_1 , x_2 , ……………… , x_n .

Similarly, if

$\xi(x_i) = x_i 2$, (i = 1 , 2 , …………. , n)

then

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i^{2/3})^3 \quad (3.38)$$

This is the cube root mean of $x_1 2$, $x_2 2$, … … … … ….. , $x_n 2$.

In general, the cube root mean of x_1k , x_2k , … … … … ….. , x_nk can be obtained as

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i^{k/3})^3 \quad (3.39)$$

pth Mean

Let the function $\phi \{\xi(x_1), \xi(x_2), \ldots\ldots\ldots, \xi(x_n)\}$ be

$$\phi \{\xi(x_1), \xi(x_2), \ldots\ldots\ldots, \xi(x_n)\} = \{\xi(x_1)\}^p + \{\xi(x_2)\}^{\mathrm{p}} + \ldots\ldots\ldots + \{\xi(x_n)\}^{\mathrm{p}} \quad (3.40)$$

Then equation (2.5) implies

$$\mu^{\mathrm{p}} + \mu^p + \ldots\ldots\ldots + \mu^{\mathrm{p}} = \{\xi(x_1)\}^p + \{\xi(x_2)\}^{\mathrm{p}} + \ldots\ldots\ldots + \{\xi(x_n)\}^{\mathrm{p}}$$

which yields,

$$\mu = [\frac{1}{n} \sum_{i=1}^{n} \{\xi(x_i)\}^{\mathrm{p}}]^{1/\mathrm{p}} \quad (3.41)$$

which is nothing but the cubic mean of $\xi(x_1)$, $\xi(x_2)$, … … … ….. , $\xi(x_n)$.

In particular if $\xi(.)$ is an identity function then

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i^{\mathrm{p}})^{1/p} \quad (3.42)$$

which is nothing but the pth mean of x_1 , x_2 , ……………… , x_n .

Similarly, if

$\xi(x_i) = x_i 2$, (i = 1 , 2 , …………. , n)

then

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i^{2p})^{1/\mathrm{p}} \quad (3.43)$$

This is the pth mean of $x_1 2$, $x_2 2$, ……………….. , $x_n 2$.

In general, the pth mean of x_1k , x_2k , ……………….. , x_nk can be obtained as

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i^{kp})^{1/p} \quad (3.44)$$

pth Root Mean

Let the function ϕ $\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots, \xi(x_n)\}$ be

$$\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots, \xi(x_n)\} = \{\xi(x_1)\}^{1/p} + \{\xi(x_2)\}^{1/p} + \ldots\ldots\ldots + \{\xi(x_n)\}^{1/p} \quad (3.45)$$

Then equation (2.5) implies

$$\mu^{1/p} + \mu^{1/p} + \ldots\ldots\ldots + \mu^{1/p} = \{\xi(x_1)\}^{1/p} + \{\xi(x_2)\}^{1/p} + \ldots\ldots\ldots + \{\xi(x_n)\}^{1/p}$$

which yields,

$$\mu = [\frac{1}{n} \sum_{i=1}^{n} \{\xi(x_i)\}^{1/p}]^{p} \quad (3.46)$$

which is nothing but the cubic mean of $\xi(x_1)$, $\xi(x_2)$, ……………. , $\xi(x_n)$.

In particular if $\xi(.)$ is an identity function then

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i^{1/p})^{p} \quad (3.47)$$

which is nothing but the pth root mean of x_1 , x_2 , ……………… , x_n .

then

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i^{2/p})^{p} \quad (3.48)$$

This is the pth root mean of $x_1 2$, $x_2 2$, ……………….. , $x_n 2$.

In general, the pth root mean of x_1k , x_2k , ……………….. , x_nk can be obtained as

$$\mu = (\frac{1}{n} \sum_{i=1}^{n} x_i^{k/p})^{p} \quad (3.49)$$

Generalized f - Mean

Let the function ϕ $\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots, \xi(x_n)\}$ be

$$\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots, \xi(x_n)\} = f\{\xi(x_1)\} + f\{\xi(x_2)\} + \ldots\ldots\ldots + f\{\xi(x_n)\} \quad (3.50)$$

Then equation (2.5) implies

$$f(\mu) + f(\mu) + \ldots\ldots\ldots + f(\mu) = f\{\xi(x_1)\} + f\{\xi(x_2)\} + \ldots\ldots\ldots + f\{\xi(x_n)\}$$

which yields,

$$\mu = f^{-1} [\{\frac{1}{n}\sum_{i=1}^{n} f\{\xi(x_i)\}] \tag{3.51}$$

This is nothing but the generalized f - mean of $\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)$.

In particular if $\xi(.)$ is an identity function then

$$\mu = f^{-1} \{\frac{1}{n}\sum_{i=1}^{n} f(x_i)\} \tag{3.52}$$

which is nothing but the generalized f - mean of x_1 , x_2 , ……………… , x_n .

The generalized f - mean of $x_1 2$, $x_2 2$, ……………… , $x_n 2$ is

$$\mu = f^{-1} \{\frac{1}{n}\sum_{i=1}^{n} f(x_i^{\,2})\} \tag{3.53}$$

while the generalized f - mean of x_1k , x_2k , ……………… , x_nk is

$$\mu = f^{-1} \{\frac{1}{n}\sum_{i=1}^{n} f(x_i^{\,k})\} \tag{3.54}$$

Generalized fH - Mean

Let the function ϕ $\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\}$ be

ϕ $\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\} = f_1{-1} + f_2{-1} + \ldots\ldots\ldots\ldots + f_n{-1}$ (3.55)

where f_1 = f$\{\xi(x_1)\}$, f_2 = f$\{\xi(x_2)\}$, , ……………… , f_n = f$\{\xi(x_n)\}$

Then equation (2.5) implies

$$\{f(\mu)\}{-1} + \{f(\mu)\}{-1} + \ldots\ldots\ldots\ldots + \{f(\mu)\}{-1} = f_1{-1} + f_2{-1} + \ldots\ldots\ldots\ldots + f_n{-1}$$

which yields,

$$\mu = f^{-1} \{(\frac{1}{n} \sum_{i=1}^{n} f_i^{\,-1})^{-1}\} \tag{3.56}$$

which is nothing but the generalized fH - mean of $\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)$.

In particular if $\xi(.)$ is an identity function then

$$\mu = f^{-1} [\frac{1}{n}\sum_{i=1}^{n} \{f(x_i)\}{-1}]{-1} \tag{3.57}$$

which is nothing but the generalized fH - mean of x_1 , x_2 , ……………… , x_n

(3.50)

Generalized fG - Mean

Let the function $\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\}$ be

$$\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\} = f\{\xi(x_1)\}.f\{\xi(x_2)\}. \ldots\ldots\ldots\ldots .f\{\xi(x_n)\} \quad (3.58)$$

Then equation (2.5) implies

$$f(\mu).f(\mu).\ldots\ldots\ldots\ldots . f(\mu) = f\{\xi(x_1)\}.f\{\xi(x_2)\}. \ldots\ldots\ldots\ldots .f\{\xi(x_n)\}$$

which yields,

$$\mu = f^{-1}[\{\tfrac{1}{n}\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} f\{\xi(x_i)\}] \quad (3.59)$$

where $f_1 = f\{\xi(x_1)\}$, $f_2 = f\{\xi(x_2)\}$, , $\ldots\ldots\ldots\ldots$, $f_n = f\{\xi(x_n)\}$.

This is nothing but the generalized fG - mean of $\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)$.

In particular if $\xi(.)$ is an identity function then

$$\mu = f^{-1}[\{\prod_{i=1}^{n} f(x_i)\}1/n] \quad (3.60)$$

which is nothing but the generalized fG - mean of x_1 , x_2 , ……………… , x_n .

Note

The generalized fG – mean can also be as follows:

Let the function $\phi\{\xi(x_1), \xi(x_2), \ldots\ldots\ldots\ldots, \xi(x_n)\}$ be

$$\phi(x_1, x_2, \ldots\ldots\ldots\ldots, x_n) = log\, f\{\xi(x_1)\} + \log f\{\xi(x_2)\} + \ldots\ldots\ldots\ldots + log f\{\xi(x_n)\}$$

Then equation (2.5) implies

$$log\, f\{\xi(\mu)\} + log\, f\{\xi(\mu)\} \ldots\ldots\ldots\ldots + log\, f\{\xi(\mu)\}$$

$$= log\, f\{\xi(x_1)\} + \log f\{\xi(x_2)\} + \ldots\ldots\ldots\ldots + log f\{\xi(x_n)\}$$

which yields,

$$\mu = f^{-1}[antilog(\tfrac{1}{n}\sum_{i=1}^{n} log\, f_i)\} \quad (3.61)$$

which is nothing but the generalized fG - mean of x_1 , x_2 , ……………… , x_n .

4. CONCLUSION

The general method derived here is based on the principle behind the concept of average. This general definition captures the important property of all averages that the average of a list of identical elements is that element itself. Various definitions/formulations of average can be derived from this general method of defining selecting different forms of the function $\phi(x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_n)$ which satisfy the properties (i) , (ii) & (iii), as mentioned above from the equation (2.5). Thus the equation (2.5) namely

$$\phi(\mu\ ,\ \mu\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ \mu) = \phi(x_1\ ,\ x_2\ ,\ \ldots\ldots\ldots\ldots\ldots\ ,\ x_n)$$

can be regarded as the characteristic equation of average.

The properties of the average defined by this method have not been studied in this attempt. Thus one problem of further research, at this stage, is to study the properties of the average defined by this method.

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