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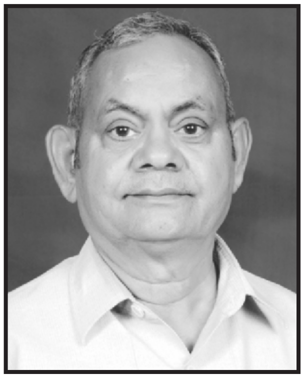

Dr. T.P. Singh
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The Journal : 10 Years of Journey

About 10 years back, a group of enthusiastic professors, mathematicians, economists and researchers from different institutes of higher learning gathered together under the registered trust/society 'Aryans Research and Educational Trust' and decided to promote research activities in interdisciplinary approach through a research journal. The title of journal was named as **'Aryabhatta Journal of Mathematics & Informatics' (AJMI)** in honor of ancient famous mathematical scientist. **'Aryabhatta'** (born in 5th century) who for the first time established the healthy tradition of scientific research in India discarding the traditional way of thinking. The responsibility for registration, title approval, getting ISSN No. and to act as Editor in chief was assigned to me. On completing all the formalities, the first issue of the journal was published in year 2009. Since then and till today the journal is publishing regularly well in time. It gives us a great pleasure to put forward before the scholars and academicians that **The Journal** from its start is making an effort to produce good quality articles. The credit goes to its editorial and reviewer team which review sincerely and furnish valuable suggestions to improve its quality. We are proud to mention that **AJMI** is among 50 Indian journals (Rank 10) based on citation per year from foreign countries (table 3.5 on page 30, a report based on Indian Citation Index 2016 under supervision of Sh. Prakash Chand, Scientist NISCAIR-CSIR and Head ICI). The Journal has 1.583 citation per paper and its impact factor is continuously increasing. The Journal has been indexed by many National and International agencies as **Copernicus, Indian Citation index(ICI), cite factor Google scholars, CNKI scholar, EBSCO Discover etc.** AJMI discourages any type of plagiarism in the paper and motivate the authors to have self plagiarism not exceed 20% while sending the paper, if at any time it is found we remove such paper from our website.

Ayrabhatta Journal of Mathematics & Informatics mainly covers area of mathematical and statistical sciences, Operational Research, data based management, economical issues and information sciences. Mathematics being an interdisciplinary approach, is the core of computer simulation, physical sciences, quantitative analysis of management and economic issues, above all it is a key of key technologies of our times while information sciences the fastest growing segment due to industrial liberation, changes business environment, globalization and the trends in world economic scenario has posed an increasing challenges for the organization to be competitive and productive. The statistics reveal that today no organization or individual is without communicating or information device. Information Technology is becoming the dominant force in our culture and will continue to transform the key and the world we live and work. Information is an asset which is currently as important as capital or work. The Journal aims to focus on all such issues in mathematical, technical and business domain using the available set of knowledge.

We are of the opinion, it is good that life should be on going search, the journey is more important than the destination.

DR. T.P. SINGH

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COUPLED FIXED POINT THEOREMS FOR FOUR MAPPINGS IN DISLOCATED QUASI b-METRIC SPACES

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ABSTRACT :

In this paper, we mainly prove two common coupled fixed point theorems for four maps satisfying more general contractive conditions in dislocated quasi b-metric spaces and obtain some existing results in the literature as corollaries. We also provide two examples to support our theorems.

Keywords: Dislocated quasi b-metric, w-compatible maps, continuity, coupled fixed points.

Mathematics Subject Classification: 54H25, 47H10.

1. INTRODUCTION AND PRELIMINARIES

Hitzler [9] and Hitzler and Seda [8] introduced the notion of dislocated metric spaces and generalized the celebrated Banach contraction principle in such spaces. Zeyada et.al [17] initiated the concept of dislocated quasi metric spaces and generalized the results of Hitzler and Seda [8] in dislocated quasi metric spaces. Bakhtin [4] introduced the concept of b-metric space. The concept of Bakhtin is extensively used by Czerwic [5] in connection with some problems concerning with the convergence of non measurable functions with respect to measure.

Recently Klin-eam and Suanoom [10] introduced the concept of dislocated quasi b-metric spaces, which generalize b-metric spaces [5] and quasi b-metric spaces [15] and proved some fixed point theorems in it by using cyclic contractions. The authors [1,7,10,12,13,14,16, 21] etc. obtained fixed, common fixed points and common coupled fixed point theorems in dislocated quasi b-metric spaces using various contraction conditions for single and two maps. In this paper, we prove two common coupled fixed point theorems for four maps in dislocated quasi b-metric spaces and we also give examples to support our theorems.

First we recall some known definitions and lemmas. Throughout this paper, we assume that $\mathbf{R}^+$ is the set of all non-negative real numbers.

Definition1.1 Let X be a non-empty set , $s \geq 1$ (a fixed real number) and $d: X \times X \to \mathrm{R}^+$ be a function. Consider the following condition on d.

(1.1.1) $d(x,x) = 0, \forall x \epsilon X$

(1.1.2) $d(x,y) = d(y,x) = 0 \Rightarrow x = y, \forall x, y \,\epsilon X$

(1.1.3) $d(x,y) = d(y,x), \forall x, y \epsilon X$

(1.1.4) $d(x,y) \leq d(x,z) + d(z,y), \forall x, y, z \,\epsilon X$

(1.1.5) $d(x,y) \leq s[d(x,z) + d(z,y)], \forall x, y, z \,\epsilon X.$

(i) If d satisfies (1.1.2), (1.1.3) and (1.1.4) then d is called a dislocated metric and (X,d) is called a dislocated metric space.

(ii) If d satisfies (1.1.1),(1.1.2) and (1.1.4) then d is called a quasi metric and (X,d) is called a quasi metric space.

(iii) If d satisfies (1.1.2) and (1.1.4) then d is called a dislocated quasi metric or dq-metric and (X,d) is called a dislocated quasi metric space.

(iv) If d satisfies (1.1.1),(1.1.2),(1.1.3) and (1.1.4) then d is called a metric and (X,d) is called a metric space.

(v) If d satisfies (1.1.1),(1.1.2),(1.1.3) and (1.1.5) then d is called a b-metric and (X,d) is called a b-metric space.

(vi) If d satisfies (1.1.2) and (1.1.5) then d is called a dislocated quasi b-metric and (X,d) is called a dislocated quasi b-metric space or dq b-metric space.

Definition 1.2 Let (X,d) be a dq b-metric space. A sequence $\{x_n\}$ in (X,d) is said to be
(i) dq b-convergent if there exists some point $x \epsilon X$ such that $\lim_{n\to\infty} \mathrm{d}(x_n, x) = 0 = \lim_{n\to\infty} d(x, x_n)$.

In this case x is called a dq b-limit of $\{x_n\}$ and we write $x_n \to x \; as \; n \to \infty$.
(ii) Cauchy sequence if $\lim_{m,n\to\infty} d\big(x_n, x_m\big) = 0 = \lim_{m,n\to\infty} d\big(x_m, x_n\big)$.

The space (X,d) is called complete if every Cauchy sequence in X is dq b-convergent.

One can prove easily the following

Lemma 1.3 Let (X,d) be a dq b-metric space and $\{x_n\}$ be dq b-convergent to $x \epsilon X$ and $y \epsilon X$ be arbitrary.Then
$\frac{1}{s}\, d(x, y) \le \lim_{n\to\infty} \inf d(x_n, y) \le lim_{n\to\infty} \sup d(x_n, y) \le s\, d(x, y)$ and
$\frac{1}{s} d(y, x) \le \lim_{n\to\infty} \inf d(y, x_n) \le \lim_{n\to\infty} \sup d(y, x_n) \le s\, d(y, x)$.
Note: $\frac{1}{2s} d(x, y) \le max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$.

Definition 1.4 ([2]) Let X be a non-empty set and $f, g: X \to X$ be mappings.If there exists $x \epsilon X$ such that fx = gx then x is called a coincidence point of of f and g and fx is called a point of coincidence f and g.

The notion of coupled fixed point is introduced by Bhaskar and Lakshmikantham [6] and studied some fixed point theorems in partially ordered metric spaces. Later Lakshmikantham and Ciric [11] defined coupled coincidence point and common coupled fixed points for a pair of maps and Abbas et.al [3] introduced the notion of w-compatible mappings.

Definition 1.5 ([6]) Let X be a non-empty set .An element $(x, y) \epsilon\, X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.6 ([11]) Let X be a non-empty set. An element $(x, y) \epsilon X \times X$ is called (1) a coupled coincidence point of mappings $F : X \times X \to X$ and $f: X \to$ X if fx = F(x,y) and fy = F(y,x).

(2) a common coupled fixed point of mappings $F : X \times X \to X$ and $f: X \to$ X if x = fx = F(x,y) and y = fy = F(y,x).

Definition 1.7 ([3]) Let X be a non-empty set. The mappings $F : X \times X \to X$ and $f: X \to$ X are called w-compatible if $f\big(F(x, y)\big) = F(fx, fy)$ and $f\big(F(y, x)\big) = F(fy, fx)$ whenever there exist $x, y \,\epsilon X$ such that fx = F(x,y) and fy = F(y,x).

We utilize the following class of functions in our main theorems.

Definition 1.8 For the integer $s \geq 1$, let $\mathbf{\Phi}_s$ denote the class of all functions $\varphi : R^+ \to R^+$ satisfying the following

(φ_1) : φ is monotonically non-decreasing ,

(φ_2): $\sum s^n \varphi^n(t) < \infty$ for all $t > 0$,

(φ_3) : $\varphi(t) < t$ for all $t > 0$.

It is clear from (φ_1) and (φ_3) that $\varphi(0) = 0$.

2. MAIN RESULTS

Our main results are the following

Theorem 2.1: Let (X,d) be a complete dislocated quasi b-metric space with fixed integer $s \geq 1$ and $F, G: X \times X \to X$ and $S, T: X \to X$ be continuous mappings satisfying

(2.1.1) $$d\big(F(x,y), G(u,v)\big) \leq \varphi\left(max \left\{ \begin{array}{c} d(Sx,Tu), d(Sy,Tv), \frac{1}{2s} d\big(Sx, F(x,y)\big), \\ \frac{1}{2s} d\big(Sy, F(y,x)\big), \frac{1}{2s} d\big(Tu, G(u,v)\big), \frac{1}{2s} d\big(Tv, G(v,u)\big), \\ \frac{1}{2s} d\big(Sx, G(u,v)\big), \frac{1}{2s} d\big(Sy, G(v,u)\big), \\ \frac{1}{2s} d\big(Tu, F(x,y)\big), \frac{1}{2s} d\big(Tv, F(y,x)\big) \end{array} \right\} \right)$$

for all $x, y, u, v \in X, where\ \varphi \in \mathbf{\Phi}_s$ **,**

(2.1.2) $$d\big(G(x,y), F(u,v)\big) \leq \varphi\left(max \left\{ \begin{array}{c} d(Tx,Su), d(Ty,Sv), \frac{1}{2s} d\big(Tx, G(x,y)\big), \\ \frac{1}{2s} d\big(Ty, G(y,x)\big), \frac{1}{2s} d\big(Su, F(u,v)\big), \\ \frac{1}{2s} d\big(Sv, F(v,u)\big), \frac{1}{2s} d\big(Tx, F(u,v)\big), \frac{1}{2s} d\big(Ty, F(v,u)\big), \\ \frac{1}{2s} d\big(Su, G(x,y)\big), \frac{1}{2s} d\big(Sv, G(y,x)\big) \end{array} \right\} \right)$$

for all $x, y, u, v \in X$, where $\varphi \in \mathbf{\Phi}_s$ **,**

(2.1.3) $F(X \times X) \subseteq T(X)$ and $G(X \times X) \subseteq S(X)$,

(2.1.4) FS = SF and GT = TG.

Then F, G, S and T have a unique common coupled fixed point in $X \times X$.

Proof: Let $(x_0, y_0) \in X \times X$.

From (2.1.3), there exist sequences $\{x_n\}, \{y_n\}, \{z_n\}$ and $\{w_n\}$ in X such that

$z_{2n} = \mathrm{F}(x_{2n}, y_{2n}) = Tx_{2n+1}$,

$w_{2n} = \mathrm{F}(y_{2n}, x_{2n}) = Ty_{2n+1}$,

$z_{2n+1} = \mathrm{G}(x_{2n+1}, y_{2n+1}) = Sx_{2n+2}$,

$w_{2n+1} = \mathrm{G}(y_{2n+1}, x_{2n+1}) = Sy_{2n+2}$, $n = 0,1,2, ...$

Case (i): Suppose

max $\{d(z_{n-1}, z_n), d(z_n, z_{n-1}), d(w_{n-1}, w_n), d(w_n, w_{n-1})\} = 0$,for some n ……(1)

Without loss of generality assume that n = 2m.

Then $z_{2m-1} = z_{2m}$ and $w_{2m-1} = w_{2m}$ from (1.1.2).

Consider from (2.1.1),

$d(z_{2m}, z_{2m+1}) = d\big(\mathrm{F}(x_{2m}, y_{2m}), \mathrm{G}(x_{2m+1}, y_{2m+1})\big)$

$$\leq \varphi\left(max\begin{Bmatrix} d(z_{2m-1},z_{2m}), d(w_{2m-1},w_{2m}), \frac{1}{2s}d(z_{2m-1},z_{2m}), \\ \frac{1}{2s}d(w_{2m-1},w_{2m}), \frac{1}{2s}d(z_{2m},z_{2m+1}), \frac{1}{2s}d(w_{2m},w_{2m+1}), \\ \frac{1}{2s}d(z_{2m-1},z_{2m+1}), \frac{1}{2s}d(w_{2m-1},w_{2m+1}), \\ \frac{1}{2s}d(z_{2m},z_{2m}), \frac{1}{2s}d(w_{2m},w_{2m}) \end{Bmatrix}\right).$$

Using Note and (φ_1), we have

$$d(z_{2m},z_{2m+1}) \leq \varphi\left(max\begin{Bmatrix} d(z_{2m-1},z_{2m}), d(w_{2m-1},w_{2m}), d(z_{2m},z_{2m+1}), \\ d(w_{2m},w_{2m+1}), \max\{d(z_{2m-1},z_{2m}), d(z_{2m},z_{2m+1})\}, \\ \max\{d(w_{2m-1},w_{2m}), d(w_{2m},w_{2m+1})\}, \\ \max\{d(z_{2m},z_{2m-1}), d(z_{2m-1},z_{2m})\}, \\ \max\{d(w_{2m},w_{2m-1}), d(w_{2m-1},w_{2m})\} \end{Bmatrix}\right)$$

$$= \varphi\left(\max\begin{Bmatrix} d(z_{2m},z_{2m+1}), d(w_{2m},w_{2m+1}), d(z_{2m-1},z_{2m}), \\ d(z_{2m},z_{2m-1}), d(w_{2m-1},w_{2m}), d(w_{2m},w_{2m-1}) \end{Bmatrix}\right).$$

Also we have

$$d(z_{2m+1},z_{2m}) \leq \varphi\left(\max\begin{Bmatrix} d(z_{2m},z_{2m+1}), d(w_{2m},w_{2m+1}), d(z_{2m-1},z_{2m}), \\ d(z_{2m},z_{2m-1}), d(w_{2m-1},w_{2m}), d(w_{2m},w_{2m-1}) \end{Bmatrix}\right),$$

$$d(w_{2m},w_{2m+1}) \leq \varphi\left(\max\begin{Bmatrix} d(z_{2m},z_{2m+1}), d(w_{2m},w_{2m+1}), d(z_{2m-1},z_{2m}), \\ d(z_{2m},z_{2m-1}), d(w_{2m-1},w_{2m}), d(w_{2m},w_{2m-1}) \end{Bmatrix}\right),$$

$$d(w_{2m+1},w_{2m}) \leq \varphi\left(\max\begin{Bmatrix} d(z_{2m},z_{2m+1}), d(w_{2m},w_{2m+1}), d(z_{2m-1},z_{2m}), \\ d(z_{2m},z_{2m-1}), d(w_{2m-1},w_{2m}), d(w_{2m},w_{2m-1}) \end{Bmatrix}\right).$$

Using (φ_1), we have

$$max\begin{Bmatrix} d(z_{2m},z_{2m+1}), d(z_{2m+1},z_{2m}), \\ d(w_{2m},w_{2m+1}), d(w_{2m+1},w_{2m}) \end{Bmatrix} \leq \varphi\left(\max\begin{Bmatrix} d(z_{2m},z_{2m+1}), d(w_{2m},w_{2m+1}), \\ d(z_{2m+1},z_{2m}), d(w_{2m+1},w_{2m}) \\ d(z_{2m-1},z_{2m}), d(z_{2m},z_{2m-1}), \\ d(w_{2m-1},w_{2m}), d(w_{2m},w_{2m-1}) \end{Bmatrix}\right)$$

$$\leq \varphi\left(max\begin{Bmatrix} d(z_{2m},z_{2m+1}), d(z_{2m+1},z_{2m}), \\ d(w_{2m},w_{2m+1}), d(w_{2m+1},w_{2m}) \end{Bmatrix}\right). \dots\dots\dots\dots (2)$$

From (φ_3) it follows that

$$max\begin{Bmatrix} d(z_{2m},z_{2m+1}), d(z_{2m+1},z_{2m}), \\ d(w_{2m},w_{2m+1}), d(w_{2m+1},w_{2m}) \end{Bmatrix} = 0$$

which in turn yields from (1.1.2) that $z_{2m} = z_{2m+1}$ and $w_{2m} = w_{2m+1}$.

Continuing in this way, we get $z_{2m-1} = z_{2m} = z_{2m+1} = \cdots$ and $w_{2m-1} = w_{2m} = w_{2m+1} = \cdots$

Hence $\{z_n\}$ *and* $\{w_n\}$ are Cauchy sequences in X.

Case (ii): Suppose

$$\max\{d(z_n,z_{n+1}), d(z_{n+1},z_n), d(w_n,w_{n+1}), d(w_{n+1},w_n)\} > 0 \text{ for all n.}$$

As in (2), we have

$$max\begin{Bmatrix} d(z_{2n},z_{2n+1}), d(z_{2n+1},z_{2n}), \\ d(w_{2n},w_{2n+1}), d(w_{2n+1},w_{2n}) \end{Bmatrix} \leq \varphi\left(\max\begin{Bmatrix} d(z_{2n-1},z_{2n}), d(z_{2n},z_{2n-1}), \\ d(w_{2n-1},w_{2n}), d(w_{2n},w_{2n-1}), \\ d(z_{2n},z_{2n+1}), d(w_{2n},w_{2n+1}), \\ d(z_{2n+1},z_{2n}), d(w_{2n+1},w_{2n}) \end{Bmatrix}\right) \dots\dots\dots(3)$$

If $max\begin{Bmatrix} d(z_{2n-1},z_{2n}), d(z_{2n},z_{2n-1}), \\ d(w_{2n-1},w_{2n}), d(w_{2n},w_{2n-1}) \end{Bmatrix} \leq max\begin{Bmatrix} d(z_{2n},z_{2n+1}), d(z_{2n+1},z_{2n}), \\ d(w_{2n},w_{2n+1}), d(w_{2n+1},w_{2n}) \end{Bmatrix}$

then from (3) and (φ_2) , we get $max\begin{Bmatrix} d(z_{2n},z_{2n+1}), d(z_{2n+1},z_{2n}), \\ d(w_{2n},w_{2n+1}), d(w_{2n+1},w_{2n}) \end{Bmatrix} = 0.$

It is a contradiction to Case (ii).

Hence from (3) we have

$$max\begin{Bmatrix} d(z_{2n},z_{2n+1}), d(z_{2n+1},z_{2n}), \\ d(w_{2n},w_{2n+1}), d(w_{2n+1},w_{2n}) \end{Bmatrix} \le \varphi\left(max\begin{Bmatrix} d(z_{2n-1},z_{2n}), d(z_{2n},z_{2n-1}), \\ d(w_{2n-1},w_{2n}), d(w_{2n},w_{2n-1}) \end{Bmatrix}\right).$$

Continuing in this way, we have

$$max\begin{Bmatrix} d(z_{2n},z_{2n+1}), d(z_{2n+1},z_{2n}), \\ d(w_{2n},w_{2n+1}), d(w_{2n+1},w_{2n}) \end{Bmatrix} \le \varphi\left(max\begin{Bmatrix} d(z_{2n-1},z_{2n}), d(z_{2n},z_{2n-1}), \\ d(w_{2n-1},w_{2n}), d(w_{2n},w_{2n-1}) \end{Bmatrix}\right)$$

$$\ldots\ldots\ldots\ldots$$

$$\le \varphi^{\,n}\left(max\begin{Bmatrix} d(z_0,z_1), d(z_1,z_0), \\ d(w_0,w_1), d(w_1,w_0) \end{Bmatrix}\right) \ldots\ldots\ldots\ldots\ldots\ldots \text{(4)}$$

Now for all positive integers n and p and using (4), consider

$$d(z_n,z_{n+p}) \le s\, d(z_n,z_{n+1}) + s^2\, d(z_{n+1},z_{n+2}) + \cdots + s^p\, d(z_{n+p-1},z_{n+p})$$

$$\le s\,\varphi^{\,n}(t) + s^2\varphi^{\,n+1}(t) + \cdots + s^p\varphi^{\,n+p-1}(t),$$

where $t = max\begin{Bmatrix} d(z_0,z_1), d(z_1,z_0), \\ d(w_0,w_1), d(w_1,w_0) \end{Bmatrix}$

$$\le s^n\,\varphi^{\,n}(t) + s^{n+1}\varphi^{\,n+1}(t) + \cdots + s^{n+p-1}\varphi^{\,n+p-1}(t), \text{ since } s \ge 1$$

$$= \sum_{i=n}^{n+p-1} s^i\varphi^i(t) \le \sum_{i=n}^{\infty} s^i\varphi^i(t).$$

Since $\sum_{i=n}^{\infty} s^i\varphi^i(t)$ converges for all t > 0, its remainder after n terms tends to zero as $n \to \infty$.
Hence we have $\lim_{n\to\infty} d(z_n,z_{n+p}) = 0$. Also in the similar way, we have

$$d(z_{n+p},z_n) \le s\, d(z_{n+p},z_{n+1}) + s\, d(z_{n+1},z_n)$$

$$\le s^2 d(z_{n+p},z_{n+2}) + s^2\, d(z_{n+2},z_{n+1}) + s\, d(z_{n+1},z_n)$$

$$\le s^3 d(z_{n+p},z_{n+3}) + s^3\, d(z_{n+3},z_{n+2}) + s^2\, d(z_{n+2},z_{n+1}) + s\, d(z_{n+1},z_n)$$

$$\ldots\ldots\ldots$$

$$\le s^{p-1} d(z_{n+p},z_{n+p-1}) + s^{p-1}\, d(z_{n+p-1},z_{n+p-2}) + \cdots + s\, d(z_{n+1},z_n)$$

$$\le s^{p-1}\varphi^{\,n+p-1}(t) + s^{p-1}\,\varphi^{\,n+p-2}(t) + \cdots + s^2\,\varphi^{\,n+1}(t) + s\,\varphi^{\,n}(t),$$

where t is as in above

$$\le s^{n+p-1}\varphi^{\,n+p-1}(t) + s^{n+p-2}\,\varphi^{\,n+p-2}(t) + \cdots + s^n\,\varphi^{\,n}(t), \text{ since } s \ge 1$$

$$= \sum_{i=n}^{n+p-1} s^i\varphi^i(t) \le \sum_{i=n}^{\infty} s^i\varphi^i(t).$$

As in above we have $\lim_{n\to\infty} d(z_n,z_{n+p}) = 0$.

Similarly we can show that $\lim_{n\to\infty} d(w_n,w_{n+p}) = 0$ and $\lim_{n\to\infty} d(w_{n+p},w_n) = 0$.

Thus $\{z_n\}$ *and* $\{w_n\}$ are Cauchy sequences in X.

Since X is complete dislocated quasi b-metric space, there exist z, w $\epsilon\, X$ such that $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} w_n = w$.

Since SF = FS and S and F are continuous, we have

Sz = $\lim_{n\to\infty} S(z_{2n}) = \lim_{n\to\infty} S(F(x_{2n},y_{2n})) = \lim_{n\to\infty} F(Sx_{2n},Sy_{2n})$

$= \lim_{n\to\infty} F(z_{2n-1},w_{2n-1}) = F\left(\lim_{n\to\infty} z_{2n-1}, \lim_{n\to\infty} w_{2n-1}\right) = F(z,w).$

Similarly, we have Sw = F(w,z).

Since TG = GT and T and G are continuous, we have

Tz = $\lim_{n\to\infty} T(z_{2n+1}) = \lim_{n\to\infty} T(G(x_{2n+1},y_{2n+1})) = \lim_{n\to\infty} G(Tx_{2n+1},Ty_{2n+1})$

$= \lim_{n\to\infty} G(z_{2n},w_{2n}) = \lim_{n\to\infty} G\left(\lim_{n\to\infty} z_{2n}, \lim_{n\to\infty} w_{2n}\right) = G(z,w).$

Similarly, we have Tw = G (w, z).

Using Note , (φ_1) and from (2.1.1), we have

$$d(Sz,Tz) = d\big(F(z,w),G(z,w)\big)$$
$$\leq \varphi\left(max\left\{ \begin{matrix} d(Sz,Tz), d(Sw,Tw), \frac{1}{2s}d(Sz,Sz), \frac{1}{2s}d(Sw,Sw), \frac{1}{2s}d(Tz,Tz), \\ \frac{1}{2s}d(Tw,Tw), \frac{1}{2s}d(Sz,Tz), \frac{1}{2s}d(Sw,Tw), \frac{1}{2s}d(Tz,Sz), \frac{1}{2s}d(Tw,Sw) \end{matrix} \right\}\right)$$
$$\leq \varphi(max\{d(Sz,Tz), d(Sw,Tw), d(Tz,Sz), d(Tw,Sw)\}).$$

Similarly we can show that

$d(Tz,Sz) \leq \varphi(max\{d(Sz,Tz), d(Sw,Tw), d(Tz,Sz), d(Tw,Sw)\})$,

$d(Sw,Tw) \leq \varphi(max\{d(Sz,Tz), d(Sw,Tw), d(Tz,Sz), d(Tw,Sw)\})$ and

$d(Tw,Sw) \leq \varphi(max\{d(Sz,Tz), d(Sw,Tw), d(Tz,Sz), d(Tw,Sw)\})$.

Thus

$$max\left\{\begin{matrix} d(Sz,Tz), d(Sw,Tw), \\ d(Tz,Sz), d(Tw,Sw) \end{matrix}\right\} \leq \varphi\left(max\left\{\begin{matrix} d(Sz,Tz), d(Sw,Tw), \\ d(Tz,Sz), d(Tw,Sw) \end{matrix}\right\}\right)$$

which in turn yields from (φ_3) and (1.1.2) that Sz = Tz and Sw = Tw.

Let $\alpha = Sz = Tz$ and $\beta = Sw = Tw$.

Since SF =FS and GT = TG, we have

S$\alpha = S^2z = S\big(F(z,w)\big) = F(Sz,Sw) = F(\alpha,\beta)$,

S$\beta = S^2w = S\big(F(w,z)\big) = F(Sw,Sz) = F(\beta,\alpha)$,

T$\alpha = T^2z = T\big(G(z,w)\big) = G(Tz,Tw) = G(\alpha,\beta)$ and

$T\beta = T^2w = T\big(G(w,z)\big) = G(Tw,Tz) = G(\beta,\alpha)$.

Using Note, (φ_1) and from (2.1.1), we have

$$d(S\alpha,\alpha) = d(F(\alpha,\beta),Tz) = d\big(F(\alpha,\beta),G(z,w)\big)$$
$$\leq \varphi\left(max\left\{ \begin{matrix} d(S\alpha,\alpha), d(S\beta,\beta), \frac{1}{2s}d(S\alpha,S\alpha), \frac{1}{2s}d(S\beta,S\beta), \frac{1}{2s}d(\alpha,\alpha), \\ \frac{1}{2s}d(\beta,\beta), \frac{1}{2s}d(S\alpha,\alpha), \frac{1}{2s}d(S\beta,\beta), \frac{1}{2s}d(\alpha,S\alpha), \frac{1}{2s}d(\beta,S\beta), \end{matrix} \right\}\right)$$
$$\leq \varphi(max\{d(S\alpha,\alpha), d(\alpha,S\alpha), d(\beta,S\beta), d(S\beta,\beta)\}).$$

Similarly, we can show that

$d(\alpha,S\alpha) \leq \varphi(max\{d(S\alpha,\alpha), d(\alpha,S\alpha), d(\beta,S\beta), d(S\beta,\beta)\})$,

$d(S\beta,\beta) \leq \varphi(max\{d(S\alpha,\alpha), d(\alpha,S\alpha), d(\beta,S\beta), d(S\beta,\beta)\})$ and

$d(\beta,S\beta) \leq \varphi(max\{d(S\alpha,\alpha), d(\alpha,S\alpha), d(\beta,S\beta), d(S\beta,\beta)\})$.

Thus we obtain

$$max\left\{\begin{matrix} d(S\alpha,\alpha), d(\alpha,S\alpha), \\ d(\beta,S\beta), d(S\beta,\beta) \end{matrix}\right\} \leq \varphi\left(max\left\{\begin{matrix} d(S\alpha,\alpha), d(\alpha,S\alpha), \\ d(\beta,S\beta), d(S\beta,\beta) \end{matrix}\right\}\right)$$

which in turn yields from (φ_3) and (1.1.2) that $S\alpha = \alpha$ and $S\beta = \beta$.

Similarly we can show that T$\alpha = \alpha$ and T$\beta = \beta$.

Thus $F(\alpha,\beta) = S\alpha = \alpha = T\alpha = G(\alpha,\beta)$ and $F(\beta,\alpha) = S\beta = \beta = T\beta = G(\beta,\alpha)$.

Hence (α,β) is a common coupled fixed point of F, G, S and T.

Let (p,q) be another common coupled fixed point of F, G, S and T.

Then $F(p,q) = Sp = p = Tp = G(p,q)$ and $F(q,p) = Sq = q = Tq = G(q,p)$.

Using Note, (φ_1) and from (2.1.1), we obtain

$d(\alpha,p) = d\big(F(\alpha,\beta),G(p,q)\big)$

$$\leq \varphi\left(max \begin{Bmatrix} d(\alpha,p), d(\beta,q), \frac{1}{2s} d(\alpha,\alpha), \frac{1}{2s} d(\beta,\beta), \frac{1}{2s} d(p,p), \\ \frac{1}{2s} d(q,q), \frac{1}{2s} d(\alpha,p), \frac{1}{2s} d(\beta,q), \frac{1}{2s} d(p,\alpha), \frac{1}{2s} d(q,\beta) \end{Bmatrix} \right)$$

$$\leq \varphi(max\{d(\alpha,p), d(p,\alpha), d(\beta,q), d(q,\beta)\}).$$

In the similar way we have

$d(p,\alpha) \leq \varphi(max\{d(\alpha,p), d(p,\alpha), d(\beta,q), d(q,\beta)\})$,

$d(\beta,q) \leq \varphi(max\{d(\alpha,p), d(p,\alpha), d(\beta,q), d(q,\beta)\})$ and

$d(q,\beta) \leq \varphi(max\{d(\alpha,p), d(p,\alpha), d(\beta,q), d(q,\beta)\})$.

Thus $max \begin{Bmatrix} d(\alpha,p), d(p,\alpha), \\ d(\beta,q), d(q,\beta) \end{Bmatrix} \leq \varphi\left(max \begin{Bmatrix} d(\alpha,p), d(p,\alpha), \\ d(\beta,q), d(q,\beta) \end{Bmatrix} \right)$

which in turn yields from (φ_3) and (1.1.2) that $p = \alpha$ and $q = \beta$.

Thus (α, β) is the unique common coupled fixed point of F, G, S and T.

Example 2.2: Let X= [0, 1] and $d(x,y) = |x-y|^2 + |x|$.

Let $F, G: X \times X \to X$ and $S, T: X \to X$ be defined by

$F(x,y) = \frac{x+y}{16}, Sx = \frac{x}{2}, G(x,y) = \frac{x+y}{24}$ and $Tx = \frac{x}{3}$., let $\varphi\ (t) = {}^{t}/_{4}$ for all t ε [0,∞)

Clearly $d(x,y) = d(y,x) = 0 \Rightarrow x = y$. Now consider

$$\begin{aligned} d(x,y) = |x-y|^2 + |x| &= |x-z+z-y|^2 + |x| \\ &\leq 2[|x-z|^2 + |z-y|^2] + |x| \\ &\leq 2[|x-z|^2 + |x| + |z-y|^2 + |z|] \\ &= s[d(x,z) + d(z,y)], \text{ where s=2.} \end{aligned}$$

Consider

$$\begin{aligned} d\big(F(x,y), G(u,v)\big) &= d\left(\frac{x+y}{64}, \frac{u+v}{96}\right) = \left|\frac{x+y}{16} - \frac{u+v}{24}\right|^2 + \left|\frac{x+y}{16}\right| \\ &= \left|\frac{3x-2u+3y-2v}{48}\right|^2 + \frac{x}{16} + \frac{y}{16} \\ &= \frac{1}{64}\left[\left|\frac{x}{2} - \frac{u}{3} + \frac{y}{2} - \frac{v}{3}\right|^2\right] + \frac{x}{16} + \frac{y}{16} \\ &\leq \frac{1}{32}\left[\left|\frac{x}{2} - \frac{u}{3}\right|^2 + \left|\frac{y}{2} - \frac{v}{3}\right|^2\right] + \frac{x}{16} + \frac{y}{16} \\ &= \frac{1}{8}\left[\frac{1}{4}\left|\frac{x}{2} - \frac{u}{3}\right|^2 + \frac{1}{4}\left|\frac{y}{2} - \frac{v}{3}\right|^2 + \frac{x}{2} + \frac{y}{2}\right] \\ &\leq \frac{1}{8}\left[\left|\frac{x}{2} - \frac{u}{3}\right|^2 + \left|\frac{y}{2} - \frac{v}{3}\right|^2 + \frac{x}{2} + \frac{y}{2}\right] \\ &= \frac{1}{8}[d(Sx,Tu) + d(Sy,Tv)] \\ &\leq \frac{1}{4}\max\{d(Sx,Tu), d(Sy,Tv)\} \end{aligned}$$

$$\leq \varphi\left(max \begin{Bmatrix} d(Sx,Tu), d(Sy,Tv), \\ \frac{1}{2s} d\big(Sx, F(x,y)\big), \frac{1}{2s} d\big(Sy, F(y,x)\big), \\ \frac{1}{2s} d\big(Tu, G(u,v)\big), \frac{1}{2s} d\big(Tv, G(v,u)\big), \\ \frac{1}{2s} d\big(Sx, G(u,v)\big), \frac{1}{2s} d\big(Sy, G(v,u)\big), \\ \frac{1}{2s} d\big(Tu, F(x,y)\big), \frac{1}{2s} d\big(Tv, F(y,x)\big) \end{Bmatrix} \right)$$

Thus (2.1.1) is satisfied. Similarly we can verify (2.1.2). Also it is clear that F, G,S and T are continuous, FS = SF ,GT = TG and $F(X \times X) \subseteq T(X), G(X \times X) \subseteq S(X)$. Thus all conditions of Theorem 2.1 are satisfied. Clearly (0,0) is the unique common coupled fixed point of F,G,S and T in $X \times X$.

One can prove the following theorems in the similar lines of Theorem 2.1.

Theorem 2.3 Assume all conditions of Theorem 2.1 except (2.1.1),(2.1.2) and further assume

(2.3.1) $d\big(F(x,y),G(u,v)\big) \le \varphi\left(max\left\{\begin{matrix} d(Sx,Tu), d(Sy,Tv), \\ d\big(Sx,F(x,y)\big), d\big(Sy,F(y,x)\big), \\ d\big(Tu,G(u,v)\big), d\big(Tv,G(v,u)\big)\end{matrix}\right\}\right)$

for all x,y,u,v ∈ X,Where $\varphi \in \mathbf{\Phi}_s$,

(2.3.2) $d\big(G(x,y),F(u,v)\big) \le \varphi\left(max\left\{\begin{matrix} d(Tx,Su), d(Ty,Sv), \\ d\big(Tx,G(x,y)\big), d\big(Ty,G(y,x)\big), \\ d\big(Su,F(u,v)\big), d\big(Sv,F(v,u)\big)\end{matrix}\right\}\right)$

for all x,y,u,v ∈ X, where $\varphi \in \mathbf{\Phi}_s$,

Then the pairs (F,S) and (G,T) have a common coupled coincidence point , i.e, there exist u,v ∈ X such that F (u,v) = Su , F(v,u) = Sv ,G (u,v) = Tu, and G(v,u) = Tv.

Theorem 2.4 Let (X,d) be a complete dislocated quasi b-metric space with fixed integer $s \ge 1$ and $f, g: X \to X$ be continuous mappings satisfying

(2.4.1) $d(fx,gy) \le \varphi\left(max\left\{d(x,y), d(x,fx), d(y,gy), \frac{1}{2s}d(x,gy), \frac{1}{2s}d(y,fx)\right\}\right)$

for all x,y ∈ X, where $\varphi \in \mathbf{\Phi}_s$ and

(2.4.2) $d(gx,fy) \le \varphi\left(max\left\{d(x,y), d(x,gx), d(y,fy), \frac{1}{2s}d(x,fy), \frac{1}{2s}d(y,gx)\right\}\right)$

for all x,y ∈ X, where $\varphi \in \mathbf{\Phi}_s$.

Then f and g have a unique common fixed point in X.

Proof: As in Theorem 2.1, the sequence $\{x_n\}$ defined by

$x_{2n+1} = fx_{2n}$, $x_{2n+2} = gx_{2n+1}$, $n = 0,1,2, \ldots$ converges to z ∈ X and fz = z = gz.

Now from (2.4.1), we have

$$\begin{aligned} d(z,z) = d(fz,gz) &\le \varphi\left(max\left\{d(z,z), d(z,z), d(z,z), \frac{1}{2s}d(z,z), \frac{1}{2s}d(z,z)\right\}\right) \\ &= \varphi\big(d(z,z)\big) \end{aligned}$$

which in turn yields from (φ_3) that $d(z,z) = 0$.

Thus $d(z,z) = 0$ if z is a common fixed point of f and g.

Suppose w is another common fixed point of f and g.

Then $d(w,w) = 0$.

Now from (2.4.1) we have

$$\begin{aligned} d(z,w) = d(fz,gw) &\le \varphi\left(max\left\{d(z,w), d(z,z), d(w,w), \frac{1}{2s}d(z,w), \frac{1}{2s}d(w,z)\right\}\right) \\ &\le \varphi(max\{d(z,w), d(w,z)\}). \end{aligned}$$

Similarly from (2.4.2), we have

$d(w,z) \le \varphi(max\{d(z,w), d(w,z)\})$.

Thus

$max\{d(z,w), d(w,z)\} \leq \varphi(max\{d(z,w), d(w,z)\})$

which in turn yields from (φ_3) and (1.1.2) that w = z.

Thus z is the unique common fixed point of f and g.

Corollary 2.5 Let (X,d) be a complete dislocated quasi b-metric space with fixed integer $s \geq 1$ and $f: X \to X$ be continuous mapping satisfying

(2.5.1) $d(fx, fy) \leq \varphi\left(max\left\{d(x,y), d(x,fx), d(y,fy), \frac{1}{2s}d(x,fy), \frac{1}{2s}d(y,fx)\right\}\right)$

for all x,y ∈ X,Where $\varphi \in \mathbf{\Phi}_s$.

Then f has a unique fixed point in X.

Theorem 2.6 Let (X,d) be a complete dislocated quasi b-metric space with fixed integer $s \geq 1$ and $f, g: X \to X$ be continuous mappings satisfying

(2.6.1) $d(fx, fy) \leq \varphi\left(max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{1}{2s}d(gx,fy), \frac{1}{2s}d(gy,fx)\right\}\right)$

for all x,y ∈ X,Where $\varphi \in \mathbf{\Phi}_s$,

(2.6.2) $f(X) \subseteq g(X)$ and $fg = gf$.

Then f and g have a unique common fixed point in X.

Proof. As in proof of Theorem 2.1, the sequence $\{gx_n\}$ defined by

$fx_n = gx_{n+1}$, $n = 0,1,2,\ldots$ converges to z ∈ X and fz = gz.

Thus fz = gz is a point of coincidence of f and g.

Also we have

$$d(gz,gz) = d(fz,fz) \leq \varphi\left(max\left\{\begin{matrix} d(gz,gz), d(gz,fz), d(gz,fz), \\ \dfrac{1}{2s}d(gz,fz), \dfrac{1}{2s}d(gz,fz) \end{matrix}\right\}\right)$$
$$= \varphi\big(d(gz,gz)\big)$$

which in turn yields from (φ_3) that $d(gz,gz) = 0$.

Thus $d(gz,gz) = 0$ whenever gz = fz.

Suppose that there exists a point w ∈ X such that fw = gw.

Then $d(fw,fw) = 0$ or $d(gw,gw) = 0$.

Now from (2.6.1),we have

$$d(gz,gw) = d(fz,fw) \leq \varphi\left(max\left\{\begin{matrix} d(gz,gw), d(gz,fz)d(gw,fw), \\ \frac{1}{2s}d(gz,fw), \frac{1}{2s}d(gw,fz) \end{matrix}\right\}\right)$$
$$= \varphi(max\{d(gz,fw), d(gw,fz)\}).$$

Similarly we can show that

$d(gw,gz) \leq \varphi(max\{d(gz,fw), d(gw,fz)\})$.

Thus we have

$max\{d(gz,gw), d(gw,gz)\} \leq \varphi(max\{d(gz,gw), d(gw,gz)\})$

which in turn yields from (φ_3) and (1.1.2) that gz = gw.

Thus the point of coincidence f and g is unique.

Let $\alpha = fz = gz$.

Since fg = gf ,we have $f\alpha = fgz = gfz = g\alpha$.

Thus $f\alpha$ is a the point of coincidence f and g. Since $\alpha = fz$ is the only point of coincidence f and g, it follows that $f\alpha = \alpha$.

Thus α is a common fixed point of f and g.

If β is another common fixed point of f and g then $\beta = f\beta$ is a point of coincidence f and g.

Hence $\beta = \alpha$. Thus α is the unique common fixed point of f and g.

Remark 2.7 (i) Theorem 2.4 is an improvements of Theorem 3.4 of [18] and Theorem 3.2 of [19].

(ii) Corollary 2.5 is an improvements of Theorem 4.1 of [16] ,Theorem 3.1 of [14] and Theorem 2.5 of [1].

Now replacing the completeness of X, continuities of F,G,S and T and commutativity of pairs (F,S) and (G,T) by w-compatible pairs (F,S) and (G,T) and completeness of one of S(X) and T(X) ,we prove a unique common coupled fixed point theorem. In fact, we prove the following Theorem.

Theorem 2.8: Let (X,d) be a complete dislocated quasi b-metric space with fixed integer $s \geq 1$ and $F, G: X \times X \to X$ and $S, T: X \to X$ be mappings satisfying

(2.8.1) $$d\big(F(x,y), G(u,v)\big) \leq \varphi\left(\frac{1}{2s^2} max \begin{Bmatrix} d(Sx,Tu), d(Sy,Tv), d\big(Sx, F(x,y)\big), \\ d\big(Sy, F(y,x)\big), d\big(Tu, G(u,v)\big), d\big(Tv, G(v,u)\big), \\ d\big(Sx, G(u,v)\big), d\big(Sy, G(v,u)\big), \\ d\big(Tu, F(x,y)\big), d\big(Tv, F(y,x)\big) \end{Bmatrix}\right)$$

for all $x, y, u, v \in X, where\ \varphi \in \mathbf{\Phi}_s$ and φ is continuous,

(2.8.2) $$d\big(G(x,y), F(u,v)\big) \leq \varphi\left(\frac{1}{2s^2} max \begin{Bmatrix} d(Tx,Su), d(Ty,Sv), d\big(Tx, G(x,y)\big), \\ d\big(Ty, G(y,x)\big), d\big(Su, F(u,v)\big), \\ d\big(Sv, F(v,u)\big), d\big(Tx, F(u,v)\big), d\big(Ty, F(v,u)\big), \\ d\big(Su, G(x,y)\big), d\big(Sv, G(y,x)\big) \end{Bmatrix}\right)$$

for all $x, y, u, v \in X, where\ \varphi \in \mathbf{\Phi}_s$ and φ is continuous,

(2.8.3) $F(X \times X) \subseteq T(X)\ and\ G(X \times X) \subseteq S(X)$,

(2.8.4) One of T(X) and S(X) is a complete subspace of X and

(2.8.5) the pairs (F, S) and (G, T) are w-compatible.

Then F, G, S and T have a unique common coupled fixed point in $X \times X$.

Proof. As in proof of Theorem 2.1, the sequences $\{z_n\}\ and\ \{w_n\}$ are Cauchy,

where $z_{2n} = \mathrm{F}(x_{2n}, y_{2n}) = Tx_{2n+1}$,

$w_{2n} = \mathrm{F}(y_{2n}, x_{2n}) = Ty_{2n+1}$,

$z_{2n+1} = \mathrm{G}(x_{2n+1}, y_{2n+1}) = Sx_{2n+2}$,

$w_{2n+1} = \mathrm{G}(y_{2n+1}, x_{2n+1}) = Sy_{2n+2}$, $n = 0,1,2, ...$

Without loss of generality assume that S(X) is a complete subspace of X.

Since $z_{2n+1} = Sx_{2n+2} \in S(X)$, and $w_{2n+1} = Sy_{2n+2} \in S(X)$, there exist z, w,u and v in X such that $z_{2n+1} \to z = Su$ and $w_{2n+1} \to w = Sv$.

By Lemma 1.3, (2.8.1), (φ_1) and continuity of φ,we have

$$\frac{1}{s} d(F(u,v), z) \leq \lim_{n\to\infty} \inf d\big(F(u,v), G(x_{2n+1}, y_{2n+1})\big)$$

$$\leq \lim_{n\to\infty} \inf \varphi\left(\frac{1}{2s^2} max\left\{\begin{array}{c} d(z,z_{2n}), d(w,w_{2n}), d\big(z,F(u,v)\big), \\ d\big(w,F(v,u)\big), d(z_{2n},z_{2n+1}) \\ d(w_{2n},w_{2n+1}), d(z,z_{2n+1}), \\ d(w,w_{2n+1}), d\big(z_{2n},F(u,v)\big), d\big(w_{2n},F(v,u)\big) \end{array}\right\}\right)$$

$$\leq \lim_{n\to\infty} \inf \varphi\left(\frac{1}{2s^2} max\left\{\begin{array}{c} d(z,z_{2n}), d(w,w_{2n}), d\big(z,F(u,v)\big), \\ d\big(w,F(v,u)\big), s[d(z_{2n},z)+d(z,z_{2n+1})], \\ s[d(w_{2n},w)+d(w,w_{2n+1})], d(z,z_{2n+1}), \\ d(w,w_{2n+1}), d\big(z_{2n},F(u,v)\big), d\big(w_{2n},F(v,u)\big) \end{array}\right\}\right)$$

$$\leq \varphi\left(\frac{1}{2s^2}\ max\left\{\begin{array}{c} 0,0,d\big(z,F(u,v)\big), d\big(w,F(v,u)\big), 0,0, \\ 0,0,s\ d\big(z,F(u,v)\big)\ , s\ d\big(w,F(v,u)\big) \end{array}\right\}\right)$$

$$\leq \varphi\left(\frac{1}{s}\max\{d\big(z,F(u,v)\big), d\big(w,F(v,u)\big)\}\right).$$

Similarly, we can show that

$\frac{1}{s}d\big(z,F(u,v)\big) \leq \varphi\left(\frac{1}{s}\max\{d\big(z,F(u,v)\big), d\big(w,F(v,u)\big)\}\right)$,

$\frac{1}{s}d(F(v,u),w) \leq \varphi\left(\frac{1}{s}\max\{d\big(z,F(u,v)\big), d\big(w,F(v,u)\big)\}\right)$ and

$\frac{1}{s}\ d\big(w,F(v,u)\big) \leq \varphi\left(\frac{1}{s}\max\{d\big(z,F(u,v)\big), d\big(w,F(v,u)\big)\}\right)$.

From (φ_1), we have

$$\frac{1}{s}\max\left\{\begin{array}{c} d(F(u,v),z), d\big(z,F(u,v)\big), \\ d(F(v,u),w), d\big(w,F(v,u)\big) \end{array}\right\} \leq\ \varphi\ \left(\frac{1}{s}\max\left\{\begin{array}{c} d(F(u,v),z), d\big(z,F(u,v)\big), \\ d(F(v,u),w), d\big(w,F(v,u)\big) \end{array}\right\}\right)$$

which in turn yields from (φ_3) and (1.1.2) that F(u,v) = z and F(v,u) = w.

Thus Su = F (u,v) = z and Sv = F (v,u) = w.

Since the pair (F,S) is w-compatible , we have

Sz = S(F(u,v)) = F(Su,Sv) = F(z,w) and Sw = F(w,z).

Now from Lemma 1.3, (2.8.1), (φ_1) and continuity of φ ,we have

$$\frac{1}{s}d(Sz,z) \leq \lim_{n\to\infty} \inf d\big(F(z,w), G(x_{2n+1},y_{2n+1})\big)$$

$$\leq \lim_{n\to\infty} \inf \varphi\left(\frac{1}{2s^2} max\left\{\begin{array}{c} d(Sz,z_{2n}), d(Sw,w_{2n}), d(Sz,Sz), \\ d(Sw,Sw), d(z_{2n},z_{2n+1}), \\ d(Tw_{2n},w_{2n+1}), d(Sz,z_{2n+1}), \\ d(Sw,w_{2n+1}), d(z_{2n},Sz), d(w_{2n},Sw) \end{array}\right\}\right)$$

$$\leq \lim_{n\to\infty} \inf \varphi\left(\frac{1}{2s^2} max\left\{\begin{array}{c} s\ d(Sz,z), s\ d(Sw,w), \\ 2s\max\{d(Sz,z), d(z,Sz)\}, \\ 2s\max\{\ d(Sw,w), d(w,Sw)\},0,0, \\ s\ d(Sz,z), s\ d(Sw,w), \\ s\ d(z,Sz), s\ d(w,Sw) \end{array}\right\}\right)$$

$$\leq\ \varphi\left(\frac{1}{s}\max\{d(Sz,z), d(z,Sz), d(w,Sw), d(Sw,w)\}\right)$$

Similarly we can prove that

$\frac{1}{s}d(z,Sz) \leq\ \varphi\left(\frac{1}{s}\max\{d(Sz,z), d(z,Sz), d(w,Sw), d(Sw,w)\}\right)$,

$\frac{1}{s}d(Sw,w) \leq\ \varphi\left(\frac{1}{s}\max\{d(Sz,z), d(z,Sz), d(w,Sw), d(Sw,w)\}\right)$ and

$\frac{1}{s}d(w,Sw) \leq\ \varphi\left(\frac{1}{s}\max\{d(Sz,z), d(z,Sz), d(w,Sw), d(Sw,w)\}\right)$.

Hence, we have

$$\frac{1}{s}\max\left\{\begin{array}{c} d(Sz,z), d(z,Sz), \\ d(w,Sw), d(Sw,w) \end{array}\right\} \leq\ \varphi\left(\frac{1}{s}\max\left\{\begin{array}{c} d(Sz,z), d(z,Sz), \\ d(w,Sw), d(Sw,w) \end{array}\right\}\right)$$

which in turn yields (φ_3) and (1.1.2) that Sz = z and Sw = w.

Thus F(z,w) = Sz = z (5)

and F(w,z) = Sw = w (6)

Since $F(X \times X) \subseteq T(X)$ there exist $\alpha, \beta \in X$ such that

$T\alpha = F(z,w) = Sz = z$ and $T\beta = F(w,z) = Sw = w$.

Now using (2.8.1), (φ_1) and $s \geq 1$, we have

$$\begin{aligned} d\big(T\alpha, G(\alpha,\beta)\big) &\leq d(F(z,w), G(\alpha,\beta)) \\ &\leq \varphi\left(\frac{1}{2s^2} max\left\{\begin{matrix} d(T\alpha,T\alpha), d(T\beta,T\beta), d(T\alpha,T\alpha), \\ d(T\beta,T\beta), d(T\alpha, G(\alpha,\beta)), \\ d(T\beta, G(\beta,\alpha)), d(T\alpha, G(\alpha,\beta)), \\ d(T\beta, G(\beta,\alpha)), d(T\alpha,T\alpha), d(T\beta,T\beta) \end{matrix}\right\}\right) \\ &\leq \varphi\left(\frac{1}{2s^2} \max\left\{\begin{matrix} 2s\max\{ d\big(T\alpha, G(\alpha,\beta)\big), d(G(\alpha,\beta),T\alpha)\}, \\ 2s\max\{ d\big(T\beta, G(\beta,\alpha)\big), d(G(\beta,\alpha),T\beta)\}, \\ d(T\alpha, G(\alpha,\beta)), d(G(\beta,\alpha),T\beta) \end{matrix}\right\}\right) \\ &\leq \varphi\left(\max\left\{\begin{matrix} d\big(T\alpha, G(\alpha,\beta)\big), d(G(\alpha,\beta),T\alpha), \\ d\big(T\beta, G(\beta,\alpha)\big), d(G(\beta,\alpha),T\beta) \end{matrix}\right\}\right). \end{aligned}$$

Similarly we have

$$d(G(\alpha,\beta),T\alpha) \leq \varphi\left(\max\left\{\begin{matrix} d\big(T\alpha, G(\alpha,\beta)\big), d(G(\alpha,\beta),T\alpha), \\ d\big(T\beta, G(\beta,\alpha)\big), d(G(\beta,\alpha),T\beta) \end{matrix}\right\}\right),$$

$$d\big(T\beta, G(\beta,\alpha)\big) \leq \varphi\left(\max\left\{\begin{matrix} d\big(T\alpha, G(\alpha,\beta)\big), d(G(\alpha,\beta),T\alpha), \\ d\big(T\beta, G(\beta,\alpha)\big), d(G(\beta,\alpha),T\beta) \end{matrix}\right\}\right) \text{ and}$$

$$d(G(\beta,\alpha),T\beta) \leq \varphi\left(\max\left\{\begin{matrix} d\big(T\alpha, G(\alpha,\beta)\big), d(G(\alpha,\beta),T\alpha), \\ d\big(T\beta, G(\beta,\alpha)\big), d(G(\beta,\alpha),T\beta) \end{matrix}\right\}\right).$$

Thus we have

$$\max\left\{\begin{matrix} d\big(T\alpha, G(\alpha,\beta)\big), d(G(\alpha,\beta),T\alpha), \\ d\big(T\beta, G(\beta,\alpha)\big), d(G(\beta,\alpha),T\beta) \end{matrix}\right\} \leq \varphi\left(\max\left\{\begin{matrix} d\big(T\alpha, G(\alpha,\beta)\big), d(G(\alpha,\beta),T\alpha), \\ d\big(T\beta, G(\beta,\alpha)\big), d(G(\beta,\alpha),T\beta) \end{matrix}\right\}\right)$$

which in turn yields from (φ_3) and (1.1.2) that $G(\alpha,\beta) = T\alpha$ and $G(\beta,\alpha) = T\beta$.

Thus $T\alpha = G(\alpha,\beta) = z$ and $T\beta = G(\beta,\alpha) = w$.

Since the pair (G,T) is w-compatible, we have

$Tz = T\big(G(\alpha,\beta)\big) = G(T\alpha,T\beta) = G(z,w)$ and Tw = G(w,z).

Now using (2.8.1),(5),(6), (φ_1) and $s \geq 1$, we have

$$\begin{aligned} d(z.Tz) &\leq d\big(F(z,w), G(z,w)\big) \\ &\leq \varphi\left(\frac{1}{2s^2}\max\left\{\begin{matrix} d(z,Tz), d(w,Tw), d(z,z), d(w,w), d(Tz,Tz), \\ d(Tw,Tw), d(z,Tz), d(w,Tw), d(Tz,z), d(Tw,w) \end{matrix}\right\}\right) \\ &\leq \varphi\left(\frac{1}{2s^2}\max\left\{\begin{matrix} d(z,Tz), d(w,Tw), 2s\max\{d(z,Tz), d(Tz,z)\}, \\ 2s\max\{d(w,Tw), d(Tw,w)\}, \\ 2s\max\{d(Tz,z), d(z,Tz)\}, \\ 2s\max\{d(w,Tw), d(Tw,w)\}, d(Tz,z), d(Tw,w) \end{matrix}\right\}\right) \\ &\leq \varphi(max\{d(z,Tz), d(Tz,z), d(w,Tw), d(Tw,w)\}) \end{aligned}$$

Similarly we have

$d(Tz,z) \leq \varphi(max\{d(z,Tz), d(Tz,z), d(w,Tw), d(Tw,w)\})$,

$d(w,Tw) \leq \varphi(max\{d(z,Tz), d(Tz,z), d(w,Tw), d(Tw,w)\})$ and

$d(Tw,w) \leq \varphi(max\{d(z,Tz),d(Tz,z),d(w,Tw),d(Tw,w)\})$.

Thus we have

$$max\begin{Bmatrix} d(z,Tz),d(Tz,z), \\ d(w,Tw),d(Tw,w) \end{Bmatrix} \leq \varphi\left(max\begin{Bmatrix} d(z,Tz),d(Tz,z), \\ d(w,Tw),d(Tw,w) \end{Bmatrix}\right)$$

which in turn yields from (φ_3) and (1.1.2) that Tz = z and Tw = w.

Thus $$G(z,w) = Tz = z \tag{7}$$
$$G(w,z) = Tw = w. \tag{8}$$

From (5),(6),(7) and (8), it follows that (z,w) is a common coupled fixed point of F,G,S and T.

Uniqueness of common coupled fixed point of F,G,S and T follows as in Theorem 2.1.

Now we give an example to illustrate Theorem 2.8.

Example 2.9 Let X= [0, 1] and $d(x,y) = |x-y|^2 + |x|$ and $F,G: X \times X \to X$ and $S,T: X \to X$ be defined by $F(x,y) = \frac{x^2+y^2}{128}$, $G(x,y) = \frac{x^2+y^2}{256}$, $Sx = \frac{x^2}{2}$ and $Tx = \frac{x^2}{4}$. Let $\varphi : [0,\infty) \to [0,\infty)$ be defined by $\varphi(t) = \frac{t}{4}$. As in Example 2.2 , d is a dislocated quasi b-metric with s = 2.

$$\begin{aligned}
d\big(F(x,y),G(u,v)\big) &= \left|\frac{x^2+y^2}{128} - \frac{u^2+v^2}{256}\right|^2 + \left|\frac{x^2+y^2}{128}\right| \\
&= \left|\frac{(2x^2-u^2)+(2y^2-v^2)}{256}\right|^2 + \frac{x^2+y^2}{128} \\
&= \left|\frac{\left(\frac{x^2}{2}-\frac{u^2}{4}\right)+\left(\frac{y^2}{2}-\frac{v^2}{4}\right)}{64}\right|^2 + \frac{x^2+y^2}{128} \\
&\leq \frac{2}{(64)^2}\left[\left|\left(\frac{x^2}{2}-\frac{u^2}{4}\right)\right|^2 + \left|\left(\frac{y^2}{2}-\frac{v^2}{4}\right)\right|^2\right] + \frac{x^2+y^2}{128} \\
&= \frac{1}{64}\left[\frac{1}{32}\left|\left(\frac{x^2}{2}-\frac{u^2}{4}\right)\right|^2 + \frac{x^2}{2} + \frac{1}{32}\left|\left(\frac{y^2}{2}-\frac{v^2}{4}\right)\right|^2 + \frac{y^2}{2}\right] \\
&\leq \frac{1}{64}\left[\left|\left(\frac{x^2}{2}-\frac{u^2}{4}\right)\right|^2 + \frac{x^2}{2} + \left|\left(\frac{y^2}{2}-\frac{v^2}{4}\right)\right|^2 + \frac{y^2}{2}\right] \\
&= \frac{1}{64}[d(Sx,Tu) + d(Sy,Tv)] \\
&\leq \frac{1}{32}\max\{d(Sx,Tu),d(Sy,Tv)\} \\
&\leq \frac{1}{4}\,\frac{1}{2s^2}\, max\begin{Bmatrix} d(Sx,Tu),d(Sy,Tv),d\big(Sx,F(x,y)\big), \\ d\big(Sy,F(y,x)\big),d\big(Tu,G(u,v)\big),d\big(Tv,G(v,u)\big), \\ d\big(Sx,G(u,v)\big),d\big(Sy,G(v,u)\big), \\ d\big(Tu,F(x,y)\big),d\big(Tv,F(y,x)\big) \end{Bmatrix} \\
&= \varphi\left(\frac{1}{2s^2}max\begin{Bmatrix} d(Sx,Tu),d(Sy,Tv),d\big(Sx,F(x,y)\big), \\ d\big(Sy,F(y,x)\big),d\big(Tu,G(u,v)\big),d\big(Tv,G(v,u)\big), \\ d\big(Sx,G(u,v)\big),d\big(Sy,G(v,u)\big), \\ d\big(Tu,F(x,y)\big),d\big(Tv,F(y,x)\big) \end{Bmatrix}\right).
\end{aligned}$$

Thus (2.8.1) is satisfied. Similarly we can verify (2.8.2).

One can easily verify all the remaining conditions of Theorem 2.8. In this Example (0,0) is the unique common coupled fixed point of F,G,S and T.

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A SURVIVAL FUNCTION OF LH AND FSH DUE TO OMEGA 3 FATTY ACID SUPPLEMENTATION IN NORMAL AND OBESE WOMEN BY USING GENERALIZATION OF UNIFORM DISTRIBUTION

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ABSTRACT :
Normally the mass of a root has a uniform distribution but some have different uniform distributions named Generalized Uniform Distribution (GUD).An application to a real data set is discussed.We have derived some properties of the Generalised Uniform distribution such as shape of probability density function, hazard rate function. Using this distribution we estimate shape of the probability density function, survival function and hazard rate function of dietary administration with omega-3 PUFA decreased FSH in normal women but not in obese women.
Keywords: Luteinizing hormone (LH), Follicle stimulating hormone (FSH),omega-3 polyunsaturated fatty acids (PUFA),Marshall-Olkin family of distribution, Truncated Negative Binomial distribution, generalized uniform distribution (GUD).

1. INTRODUCTION

Many researchers are interested in search that introduces new families of distributions or generalization of distributions which can be used to describe the lifetimes of some devices or to describe sets of real data. Exponential, Rayleigh, Weibull and linear failure rate are some of the important distributions widely used in reliability theory and survival analysis. However, these distributions have a limited range of applicability and cannot represent all situations found in application. For example, although the exponential distribution is often described as flexible, its hazard function is constant. The limitations of standard distributions often arouse the interest of researchers in finding new distributions by extending ones. The procedure of expanding a family of distributions for added flexibility or constructing covariates models is a well-known technique in the literature. Uniform distribution is regarded as the simplest probability model and is related to all distributions by the fact that the cumulative distribution function, taken as a random variable, follows Uniform distribution over (0,1) and this result is basic to the inverse method of random variables generation. This distribution is also applied to determine power functions of tests of randomness. It is also applied in a power comparison of tests of non-random clustering. There are also numerous applications in non-parametric inference, such as Kolmogorov-Smirnov test for goodness of fit. It is well known that Uniform distribution can be used as a representation distribution of round-off errors, and it is also connected to probability integral transformations. José and Krishna [5] introduced Marshall-Olkin extended uniform distribution as a generalization of uniform distribution and studied its properties.

1.2. Marshall-Olkin family of distribution

Marshall and Olkin[11],(1997) introduced a new family of distribution by adding a parameter to a family of distribution. They started with a survival function $\bar{F}(x)$ and consider a family of survival functions given by

$$\bar{G}(x) = \frac{\alpha\, \bar{F}(x)}{\bar{F}(x) + \alpha\bar{F}(x)}$$

Let $X_1, X_2, \ldots, X_N$ be a sequence of independent and identically distributed random variables with survival function $\bar{F}(x)$. Let N be a geometric random variable with probability mass function P(N= n) = $(\alpha)(1-\alpha)^{n-1}$, for n= 1,2,…. and 0<α<1. Then $U_N = \min(X_1, X_2, \ldots, X_N)$ has the survival function given by above equation. If α > 1 and N is a geometric random variable with probability mass function

P(N= n) = $\left(\frac{1}{\alpha}\right)\left(1-\frac{1}{\alpha}\right)^{n-1}$, for n = 1,2,….

Then $V_N = \max(X_1, X_2, \ldots, X_N)$ also has the survival function as above. Jayakumar and Thomas [3] proposed a new generalization of the family of Marshall-Olkin distribution as

$\bar{G}(x;\alpha,\gamma) = \left[\frac{\alpha\,\bar{F}(x)}{1-(1-\alpha)\bar{F}(x)}\right]^{\gamma}$, for α> 0, γ> 0, x∈ R

1.3. Truncated Negative Binomial distribution

Nadarajah,et, al. [12] (2013) introduced a new family of distributions as follows:

Let $X_1, X_2, \ldots$ Be a sequence of independent and identically distributed random variables with survival function $\bar{F}(x)$.Let N be a truncated negative binomial random variable with parameters α∈(0,1) and θ> 0.

That is, P(N= n) = $\frac{\alpha^\theta}{1-\alpha^\theta}\binom{\theta+n-1}{\theta-1}(1-\alpha)^n$ for n= 1,2,…

Consider $U_N = \min(X_1, X_2, \ldots, X_N)$. We have

$$\bar{G}_{U_N}(x) = \frac{\alpha^\theta}{1-\alpha^\theta}\sum_{n=1}^{\infty}\binom{\theta+n-1}{\theta-1}\left((1-\alpha)\bar{F}(x)\right)^n$$

$$\bar{G}_{U_N}(x) = \frac{\alpha^\theta}{1-\alpha^\theta}\left[\left(F(x)+\alpha\bar{F}(x)\right)^{-\theta}-1\right] \qquad (1.3.1)$$

Similarly if α>1 and N is a truncated negative binomial random variable with parameters $\frac{1}{\alpha}$and θ> 0 then $V_N = \max(X_1, X_2, \ldots, X_N)$ also has the survival function given by (1.3.1). This implies that we consider a new family of distribution given by the survival function

$\bar{G}(x;\alpha,\theta) = \frac{\alpha^\theta}{1-\alpha^\theta}\left[\left(F(x)+\alpha\bar{F}(x)\right)^{-\theta}-1\right]$ for α > 0, θ> 0, x∈ R .

Note that if α→1 then $\bar{G}(x;\alpha,\theta) \to \bar{F}(x)$. The family of distributions (1.3.1) is a generalization of the family of Marshall-Olkin distributions. If θ= 1, then (1.3.1) reduces to the family of Marshall-Olkin distributions.

2. A NEW FAMILY OF UNIFORM DISTRIBUTION

Let $\bar{F}(x) = 1 - x$, 0< x < 1 and introduce a new family distributions given by the Survival function

$$\bar{G}(x;\alpha,\theta) = \frac{\alpha^\theta}{1-\alpha^\theta}\left[\left((x(1-\alpha)+\alpha\right)^{-\theta}-1\right], \quad \alpha>0,\ \theta>0 \qquad (2.1)$$

Therefore, the distribution function is given by

$$G(x;\alpha,\theta) = \frac{1-\alpha^\theta(x(1-\alpha)+\alpha)^{-\theta}}{1-\alpha^\theta} \qquad (2.2)$$

The probability density function is given by

$$g(x;\alpha,\theta) = \frac{(1-\alpha)\theta\alpha^\theta}{(1-\alpha^\theta)(x(1-\alpha)+\alpha)^{\theta+1}} \qquad (2.3)$$

for $0 < x < 1, \alpha > 0$ and $\theta > 0$.This new distribution is the generalised uniform distribution with parameters α and θ.We write it as GUD (α, θ).

2.1. Shape of the density function

We consider the function

$$(\log g)' = \frac{g'(x)}{g(x)} = -\frac{(1-\alpha)(\theta+1)}{x(1-\alpha)+\alpha}$$

Let $s(x) = \frac{(1-\alpha)(\theta+1)}{x(1-\alpha)+\alpha}$

i) If $\alpha \in (0, 1)$ then the function s(x) is positive and this implies that g is a decreasing function with $g(0) = \frac{(1-\alpha)\theta}{\alpha(1-\alpha^{\theta})}$ and $g(1) = \frac{\alpha^{\theta}(1-\alpha)\theta}{(1-\alpha^{\theta})}$.

ii) If $\alpha > 1$ then the function s(x) is negative and this implies that g is an increasing function with $g(0) = \frac{(\alpha-1)\theta}{\alpha(\alpha-1)}$ and $g(1) = \frac{\alpha^{\theta}(\alpha-1)\theta}{(\alpha^{\theta}-1)}$.

2.2 Hazard rate function

The hazard function of a random variable X with density g(x) and a cumulative distribution function G(x) is given by

$$h(x) = \frac{g(x)}{\bar{G}(x)} = \frac{(1-\alpha)\theta}{(x(1-\alpha)+\alpha)(1-(x(1-\alpha)+\alpha)^{\theta})}$$

For $\alpha \leq 0.6$, hazard rate is initially decreasing and there exists an interval where it changes to be IFR. For $\alpha > 0.6$, the hazard function is evidently IFR.

3. APPLICATION

3.1 Introduction

A study has the highest prevalence of obesity among all countries surveyed in 2012 by the Organization for Economic Cooperation and Development [15]. It is estimated that by the end of 2015, 41% of adults will be obese [20], as defined by a body mass index (BMI) > 30 kg/m2. In women, reproductive morbidity associated with obesity is considerable and involves anovulation, menstrual cycle abnormalities, sub-fertility, fetal loss, obstetrical complications, and congenital anomalies [13]. Obesity has been associated with a state of relative hy-pogonado tropichypogonadism, as was shown by animal models [18] and clinical studies in both women and men [14, 19]. A gap in knowledge exists because the mechanisms underlying these harmful effects are poorly understood, and no specific treatments exist.

Increasing prevalence of obesity in recent decades has been preceded by dramatic dietary changes in industrialized societies. Over the past 100 years, there has been a considerable shift in the human diet, particularly with respect to the amount and type of consumed fat. Intake of omega-3 polyunsaturated fatty acids (PUFAs), previously consumed in large quantities by humans from vegetable and fish sources, has dwindled [16]. The contemporary ratio of dietary omega-6 to omega-3 PUFA is estimated to be as high as 25:1, a drastic change from the 1:1 ratio for previously estimated consumption [17]. Dietary fat can have inflammatory or anti-inflammatory properties based upon its composition. Saturated fats increase plasma free fatty acids and promote inflammation, whereas omega-3 PUFAs reduce macrophage-induced cytokine production [10].We sought to test whether dietary supplementation with omega-3 PUFA affects reproductive hormones in women and/or whether this effect is modulated by body mass.

3.2 Participants

Twenty-seven regularly menstruating obese and normal- weight (NW) women (15 obese and 12 NW controls) were recruited from the community through campus-wide advertisement and completed the study. A signed informed consent was obtained from each participant before participation. Study criteria included: 1) age, 18 – 42 years; 2) BMI, ≥30 kg/m^2 (obese) or 18 –25 kg/m^2 (NW); 3) history of regular menses every 25– 40 days; and 4) normal baseline prolactin, TSH, and blood count. Polycystic ovary syndrome was prospectively ruled out because all participants were required to have regular menstrual cycles Participants were excluded if they had allergies to sea-food, used medications known to affect reproductive hormones, used exogenous sex steroids within the last 3 months, exercised vigorously more than 4 hours weekly, or were attempting pregnancy.

3.3 Result

At baseline, the ratio of omega-6 to omega-3 PUFA was similar between the two groups in both plasma and RBC. After omega-3 PUFA supplementation, this ratio decreased significantly in both NW and obese women by 58 and 55%, respectively, in plasma similarly, the ratio in the RBC component decreased by 40 and 36%, respectively. The increase in EPA and DHA (omega-3 PUFA that were directly supplemented) was the greatest. Plasma EPA increased by 557% in NW and by 443% in obese women, whereas plasma DHA increased by 79% in NW and by 130% in obese women. Two subjects in the obese group and one in the NW group did not have a measurable change in omega-6 to omega-3 PUFA ratio after supplementation. Of note, these three subjects had the highest and lowest BMI values for our cohort. The two morbidly obese participants had BMI values of 42.2 and 51.2 kg/m2, respectively, and the NW participant had a BMI of 18.9 kg/m2 (lowest BMI in the cohort).

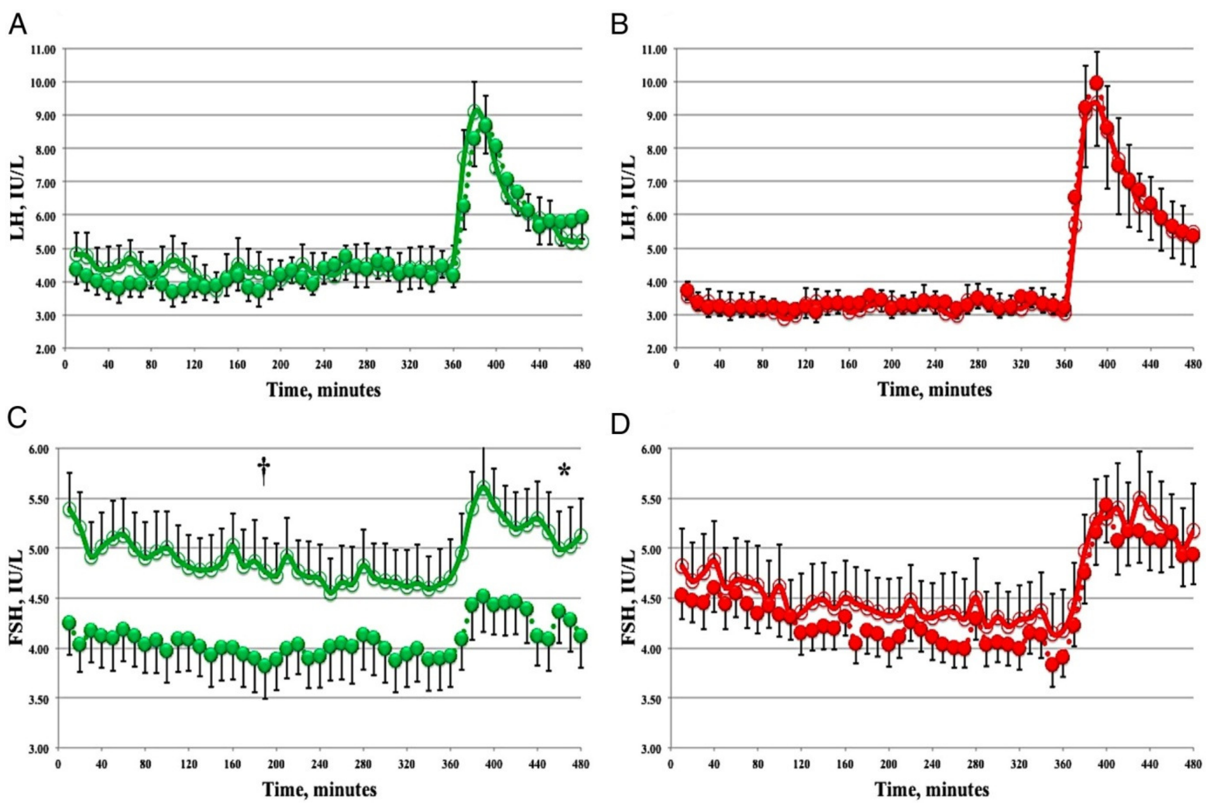


Figure 3.3.1. *Effect of omega-3 PUFA supplementation on serum gonadotropins. LH (A and B) and FSH (C and D) during unstimulated and GnRH- stimulated portions of frequent blood sampling (GnRH was given after 360 min). Data are from NW women (green; n = 12) and obese women (red; n = 15) at month 1 (open circles) and month 3 (filled circles). Error bars indicate standard error of the mean for group composites.*

At baseline (month 1), during the first 6 hours of unstimulated testing, mean serum LH was lower but was not significantly different in obese women when compared to NW controls (3.3 ± 0.3 vs 4.4 ± 0.6 IU/L, respectively). Similarly, no significant difference was observed for LH pulse amplitude (1.3 ± 0.1 vs 1.6 ± 0.2 IU/L, respectively), LH pulse frequency (3.1 ± 0.4 vs 2.6± 0.3 pulses/h), or GnRH-stimulated gonadotropin levels. Baseline (month 1) FSH parameters were not statistically different between obese and NW women before or after GnRH. At month 3 in NW women, both FSH mean level and peak were decreased by 17% vs month 1 studies (4.8 ± 0.3 to 4.0 ± 0.3 IU/L, and 5.5 ± 0.4 to 4.6 ± 0.4 IU/L, respectively), with a trend to statistical significance (Figure 3.3.1C). In contrast, no observable change was seen in FSH parameters before and after dietary supplementation in obese women (Figure 3.3.1D). Upon GnRH stimulation, FSH response has significantly decreased in NW women after omega-3 PUFA treatment (17% for mean serum level, 17% for AUC, and 19% for peak; Figure 3.3.1C). Again, no significant changes were seen in FSH parameters for obese women (Figure 3.3.1D). LH response was not different before or after omega-3 PUFA treatment for either group (Figure 3.3.1 A and B). This association of the reported LH and FSH parameters remained similar, and no changes in the level of significance were detected after adjusting for age.

3.4 Discussion

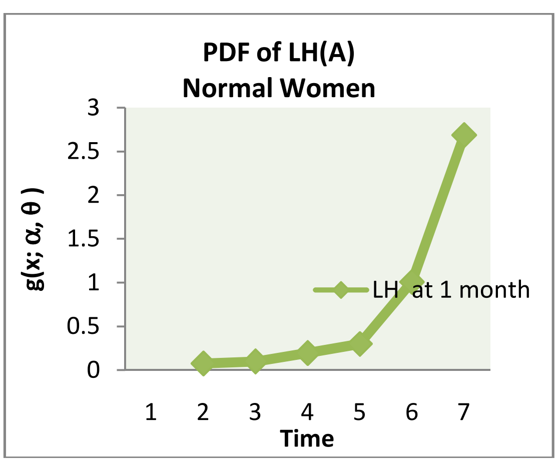
Reproductive aging in women is characterized by shortening of menstrual cycles [9], increased oocyte meiotic spindle abnormalities, and aneuploidy [4,7], along with declining fertility. This phenotype is classically associated with elevated levels of serum FSH in humans [6, 8]. To investigate a similar positive effect of an acute dietary treatment with omega-3 PUFA in women, FSH may be used in proxy to study ovarian reserve in women because circulating FSH levels are responsive to the endocrine input from the ovary [2]. Thus,the study is the first human investigation, to the best of our knowledge, to test the impact of omega-3 PUFA supplementation on gonadotropins in women. We have observed a combination of baseline serum FSH level reduction along with the FSH response to GnRH, which may indicate a possible reflection of modulated ovarian feedback effect on the central components of the HPO axis.

These results suggest that omega-3 PUFAs were associated with consistent reduction of serum FSH in NW women. Elevated serum FSH is a well-known feature of ovarian aging. Also this line of reasoning supports a testable hypothesis that increased dietary omega-3 PUFAs are to be tested in women in an attempt to delay ovarian aging.

In conclusion, Dietary administration with omega-3 PUFA decreased serum FSH levels in NW but not in obese women with normal ovarian reserve. The results imply that this nutritional intervention should be tested in women with diminished ovarian reserve in an attempt to delay ovarian aging.

4. MATHEMATICAL RESULT

Figure 4.1(a)



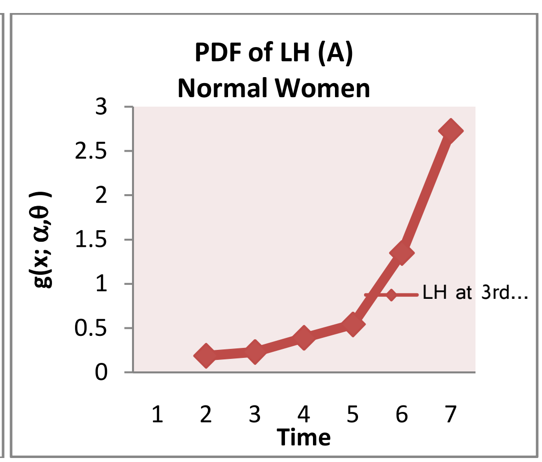
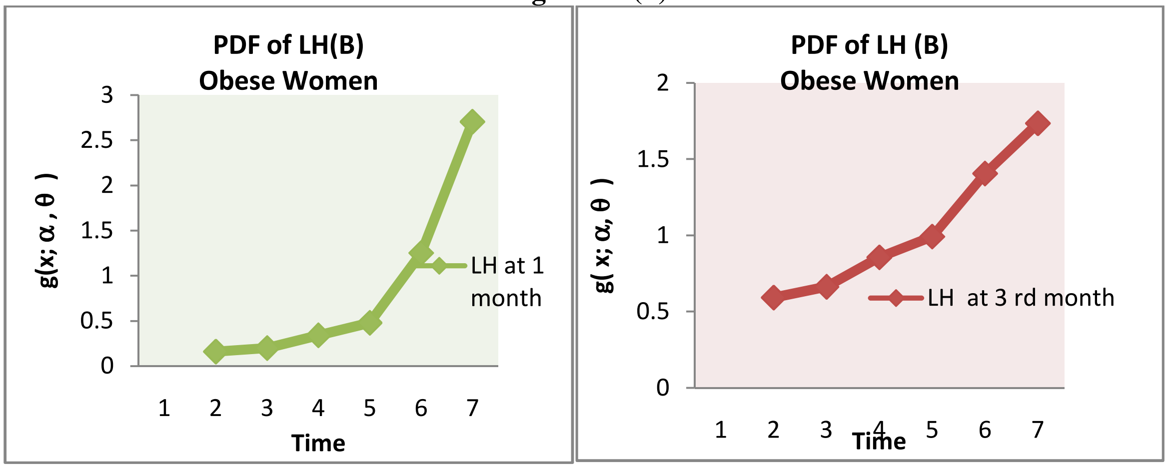


Figure 4.1(b)



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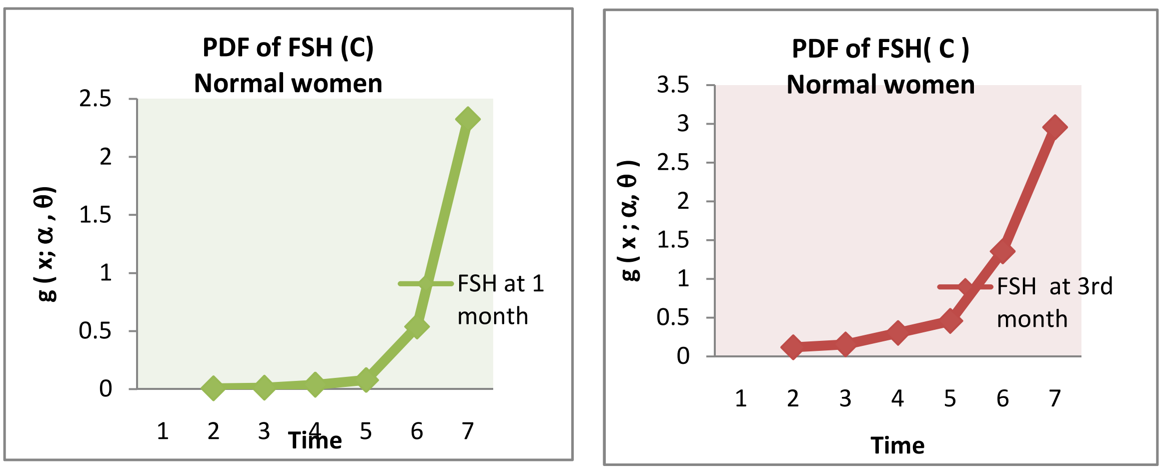

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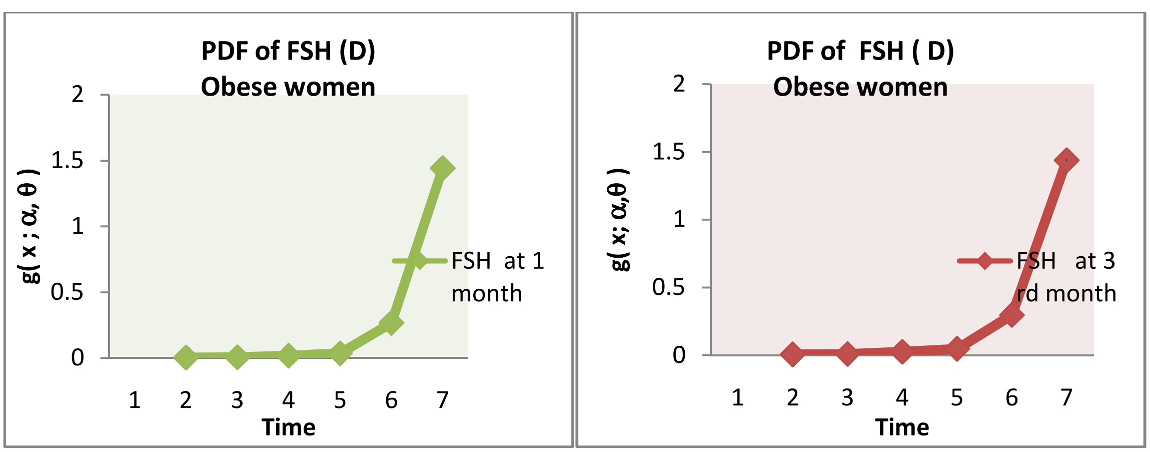

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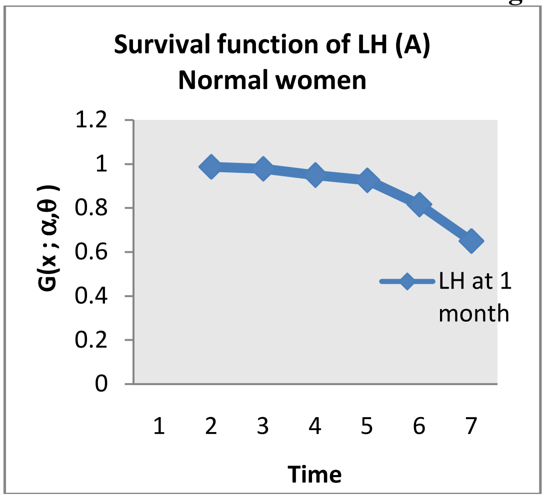

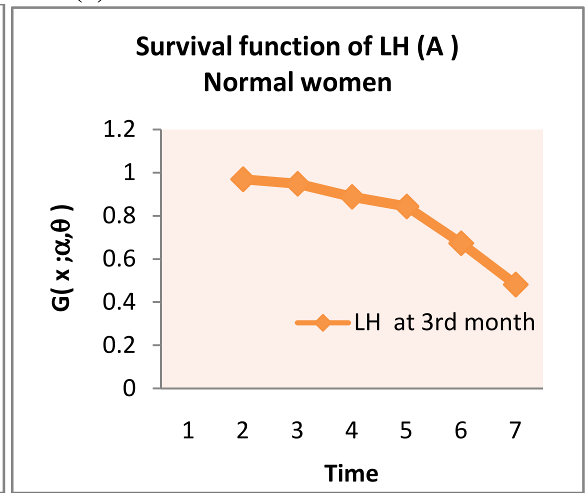

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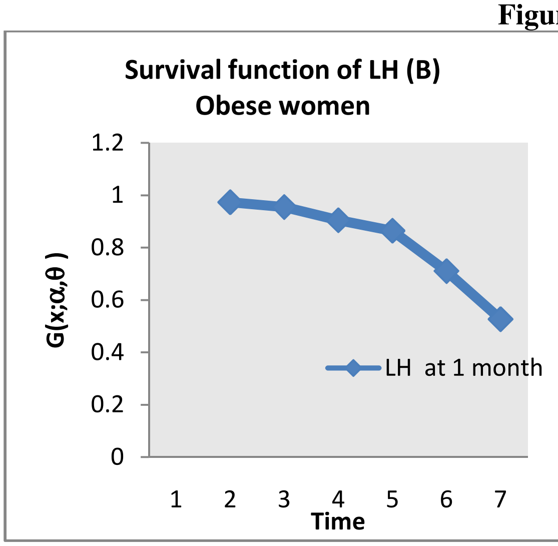


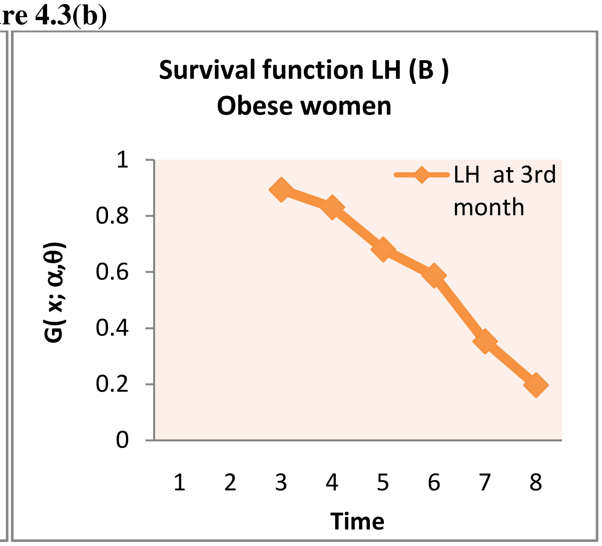
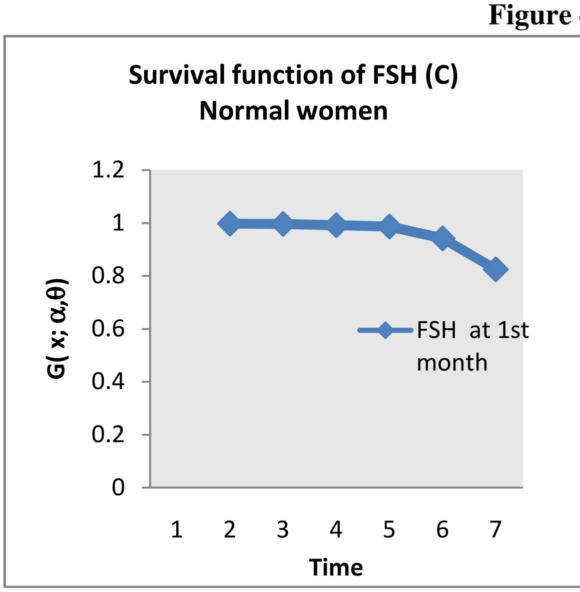
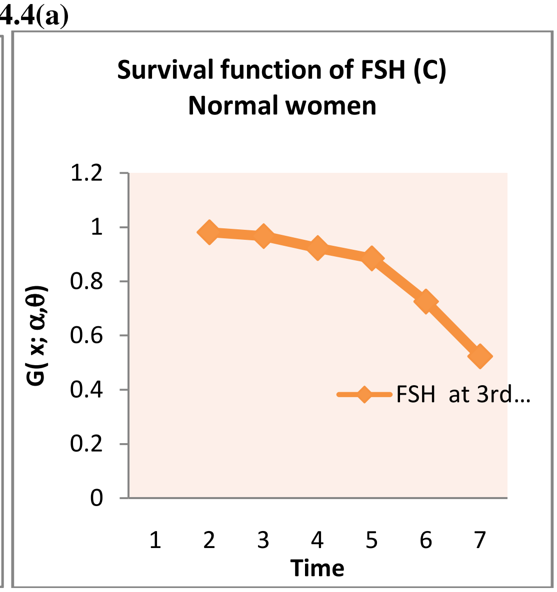


Figure 4.4(a)





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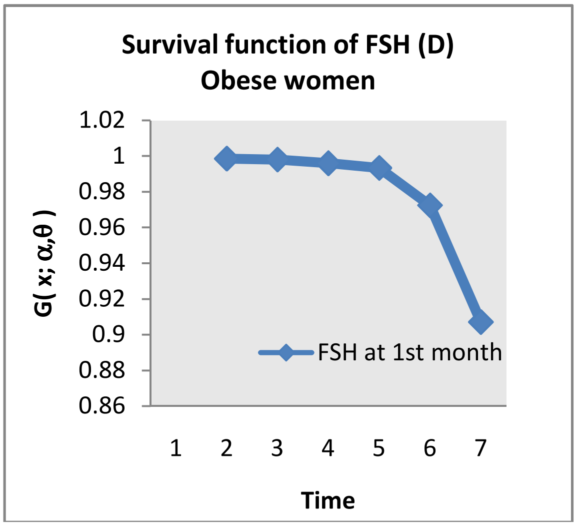

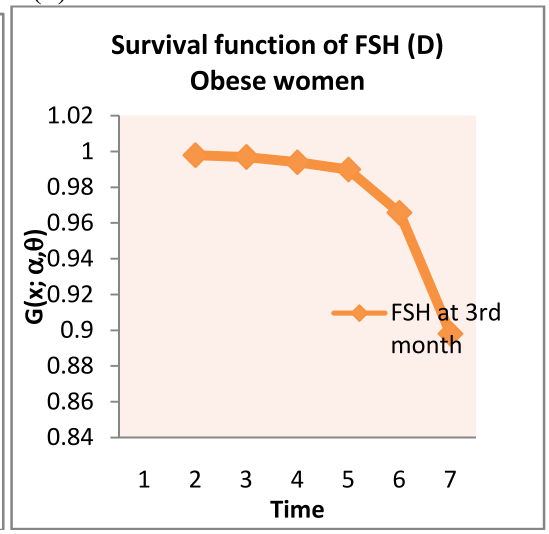

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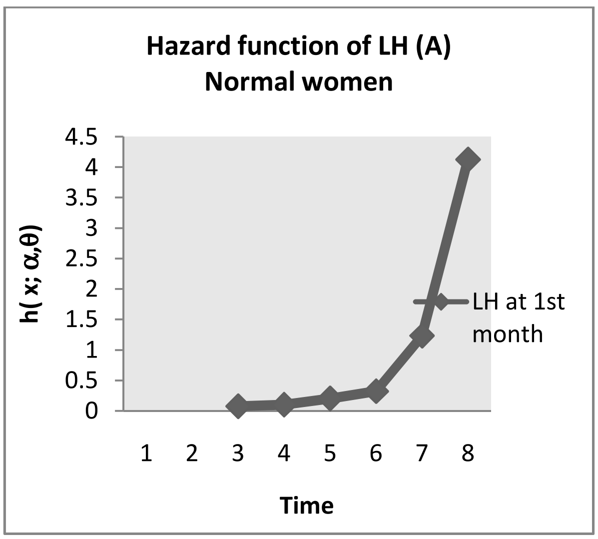


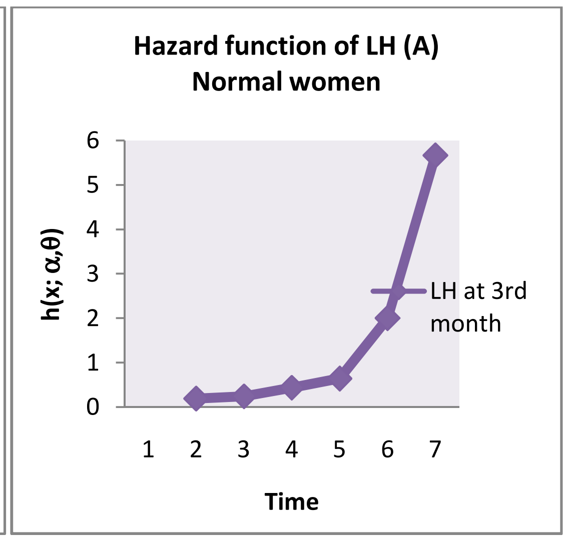
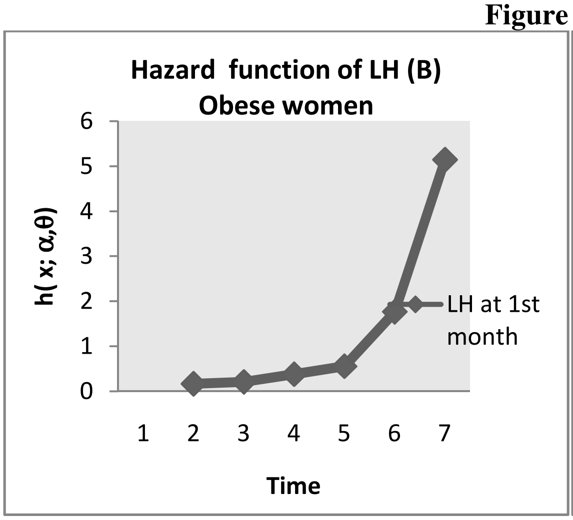


Figure 4.5(b)



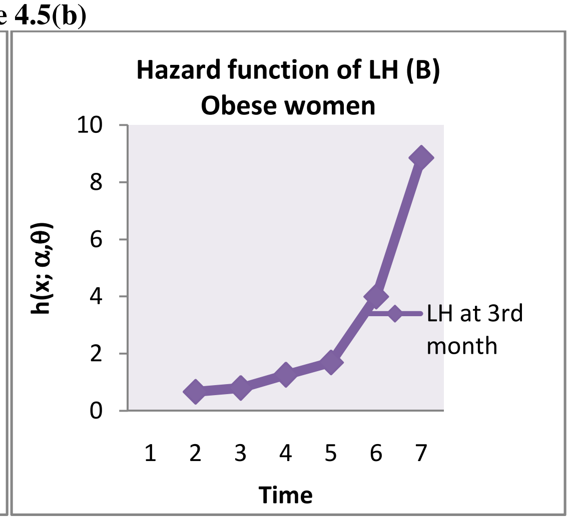


Figure 4.6(a)

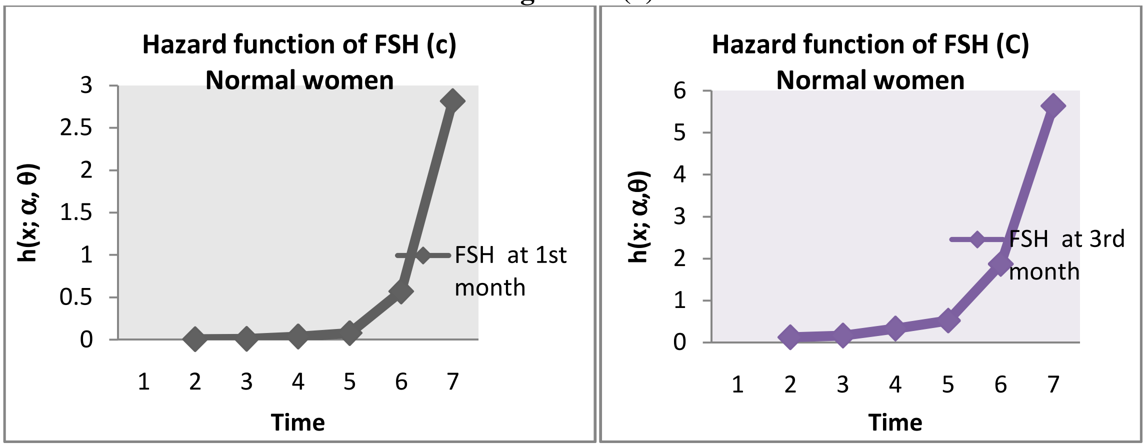
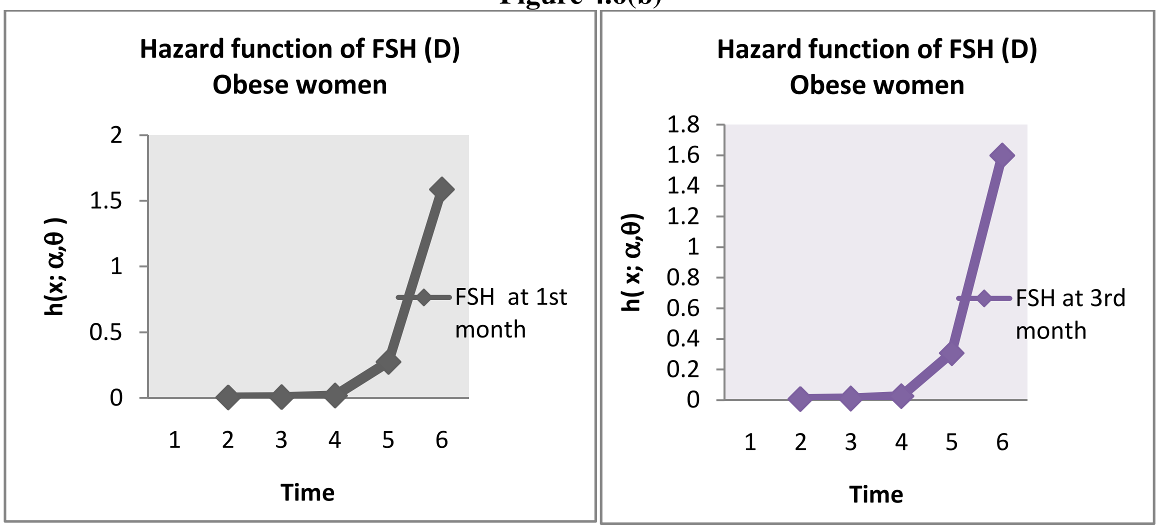


Figure 4.6(b)



5. CONCLUSION

A new family of distributions or generalization of distributions which can be used to describe the lifetimes of some devices or to describe sets of real data.Uniform distribution is regarded as the simplest probability model and is related to all distributions by the fact that the cumulative distribution function, taken as a random variable, follows Uniform distribution over (0,1) and this result is basic to the inverse method of random variablesgeneration. In Medical Part: Dietary administration with omega-3 PUFA decreases serum FSH levels in NW but not in obese women with normal ovarian reserve. The results imply that this nutritional intervention should be tested in women with diminished ovarian reserve in an attempt to delay ovarian aging. In Mathematical Part: If $\alpha>1$ then the shape of density function of LH and FSH is an increasing function for normal and obese women. But there is no difference between normal and obese women in LH at first month. At the 3rdmonth, there is a difference between normal and obese women in PDF of LH.At the first month, PDF of LH increases at 4min in Normal women, At the 3rd month, PDF of LH increases at 3min in N.W.At the first month, PDF of LH increases at 4min in obese women, At the 3rd month, PDF of LH increases at 2min in N.W.At the first month, PDF of FSH increases at 6min in Normal women. At the 3rd month, PDF of FSH increases at 4min in N.W.At the first month, PDF of FSH increases at 6min in obese women. At the 3rd month, PDF of FSH increases at 6min in obese women.At the 1st and 3rd month, Survival function of LH decreases in both cases. At the 1st and 3rd month, there is no change in survival function of FSH for obese women before and after dietary supplementation.At the 1st and 3rd month, there is a change in survival function of FSH for normal women before and after dietary supplementation.When compared to LH, hazard function of FSH response has decreased in normal women but no change in obese women.

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INBUILT MATHEMATICS IN SCRATCHES ON ISHANGO BONE BY UPPER PALEOLITHIC PEOPLE OF AFRICA

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ABSTRACT :
It was as early as 35000 BCE when people used bones and stones for writing. Then women were the torch bearers. They scratched on any surface available to them and noted the events relating to their daily life through lines or dots. Now archaeologists are trying to decipher the pristine symbols. In this paper we have collected few items of that age from different sources and tried to decode mathematically the meaning of primitive dots and lines engraved on baboon bones which is much earlier than the hieroglyphic days of the known civilizations.

INTRODUCTION

Upper Paleolithic[1] people started to keep records on bones, soft-stones and bas reliefs. Their signs were not easy to read but their counting had started by marking Archeologists had given explanation in different ways and we are expressing their mathematical views. These were as early as 35000 BCE and as late as 10000 BCE. These were earliest prehistoric arts. Afterwards we find scratches on bone which were read by archeologists as count of lunar periods but we read these as mathematical phenomena. These scratches were deep which indicate these are done by prehistoric women. Prehistoric women were basically home makers. They wanted to express and keep records of whatever they observed. The style of putting *Scratches* on flat surface or collecting pellets as memoir or count fingers / digit-sections for record of an event was probably forerunner of hieroglyphic inscriptions. Counting or mathematics was probably initiated by women in this way: discussed in later. The language was rudimentary mathematical. Deciphering these codes need play of imagination.

DISCUSSION

According to *d'Errico et al. 2012* [Ref.8] recent archaeological discoveries have revealed that beads, engravings and bone tools were already present in southern Africa 75,000 years ago. Many of these artifacts disappeared by 60,000 years ago showing that modern behaviour appeared in the past and was subsequently lost before firmly established. Most archaeologists think that San[2] hunter–gatherer cultural adaptation emerged in 20,000 BCE. However, reanalysis of organic artifacts from South Africa shows that inhabitants of the cave in the Early Later Stone Age used notched bones for notational purposes.

The Kemetic God, Djehuty (Tehuty or Toth), was later depicted as a baboon (also an ibis), and is usually associated with moon, math, writing and science. Use of baboon bones as mathematical devices has been continuous throughout Africa, suggesting Africans always held the baboon as sacred and associated with the moon, math and time.

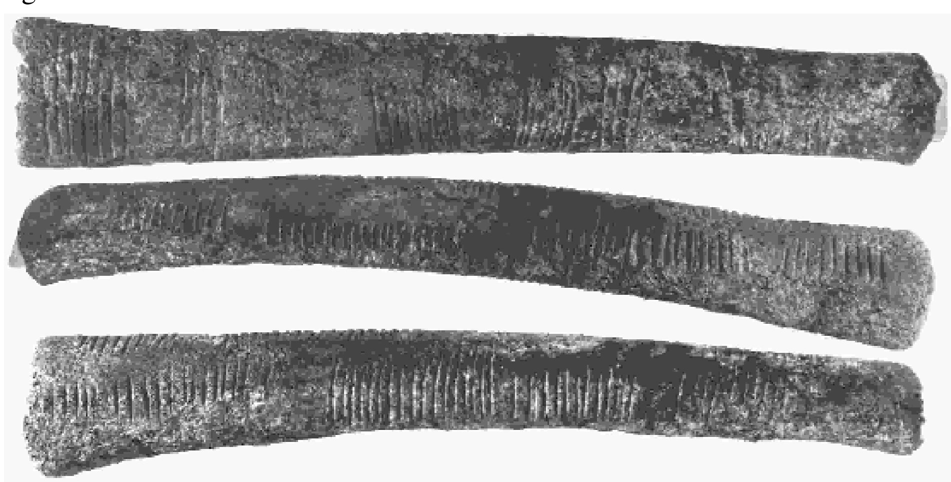
The Ishango bone is a bone-tool in Upper Paleolithic era. It is a dark brown fibula of baboon. The scratches were intentionally made and these were not tooth marks either by man or beast. We demonstrate it as following ways:

1. It was first thought to be a tally stick, as it has a series of tally marks carved in three columns running along the length of the tool.
2. It has also been suggested that the scratches might have been created for better grip or for some other non-mathematical reason.
3. The marking also may be the product of New Stone Age to express their existence of *homo-sapiens*.
4. The bone may also has so many new speculations about hunter-gatherers.
5. Notches have been marked to form a pattern-tool on some logical reasoning.
6. It may be recognised as ***slide-rule*** in contrast with ***arithmetical game*** or ***calendar*** explanation.
7. It is a primitive ***Mathematical tool***.
8. This bone scratches indicates ***physiological events*** in life of primitive women.

The Ishango bone was found in 1960 by Belgian Professor Jean de Heinzelin de Braucourt[3] in Congo. It was discovered in the area of Ishango[4] near the Semliki River[5]. The bone was found among the remains of a small community that fished and gathered in this area of Africa. The settlement had been buried in a volcanic eruption.

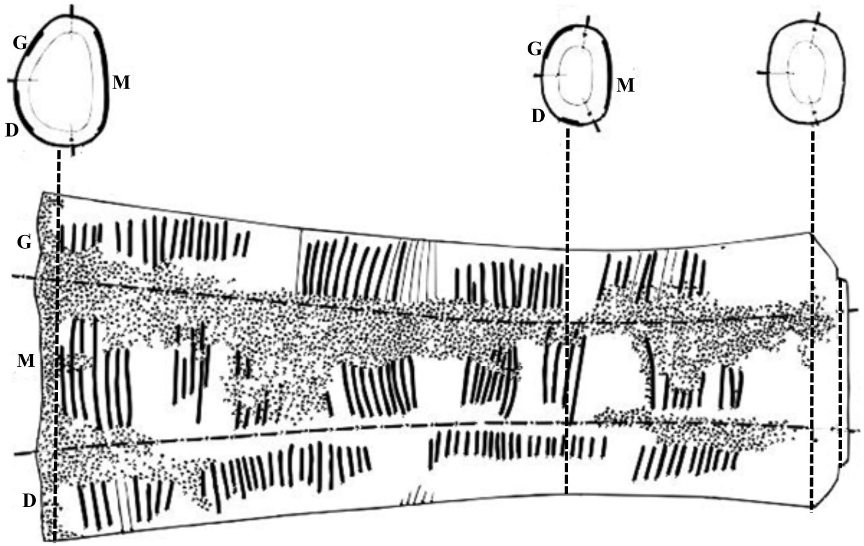
The artifact was first estimated to have originated between 9,000 BCE and 6,500 BCE. However, the dating of the site where it was discovered was re-evaluated with the help of anthropologists and it is now believed to be more than 20,000 years old.

The Ishango bone is on permanent exhibition at the Royal Belgian Institute of Natural Sciences, Brussels, Belgium.



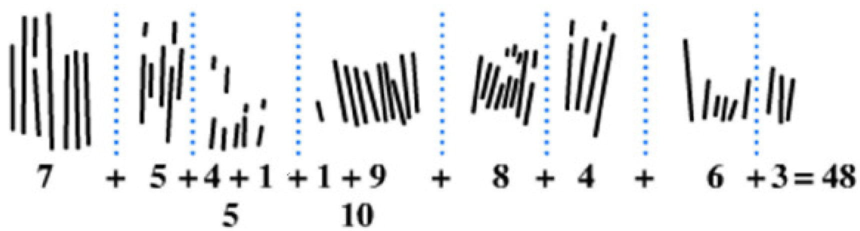
Demonstration-I: 10 cm Marked bone of 20000 BCE, its extremity fitted with a fragment of quartz, protruding by 2 mm, most probably for tattooing or engraving purposes found in Congo excavation

Bone carries 168 notches distributed in three columns along the bone length. Notches are nearly parallel within each group but of different length and orientation.

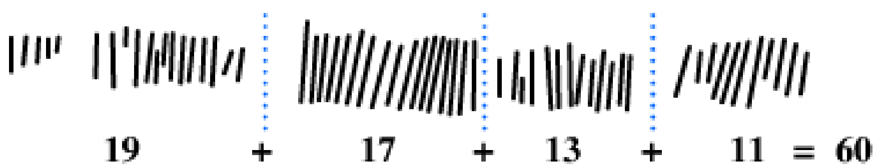


Demonstration-IA: Structural presentation of scratches on Ishango bone.

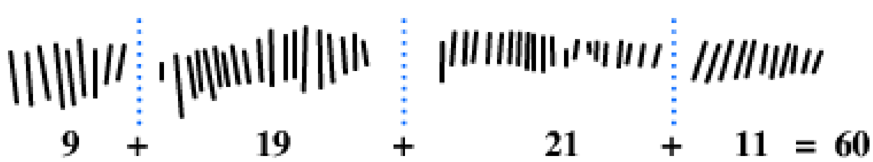
Upper part of the demonstration-IA by Prof. Heinzelin are proximal end structure of bone which also indicating the three faces of bone whereas lower part showing scratches on flattening it. He marked the columns in erected position as (1) Middle one as ‘M’ (from the french word *Milieu*); (2) Left one as ‘G’ (from the french word *Gauche*); (3) Right one as ‘D’ (from the french word *Droite*).



Demonstration-IB: M-column (top to bottom — shown from right to left).



Demonstration-IC: G-column (top to bottom — shown from right to left).



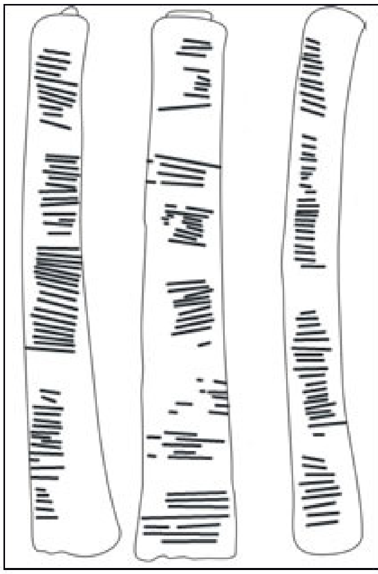
Demonstration-ID: D-column (top to bottom — shown from right to left).

Within each column notches are grouped like depicted but not random which proves that notches are done by primitive women as they write or scratch deeper that men.

M-column exhibits digits less than or equal to 10, G-column shows numbers between 10 and 20 whereas D-column holds numbers between 9 and 21.

Within M-column in (9 + 1) =10 & (4 + 1) = 5 there exist small scratches. Though we have conserved these in a group even there exists (?) as shown in Demonstration-IIE.

We may consider that in some purposes some columns ‘G’ and ‘D’ have oriented from column ‘M’ under mathematical know how as:



	3	11
11	6	
13	4	21
	8	
17	10 (9+1 ?)	
		19
	5 (1+4 ?)	
19		9
	5	
	7	

G	**M**	**D**	**G**	**M**	**D**
Total number of notches			**60**	**48**	**60**

Demonstration-IE: G, M, D-columns.

- Column ‘M’ has 8-groups (Number of notches are respectively: 3, 6, 4, 8, 10, 5, 5 and 7) whereas columns ‘G’ & ‘D’ have 4-groups each (Number of notches are respectively: 11, 13, 17, 19 & 11, 21, 19, 9) i.e. middle column group division has been doubled than the side columns.
- Digits in the M-column are arranged as:

a) Duplicated by 2: $3 \times 2 = 6$ and $4 \times 2 = 8$

b) Repeated addition makes double: $5 + 5 = 10 = 5 \times 2$

c) Division by 2 makes half of a digit: $10 \div 2 = 5$

d) Subtraction: $10 - 5 = 5$

e) Addition by 2: $5 + 2 = 7$

f) Here odd primes 3, 5, 7 have been used but operated with only even prime i.e. 2.

g) Base numbers are 3 and 4; composed numbers are 5 and 7 i.e. $3 + 2 = 5$, $5 + 2 = 7$; associated numbers are 6, 8, and 10 i.e. $3 \times 2 = 6$, $4 \times 2 = 8$ and $5 \times 2 = 10$

- Digits in the G-column can be expressed as:

➢ Prime numbers between 10 and 20: 11, 13, 17, 19

➢ It can also be expressed under the scale of 12 as: $12 \pm 1 = 11, 13$ and $18 \pm 1 = 17, 19$ where $18 = 12 \times 1\frac{1}{2}$

- Digits in the D-column can be demonstrated as:

a) Digits are more or less than 10 or 20 by 1: $10 + 1 = 11$; $20 + 1 = 21$

$10 - 1 = 9$; $20 - 1 = 19$

b) Digits can be generalised as: $10 \pm 1 = 11, 9$ and $20 \pm 1 = 21, 19$ i.e. $2 \times 10 = 20 \pm 1$ whereas it is under the scale of 10.

- Total number of scratches in the columns are multiple of 12 i.e. $4 \times 12 = 48$ and $5 \times 12 = 60$. So, two side columns are oriented from middle column i.e. extending 12 numbers of scratches under the scale of 12.
- Sum of consecutive numbers of M-column yield middle digits of G-column and D-column.

M-column		G-column	D-column
3 + 6 + 4	→	13	
4 + 8 + 9	→		21
8 + 9	→	17	
9 + 5 + 5	→		19

Here tenth notch of fifth group of has not been considered.

- Now considering the tenth notch it can be demonstrated as:

M-column		G-column	D-column
4 + 8 (**+1**)	→	13	
4 + 8 + 10 (**–1**)	→		21
8 + 10 (**–1**)	→	17	
10 + 5 + 5 (**–1**)	→		19

This one is possibly less elegant.

RESULT:

In our perception we see the scratches are expressions of: Prime numbers, relations between numbers and elementary operations had been involved within it.

CONCLUSION:

Every discoverer of an archeological artefact wants to count century. Thus, we see that the discoverers and interpreters of the ISHANGO BONE tried to express their artefact to a standard more than it was worth. Thus, without mathematical perspective, they did not realise that concepts such as a number system with base, prime numbers and even multiplication and elementary operations had not come into picture rigorously before twenty thousand BCE.

Discoverer wrote: *Scratches on the Ishango Bone may represent an arithmetical game of some sort, devised by a people who had a number system based on 10 as well as a knowledge of duplication and prime numbers.*

The suggestive marking on the bone may be treated as possibilities of a highly developed sense of arithmetic awareness. It may be said Ishango numeration system may have the earliest decimal base system of the world.

According to archeologist, particularly Alexander Marshak, the marks were of lunar observations in the Ishango culture i.e. series of notches matches the number of days contained in successive phases of the moon.

Ethnomathematicians, who commendably seek to instill pride about their African heritage, have made the ISHANGO BONE into a symbol of precocious African mathematics. This is an unfortunate choice because of its dubious interpretation. Other artefacts of comparable age, less ambiguously marked as tallies, have been found in Europe and Africa. *Homo sapiens,* whether in Africa or Europe, in a Stone Age were hunter-gatherer learned to express his quantitative instinct as discrete number symbols: fingers, scratches and sounds. They had ability to count.

REFERENCES TO INDICES:

1. The Upper Paleolithic (or Upper Paleolithic, *Late Stone Age*) is the third and last subdivision of the Paleolithic or Old Stone Age. Very broadly, it dates to between 50,000 BCE and 10,000 BCE (the

beginning of the Holocene meaning entirely recent), roughly coinciding with the appearance of behavioral modernity and before the advent of agriculture.

Anatomically modern humans (i.e. *Homo sapiens*) are believed to have emerged around 2,00,000 BCE, although these lifestyles changed very little from that of Archaic humans of the Middle Paleolithic, until about 50,000 BCE, when there was a marked increase in the diversity of artifacts. This period coincides with the early human migration throughout Eurasia, which contributed to the extinction of the Neanderthals.

2 The Sanpeople, also recognised as Bushman, are members of various Khoisan speaking indigenous hunter-gatherer people of South Africa, whose territories span Botswana, Namibia, Angola, Zambia, Zimbabwe, Lesotho and South Africa.The ancestors of the hunter-gatherer San people are considered to have been the first inhabitants of what is now Botswana and South Africa.In this area, stone tools and rock art paintings date back over 70,000 years and are by far the oldest known art.A set of tools almost identical to that used by the modern San and dating to 44,000 BCE was discovered at Border Cave in KwaZulu-Natal in 2012. The San tribe, one of the most intriguing people in this world, are the region's earliest inhabitants (it is estimated that they have been living here for the last 30,000+ years) and are still settled in many parts of Southern Africa.The San people are the first people of Africa. This means the San are descendants of the first people who ever lived here, before black or white people migrated into the African region.Archaeologists and geneticists agree, that the San are the descendants of the original *Homo sapiens* groupings, who occupied Southern Africa, for at least 150 000 years. According to genetic studies and the geneticists: That one of the oldest gene patterns found in some modern humans, is that of the Khoe-San, and it dates back around 80 0000 years.

3 **Jean de Heinzelin de Braucourt** (6 August 1920 – 4 November 1998) was a Belgian Geologist as well as anthropologist who worked mainly in Africa. He worked at the Universities of Ghent and Brussels. He gained international fame in 1960 when he discovered the Ishango Bone.

4 It was discovered from a fisherman village called Ishango whereas it is at the furthest source of Nile on the Border of Congo and Uganda.

5 **Ishango** is a sub-station of Virunga National Park, situated on the Northern Shores of Lake Edward in the Democratic Republic of Congo. The station was created in the 1950s. Lake Edward empties into the Semliki which forms part of the headwaters of the Nile River. The lake was formerly held the biggest hippo population in the world.Now it ison the border between modern-day Uganda and Democratic Republic of Congo.

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AVAILABILITY AND COST-BENEFIT ANALYSIS OF TWO SIMILAR COLD STANDBY SYSTEM UNDER THE INFLUENCE OF SNOWSTORM CAUSING RESCUE OPERATION

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ABSTRACT :
This paper explores the effects of snowstorm as abnormal weather on two unit not similar cold standby system. In this system after snowstorm has ended, rescue operation starts first digging out start after complete the digging out snow removing start and then hospitalization of the system (human) starts as a repair. The system has only a single repairman which has different types of repair rates and the failure rate due to snowstorm follow exponential distributions and different repair rates follow different general time distributions. By using the concept of semi-Markov process the efficiency of system such as expected profit, availability analysis and mean time to system failure are obtained. In the end, numerical results, various graphs are presented to illustrate the behaviour of the various reliability characteristic consequent.
Keywords: Cold stand by system; Rescue operation; Snow removing; Hospitalization

1. INTRODUCTION

A large number of articles have been analyzed by different researchers in the field of reliability to improve the reliability of systems. Various researchers such as Osaki, 1972; Rudin, 1976; Taneja et al., 1991; Chandrasekhar and Natrajan, 2004; Malhotra and Taneja, 2014 have studied reliability of standby systems with different types of repairs rates and failure rates [1, 4, 6, 7, 12,13]. Subramanian and Anantharaman analysed the reliability of a complex reductive wait system [10]. Singh and Taneja [9] and Malhotra and Taneja [5] have studied a comparative study of systems. Taj et al. [11] analyzed the reliability and modeling of a single subsystem of a cable machine. In abnormal climatic conditions expect conditions are very interested. Environmental conditions can not be the control due to variation in climate changes. Therefore, L.R. Goel, Ashok Kumar, A. K. Rastogi [2] and Gupta and Goel [3] have obtained reliability measures of standby systems with different climatic situations. Singh et al. [8] analyzed availability of warm standby systems failure due to heavy rain.

On 17th march 2017 Friday thousands of people across the state experienced travel woes as airlines and airports struck into a snow storm, which dumped nearly two feet of snow and brought heavy winds throughout the northern region. The whole Jammu and Kasmir was covered under about 20 inches of snow cover. Houses, roads, streets, parks, gardens every place was covered by thick layers of snow. Intence cold wave conditions continued far two or three days as the minimum temperature was dipped further to minus 6.8˚C recording the season's coldest weather so far. Thousands of cars and trucks were stranded on the highway for 3 days, in the middle of forest area. Several soldires were trapped in the snow storm camp of Gurez sector of Bandipora district near Line of Control. To rescue the soldiers a team including a junior commissioned officer with seven soldiers swas launched for operation immediately and total 10 bodies were retrained 10 missing bodies by team from the spot of the incident. Due to snow storm in the region of land sliding some people were trapped under the snow. After the storm over, the rescue team was reached at the incident to remove the snow for digging out the bodies and their hospitalization.

By using the semi-Markov regenerative point technique, considering the facts and practical situations we measured the reliability in a system of two identical units of cold standby system operating under the snowstorm. In that type of situations to improve the reliability of operating system is better when we start to dig out the bodies under snow and the path should be cleared by JCB and then hospitalize the suffered people immediately under the supervision of specialized medical team. The model has the following assumations:

- Both units are similar
- System is cold standby
- The unit of the system fail due to snow strom
- First the failed unit digging out from snow storm after the excavation snow removing starts after then hospitalization for repairs
- The unit becomes operative after complete the hospitalization
- The system has only a single repairman

1.1 Notations

•• : Failure rate of operative unit due to snowstrom

⬭ : Up state

□ : Failed state

$G_1(t)$, $G_2(t)$, $G_3(t)$: Cumulative density function of the repair rate of digging out, snow removing and hospitalization of failed unit respectively.

$g_1(t)$ $g_2(t)$ $g_3(t)$: Probability density function of the repair rate of digging out, snow removing and hospitalization of failed unit respectively.

Op : Operative unit

cs : Cold standby

Fd : Failed unit is under digging out

FD : Failed unit is under digging out continuing on the unit

Fsr : Failed unit is under snow removing

FSR : Failed unit is under snow removing continuing on the unit

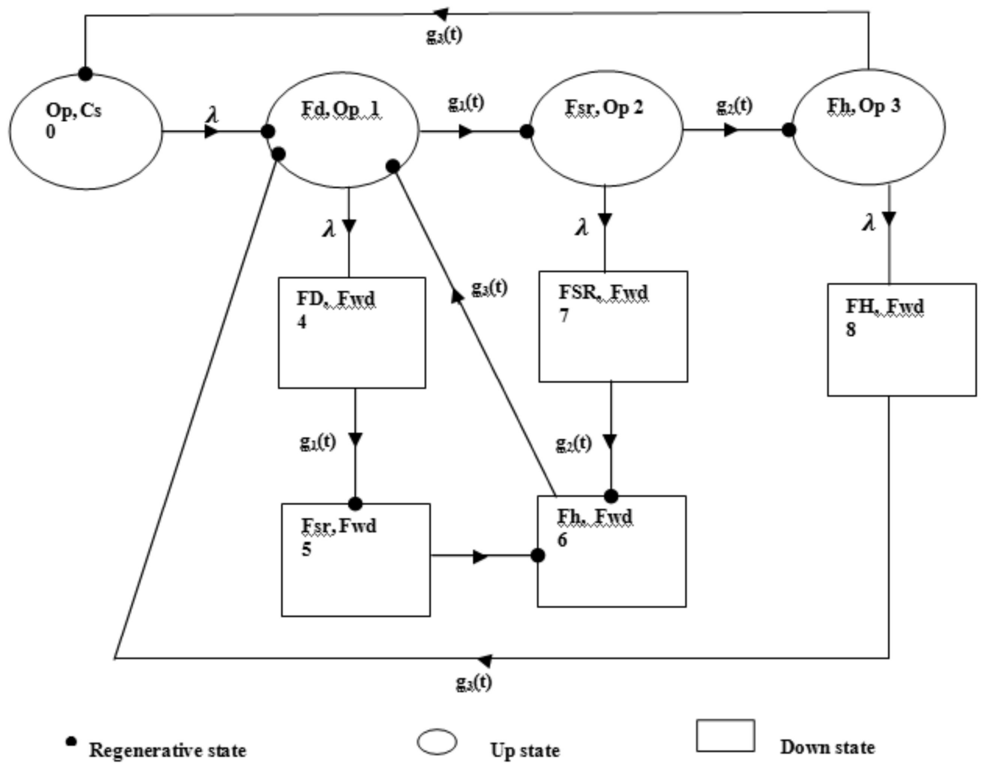
Fh : Failed unit is under hospitalization after snow removing

FH :Failed unit is under hospitalization continuing after snow removing

Fwd :Waiting for digging out

2. MODEL AND TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES

2.1 Model



2.2 Transition probabilities

In the model, the stages 0, 2, 1 and 5 regenerative stages and non regenerative states are 3 and 4 states as in the **Fig.1**.

$dQ_{01}(t) = \lambda\, e^{-\lambda t}\, dt$

$dQ_{12}(t) = e^{-\lambda t} g_1(t)dt$

$dQ_{14}(t) = \lambda\, e^{-\lambda t}\, \overline{G_1}(t)\, dt$

$dQ_{15}^{(4)}(t) = (\lambda\, e^{-\lambda t} © 1)\, g_1(t)dt$

$dQ_{23}(t) = e^{-\lambda t}\, g_2(t)dt$

$dQ_{27}(t) = \lambda\, e^{-\lambda t}\, \overline{G_2}(t)\, dt$

$dQ_{26}^{(7)}(t) = (\lambda\, e^{-\lambda t} © 1)\, g_2(t)dt$

$dQ_{30}(t) = e^{-\lambda t}\, g_3(t)dt$

$dQ_{38}(t) = \lambda\, e^{-\lambda t}\, \overline{G_3}(t)dt$

$dQ_{31}^{(8)}(t) = (\lambda e^{-\lambda t}©1) g_3(t)dt$

$dQ_{61}(t) = g_3(t)dt$

$dQ_{56}(t) = g_2(t)dt$

Taking Laplace Stieltjes Transformation we get

$p_{ij} = \lim_{t\to\infty} Q_{ij}(t) = \lim_{s\to 0} Q_{ij}^{**}(s)$

$p_{01} = 1$,

$p_{12} = g_1^*(\lambda)$, $p_{14} = (1 - g_1^*(\lambda))$, $p_{15}^4 = (1 - g_1^*(\lambda))$

$p_{23} = g_2^*(\lambda)$, $p_{27} = (1 - g_2^*(\lambda))$, $p_{15}^4 = (1 - g_2^*(\lambda))$

$p_{30} = g_3^*(\lambda)$, $p_{38} = (1 - g_3^*(\lambda))$, $p_{31}^8 = (1 - g_3^*(\lambda))$

$p_{56} = g_2^*(0) = 1$, $p_{61} = g_3^*(0) = 1$

The transition probabilities, it can be verified as

$p_{12} + p_{15}^{(4)} = p_{12} + p_{14} = 1$

$p_{23} + p_{26}^{(7)} = p_{23} + p_{27} = 1$

$p_{30} + p_{31}^{(8)} = p_{30} + p_{38} = 1$

$p_{61} == p_{01} = p_{56} = 1$

2.3 Mean sojourn times

The mean sojourn time (μ_i) can be defined at the regenerative state ‘i’ by using the formula μ_i=E(T)=Pr(T>t)

Where, T denotes the sojurn at the regenerative state.

$\mu_0 = 1/\lambda$, $\mu_1 = (1 - g_1^*(\lambda))/\lambda$, $\mu_2 = (1 - g_2^*(\lambda))/\lambda$, $\mu_3 = (1 - g_3^*(\lambda))/\lambda$,

$\mu_5 = -g_2^{*\prime}(0)$, $\mu_6 = -g_3^{*\prime}(0)$

The unconditional mean time defined in the regenerative state ‘i’ when it transit into the another state ‘j’ mathematically as:

$m_{ij} = \int_0^\infty t q_{ij}(t)dt = -q_{ij}^{*\prime}(0)$

$m_{01} = \mu_0 = \frac{1}{\lambda}$

$m_{12} + m_{15}^4 = -g_1^{*\prime}(0) = k_1(\text{say})$, $m_{12} + m_{14} = \mu_1$

$m_{23} + m_{26}^7 = -g_2^{*\prime}(0) = k_2(say)$, $m_{27} + m_{23} = \mu_2$

$m_{30} + m_{31}^8 = -g_3^{*\prime}(0) = k_3(say)$, $m_{30} + m_{38} = \mu_3$

$m_{61} = k_3$

$m_{51} = k_2$

3. ANALYSIS OF EXPECTED TIME TO SYSTEM FAILURE

Recursive relations for $\phi_i(t)$ obtained by considering the failed states as absorbing state and the arguments were used for regenerative process as follow:

$\Phi_0(t) = Q_{01}(t) \circledS \Phi_1(t)$

$\Phi_1(t) = Q_{14}(t) + Q_{12}(t) \circledS \Phi_2(t)$

$\Phi_2(t) = Q_{27}(t) + Q_{23}(t) \circledS \Phi_3(t)$

$\Phi_3(t) = Q_{38}(t) + Q_{30}(t) \circledS \Phi_0(t)$

By using mathematical concept of Laplace Stieltjes transformation and crammer rule we find

$$\Phi_0^{**}(s) = \frac{N(S)}{D(S)}$$

where,

$$N(S) = Q_{01}^{**}(s)[Q_{14}^{**}(s) + Q_{12}^{**}(s)Q_{23}^{**}(s)\, Q_{38}^{**}(s) + Q_{12}^{**}(s)Q_{27}^{**}(s)]$$

$$D(S) = 1 - Q_{01}^{**}(s)\, Q_{12}^{**}(s)Q_{23}^{**}(s)\, Q_{38}^{**}(s)$$

At the initial stage '0', the mean time to system failure is:

$$T_0 = \lim_{S\to 0}\frac{1-\Phi_0^{**}(s)}{S} = \lim_{S\to 0}\frac{1-\frac{N(S)}{D(S)}}{S}$$

$$= \lim_{S\to 0}\frac{D(S)-N(S)}{SD(s)} = \lim_{S\to 0}\frac{D^1(0)-N^1(0)}{D(0)} = \frac{N}{D}$$

Where

$$N = \mu_0 + p_{12}\mu_2 + \mu_1 + p_{12}p_{23}\,\mu_3$$

And

$$D = 1 - p_{12}p_{23}p_{30}$$

4. ANALYSIS OF AVAILABILITY

Following recursive relation are obtained by using the probabilistic theory and probability $A_i(t)$ of unit entering into upstate at given time t when the unit entered in regenerative state 'i' at t=0.

$$A_0(t) = M_0(t) + q_{01}(t) \quad A_1(t)$$

$$A_1(t) = M_1(t) + q_{12}(t) \quad A_2(t) + q_{15}^{(4)}(t) \quad A_5(t)$$

$$A_2(t) = M_2(t) + q_{23}(t) \quad A_3(t) + q_{26}^{(7)}(t) \quad A_6(t)$$

$$A_3(t) = M_3(t) + q_{30}(t)E\, A_0(t) + q_{31}^{(8)}(t) \quad A_1(t)$$

$$A_5(t) = q_{56}(t) \quad A_6(t)$$

$$A_6(t) = q_{61}(t) \quad A_1(t)$$

Where

$M_0(t) = e^{-\lambda t}dt,\qquad M_1(t) = e^{-\lambda t}\,\overline{G_1(t)}dt$

$M_2(t) = e^{-\lambda t}dt\,\overline{G_2(t)}dt$ and $M_3(t) = e^{-\lambda t}\,\overline{G_3(t)}dt$

$A^*_0(s)$ was calculated by Laplace transforms and crammer's rule we obtained:

$$A_0^*(s) = \frac{N_1(s)}{D_1(s)}$$

$$N_1(s) = M_0^*(s)[1 - q_{15}^{*4}(s)q_{56}^*(s)q_{61}^*(s)] - q_{01}^*(s)q_{12}^*(s)\,q_{26}^{*7}(s)q_{61}^*(s)$$
$$- q_{12}^*(s)q_{23}^*(s)\,q_{31}^{*(8)}(s)M_0^*(s) + M_1^*(s)q_{01}^*(s) + q_{01}^*(s)q_{12}^*(s)M_2^*(s)$$
$$+ q_{01}^*(s)q_{12}^*(s)q_{23}^*(s)M_3^*(s)$$

$$D_1(s) = 1 - q_{15}^{*4}(s)q_{56}^*(s)q_{61}^*(s) - q_{12}^*(s)q_{26}^{*7}(s)q_{61}^*(s)$$
$$- q_{01}^*(s)q_{12}^*(s)q_{23}^*(s)q_{30}^*(s) - q_{12}^*(s)q_{23}^*(s)q_{31}^{*8}(s)$$

The steady state availability of system is given by

$$A_0 = \lim_{S\to 0}[SA_0^*(s)] = \lim_{S\to 0}\left(S\frac{N_1(s)}{D_1(s)}\right) = \left(\frac{N_1(0)}{D_1(0)}\right) = \frac{N_1}{D_1}$$

Where

$$N_1 = \mu_3\, p_{12}\, p_{23} + \mu_1 + \mu_2\, p_{12}\, p_{23}\, p_{30}$$

$$D_1 = K_1 + p_{12}\, K_2 + \mu_0 p_{12} p_{20} + \mu_0\, p_{12}\, p_{23}\, p_{30} + p_{12}p_{23}K_3$$
$$+ p_{15}^4(\mu_5 + \mu_6) + p_{12}\, p_{26}^{(7)}\,\mu_6$$

Where, K_1, K_2 and $K_3$is already mentioned.

5. ANALYSIS OF BUSY PERIOD OF REPAIRMAN IN RESCUE OPERATION DURING DIGGING OUT

$$B_0^D(t) = q_{01}(t)\quad B_1^D(t)$$

$$B_1^D(t) = w_0(t) + q_{12}(t)\quad B_2^D(t) + q_{15}^{(4)}(t)\quad B_5^D(t)$$

$$B_2^D(t) = q_{23}(t)\quad B_3^D(t) + q_{26}^{(7)}(t)\quad B_6^D(t)$$

$$B_3^D(t) = q_{30}(t)\quad B_0^D(t) + q_{31}^{(8)}(t)\quad B_1^D(t)$$

$$B_5^D(t) = q_{56}(t)\quad B_6^D(t)$$

$$B_6^D(t) = q_{61}(t)\quad B_1^D(t)$$

Where

$$W_1(t) = e^{-\lambda t}\,\overline{G_1(t)}dt + \lambda\, e^{-\lambda t}\,\overline{G_1(t)}dt$$

We obtain the B_0^{*D}(s), by using Laplace transformation and crammer's rule on the above system of equations:

$$B_0^{*D}(s) = \frac{N_2(s)}{D_1(s)}$$

Where

$$N_2(s) = q_{01}^*(s)\, w_1^*(s)$$

Total fraction of time in steady state when system is under repair during digging out

$$B_0^* = \lim_{S\to 0}[SB_0^{*D}(s)] = \lim_{S\to 0}\left(S\frac{N_2(s)}{D_1(s)}\right) = \left(\frac{N_2(0)}{D_1(0)}\right) = \frac{N_2}{D_1}$$

N_2=W_1

WhereW_1=W_1^*(0) and D_1 is given above.

6. ANALYSIS OF BUSY PERIOD OF REPAIRMAN IN RESCUE OPERATION DURING SNOW REMOVING

$$B_0^{SR}(t) = q_{01}(t)\quad B_1^{SR}(t)$$

$$B_1^{SR}(t) = q_{12}(t)\quad B_2^{SR}(t) + q_{15}^{(4)}(t)\quad B_5^{SR}(t)$$

$$B_2^{SR}(t) = w_2(t) + q_{23}(t)\quad B_3^{SR}(t) + q_{26}^{(7)}(t)\quad B_6^{SR}(t)$$

$$B_3^{SR}(t) = q_{30}(t)\quad B_0^{SR}(t) + q_{31}^{(8)}(t)\quad B_1^{SR}(t)$$

$$B_5^{SR}(t) = w_5(t) + q_{56}(t)\quad B_6^{SR}(t)$$

$$B_6^{SR}(t) = q_{61}(t)\quad B_1^{SR}(t)$$

Where

$$W_2(t) = e^{-\lambda t}\,\overline{G_2(t)}dt + \lambda\, e^{-\lambda t}\,\overline{G_2(t)}dt \text{ and } W_5(t) = \overline{G_2(t)}dt$$

We obtain the B_0^{*SR}(s), by using Laplace transformation and crammer's rule on the above system of equations:

$$B_0^{*SR}(s) = \frac{N_3(s)}{D_1(s)}$$

Where,

$$N_3(s) = q_{01}^*(s) q_{15}^{*4}(s) w_5^*(s) + q_{01}^*(s) q_{12}^*(s) w_2^*(s)$$

Total fraction of time in steady state when system is under repair during snow removing

$$B_0^{SR} = \lim_{S\to 0}[SB_0^{*SR}(s)] = \lim_{S\to 0}\left(S\frac{N_3(s)}{D_1(s)}\right) = \left(\frac{N_3(0)}{D_1(0)}\right) = \frac{N_3}{D_1}$$

Where $N_3 = W_5\, p_{15}^{(4)} + p_{21}\, W_2$

Where $W_2 = W_2^*$ (0), $W_5 = W_5^*$ (0) and D_1 in mentioned already.

7. ANALYSIS OF BUSY PERIOD OF REPAIRMAN IN RESCUE OPERATION DURING HOSPITALIZATION IN RESCUE OPERATION

$$B_0^H(t) = q_{01}(t) \quad B_1^H(t)$$

$$B_1^H(t) = q_{12}(t) \quad B_2^H(t) + q_{15}^{(4)}(t) \quad B_5^H(t)$$

$$B_2^H(t) = q_{23}(t) \quad B_3^H(t) + q_{26}^{(7)}(t) \quad B_6^H(t)$$

$$B_3^H(t) = w_3(t) + q_{30}(t) \quad B_0^H(t) + q_{31}^{(8)}(t) \quad B_1^H(t)$$

$$B_5^H(t) = q_{56}(t) \quad B_6^{HR}(t)$$

$$B_6^H(t) = w_6(t) + q_{61}(t) \quad B_1^H(t)$$

Where

$$W_3(t) = e^{-\lambda t}\,\overline{G_3(t)}dt + \lambda\, e^{-\lambda t}\,\overline{G_3(t)}dt \text{ and } W_6(t) = \overline{G_3(t)}dt$$

We obtain the B_0^{*H}(s), by using Laplace transformation and crammer's rule on the above system of equations:

$$B_0^{*H}(s) = \frac{N_4(s)}{D_1(s)}$$

$$N_4(t) = q_{01}^*(s)\, q_{12}^*(s) q_{23}^*(s) w_3^*(s) + q_{01}^*(s)\, q_{15}^{*4}(s) q_{56}^*(s)$$
$$+ q_{01}^*(s)\, q_{12}^*(s) q_{26}^{*(7)}(s) w_6^*(s)$$

$and\ D_1(s)$ is already mentioned.

Total fraction of time in steady state when system is under repair during hospitalization in rescue operation:

$$B_0^H = \lim_{S \to 0}[SB_0^{*H}(s)] = \lim_{S \to 0}\left(S\frac{N_4(s)}{D_1(s)}\right) = \left(\frac{N_4(0)}{D_1(0)}\right) = \frac{N_4}{D_1}$$

Where

$$N_4 = p_{12}\, p_{23}\, w_3 + p_{15}^4\, p_{56}\, w_6 + p_{12}\, p_{26}^7\, w_6$$

And $W_3 = W_3^*(0), W_6 = W_6^*(0)\ and\ D_1$ is mentioned already.

8. EXPECTED NUMBER OF VISITS BY THE REPAIRMAN

$V_0(t)$ = The expected number of visits by the repairman when the system state in regenerative i.e. 'i' at t=0

$$V_0(t) = Q_{01}(t) ⓢ (1 + V_1(t)$$

$$V_1(t) = Q_{12}(t) ⓢ V_2(t) + Q_{15}^{(4)}(t) ⓢ V_5(t)$$

$$V_2(t) = Q_{23}(t) ⓢ V_3(t) + Q_{26}^7(t) ⓢ V_6(t)$$

$$V_3(t) = Q_{30}(t) ⓢ V_0(t) + Q_{31}^{(8)}(t) ⓢ V_1(t)$$

$$V_5(t) = Q_{56}(t) ⓢ V_6(t)$$

$$V_6(t) = Q_{61}(t) ⓢ V_1(t)$$

We obtain the $V_0^*(s)$, by using Laplace –Stielltjes Transformation and crammer's rule on the above system of equations:

$$V_0^{**}(s) = \frac{N_5(S)}{D_1(S)}$$

Where

$$N_5(S) = Q_{01}^{**}(s) - Q_{01}^{**}(s)Q_{12}^{**}(s)Q_{23}^{**}(s)\,Q_{31}^{**8}(s) - Q_{01}^{**}(s)Q_{15}^{**(4)}(s)Q_{56}^{**}(s)\,Q_{61}^{**}(s)$$
$$-Q_{01}^{**}(s)Q_{12}^{**}(s)Q_{26}^{**7}(s)\,Q_{61}^{**}(s)$$

And D_1 (s) is already specified

$$V_0 = \lim_{S\to 0}(SV_0^*(s)) = \lim_{S\to 0}\frac{SN_5(s)}{D_1(s)}$$
$$= \frac{N_1(0)}{D_1(0)} = \frac{N_5}{D_1}$$

Where $N_5 = p_{12}\,p_{23}\,p_{30}$ and D_1 is already mentioned.

9. ANALYSIS OF COST-BENEFIT

The expected profit of the system incurred in steady state is given by

$P=C_0-C_{11}B_0^{D}-C_{12}B_0^{SC}-C_{13}B_0^{H}-C_2V_0$

where,

C_0=total cost per unit up time of the system

C_{11}=cost of per unit time of repairman when it is busy during digging out in rescue operation

C_{12}= cost of per unit time of repairman when it is busy during snow removing in rescue operation

C_{13}= cost of per unit time of repairman when it is busy during hospitalization in rescue operation

C_2=Cost taken by the repairman for each visit

10. PARTICULAR CASES

Numerical result for the particular cases the following case is considered. we take repair rate as exponentially distributed.

$g_1(t)=\alpha_1 e^{-\alpha_1 t}$, $g_2(t)=\alpha_2 e^{-\alpha_2 t}$ and $g_3(t)=\alpha_3 e^{-\alpha_3 t}$

$p_{01} = 1$, $p_{12}=\frac{\alpha_1}{\lambda+\alpha_1}$, $p_{14} = \lambda/(\lambda+\alpha_1)$, $p_{15}^{(4)}=\frac{\lambda}{\lambda+\alpha_1}$,

$p_{23} = \alpha_2/(\lambda+\alpha_2)$, $p_{27} = \lambda/(\lambda+\alpha_2)$, $p_{21}^{(7)} = \frac{\lambda}{\lambda+\alpha_2}$,

$p_{30} = \alpha_3/(\lambda+\alpha_3)$, $p_{38} = \lambda/(\lambda+\alpha_3)$, $p_{31}^{(8)} = \frac{\lambda}{\lambda+\alpha_3}$,

$p_{56} = 1, p_{61} = 1$

$\mu_5=1/\alpha_2$, $\mu_6=1/\alpha_3$ $k_1=1/\alpha_1$, $k_2=1/\alpha_2$, $k_3=1/\alpha_3$, $\mu_0=1/\boldsymbol{\lambda}$, $\mu_1=1/(\lambda+\alpha_1)$, $\mu_2=1/(\lambda+\alpha_2)$

11. GRAPHICAL EXPLANATION

Parameters are replaced by numerical values then we were plotting multiple ghaphs with a particular case and derived the following interpretations.

The operating behaviour unit for several repair rate value ($\boldsymbol{\alpha}_1$) from the chart we can see MTSF, regarding the failure rate ($\boldsymbol{\lambda}$) MTSF decreases as it increases, but with higher values for higher values for higher values for higher values of repair rate ($\boldsymbol{\alpha}_1$) as shown in fig.2.

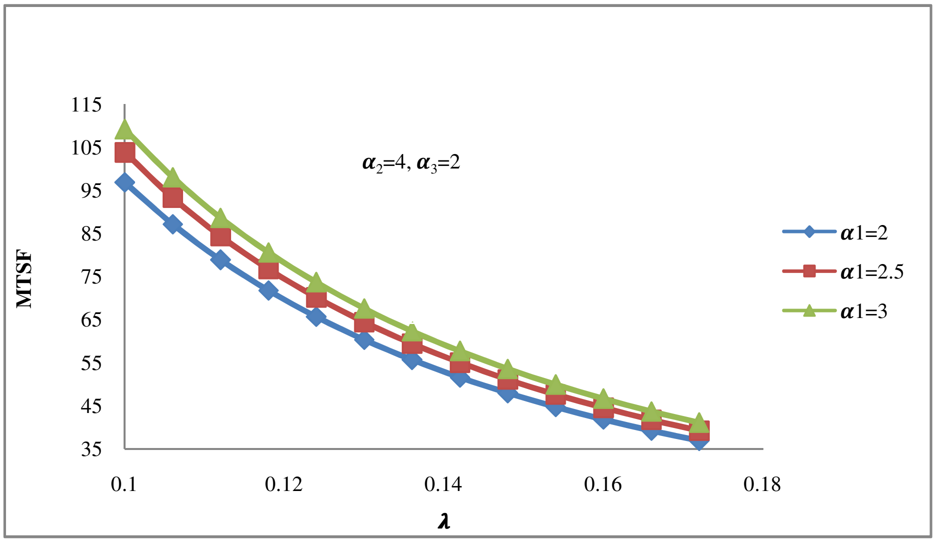


Fig.2: MTSF versus Failure Rate (λ) for Different Value of Repair Rate (α1)

Availability behaviour (A_0) with respect to failure rate (λ) of the operating unit for different repair rate value (α_1). From the chart, we can see that A_0 decreases as it increases failure rate (λ), but has more values for more value of repair rate (α_1) as shown in **fig.3.**

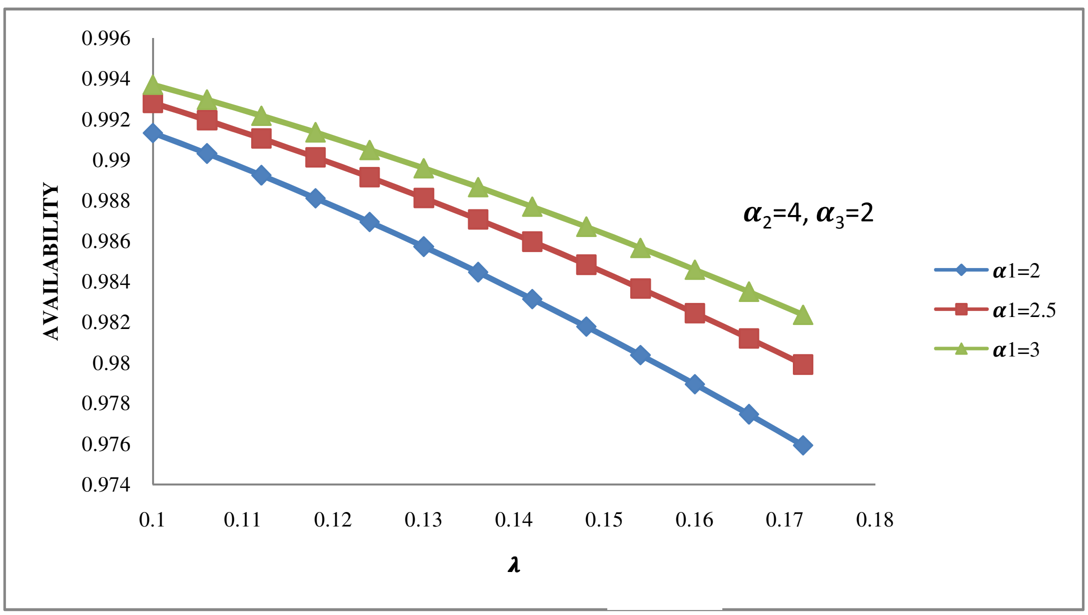


Fig 3: Failure rate (λ) vs availability (A_0) at various values of repair rate (α_1)

Improvement in performance of benefit (P) relative to revenue cost (C_0) for different values of cost per visit (C_2) of the repairman. It can be seen that benefit (P) increases to increase revenue cost C_2 but decreases for higher values of C_0 as shown in **fig.4.**

The following result are found from the fig.4

For C_2=400, C_2=700 and C_2=1000 we fixed revenue cost atleast 1667.967, 1763.978 and 1864.987 respectively so that the profit will be always positive.

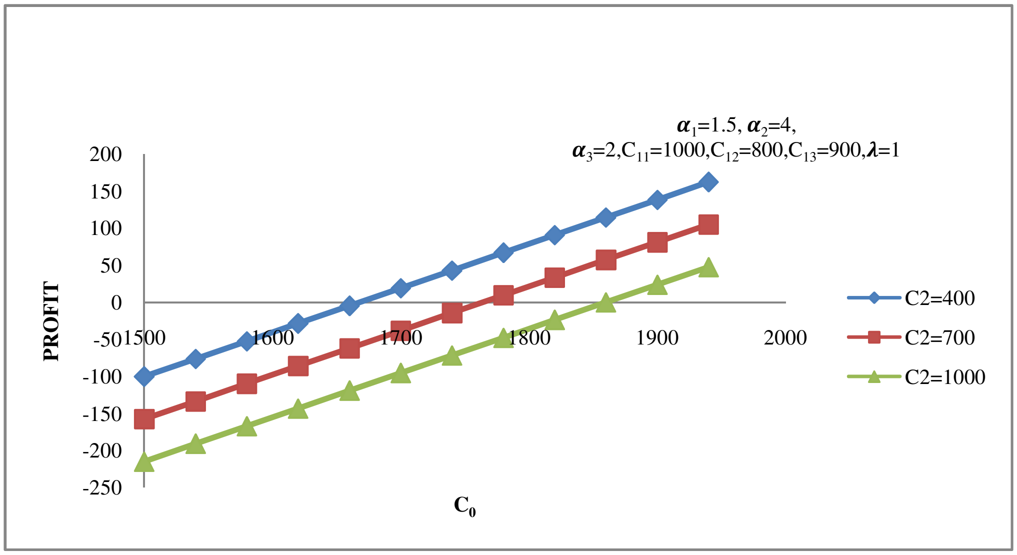


Fig.4: Profit (p) versus revenue (C_0) for different values of cost per visit of repairman (C_2)

Describe the improvement in performance of benefit (P) relative to the failure rate (λ) of the operating unit for different repair rate values α_1, it can be seen that benefit (P) decreases to increase λ but has higher values for higher value α_1 from the **fig.5.**

The fig.5 display the following result.

For α_1=2, α_1=2.5 and α_1=3 we fixed failure rate atmost 1.2262, 1.2816 and 1.3267 respectively so that the profit will be always positive.

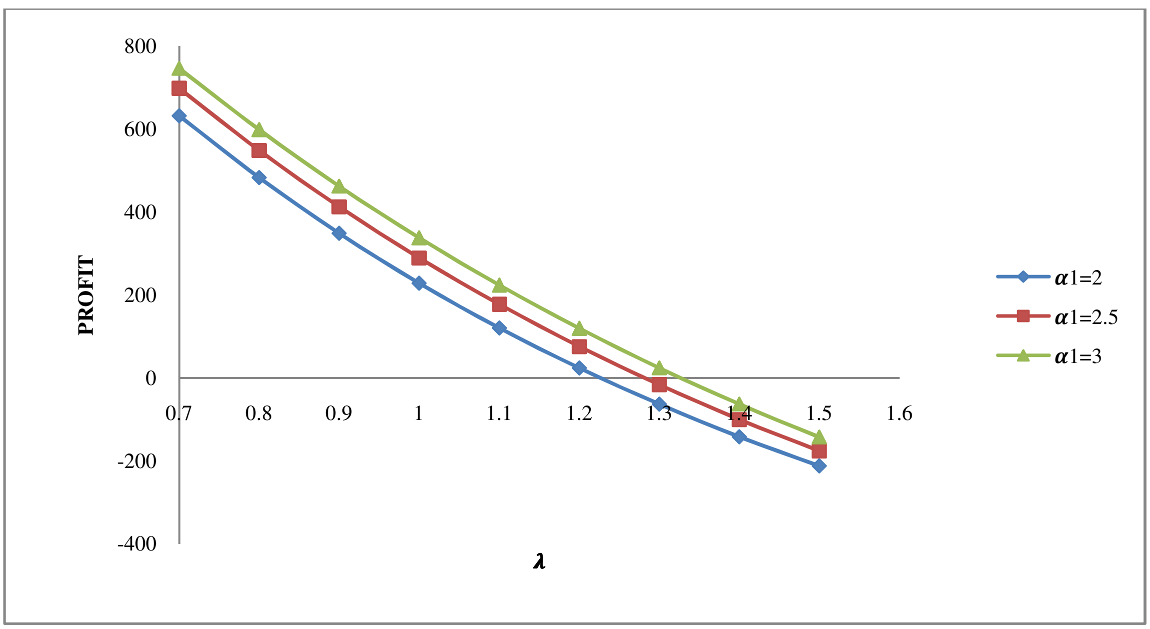


Fig 5: Failure rate (λ) vs profit (p) at various values of repair rate (α_1)

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3-COLORABLE PLANAR GRAPHS

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ABSTRACT :

In this paper, we consider 3-colorable planar graphs. The Four Color Theorem (4CT) gives an O(n2) time algorithm to 4-color any planar graph. Graph coloring for 3-colorable graphs receives very much attention by many researchers in theoretical computer science. Deciding 3-colorablility of a graph is a well-known NP-complete problem. We give a very simple O (n2) time algorithm to 4-color 3-colorable planar graphs.

Keywords: The Four color Theorem, 3-colorable graph.
MSC Code: 05C15

1. INTRODUCTION

Graph coloring is arguably the most popular subject in graph theory. Also, it is one of the central problems in combinatorial optimization, since it is one of the hardest problems to approximate. In general, the chromatic number is in approximable in polynomial time within factor $n^{1-\epsilon}$ for any $\epsilon > 0$, unless coRP = NP[8],[9].

Graph coloring for 3-colorable graphs receives much attention by many researchers in theoretical computer science. Let us first observe that deciding 3-colorability is one of the well-known NP-complete problems. The first nontrivial result was due to gave an $O(\sqrt{n})$ -approximation algorithm. The factor $O(\sqrt{n})$ was improved to $O(n^{3/8})$ [3]. Then it[4] observed that it is possible to use the method[7] to improve to $O(n^{3/14})$, which is where things stood for a decade. Currently, the best known polynomial approximation algorithm achieves a factor of $O(n^{0:2072})$[5]. For the negative side, gave some evidence[6] that assuming the well-known Khot's unique game conjecture [10]. It is not possible to give a coloring of 3-colorable graph, using at most c colors for some absolute constant c in polynomial time. It is a well-known open problem whether or not a 3-colorable graph can be colored using at most O(poly(logn)) colors in polynomial time.

The fact that an input graph is 3-colorable does not help much for graph coloring. The Four Color Theorem (4CT) [1].[2]gives an $O(n^2)$ time algorithm to 4-color any planar graph. However the current known proof for the 4CT is computer assisted. In fact, for both the unavoidability.

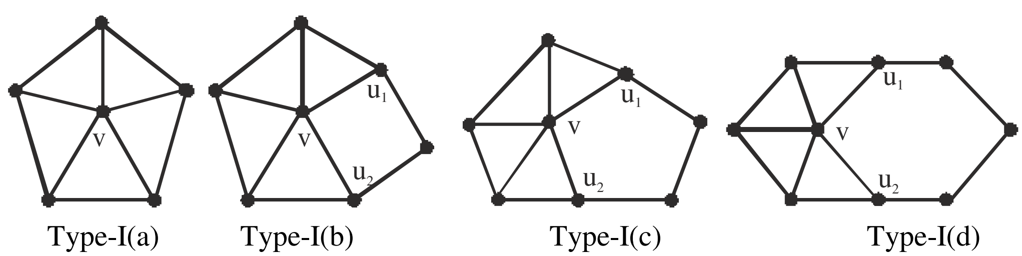

Fig.1. A bad vertex

Our motivation is that if we restrict our attention to 3-colorable planar graphs, we may be able to find a very simple algorithm and a very easy correctness.

DEFINITION1.1

A graph *G* is planar if it can be represented by a drawing in the plane so that no edges cross.

2. ALGORITHM

Let G be a plane graph. We define V(G), F(G) by the set of vertices of G and the set of faces of G, respectively. We also define G^* by the dual graph of G. Let v be a vertex of G, and let f be a face of G. Let us denote the degree of v in G by $d_G(v)$ Since G is a plane graph and the size of the face f is exactly the same as the degree of f in the dual graph G^*, we can define $d_{G^*}(f)$ as the size of the face f. Note that the dual graph G^* may have multiple edges or loops, but we allow them and consider the degree in G containing them. A quadrangle is a face of size 4 ,a pentagon is a face of size 5and a hexagon is a face of size 6 . We need to define some special configurations. For a simple plane graph G, we call a vertex v of degree five bad if all faces incident with v, except for at most one, are triangles and the exceptional face has size at most five. Moreover, v is Type I(a), Type I(b) ,Type I(c) and Type I(d) if the exceptional face is a triangle, a quadrangle, and a pentagon, and a hexagon respectively. From Fig.1. We have the following restrictions for a 3-coloring around a bad vertex.

Case 1

If a plane graph G has a bad vertex of Type I(a), then G is not 3-colorable.

Case 2

Suppose that a plane graph G has a bad vertex v of Type I(b) or Type I(c) or Type I(d), and G is 3-colorable. Then u_1 and u_2 are contained in the same color class for any 3-coloring of G, where u_1 and u_2 are two neighbors of v which are contained in a quadrangle or pentagon or hexagon In other words, G is 3-colorable if and only if G' is, where G' is obtained from G by identifying u_1 and u_2.

Lemma 2.1

Every simple planar graph G contains (i) a vertex of degree at most 4 or (ii) a bad vertex.

Proof

We define the initial charge function w for each $V(G) \cup F(G)$ as $w(v) := d_G(v) - 6$ for $v \in V(G)$ and $w(f) := 2d_{G^*}(f) - 6$ for $f \in F(G)$. By Euler's formula, we have $\sum_{v\in V(G)} w(v) + \sum_{f\in F(G)} w(f) = -12$. The new function w^* is obtained by the following discharging rules:

(R1) Send ½ from each quadrangle or pentagon or hexagon to each of incident vertex of degree five.

(R2) Send 1 from each face of size at least six to each of incident vertex of degree five.

We will show that if G has no bad vertex nor a vertex of degree at most 4, then $w^*(v) \geq 0$ for any $v \in V(G)$ and $w^*(f) \geq 0$ for any $f \in F(G)$. This would contradict that

$$\sum_{v\in V(G)} w^*(v) + \sum_{f\in F(G)} w^*(f) = \sum_{v\in V(G)} w(v) + \sum_{f\in F(G)} w(f) = -12.$$

Let $f \in F(G)$. Since G is simple, $d_{G^*}(f) \geq 3$ for any $f \in F(G)$. Then we have

$$w^*(f) \geq \begin{cases} w(f) = 0 & \text{if } d_{G^*}(f) = 3, \\ w(f) = -4 \,.\, \dfrac{1}{2} = 0 & \text{if } d_{G^*}(f) = 4, \\ w(f) = -5 \,.\, \dfrac{1}{2} = \dfrac{3}{2} & \text{if } d_{G^*}(f) = 5, \\ w(f) - d_{G^*}(f) \geq 0 & \text{otherwise.} \end{cases}$$

Thus for each face f, $w^*(f) \geq 0$.

It remains to prove that $w^*(v) \geq 0$ for every vertex v. Take a vertex $v \in V(G)$. We may assume that $d_G(v) \geq 5$, or there is a vertex of degree at most four (in this case, we are done). If $d_G(v) \geq 6$, then $w^*(v) = w(v) \geq 0$. Thus, we may assume that $d_G(v) = 5$. If v is incident with a face of size at least 6, then $w^*(v) \geq w(v) + 1 = 0$. This implies that v is only incident with a face of size at most 5. So, if G has no bad vertex, then v must be incident with at least two faces of size 4 or 5, In either case, we have $w^*(v) \geq 0$. Hence the proof.

Proposition 2.1

Let G be a plane graph with a 4-coloring c' and let f be a face of size at least four. Take four vertices x_1, x_2, x_3, x_4 in f. Then G also has a 4-coloring such that at most three colors are used for x_1, x_2, x_3, x_4. Moreover, given the graph G and the coloring c', we can find such a 4-coloring in O(n) time, where n = |G|.

Proof

Suppose that a 4-coloring c' of G uses 4 colors for x_1, x_2, x_3, x_4, say x_i is colored by the color i. Let us recall that an (i, j)-Kempe chain containing a vertex v is the component induced by the color classes i and j that contains the vertex v. Consider the (1, 3)-Kempe chain containing x_1. If it does not contain x_3, then by changing colors 1 and 3 in the Kempe chain, we can color x_1 by the color 3, which gives rise to a desired coloring. If it contains x_3, then by the planarity, the (2, 4)-Kempe chain containing x_2 does not contain x_4, and hence by changing colors 2 and 4 in the (2, 4)-Kempe chain containing x_2, we can color x_2 by the color 4, which gives rise to a desired 4-coloring.Clearly, given the graph G and the coloring c', this change takes only O(n) time.

Theorem 2.1

Let G be a 3-colorable planar graph. Then there exists an $O(n^2)$-time algorithm to find a 4-coloring of G, where n = |G|.

Proof:

Given a plane graph G with n vertices, which is 3-colorable, we have to give a 4-coloring. We may assume that G is simple.

Step 1

If the current graph has at most three vertices, we just give a 3-coloring. Otherwise, delete a vertex of degree at most 4 in the current graph G.

Step 2

Apply the linear time algorithm to find a bad vertex v. By Lemma 2.1, G has (i) a vertex of degree at most 4 or (ii) a bad vertex. At the moment, there is no vertex of degree at most 4. Thus we may assume that there is such a bad vertex v. By Case1, v is not Type I(a), so is Type I(b) or I(c) and Type I(c) or I(d). The linear time algorithm will find a subgraph isomorphic to one of two graphs (Type I(b) and Type I(c), Type I(c) and Type I(d)) drawn in Fig. 1. We then construct the new graph G' by identifying two vertices u_1 and u_2, as in Case 2. It follows from Case 2 that G' is also 3-colorable. Then rerun the algorithm on G'.

Step 3

Extend the coloring c' of the current graph to the original graph G in O(n) time.

At the moment, we have a 4-coloring c of the current graph G'. We need to extend the c to the original input graph G .In Step 2, we identify two vertices u_1; u_2. The reverse operation clearly extends the coloring c. In Step 1, we delete a vertex v of degree at most 4. We now need to put a vertex v back to G' to obtain the resulting graph G. If $d_G(v) \geq 3$, then the coloring c can be easily extended for v. If $d_G(v) = 4$, then we can also change the coloring c' of G' to color v by Proposition 2.1. All of these processes can be done in O(n) time by Proposition 2.1.

Both steps 1, 2 and 3 can be implemented in linear time. Another factor of n pops up because of applying the recursion. Hence we can find a 4-coloring in $O(n^2)$ time.

CONCLUSION

Hence, we conclude that Four Color Theorem (4CT) gives an $O(n^2)$ time algorithm to 4-color any planar graph and then we can apply to this plane graph. The study of theoretical and algorithmic problems related to this field, known as graph drawing is an area of growing interest with several publications

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NUMERICAL DATA ON A PAIR OF VARIABLES: REPRESENTATION BY MAKEHAM'S CURVE

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ABSTRACT :
Some methods have already been developed for representing a set of numerical data on a pair of variables by some mathematical curves namely polynomial curve, exponential curve, modified exponential curve and logistic curve. A set of available numerical data on a pair of variables are required to be represented by different types of mathematical curves in different situations. In some situation, it is required to represent the data as mentioned by a Makeham's curve. In this study, attempt has been made on the representation of numerical data by this mathematical curve. This paper describes (i) the method of representing numerical data on a pair of variables by Makeham's curve and (ii) application of the method in representing total population of India by this curve as an example of showing the application of the method to numerical data.

Key Words:Pair of variables, numerical data, representation by Makeham's curve

1. INTRODUCTION:

The existing formulae for numerical interpolation {Hummel (1947), Erdos & Turan (1938), Bathe & Wilson (1976), Jan (1930), Hummel (1947) et al} carry the task of repetitive computations i.e. if it is wanted to interpolate the values of the dependent variable corresponding to a number of values of the independent variable by a suitable existing interpolation formula then it is required to apply the formula for each value separately and thus the numerical computation of the value of the dependent variable based on the given data are to be performed in each of the cases. In order to get rid of these repeated numerical computations from the given data, one can think of an approach which consists of the representation of the given numerical data by a suitable polynomial and then to compute the value of the dependent variable from the polynomial corresponding to any given value of the independent variable. However, a method is necessary for representing a given set of numerical data *on a pair of variables* by a suitable polynomial. Das & Chakrabarty (2016*a*, 2016*b*, 2016*c* & 2016*d*) derived four formulae for representing numerical data on a pair of variables by a polynomial curve. They derived the formulae from Lagranges Interpolation Formula, Newton's Divided Difference Interpolation Formula, Newton's Forward Interpolation Formula and Newton's Backward Interpolation respectively. In another study, Das & Chakrabarty (2016*e*) derived one method for representing numerical data on a pair of variables by a polynomial curve. The *method derived is based on the inversion of a square matrix by* Caley-Hamilton theorem on characteristic equation of matrix *[*Cayley *(1858 , 1889) & Hamilton (1864a , 1864b , 1862)]. In a separate study* Das & Chakrabarty (2016*f* , 2016*g*) composed two methods, *based on the inversion of matrix by elementary transformation,* for the same purpose. The studies, made so far, are on the representation of numerical data on a pair of variables by polynomial curve It is be possible to represent the numerical data on a pair of variables by non-polynomial curve besides the representation of the said data by polynomial curve. Some methods have already been developed for representing the said data by some mathematical curves namely polynomial curve [Das & Chakrabarty (2016*a*, 2016*b* , 2016*c* , 2016*d* , 2016*e* , 2016*f* , 2017*a,* 2017*b*)], exponential curve [Das & Chakrabarty (2017*c*)] and modified exponential curve [Das & Chakrabarty (2017*d*)]. A set of available numerical data on a pair of variables are required to be represented by different types of mathematical curves in different situations. In some situation, it is required to represent the data as mentioned by a Makeham's curve. *In this study,* attempt has been made on the representation of numerical data by this mathematical curve. *This paper describes (i) the method of representing numerical data on a pair of variables by* Makeham's curve and (ii) application of the method in representing total population of India by this curve as an *example of showing the application of the method to numerical data.*

2. REPRESENTATION OF NUMERICAL DATA BY *MAKEHAM'S CURVE*:

The Makeham's curve is of the form

$$y = a\, b^x c^{d^x} \tag{2.1}$$

where a , b, c and d are parameters

Let

$$y_0 ,\ y_1 ,\ y_2 ,\ y_3$$

be the values of y corresponding to the values of x_0 , x_1 , x_2 *and* x_3 *of* x *respectively.*

Then the points

$$(x_0, y_0) , (x_1, y_1) , (x_2, y_2) \text{ and } (x_3, y_3)$$

lie on the curve described by equation (2.1).

Therefore,

$$y_0 = a\, b^{x_0}\, c^{d^{x_0}} \tag{2.2}$$
$$y_1 = a\, b^{x_1}\, c^{d^{x_1}} \tag{2.3}$$
$$y_2 = a\, b^{x_2}\, c^{d^{x_2}} \tag{2.4}$$
$$y_3 = a\, b^{x_3}\, c^{d^{x_3}} \tag{2.5}$$

$$\therefore\ (2.2) \Rightarrow log\ y_0 = log\ a + x_0\ log\ b + d^{x_0}\ log\ c \tag{2.6}$$
$$(2.3) \Rightarrow log\ y_1 = log\ a + x_1\ log\ b + d^{x_1}\ log\ c \tag{2.7}$$
$$(2.4) \Rightarrow log\ y_2 = log\ a + x_2\ log\ b + d^{x_2}\ log\ c \tag{2.8}$$
$$(2.5) \Rightarrow log\ y_3 = log\ a + x_3\ log\ b + d^{x_3}\ log\ c \tag{2.9}$$

If x_0 , x_1 , x_2 , x_3 are equally spaced then

$$x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = h$$

i.e. $x_1 = x_0 + h \quad x_2 = x_0 + 2h$ and $x_3 = x_0 + 3h$

Equation (2.7) – Equation (2.6) $\Rightarrow log\ y_1 - log\ y_0 = (x_1 - x_0)\ log\ b + (d^h - 1)\ log\ c\ d^{x_0}$

$$\Rightarrow\ \Delta\ log\ y_0 = h\ log\ b + (d^h - 1)\ log\ c\ d^{x_0} \tag{2.10}$$

Equation (2.8) – Equation (2.7) $\Rightarrow log\ y_2 - log\ y_1 = (x_2 - x_1)\ log\ b + (d^h - 1)\ log\ c\ d^{x_1}$

$$\Rightarrow\ \Delta\ log\ y_1 = h\ log\ b + (d^h - 1)\ log\ c\ d^{x_1} \tag{2.11}$$

Equation (2.9) – Equation (2.8) $\Rightarrow log\ y_3 - log\ y_2 = (x_3 - x_2)\ log\ b + (d^h - 1)\ log\ c\ d^{x_2}$

$$\Rightarrow\ \Delta\ log\ y_2 = h\ log\ b + (d^h - 1)\ log\ c\ d^{x_2} \tag{2.12}$$

Again ,

Equation (2.11) – Equation (2.10) $\Rightarrow \Delta\ log\ y_1 - \Delta\ log\ y_0 = (d^h - 1)\ log\ c\ d^{x_1} - (d^h - 1)\ log\ c\ d^{x_0}$

$$\Rightarrow \Delta^2\ log\ y_0 = (d^h - 1)\ log\ c\ (d^{x_1} - d^{x_0})$$
$$\Rightarrow \Delta^2\ log\ y_0 = (d^h - 1)\ log\ c\ (d^{x_0 + h} - d^{x_0})$$

$$\Rightarrow \Delta^2 \log y_0 = (d^h - 1) \log c \, (d^h - 1) \, d^{x_0}$$

$$\Rightarrow \Delta^2 \log y_0 = (d^h - 1)^2 \log c \, d^{x_0} \tag{2.13}$$

Similarly,

$$\text{Equation (2.12)} - \text{Equation (2.11)} \Rightarrow \Delta^2 \log y_1 = (d^h - 1)^2 \log c \, d^{x_1} \tag{2.14}$$

$$\therefore \frac{(2.14)}{(2.13)} \Rightarrow \frac{\Delta^2 \log y_1}{\Delta^2 \log y_0} = \frac{d^{x_1}}{d^{x_0}} = \frac{d^{x_0 + h}}{d^{x_0}} = \frac{d^{x_0} . d^h}{d^{x_0}} = d^h \tag{2.15}$$

If $h = 1$, $\boldsymbol{d = \frac{\Delta^2 \log y_1}{\Delta^2 \log y_0}}$

$\therefore$ (2.13) $\Rightarrow \Delta^2 \log y_0 = (d - 1)^2 \log c$ (If $x_0 = 0$)

$$\Rightarrow \log c = \frac{\Delta^2 \log y_0}{(d-1)^2}$$

$$\Rightarrow \boldsymbol{c = antilog \left\{ \frac{\Delta^2 \log y_0}{(d-1)^2} \right\}} \tag{2.16}$$

(2.10) $\Rightarrow \Delta \log y_0 = \log b + (d - 1) \log c$

$$\Rightarrow \Delta \log y_0 = \log b + (d - 1) . \frac{\Delta^2 \log y_0}{(d-1)^2}$$

$$\Rightarrow \Delta \log y_0 = \log b + \frac{\Delta^2 \log y_0}{(d-1)}$$

$$\Rightarrow \log b = \Delta \log y_0 - \frac{\Delta^2 \log y_0}{(d-1)}$$

$$\Rightarrow \boldsymbol{b = antilog \left[\Delta \log y_0 - \frac{\Delta^2 \log y_0}{(d-1)}\right]} \tag{2.17}$$

(2.6) $\Rightarrow \log y_0 = \log a + x_0 \log b + \log c$

$$\Rightarrow \log y_0 = \log a + \log c$$

$$\Rightarrow \log a = \log y_0 - \log c$$

$$\Rightarrow \log a = \log y_0 - \frac{\Delta^2 \log y_0}{(d-1)^2}$$

$$\Rightarrow \boldsymbol{a = antilog \left\{\log y_0 - \frac{\Delta^2 \log y_0}{(d-1)^2}\right\}} \tag{2.18}$$

Note: It is to be noted that the parameters $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ & $\boldsymbol{d}$ are to computed in the following order:

Step-1: Compute parameter a by equation (2.18).

Step-2: Compute parameter b by equation (2.17) substituting the value of a obtained in Step-1.

Step-3: Compute parameter c by equation (2.16) substituting the values of a & b obtained in Step-1.& Step-2 respectively.

Step-4: Compute parameter d by equation (2.15) substituting the values of a, b & c obtained in Step-1, Step-2 & Step-3 respectively.

3. EXAMPLE OF DATA REPRESENTATION BY MAKEHAM'S CURVE*:*

Example -*3.1:* The following table shows the data on total population of India corresponding to the years:

Table-3.1.1
(Total Population of India)

Year	1951	1961	1971	1981
Total Population	361088090	439234771	548159652	683329097

Taking 1971 as origin and changing scale by 1/10, one can obtain the following table for independent variable x (representing time) and $f(x)$ (representing total population of India):

Table-3.1.2

Year	1951	1961	1971	1981
x_i	0	1	2	3
$y_i = f(x_i)$	361088090	439234771	548159652	683329097

Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$

$f(x_0) = 361088090$, $f(x_1) = 439234771$, $f(x_2) = 548159652$, $f(x_2) = 683329097$

Table-3.1.3

x_i	y_i	$log\ y_i$	$\Delta\ log\ y_i$	$\Delta^2\ log\ y_i$
0	361088090	19.704632503150		
			0.195912110819	
1	439234771	19.900544613969		0.02562041346
			0.221532524279	
2	548159652	20.122077138248		− 0.00112252063
			0.220410003649	
3	683329097	20.342487141897		

Now, $d = \frac{\Delta^2\ log\ y_1}{\Delta^2\ log\ y_0} = \frac{-\ 0.00112252063}{0.02562041346} = -\ 0.043813525170$

$c = antilog\ \{\ \frac{\Delta^2\ log\ y_0}{(d-1)^2}\ \}$

$= antilog\ \{\ \frac{0.02562041346}{(-\ 0.043813525170 - 1\ )^2}\ \}$

$= antilog\ \{\ \frac{0.02562041346}{(-\ 1.04381352517\ )^2}\ \}$

$= antilog\ \{\ \frac{0.02562041346}{1.089546675327}\ \}$

$= antilog\ \{0.023514746123\}$

$= 1.023793397619$

$\Rightarrow log\ c = 0.010212324484$

$b = antilog\ [\Delta\ log\ y_0 - \frac{\Delta^2\ log\ y_0}{(d-1)}\ ]$

$= antilog\ [0.195912110819 + \frac{0.02562041346}{1.04381352517}\ ]$

$= antilog\ [0.195912110819 + 0.024545010044\ ]$

$= antilog\ [0.220457120863]$

$= 1.246646468467$

$\Rightarrow log\ b = 0.095743311086$

$$a = antilog\ \{log\ y_0 - \frac{\Delta^2\ log\ y_0}{(d-1)^2}\ \}$$

$$= antilog\ \{\ 19.704632503150 - \frac{0.02562041346}{(-0.043813525170-1)^2}\ \}$$

$$= antilog\ \{\ 19.704632503150 - \frac{0.02562041346}{(-1.04381352517)^2}\ \}$$

$$= antilog\ \{\ 19.704632503150 - \frac{0.02562041346}{1.089546675327}\ \}$$

$$= antilog\ \{\ 19.704632503150 - 0.023514746123\ \}$$

$$= antilog\ \{\ 19.681117757027\ \}$$

$$= 352696247.934125408029$$

$\Rightarrow log\ a = 8.547400839564$

Thus the Makeham's curve satisfying the data is

$$y = a\ b^x c^{d^x}$$

$$= (352696247.934125408029)\ (1.246646468467)^x (1.023793397619)^{(-0.043813525170)^x}$$

This curve yields,

$$y_0 = 352696247.934125408029 \times 1.023793397619 = 361088089$$

$$y_1 = 352696247.934125408029 \times 1.246646468467 \times (1.023793397619)^{-0.043813525170}$$

$$= 439687531.92863888437364825712154 \times \frac{1}{(1.023793397619)^{0.043813525170}}$$

$$= \frac{439687531.92863888437364825712154}{1.00103079482532138358987258291 33} = 439234771$$

$$y_2 = 352696247.934125408029 \times (1.246646468467)^2 \times (1.023793397619)^{(-0.043813525170)^2}$$

$$= 352696247.934125408029 \times 1.554127417341 \times (1.023793397619)^{0.001919624987}$$

$$= 352696247.934125408029 \times 1.554127417341 \times 1.000045140513$$

$$= 548159652$$

Again, we have

$$log\ y_3 = log\ a + x_3\ log\ b + d^{x_3}\ log\ c$$

$$= 8.547400839564 + 3 \times 0.095743311086 + (-0.043813525170)^3 \times 0.010212324484$$

$$= 8.547400839564 + 0.287229933258 - 0.0000841055.377209099340 \times 0.010212324484$$

$$= 8.834630772822 - 0.00000085891304210723407774702 4056$$

$$= 8.83462991390895789276592225297 6$$

$$\Rightarrow y_3 = antilog\ (8.834629913908957892765922252976)$$

$$= 683329097$$

The data on total population corresponding to three consecutive years (at a gap of 10 years) can be represented by the makeham's curve. The curves obtained have been shown in the following table.

Table-3.1.4
(Mathematical curve representing total population at a gap of 10 years)

Years	x	Makeham's curve representing $y_i = f(x_i)$
1951 1961 1971 1981	0 1 2 3	$y = (352696247.934125408029)(1.246646468467)^x(1.023793397619)^{(-0.043813525170)^x}$
1961 1971 1981 1991	0 1 2 3	$Y = (439256180.12431468776914205008708)(1.24827978453195169335549656 2275)^x(0.99995125931740690919515733489522)^{(5.7990067652345792978661532145229)^x}$
1971 1981 1991 2001	0 1 2 3	$y = (548947760.20731398904366203283964)(1.2504058062391221439861870583993)^x(0.99856432931429330465460843078 6)^{(3.1285829926225342920605116690177)^x}$
1981 1991 2001 2011	0 1 2 3	$y = (757632444.35501800141247188856791)(1.2966057408050614219686125621 31)^x(0.90192691996146842340766096240554)^{(1.4441832086082581615597161703986)^x}$

Similarly, the data on total population corresponding to two consecutive years (at a gap of 20 years) can be represented by the makeham's curve. The curves obtained have been shown in the following table:

Table-3.1.5
(Mathematical curve representing total population at a gap of 20 years)

Years	x	Makeham's curve representing $y_i = f(x_i)$
1951 1971 1991 2011	0 1 2 3	$y = (360890060.48806901244027752214465)(1.5227019818642108469676443063929)^x(1.0005487253144715920892566206466)^{(-4.5448066550163454958848230634923)^x}$

Example -3.2: The following table shows the data on total population of Assam corresponding to the years:

Table-3.2.1
(Total Population of *Assam*)

Year	1951	1961	1971	1981
Total Population	8028856	10837329	14625152	18041248

Taking 1971 as origin and changing scale by 1/10, one can obtain the following table for independent variable x (representing time) and $f(x)$ (representing total population of Assam):

Table-3.2.2

Year	1951	1961	1971	1981
x_i	0	1	2	3
$y_i = f(x_i)$	8028856	10837329	14625152	18041248

Here $x_0 = 0, x_1 = 1, x_2 = 2, \; x_3 = 3$
$f(x_0) = 8028856, \; f(x_1) = 10837329, f(x_2) = 14625152, \; f(x_2) = 18041248$

Table-3.1.3

x_i	y_i	log y_i
0	8028856	15.8985526100203103476428928835553
1	10837329	16.1985071213997389633425273822238
2	14625152	16.4982533441905789190407504465592
3	18041248	16.7081712498068403859403516982088

x_i	Δ log y_i	Δ^2 log y_i
0		
	0.29995451137942861569963449868 5	$-$ 0.00020828858858866000141143433 1
1		
	0.29974622279083995569822306435 4	$-$ 0.08982831717457848879862181273 8
2		
	0.20991790561626146689960125161 6	
3		

Now, $d = \frac{\Delta^2 \, log \, y_1}{\Delta^2 \, log \, y_0} = \frac{-\,0.089828317174578488798621812738}{-0.000208288588588660001411434331}$

$= 431.26854804310231676668703693725$

$c = antilog \; \{ \; \frac{\Delta^2 \, log \, y_0}{(d-1)^2} \; \}$

$= antilog \; \{ \; \frac{-\,0.000208288588588660001411434331}{(\,431.26854804310231676668703693725 - 1\,)^2} \; \}$

$= antilog \; \{ \; \frac{-\,0.000208288588588660001411434331}{(\,430.26854804310231676668703693725\,)^2} \; \}$

$= antilog \; \{ \; \frac{-0.000208288588588660001411434331}{185131.02343511944650313478201748} \; \}$

$= antilog \; \{-\, 0.0000000011250874365832925729453 8\}$

$= 0.99999999887491256404961829679604$

$b = antilog \; (\Delta \; log \; y_0 - \; \frac{\Delta^2 \, log \, y_0}{(d-1)} \,)$

$= antilog \; (0.29995451137942861569963449868 5 - \frac{-\;0.000208288588588660001411434331}{431.26854804310231676668703693725 - 1} \,)$

$= antilog\ (0.29995451137942861569963449868 5 + \frac{0.00020828858858866000141143433 1}{430.26854804310231676668703693725})$

$= antilog\ (0.299954511379428615699634498685 +$
$0.0000004840897377602292515022280246107 8)$

$= antilog\ (0.29995499546916637592888600091302)$

$= 1.3497980591806632796684482158043$

$a = antilog\ \{log\ y_0 - \frac{\Delta^2\ log\ y_0}{(d-1)^2}\ \}$

$= antilog\ \{15.898552610020310347642892883553\ - \frac{-0.000208288588588660001411434331}{(431.268548043102316766687036937 25-1)^2}\}$

$= antilog\ \{15.898552610020310347642892883553 + \frac{0.000208288588588660001411434331}{(430.26854804310231676668703693725)^2}\ \}$

$= antilog\ \{15.898552610020310347642892883553 + \frac{0.000208288588588660001411434331}{185131.02343511944650313478201748}\ \}$

$= antilog\ \{15.898552610020310347642892883553$
$+ 0.0000000011250874365832925729453853428345\}$

$= antilog\ \{15.898552611145397784226185456498\}$

$= 8028856.0090331650208179383118444$

$y = a\ b^x c^{d^x}$

$= (8028856.0090331650208179383118444)\ (1.3497980591806632796684482158043)^x\ (0.99999999887491256404961829679604)^{(431.26854804310231676668703693725)^x}$

This curve yields,

$y_0 = 8028856.0090331650208179383118444 \times 0.99999999887491256404961829679604$

$= 8028856$

$y_1 = 8028856.0090331650208179383118444 \times 1.3497980591806632796684482158043 \times (0.99999999887491256404961829679604)^{(431.26854804310231676668703693725)}$

$= 8028856.0090331650208179383118444 \times 1.3497980591806632796684482158043 \times 0.99999951478529251988171201565936$

$= 10837329$

$y_2 = (8028856.0090331650208179383118444)\ (1.3497980591806632796684482158043)^2\ (0.99999999887491256404961829679604)^{(431.26854804310231676668703693725)^2}$

$= (8028856.0090331650208179383118444) \times 1.8219548005678853694907303510899 \times (0.99999999887491256404961829679604)^{185992.56053120565113666815609136}$

$= 8028856.0090331650208179383118444 \times 1.8219548005678853694907303510899 \times 0.99979076399975419167285478301286$

$= 14625152$

$y_3 = (8028856.0090331650208179383118444)\ (1.3497980591806632796684482158043)^3\ (0.99999999887491256404961829679604)^{(431.26854804310231676668703693725)^3}$

$= 8028856.0090331650208179383118444 \times 2.4592710537214240991513533150575 \times (0.99999999887491256404961829679604)^{80212741.527111880115368664197677}$

$= 8028856.0090331650208179383118444 \times 2.4592710537214240991513533150575 \times 0.91370606811337654576816104036617 = 18041248$

The data on total population corresponding to three consecutive years (at a gap of 10 years) can be represented by the makeham's curve. The curves obtained have been shown in the following table.

Table-3.2.4
(Mathematical curve representing total population at a gap of 10 years)

Years	x	Makeham's curve representing $y_i = f(x_i)$
1951 1961 1971 1981	0 1 2 3	$y = (8028856.00903316502081793831184444)$ $(1.34979805918066327966844821580 43)^x$ $(0.99999999887491256404961829679604)^{(431.26854804310231676668703693725)^x}$
1961 1971 1981 1991	0 1 2 3	$y = (11706131.268736465043580263780311)$ $(1.24174334997093918076907459870 52)^x$ $(0.92578228888849272950660199696126)^{(-0.079279366607908952553455518446 65)^x}$
1971 1981 1991 2001	0 1 2 3	$y = (14623161.062015285536535167714516)$ $(1.23479202085134388318584248527 22)^x$ $(1.00013614963114138627167582134 79)^{(-6.232580308730710114094407383181 6)^x}$
1981 1991 2001 2011	0 1 2 3	$y = (19885397.451419812635631050628414)$ $(1.16336201595157605257761936442 59)^x$ $(0.90726112183952648104579062153581)^{(0.32468171182224469440292561965373)^x}$

Similarly, the data on total population corresponding to two consecutive years (at a gap of 20 years) can be represented by the makeham's curve. The curves obtained have been shown in the following table:

Table-3.2.5
(Mathematical curve representing total population at a gap of 20 years)

Years	x	Makeham's curve representing $y_i = f(x_i)$
1951 1971 1991 2011	0 1 2 3	$y = (19291372.750038916219903772106324)$ $(1.23437054327688480145326106247 06)^x$ $(0.41618894124493050913865080936805)^{(0.55608886453778967105967328231864)^x}$

4. CONCLUSION:

The method of representing numerical data on a pair of variables by makeham's curve described by equation (2.1) has been discussed in section 2. Its application to the total population of India and Assam have been shown in section 3.

It is to be mentioned here that the method discussed here is applicable in representing a set of numerical data on a pair of variables if the given values of the independent variable are equidistant. In the case where the given values of the independent variable are not equidistant, the method fails to represent the given data by makeham's curve. The makeham curve contains four parameters. Accordingly, when four pairs of numerical data are available, they can be represented by makeham curve.

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MSMEs IN INDIA: GROWTH AND PERFORMANCE ANALYSIS

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ABSTRACT :
Small and Medium Enterprises play a vital role for the growth of Indian economy. The Sector consisting of 51 million units, as of today, provides employment to over 117 million persons. The share of Micro, Small & Medium Enterprise (MSME) products in the exports from the country reaches 49.86 percent in the FY 2015-16.This research analyses the growth and performance of MSMEs in India from 1995 to 2015 using Growth Rate, CAGR and t-test statistics.
Key words: MSMEs, CAGR, Economic Development, GDP, Indian Economy.

I. INTRODUCTION

Micro, Small and medium-sized enterprises (MSMEs) are the world's most concentrated, booming and innovative engine for world trade and growth. With 95 per cent of global enterprises comprising SMEs, they serve as key drivers of innovation, social integration and employment representing 60 per cent of private sector jobs – which fuel economic growth.

Small and Medium Enterprises play a vital role for the growth of Indian economy. The share of Micro, Small & Medium Enterprise (MSME) products in the exports from the country during last three years is 42.42 percent in 2013-14; 44.76 percent in 2014-15 and 49.86 percent in 2015-16. As per 4th All India Census of MSME, the numbers of functional and non-functional registered MSMEs in the country are 15, 63,974 in manufacturing sector and 4, 96,355 in service sector. According to DC MSME, the MSME Sector plays a major role in India's present export performance. 45-50 percent of the Indian Exports is contributed by MSME Sector.

The non-traditional products account for more than 95 percent of the MSME exports. The exports from MSME sector have been clocking excellent growth rates in this decade. The excellent growth rates have been mostly fueled by the performance of garments, leather and gems and jewellery units from this sector. The product groups where the MSME sector dominates in exports are sports goods, readymade garments, woolen garments and knitwear, plastic products, processed food and leather products, it added. As a result, MSMEs are today exposed to greater opportunities for expansion and diversification across the sectors. The Indian market is growing rapidly and Indian industry is making remarkable progress in various Industries like Manufacturing, Precision Engineering, Food Processing, Pharmaceuticals, Textile & Garments, Retail, IT, Agro and Service sectors. SMEs are finding increasing opportunities to enhance their business activities in core sectors. To fully unlock their potential within the multilateral trading system, there is still some work to be done to help address market failures which prevent SMEs from becoming the engines of growth that the international economy needs.

ABOUT MSMES IN INDIA

Indian Small and Medium Enterprises sector has emerged as a highly vibrant and dynamic sector of the Indian economy over the last five decades. SMEs not only play crucial role in providing large employment opportunities at comparatively lower capital cost than large industries but also help in industrialization of rural areas. SMEs are complementary to large industries as ancillary units and this sector contributes enormously to the socio-economic development of the country. The Sector consisting of 51 million units, as of today, provides employment to over 117 million persons.

TABLE 1: DEFINITION OF MSMES IN INDIA

	Manufacturing Enterprises (Investment in Plant & Machinery)		**Service Enterprises (Investment in Equipments)**	
I	**II**	**III**	**IV**	**V**
Description	**INR**	**USD($)**	**INR**	**USD($)**
Micro Enterprises	Upto `25Lakh	upto $ 62,500	Upto `10Lakh	upto $ 25,000
Small Enterprises	Above `25 Lakh & upto `5 Crore	above $ 62,500 & upto $ 1.25 million	above `10 Lakh & upto `2 Crore	above $ 25,000 & upto $ 0.5 million
Medium Enterprises	above `5 Crore & upto `10 Crore	above $ 1.25 million & upto $ 2.5 million	above `2 Crore & upto `5 Crore	above $ 0.5 million & upto $ 1.5 million

Source: Micro, Small & Medium Enterprises Development (MSMED) Act, 2006)

MSMEs also play a significant role in Nation development through high contribution to Domestic Production, Significant Export Earnings, Low Investment Requirements, Operational Flexibility, Location Wise Mobility, Low Intensive Imports, Capacities to Develop Appropriate Indigenous Technology, Import Substitution, Contribution towards Defense Production, Technology Oriented Industries, Competitiveness in Domestic and Export Markets thereby generating new entrepreneurs by providing knowledge and training.

Despite their high enthusiasm and inherent capabilities to grow, SMEs in India are also facing a number of problems like sub-optimal scale of operation, technological obsolescence, supply chain inefficiencies, increasing domestic & global competition, working capital shortages, not getting trade receivables from large and multinational companies on time, insufficient skilled manpower, change in manufacturing strategies and turbulent and uncertain market scenario. To survive with such issues and compete with large and global enterprises, SMEs need to adopt innovative approaches in their operations. SMEs that are innovative, inventive, international in their business outlook, have a strong technological base, competitive spirit and a willingness to restructure themselves can withstand the present challenges and come out successfully to contribute 22% to GDP. Indian SMEs are always ready to accept and acquire new technologies, new business ideas and automation in industrial and allied sectors.

II. LITERATURE REVIEW

The small scale units, using modern machinery and hiring up to 50 workers were the most capital intensive type of enterprises (Dhar and Lydall, 1966). The small scale industries engaged in the export trade are providing more employment and production by effectively using local resources and also earns foreign exchanges for the country (H.K. Bedbak, 1980) and employment growth in small scale sector is considerably higher than the employment growth in the large scale sector (Bishwanath Goldar, 1993). After globalization, the small scale sector has also responded well to the challenge of opening up of the economy and in fact, its rate of growth has generally two to three percentage points higher than the industrial sector as whole (B.D. Jethra, 2000). The success or failure of small and medium enterprises is highly influenced by the personality and leadership characteristics of owner or board of directors (Adriana, 2011). MSME in India has the potential to increase the share of contribution to GDP from the current 8 per cent to about 15 per cent by the year 2020 (KPMG, 2014).

III. OBJECTIVE

The objective of the research is to study the performance of MSMEs in India in respect of total number of working units, investment, production achieved, magnitude of exports and employment generated over a period of 20 years i.e. from 1995-96 to 2014-15.

IV. RESEARCH METHODLOGY

The present study deals with performance evaluation of micro, small and medium enterprises (MSMEs) in India and cover a period of 20 years i.e. from 1995-96 to 2014-15. This study based on secondary data which is obtained from MSMEs annual reports of different years, Economic Survey of India and Handbook of Statistics on the Indian Economy, RBI. The study period is divided into two parts; 1995-05 and 2005-06. To analyze data descriptive statistics like arithmetic mean, standard deviation and coefficient of variation has been used. The arithmetic mean of two sub-periods has been compared using the t-test for significance of difference of means and single star (*) indicates the difference is significant at 5% level of significance. The two type of growth rate have been computed: first year to year growth rate and second the trend growth rate (that is percent per annum during the specified period).

HYPOTHESIS

The null hypothesis of the study is that there is no significant difference between the number of units, production, employment level, fixed investment and export of MSMEs up to year 2005 and post 2005.

STATISTICAL TECHNIQUES

To analyze and interpret the data following statistical method are used:

Growth Rate Percentage = $[(\text{Value}_{\text{Present Year}} - \text{Value}_{\text{Past year}}) / \text{Value}_{\text{Past Year}}]*100$

Compound Growth Rate = $[\text{Ending Value/Beginning Value}]^{[1/\text{No. of years}]} - 1$

V. DATA ANALYSIS AND INTERPRETATION

The data are classified, tabulated and analyzed using arithmetic mean, standard deviation, coefficient of variation and t-test statistics. The progress of MSMEs on the basis of number of units has been shown in table 2 from 1995-96 to 2014-15 which revealed that number of MSME in India increased from 8.28 million in 1995-96 to 51.06 million in 2014-15 with an average growth rate of 10 percent per annum. During the period 1996 to 2005 (up to 2005) SSI units increased by 4.1 percent from 8.28 million to 11.86 million and showing an increasing trend in terms of absolute number of units but actually in terms of yearly percentage growth rate, the number of MSMEs have shown a decreasing trend. This may be due to privatization, globalization and the resultant competition. The trend growth rate in post 2005 period is 17.1 percent which shows higher growth than 1996-05 sub period and yearly percentage growth rate showing increasing trend. To examine the hypothesis that there is no significant difference in the number of units in MSMEs up to 2005 and post 2005, t-test is applied at 5 percent level. The results reject the hypothesis. Hence, there is significant difference between the numbers of units in two periods. The employment is increased from 19.79 million in 1996 to 117.13 million in 2015. The overall growth rate of Employment is 9.8 percent, in 1996-05 growth rate is 4.04 percent and in Post 2005 is 16.6 percent. In terms of yearly percentage growth rate, up to 2005 sub period it is decreasing but after 2005 it shows increasing trend which is good sign. The results of t-test statistics at 5 percent level of significance reject the hypothesis. Hence, there is significant difference between the employments provided by MSMEs in two periods. The fixed assets are increased from 1, 25,750 crore in 1996 to 14, 71,913 crore in 2015. The overall growth rate of fixed assets is 13.82 percent, in 1996-05 growth rate is 3.98 percent and in Post 2005 is 25.68 percent. In terms of yearly percentage growth rate, up to 2005 sub period it is fluctuating but after 2005 it shows increasing trend. The results of t-test statistics at 5 percent level of significance reject the hypothesis. Hence, there is significant difference between the fixed assets of MSMEs in two periods. The overall growth rate of production per employee is 6.25 percent, in 1996-05 growth rate is 8.77 percent and in Post 2005 is 2.53 percent. In terms of yearly percentage growth rate, it is continuous fluctuating and negative in 2009 and 2013. The results of t-test statistics at 5 percent level of significance reject the hypothesis. Hence, there is significant difference between the productions per employee of MSMEs in two periods. Data shows that though there is increasing trend in terms of absolute value of production per employee but the improvement in production per employee preceded so slow that overall level of production per employee still remain slow.

TABLE 2: EMPLOYMENT AND FIXED ASSETS PERFORMANCE OF MSMES IN INDIA

Year/ Statistic	Total Units (Million)		Employment (Million Nos.)		Fixed Assets (`Crore)		Production Per Employee (`Thous.)	
	No.	Annual GR (%)	No.	Annual GR (%)	Value	Annual GR (%)	Value	Annual GR (%)
I	II	III	IV	V	VI	VII	VIII	IX
1995-96	8.28	-	19.79	-	125750	-	61	-
1996-97	8.62	4.11	20.59	4.04	130560	3.83	66	8.20
1997-98	8.97	4.06	21.32	3.55	133242	2.05	69	4.55
1998-99	9.34	4.12	22.06	3.47	135482	1.68	71	2.90
1999-00	9.72	4.07	22.91	3.85	139982	3.32	74	4.23
2000-01	10.11	4.01	23.87	4.19	146845	4.90	77	4.05
2001-02	10.52	4.06	24.93	4.44	154349	5.11	112	45.45
2002-03	10.95	4.09	26.02	4.37	162317	5.16	116	3.57
2003-04	11.36	3.74	27.14	4.30	170219	4.87	122	5.17
2004-05	11.86	4.40	28.26	4.13	178699	4.98	130	6.56
2005-06	12.34	4.06	29.49	4.36	188113	5.27	140	7.69
2006-07	36.18	193.2	80.52	173.1	868544	361.7	149	6.43
2007-08	37.74	4.30	84.22	4.60	920460	5.98	157	5.37
2008-09	39.37	4.33	88.08	4.59	977115	6.16	156	-0.64
2009-10	41.08	4.34	92.18	4.65	1038546	6.29	161	3.21
2010-11	42.87	4.36	96.52	4.70	1105934	6.49	171	6.21
2011-12	44.77	4.42	101.18	4.83	1182758	6.95	177	3.51
2012-13	46.76	4.45	106.15	4.91	1268764	7.27	171	-3.39
2013-14	48.85	4.47	111.43	4.97	1363701	7.48	-	-
2014-15	51.06	4.53	117.13	5.12	1471913	7.94	-	-
up to 2005 Trend GR	**4.1**		**4.04**	-	**3.98**	-	**8.77**	-
Mean	9.97	-	23.69	-	147744.5	-	89.80	-
S.D.	1.20	-	2.86	-	18012.6	-	26.72	-
C.V.	12.02	-	12.06	-	12.2	-	29.76	-
Post 2005 Trend GR	**17.1**	-	**16.6**	-	**25.68**	-	**2.53**	-
Mean	40.10	-	90.69	-	1038585	-	160.25	-
S.D.	10.87	-	24.52	-	356171.9	-	12.41	-
C.V.	27.10	-	27.04	-	34.3	-	7.74	-
Overall Trend GR	**10**	-	**9.8**	-	**13.82**	-	**6.25**	-
t-test	8.71*	-	8.58*	-	7.90*	-	6.86*	-

Source: Ministry of Micro, Small & Medium Enterprises, Government of India, New Delhi.

Notes: 1.Data from 2006-07 includes activities of wholesale/retail trade, legal, education & social services, hotel & restaurants, transports and storage & warehousing (except cold storage) for which data were extracted from Economic Census 2005, Central Statistics Office, MoSPI.
2. Data for 2013-14 onwards are projected and production per employee at constant prices.
3. The data for the period up to 2005-06 is of Small Scale Industries (SSI). Subsequent to 2005-06, data with reference to Micro, Small and Medium Enterprises (MSMEs) are being compiled.

The progress of MSMEs on the basis of production and export has been shown in table 3 from 1995-96 to 2014-15. The production at constant prices is increased from 1211 `billion in 1996 to 18099 ` billion in 2015. The overall growth rate of production at constant prices is 17.2 percent, in 1996-05 growth rate is 13.3 percent and in Post 2005 is 23.3 percent. In terms of yearly percentage growth rate, it shows increasing trend but fluctuating rate and a very low production performance level when growth rate show all time low in 2013. The results of t-test statistics at 5 percent level of significance reject the hypothesis. Hence, there is significant difference between the productions at constant prices by MSMEs in two periods. The total production (at current price) by the MSMEs is increased from 1477 billion in 1996 to 18343 billion by the end of year 2012 with an average growth rate of 17.05

percent. The data also revealed that though the value of production (at current price) of MSMEs is increasing, yet actually in terms of percentage, the performance level has shown a downward trend. The results of t-test statistics at 5 percent level of significance reject the hypothesis. So, there is significant difference between the productions (current price) of MSMEs in two periods.

Exports are an important investment for the economic development of a country and play an important role in influencing the level of economic growth, employment and the balance of payments. The overall export (`Billion) performance is 18.02 percent ranging from 364 Billion in 1996 to 8492 Billion in 2015 and in one sub period i.e. up to 2005 growth rate is 14.61 percent and during the period post 2005 growth rate is 21.22 percent. During the period 2009-10, the export of MSMEs is 76 percent which is highest in this period. The analysis of export data shows that there has been a continuous increase in absolute value of exports during the 1995-2015 study periods but growth rate has fluctuating trend over the period of study. The results of t-test statistics at 5 percent level of significance reject the hypothesis. So, there is significant difference between the exports (`Billion) of MSMEs in two periods.

TABLE 3: PRODUCTION AND EXPORT PERFORMANCE OF MSMES IN INDIA

Year/ Statistic	Production Constant Prices (`Billion)		Production Current Prices (`Billion)		Export (`Billions)		Export (US Millions)	
	Value	Annual GR (%)	No.	Annual GR (%)	Value	Annual GR (%)	Value	Annual GR (%)
I	**II**	**III**	**IV**	**V**	**VI**	**VII**	**VIII**	**IX**
1995-96	1211.8	-	1477.1	-	364.70	-	10903	-
1996-97	1348.9	11.32	1678.1	13.60	392.48	07.62	11056	1.40
1997-98	1462.6	8.43	1872.2	11.57	444.42	13.23	11958	8.16
1998-99	1575.3	7.70	2104.5	12.41	489.79	10.21	11642	-2.64
1999-00	1703.8	8.16	2337.6	11.07	542.00	10.66	12508	7.44
2000-01	1844.0	8.23	2613.0	11.78	697.97	28.78	15278	22.15
2001-02	2822.7	53.07	2822.7	8.03	712.44	2.07	14938	-2.23
2002-03	3067.7	8.68	3148.5	11.54	860.13	20.73	17773	18.98
2003-04	3363.4	9.64	3645.5	15.78	976.44	13.52	21249	19.56
2004-05	3729.4	10.88	4298.0	17.90	1244.17	27.42	27690	30.31
2005-06	4188.8	12.32	4978.4	15.83	1502.42	20.76	33935	22.55
2006-07	11988.2	186.2	135.1	-97.29	1825.38	21.50	40309	18.78
2007-08	13227.8	10.34	14351.8	10520	2020.17	10.67	50202	24.54
2008-09	13755.9	3.99	15242.3	6.21	2221.23	09.95	55531	10.62
2009-10	14883.5	8.20	16193.6	6.24	3911.59	76.10	82494	48.55
2010-11	16536.2	11.10	17215.5	6.31	5077.39	29.80	111403	35.04
2011-12	17885.8	8.16	18343.3	6.55	6301.05	24.10	131483	18.02
2012-13	18099.8	1.20	-	-	6981.66	10.80	128316	-2.41
2013-14	-	-	-	-	8068.78	15.57	133364	3.93
2014-15	-	-	-	-	8492.48	05.25	138894	4.15
upto 2005 Trend GR	**13.3**		**12.6**	-	**14.6**	-	**10.91**	-
Mean	2212.96	-	2599.7	-	672.45	-	15499.5	-
S.D.	933.31	-	899.48	-	285.23	-	5420.56	-
C.V.	42.17	-	34.6	-	42.42	-	34.97	-
Post 2005 Trend GR	**23.3**	-	**24.3**	-	**21.2**	-	**16.95**	-
Mean	13820.8	-	12351.5	-	4640.22	-	90593.1	-
S.D.	4469.36	-	6955.6	-	2708.89	-	42629.4	-
C.V.	32.34	-	56.3	-	58.38	-	47.1	-
Overall Trend GR	**17.2**	-	**17.05**	-	**18.02**	-	**14.33**	-
t-test	8.06*	-	4.44*	-	4.61*	-	5.53*	-

Source: Ministry of Micro, Small & Medium Enterprises, Government of India, New Delhi.

Notes: 1. Data from 2006-07 includes activities of wholesale/retail trade, legal, education & social services, hotel & restaurants, transports and storage & warehousing (except cold storage) for which data were extracted from Economic Census 2005, Central Statistics Office, MoSPI.

2. Data for 2013-14 onwards are projected and '-'shows non availability of data.

3. The data for the period up to 2005-06 is of Small Scale Industries (SSI). Subsequent to 2005-06, data with reference to Micro, Small and Medium Enterprises (MSMEs) are being compiled.

The contribution of MSMEs to Indian economy has been shown in table 4 from 1995-96 to 2014-15.The MSMEs generating 144.23 million employment opportunities in 2015-16, account for `8553.52 billion exports in 2015-16, contributing 8 percent of the GDP, 37 percent of total manufacturing output. The share of MSME products in the exports from the country during last three years is 42.42 percent in 2013-14; 44.76 percent in 2014-15 and 49.86 percent in 2015-16. The share of MSMEs in total GDP and in total industrial production is showing fluctuating trend. The growth rate of MSMEs sector is more than the growth rate of overall industrial sector in India. The most important contribution of SMEs in India is promoting the balanced regional development.

TABLE 4: CONTRIBUTION OF MSMES TO INDIAN ECONOMY

Year/ Statistic	Share of MSME sector in Total GDP (%)	Share of MSME sector in Total Industrial Production (%)	Growth rate of Overall Industrial Sector (%)	Growth rate of SSI Sector (%)
I	II	III	IV	V
1995-96	-	-	13	11.50
1996-97	-	-	6.1	11.30
1997-98	7.02	39.7	6.7	9.20
1998-99	6.81	39.94	4.1	7.80
1999-00	5.86	40.02	6.7	7.10
2000-01	6.04	39.91	5.0	8.00
2001-02	5.77	39.63	2.7	6.10
2002-03	5.91	39.48	5.7	8.68
2003-04	5.79	39.42	6.9	9.64
2004-05	5.84	38.62	8.4	10.88
2005-06	5.83	38.56	8.2	12.32
2006-07	7.73	45.62	11.6	12.60
2007-08	7.81	45.24	8.5	13.00
2008-09	7.52	44.86	2.8	11.77
2009-10	7.45	36.05	10.4	10.45
2010-11	7.42	36.69	-	11.83
2011-12	7.28	37.97	-	18.45
2012-13	7.04	37.54	-	14.30
2013-14	-	-	-	12.44
2014-15	-	-	-	17.18

Source: Ministry of Micro, Small & Medium Enterprises, Government of India, New Delhi.

PROBLEMS OF MSMEs AND GOVT. MEASURES TAKEN FOR MSMEs GROWTH

MSME entrepreneurs are facing several problems which are as:

- Lack of adequate capital

- Poor infrastructure
- Access to modern technology
- Access to markets
- Getting statutory clearances related to power, environment, labour etc

Some of the measures taken by government to improve the performance of MSMEs are as:

- Government has simplified the procedure through e-governance system and developed a govt. portal to save time and minimize the bureaucratic formalities.
- To improve the productivity, competitiveness and capacity building of MSMEs, the Government of India has adopted a cluster based approach.
- Setting up MSME tool room and training centre.
- Credit disbursed to MSME is considered part of priority sector lending by banks.
- Central government has initiated National Manufacturing Competitiveness Programme (NMCP). It would help them in facing stiff competition from global MNCs.
- Credit Linked Capital Subsidy Scheme for Technology Up gradation.
- Credit Guarantee Scheme.
- Scheme of Micro Finance Programme

VI. CONCLUSION AND SUGGESTIONS

The MSMEs has played a major role in the industrial growth achieved by our country, employment generation, and decentralization, less pressure of population on agriculture, industrial peace, complementary to large industries, promotion of artistic good and use of local resources. The contribution of MSMEs towards the planned economic development of the country is explained by means of their role in terms of value of output, employment creation, number of units working and exports. The study reveals that in terms of number of units registered there has been increasing trend and growth rate is declining up to 2006 but steadily increasing after 2007. The employment in the MSMES increased from 19.79 million persons in 1995-96 to 117.13 million persons in 2014-15 with an average annual growth rate of 9.8 percent. India needs to create 10 to 15 million job opportunities per year over the next decade to provide gainful employment to its population. Current MSME employment is at 28 per cent of the overall employment. The production at constant prices is increased from 1211 `billion in 1996 to 18099 ` billion in 2015 with an average annual growth rate of 17.2 percent. The MSMEs share in India's exports near 50 percent in 2015-16. MSMEs contributions are considerably high in the balanced economic growth. MSMEs need more supports from the government in the form of priority sector lending, government procurement programme, credit and performance ratings and marketing support. There is a need for strong commitment from the government as well as the MSMEs for the sector to enter the next orbit of high growth and employment generation.

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APPLICATION OF EULERIAN GRAPH

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ABSTRACT :
The genius Swiss Mathematician Leonhard Euler is considered as the father of graph theory. His solution on "Königsberg problem", led to the invention of graph theory. Graphs are used in modeling real-life situations in many disciplines. In this paper, we have discussed applications of Eulerian Graph in floor design, DNA sequencing and platonic solids etc.
Keywords: Graph, Eulerian graph, Platonic Solid.

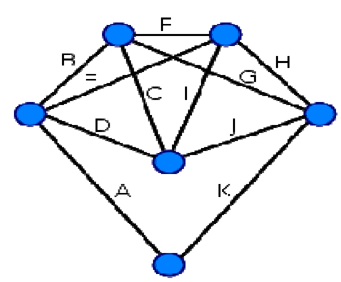
1. INTRODUCTION :

Graph theory has developed into an extensive and popular branch of mathematics, which has been applied to many problems in mathematics, computer science, biochemistry, electrical engineering, operational research and other scientific and not-so-scientific areas. There are many games and puzzles which can be analyzed by graph theoretic concepts.

2. SOME DEFINITIONS:

2.1 Eulerian path (or Eulerian trail): In graph theory, an eulerian path is a path (in graph), which visits every edge exactly once.

2.2 Example: We have following example to illustrate eulerian path:-



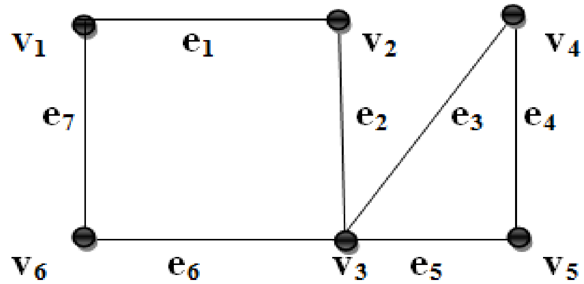
Eulerian path is, A→B→C→D→E→F→G→H→I→J→K.

2.3 Eulerian Circuit (or Eulerian cycle):

An Eulerian circuit is an Eulerian trail which starts and ends on the same vertex.

2.4 Euler Graphs: A closed walk in a graph G containing all the edges of G is called an Euler line in G. A graph containing an Euler line is called an Euler graph. We know that a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an Euler graph is connected except for any isolated vertices the graph may contain. As isolated vertices do not contribute anything to the understanding of an Euler graph, it is assumed now onwards that Euler graphs do not have any isolated vertices and are thus connected.

2.5 Example: Consider the graph shown in Figure. Clearly, $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_4\ v_5\ e_5\ v_3\ e_6\ v_6\ e_7\ v_1$ is an Euler line.



2.6 Unicursal Graphs: An open walk that includes (or traces) all edges of a graph without retracing any edge is called a unicursal line or open Euler line. A connected graph that has a unicursal line is called a unicursal graph. Clearly by adding an edge between the initial and final vertices of a unicursal line, we get an Euler line.

2.7 Platonic solid: A 3D shape in which each face is a regular polygon, and all faces are identical such that in which each edges – vertices looks same.

3. SOME THEOREM ON EULERIAN GRAPH:

3.1 Theorem A connected graph G is an Euler graph if and only if all vertices of G are of even degree.

Proof :

Necessity: Let G(V, E) be an Euler graph. Thus G contains an Euler line Z, which is a closed walk. Let this walk start and end at the vertex $u \in V$. Since each visit of Z to an intermediate vertex v of Z contributes two to the degree of v and since Z traverses each edge exactly once, d(v) is even for every such vertex. Each intermediate visit to u contributes two to the degree of u, and also the initial and final edges of Z contribute one each to the degree of u. So the degree d(u) of u is also even.

Sufficiency: Let G be a connected graph and let degree of each vertex of G be even. Assume G is not Eulerian and let G contain least number of edges. Since $\delta \geq 2$, G has a cycle. Let Z be a closed walk in G of maximum length. Clearly, G–E(Z) is an even degree graph. Let C1 be one of the components of G–E(Z). As C1 has less number of edges than G, it is Eulerian and has a vertex v in common with Z. Let Z0 be an Euler line in C1. Then Z0 ∪Z is closed in G, starting and ending at v. Since it is longer than Z, the choice of Z is contradicted. Hence G is Eulerian.

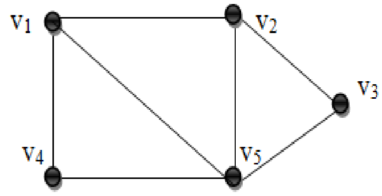
3.2 Theorem: A connected graph G is Eulerian if and only if its edge set can be decomposed into cycles.

Proof: Let G(V, E) be a connected graph and let G be decomposed into cycles. If k of these cycles are incident at a particular vertex v, then d(v) = 2k. Therefore the degree of every vertex of G is even and hence G is Eulerian. Conversely, let G be Eulerian. We show G can be decomposed into cycles. To prove this, we use induction on the number of edges. Since $d(v) \geq 2$ for each $v \in V$, G has a cycle C. Then G – E(C) is possibly a disconnected graph, each of whose components C1, C2, ..., Ck is an even degree graph and hence Eulerian. By the induction hypothesis, each Ci is a disjoint union of cycles. These together with C provide a partition of E(G) into cycles.

3.3 Theorem: A connected graph is unicursal if and only if it has exactly two vertices of odd degree.

Proof : Let G be a connected graph and let G be unicursal. Then G has a unicursal line, say from u to v, where u and v are vertices of G. Join u and v to a new vertex w of G to get a graph H. Then H has an Euler line and therefore each vertex of H is of even degree. Now, by deleting the vertex w, the degree of vertices u and v each get reduced by one, so that u and v are of odd degree. Conversely, let u and v be the only vertices of G with odd degree. Join u and v to a new vertex w to get the graph H. So every vertex of H is of even degree and thus H is Eulerian. Therefore, G = H−w has a u−v unicursal line so that G is unicursal.

3.4 For example: This example satisfies the above3.3 theorem (v_1 and v_2 are of odd degree).



4 APPLICATION OF EULERIAN GRAPH:

4.1 DNA (deoxyribonucleic acid): DNA sequencing and fragment assembly is the problem of reconstructing full strands of DNA based on the pieces of data recorded. It is of interest to note that ideas from graph theory, especially Eulerian circuits have been used in a recently proposed approach to the problem of DNA fragment assembly.

4.2 Floor Designs and Eulerian graphs : Traditional interesting floor designs, known as "kolam", are drawn as decorations in the floor and in large sizes and in interesting shapes, during festivals and weddings, with the drawing done with rice flour or rice paste especially in South India.

A kolam drawing can be treated as a special kind of a graph with the crossings considered as vertices and the parts of the kambi (the Tamil word kambi means wire) between vertices treated as edges. The only restriction is that unlike in a graph, these edges can not be freely drawn as there is a specific way of drawing the kolam.

The single kambi kolam will then be an Eulerian graph with the drawing starting and ending in the same vertex and passing through every edge of the graph only once.

4.3 Theorem: There are at most five platonic solids.

Proof: Assume that we have a regular polygon with

n = edges on each faces

m = edges that meet on a vertex

Then we count edges, $E = F*n / 2$

and count vertices, $V = F*n / m$

Also, $V - E + F = 2$

$(F*n /m) - (F*n/2) + F = 2$

So, we have

$F [(n/m) - (n/2) +1] = 2$

This implies,

$$F = 4m / (2n - mn + 2m)$$

Then we must have,

$(2n - mn + 2m) > 0$

Or $2(n + m) > mn$

$2n > m(n -2)$

This implies, $m,n \geq 3$

And $[2n / (n -2)] > m \geq 3$

This implies, $2n > 3n - 6$

This implies, $n < 6$

Also, we have , $m < 6$

So, we have the following possibilities:

[a.] (3,3) → Tetrahedron

[b.] (3,4) → Octahedron

[c.] (3,5) → Icosahedron

[d.] (4,3) → Cube

[e.] (5,3) → Dodecahedron

CONCLUSION:

Graph Theory is becoming significant in various fields such as mathematics, science and technology. In this paper we have highlighted some applications of Eulerian Graph through many more application can be search out. If we introduce algebraic methods in Eulerian Graph then we can design algebraic graph theory for the said graph. It has been observed that if the location or properties of a member is altered then the properties of structure will be different i.e. the connectivity influence the performance of the entire structure.

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FIXED POINT THEOREM FOR COMPATIBLE MAPPINGS OF TYPE (P)

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ABSTRACT :
In this paper, we prove fixed-point theorems for compatible mappings of type (P) in metric spaces. The result of D. Prasad [5] has been generalized for three mappings satisfying a new functional inequality defined by Singh and Meade [7].
Keywords: Compatible mappings, Compatible mappings of type (P).

1. INTRODUCTION AND PRELIMINARIES

Jungck [2] introduced the concept of compatible maps which is weaker than weakly commuting maps. With the evolution of compatible maps researches and academicians start to relax the condition of commutativity and continuity to introduce the notion of compatible mapping and its variants. Compatibility was defined by Jungck [2] in the following way.

COMPATIBLE MAPPINGS

Let (X,d) be a metric space and { x_n} is a sequence in X such that

$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n$ =u`for some u in X .

Then P,Q : (X,d) →(X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(PQx_n, QQ\, x_n)$ =0

Compatible mapping of type (P) was defined by Pathak-Cho-chang –Kang [3] in the following manner :

COMPATIBLE MAPPINGS OF TYPE (P)

Let (X,d) be a metric space and { x_n} is a sequence in X such that

$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n$ =u`for some u in X

Then P,Q : (X,d) →(X,d) are said to be compatible mappings of type (P)

If $\lim_{n\to\infty} d\left(QQx_n, PPx_n\right) = 0$

Now, we give propositions which show the equivalent relation between compatibility, compatibility of type (A) and compatibility of type(P).

Proposition (1.1) [3] let P, Q : (X, d) → (X, d) be continuous mappings . Then P and Q are compatible if and only if they are compatible of type (P).

Proposition (1.2) [3] let P, Q : (X, d) → (X, d) be compatible mappings of type (A). If one of P and Q is continuous, then P and Q are compatible of type (P).

Proposition (1.3) [3] let P, Q : (X, d) → (X, d) be compatible mappings of type (P) if Pu = Qu for some u in X , then

PQu = PPu = QQu = QPu

Proposition (1.4) [3] let P, Q : (X, d) → (X, d) be compatible mappings of type (P) suppose

$\lim_{n\to\infty} Px_n = \lim_{n\to\infty} Qx_n$ =u`, for some u in X .

then

(i) $\lim_{n\to\infty} QQx_n$ = Pu if P is continuous at u.

(ii) $\lim_{n\to\infty} PPx_n$ = Qu if Q is continuous at u.

(iii) $\lim_{n\to\infty} PQu$ = QPu and Pu = Qu if P and Q are continuous at u.

We see that if Q and P are continuous, then

Compatibility ⇔ compatibility of type (A) ⇔ compatibility of type (P)

If both P and Q are not continuous then the above result is not true. The following three examples prove this result :

Example (1.1) : Let X =[0,4] with the usual metric d (x, y) = $|x-y|$

Define P,Q : X → X by

$$Px = \begin{cases} x & if\ x\epsilon\ [0,2) \\ 4 & if\ x\ \epsilon\ [2,4] \end{cases} \qquad Qx = \begin{cases} 4-x\ if\ x\epsilon[0,2) \\ 4 \quad if\ x\ \epsilon\ [2,4] \end{cases}$$

Then P and Q are not continuous at x=2. Now, we suppose that $\{x_n\} \subseteq [0,4]$ is a sequence converging to 2 such that, $x_n < 2$ for all n . Then $Qx_n = 4-x_n \to 2$ from the right and $Px_n = x_n \to 2$ from the left.

Also $PQx_n = P(4-x_n) = 4$, $Qpx_n = Q(x_n) = 4-x_n$,

Therefore if $\lim_{n\to\infty} d(PQx_n, QP\,x_n) = \lim_{n\to\infty}|4-4+x_n| \to 2$and

$PPx_n = Px_n = x_n$, $QQx_n = Q(4-x_n) = 4$gives ,

$\lim_{n\to\infty} d(PPx_n, QQ\,x_n) = \lim_{n\to\infty}|x_n-4| \to 2.$

So, P and Q are neither compatible nor compatible of type (P).

But $\lim_{n\to\infty} d(PQ\,x_n, QQx_n) = |4-4| = 0.$

And $\lim_{n\to\infty} d(QPx_n, PP\,x_n) = \lim_{n\to\infty}|4-x_n-x_n| = 0.$

Implies that P and Q are compatible of type (A).

Example (1.2) Let X=R , with the usual metric d the set of real numbers.

Define P,Q : R → R by

$$Px = \begin{cases} 1/x^2 & if\ x \neq 0 \\ 2 & if\ x = 0 \end{cases} \quad ; \qquad Qx = \begin{cases} 1/x^2 & if\ x \neq 0 \\ 3 & if\ x = 0 \end{cases}$$

Then P and Q are not continuous at x=0. Define a sequence $\{x_n\} \subseteq R$ by $x_n = n^2$,

n = 1,2,….., then as n→∞ we get

$$Px_n = 1/n^4 \to 0 \ ; \ Qx_n = 1/n^6 \to 0$$

$\lim_{n\to\infty} d(PQ\,x_n, QPx_n) = \lim_{n\to\infty}|n^{12}-n^{12}| = 0.$

$\lim_{n\to\infty} d(PP\,x_n, QQx_n) = \lim_{n\to\infty}|n^{8}-n^{18}| = \infty.$

$\lim_{n\to\infty} d(PQ\,x_n, QQx_n) = \lim_{n\to\infty}|n^{12}-n^{8}| = \infty$ and

$\lim_{n\to\infty} d(QP\,x_n, PPx_n) = \lim_{n\to\infty}|n^{12}-n^{8}| = \infty$

So P and Q are compatible but P and Q are neither compatible of type (A) nor compatible of type (P).

Example (1.3) Let X =[0,3] with the usual metric in real line.

Define P,Q : [0,3]→ [0,3] by

$$Px = \begin{cases} 4 & if\ x\epsilon\ [0,3/2) \\ x & if\ x\ \epsilon\ [3/2,3] \end{cases} \qquad Qx = \begin{cases} 3/2\ \ if\ x\epsilon[0,3/2) \\ 3-x\ \ if\ x\ \epsilon\ [3/2,3] \end{cases}$$

Then P is not continuous at x=3/2. Let $\{x_n\} \subseteq [0,3]$ be a sequence converging to 3/2 such that $x_n \geq 3/2$ for all n=1,2,….. then as $n \to \infty$, we get that

$$P X_n = X_n \to 3/2 , Qx_n = 3-x \to 3/2$$

Hence, $\lim_{n\to\infty} d(QP\, x_n, PQx_n) = \lim_{n\to\infty} |3 - x_n - 1| = 1/2.$

$\lim_{n\to\infty} d(QP\, x_n, PPx_n) = \lim_{n\to\infty} |3 - x_n - x_n| = 0$

But $\lim_{n\to\infty} d(PQ\, x_n, QQx_n) = |1 - 3/2| = 1/2.$

So, P and Q are neither compatible nor compatible of type (A) but P and Q are compatible of type (P).

The following theorem for three mappings was proved by D. Prasad [5] satisfying a new functional inequality defined by Singh and Meade [7].

Theorem 1.1 Let P,Q and R be three self mappings from a complete metric space(X,d) into itself satisfying the following :

(1.1) PR = RP, QR = RQ, $P(X) \subseteq R(X)$ and $Q(X) \subseteq R(X)$

(1.2) $[d(P x, R y)]^2 \leq \varphi$ (d (R x, Px)d(R y, Q y),d (R x, Q y)d(R y,P x),
d (Rx, P x) d (R x, Q y), d (R y, P x) d (R y, Q y))

for all x, y ϵ X where $\varphi : (R^+)^4 \to R^+$ is upper semi continuous function which is non decreasing in each co-ordinate variable and satisfy the following conditions:

(1.3) $\varphi(u, u, b_1 u, b_2 u) < u$, for any $u > 0$ and $b_i \epsilon \{0,1,2\}$ such that $b_1 + b_2 = 2$

(1.4) R is continuous.

Then P, Q and R have a unique common fixed point in X.

2. MAIN RESULTS :

In this paper, the above result of D. Prasad [5] for three commuting mappings is generalized by us by using compatibility of type (P) for four mappings.

In order to prove our main result, first, we prove the following lemma:

Lemma 2.1 let P, Q, R, and S be four self maps of complete metric space (X, d) such that

(1.5) $P(X) \subseteq S(X)$; $Q(X) \subseteq R(X)$

(1.6) $[d(P x, Q y)]^2 \leq \varphi$ (d (R x, P x) d(S y, Q y),d (R x, Q y)d(S y,P x),
d (R x, P x) d (R x, Q y), d (Sy, P x) d (S y, Q y),
$[d(Sx, R y)]^2$,d (R x, P x) d (S y, P x), d (S y, Q y) d (Rx,Q y),
d (R x, S y) d (R x, P x), d (R x, S y) d (S y, P x), d (R x, Sy)
d (S y,Q y), d (R x, S y) d (R x, Q y), d (R x, S y) d (P x, Q y),
d (R x, P x) d (P x,Q y), d (S y, Q y), d (P x, Q y)

for all x, y ϵ X, where φ satisfies (i) and (ii).Then there is a cauchy sequence $\{y_n\}$ in X defined by

$y_{2n-1} = Sx_{2n-1} = Px_{2n-2}$ and $y_{2n} = R\, x_{2n} = Qx_{2n-1}$

for n = 1,2,3,……..

Proof: let $x_0 \epsilon X$ be arbitrary since

$P(X) \subseteq S(X)$ and $Q(X) \subseteq R(X)$, we can choose, x_1, x_2 in X such that

$Y_1 = Sx_1 = P\,x_0$ and $y_2 = Rx_2 = Q\,x_1$

In general we can choose x_{2n-1} , x_{2n} in X such that

(1.7) $Y_{2n-1} = Sx_{2n-1} = Px_{2n-2}$ and $y_{2n} = Rx_{2n} = Qx_{2n-1}$

Thus the indicated sequence $\{y_n\}$ exists.

To see that $\{y_n\}$ is Cauchy,note that (1.6) and (1.7) imply

$$[d(y_{2n+1}, y_{2n+2}]^2 = [d(Px_{2n}, Qx_{2n+1}]^2$$
$$\leq \varphi\, d(Rx_{2n} , Px_{2n}) d(Sx_{2n+1} , Qx_{2n+1}), d(Rx_{2n} , Qx_{2n+1}) d(Sx_{2n+1} , Px_{2n}) ,$$
$$d(Rx_{2n} , Px_{2n}) d(Rx_{2n} , Qx_{2n+1}), d(Sx_{2n+1} , Px_{2n}) d(Sx_{2n+1} , Qx_{2n+1}),$$
$$[d(Rx_{2n} , Sx_{2n+1})]^2 , d(Rx_{2n} , Px_{2n}) d(Sx_{2n+1} , Px_{2n}), d(Sx_{2n+1} , Qx_{2n+1})$$
$$d(Rx_{2n} , Qx_{2n+1}), d(Rx_{2n} , Sx_{2n+1}) d(Rx_{2n} , Px_{2n}) ,d(Rx_{2n} , Sx_{2n+1})$$
$$d(Sx_{2n+1} , Px_{2n}), d(Rx_{2n} , Sx_{2n+1}) d(Sx_{2n+1} , Qx_{2n+1}), d(2Rx_{2n} , Sx_{2n+1})$$
$$d(Rx_{2n} , Qx_{2n+1}), d(Rx_{2n} , Sx_{2n+1}) d(Px_{2n} , Qx_{2n+1}), d(Rx_{2n} , Px_{2n})$$
$$d(Px_{2n} , Qx_{2n+1}),d(Sx_{2n+1} , Qx_{2n+1}) d(Px_{2n} , Qx_{2n+1})).$$

$$[d(y_{2n+1}, y_{2n+2}]^2 \leq \varphi\, (d(y_{2n} , y_{2n+1}) ,d(y_{2n+1} , y_{2n+2}),0 , d(y_{2n} , y_{2n+1}), d(y_{2n} , y_{2n+2}),$$
$$[d(y_{2n} , y_{2n+1})]^2 ,0 , d(y_{2n+1} , y_{2n+2}), d(y_{2n} , y_{2n+2}), [d(y_{2n} , y_{2n+1})]^2 ,$$
$$0, d(y_{2n} , y_{2n+1}), d(y_{2n+1} , y_{2n+2}) , d(y_{2n} , y_{2n+1}) ,d(y_{2n} , y_{2n+2}),$$
$$d(y_{2n} , y_{2n+1}), d(y_{2n+1} , y_{2n2}), d(y_{2n} , y_{2n+1}), d(y_{2n+1} , y_{2n+2}) ,$$
$$[d(y_{2n+1} , y_{2n+2})]^2)$$

Let $d(y_{2n} , y_{2n+1}) = d_n$, so we obtain,

(1.8) $$d^2_{\ 2n+1} \leq \varphi\, (d_{2n} , d_{2n+1} ,0, d_{2n} (d_{2n} + d_{2n+1}),0 , d^{\,2}_{\ 2n} , 0 , d_{2n+1}(d_{2n} + d_{2n+1}),$$
$$d^2_{2n} , 0, d_{2n}\, d_{2n+1} , d_{2n}(d_{2n} + d_{2n+1}), d_{2n}\, d_{2n+1}, d_{2n+1}\, d_{2n}, d^2_{2n+1})$$

we want to prove that $d_{2n+1} < d_{2n}$.

for this ,now suppose that for some n,

$d_{2n+1} > d_{2n}$,now (1.8) provides

$$d^2_{\ 2n+1} \leq \varphi\, (d^2_{2n+1} ,0 , 2d^2_{\ 2n+1} ,0, d^2_{\ 2n+1}, 0 , d^{\,2}_{\ 2n+1} ,$$
$$d^2_{2n+1} ,2\, d^2_{2n+1} , d^2_{2n+1} , d^2_{2n+1} , d^2_{2n+1}) < d^2_{2n+1}$$

which is a contradiction.

Hence , $d_{2n+1} < d_{2n}$.

Similarly, we can show that $d_{2n+1} < d_{2n}$. now by equations (1.6) and (1.7) , we get,

(1.9) $$d^2_{\ 2n} \leq \varphi\, (d_{2n} , d_{2n-1} ,0,0, d_{2n-1} (d_{2n-1} + d_{2n}) , d^{\,2}_{\ 2n-1} , d_{2n}(d_{2n-1} + d_{2n}), 0,$$
$$d_{2n-1} , d_{2n} ,d_{2n-1} (d_{2n-1} + d_{2n}), d^2_{2n-1} , 0 , d_{2n-1}\, d_{2n}, d^2_{\ 2n}, d_{2n-1}\, d_{2n})$$

If possible , let $d_{2n} > d_{2n-1}$ for some n. then (1.9) provides

$$d^2_{\ 2n} \leq \varphi\, (d^2_{2n} ,0 ,0 , 2d^2_{\ 2n} , d^2_{\ 2n}, 2\, d^{\,2}_{\ 2n} , 0,$$
$$d^2_{2n} , 2\, d^2_{2n} , d^2_{2n} ,0 , d^2_{2n} , d^2_{2n} , d^2_{\ 2n}) < d^2_{2n}$$

which gives a contradiction. So $d_{2n} < d_{2n-1}$.

therefore $\{d_n\}$ is a decreasing sequence in R^+ . Let $d_n \rightarrow t \in R^+$. If $t \neq 0$ then since φ is upper semi continuous , we get from (1.8) and (1.9) $n \rightarrow \infty$.

$t^2 \leq \varphi(t^2, 0, 2t^2, 0, t^2, 0, 2t^2, t^2, 0, t^2, 2t^2, t^2, t^2, t^2) < t^2$

and

$t^2 \leq \varphi(t^2, 0, 0, 2t^2, t^2, 2t^2, 0, t^2, 2t^2, t^2, 0, t^2, t^2, t^2) < t^2$

which is a contradiction and therefore $d_n \to 0$. now we shall prove that sequence $\{y_n\}$ is Cauchy . If it is not so there exist an $\epsilon > 0$ and positive integer's sequence $\{r(m)\}$ and $\{s(m)\}$ with $m \leq s(m) < r(m)$ such that

(1.10) $K_m, = d(y_{r(m)}, y_{s(m)}) \geq \epsilon$ $m=1,2,\ldots\ldots$

Let r(m) be the least integer exceeding s(m) for which (1.10) is true , then by the well ordering principle

$d(y_{r(m)-1}, y_{s(m)}) < \epsilon$. Now

$\epsilon \leq K_m \leq d(y_{r(m)}, y_{r(m)-1}) + d(y_{r(m-1)}, y_{s(m)}) < d_{r(m)-1} + \epsilon \to \epsilon$ as $m \to \infty$.

And thus $K_m \to \epsilon$. Further under the following four cases, K_m can have different values vi.z, (i) r is even and s is odd ; (ii) r and s are odd ; (iii) r is odd and s is even ; (iv) r and s are even . Now in case (i) we have

$K_m = d(y_{2r}, y_{2s-1}) \leq d(y_{2r}, y_{2r+1}) + d(y_{2r\,1}, y_{2r}) + d(y_{2s}, y_{2s-1})$

Letting $s \to \infty$, we obtain ,

(1.11) $\epsilon \leq 0 + 0 + \lim_{n\to\infty} d(y_{2r+1}, y_{2s})$

Now by equation (1.6) and (1.7) , we get

$$\begin{aligned}
[d(y_{2r+1}, y_{2s}]^2 = [d(Px_{2r}, Qx_{2s-1}]^2 \leq\ & \varphi(d(y_{2r}, y_{2r+1}), d(y_{2s-1}, y_{2s}), d(y_{2r}, y_{2s}),\\
& d(y_{2s-1}, y_{2r+1}), d(y_{2r}, y_{2r+1}), d(y_{2r}, y_{2s}), d(y_{2s-1}, y_{2r+1}), d(y_{2s-1}, y_{2s}),\\
& [d(y_{2r}, y_{2s-1})]^2, d(y_{2r}, y_{2r+1}), d(y_{2s-1}, y_{2r+1}), d(y_{2s-1}, y_{2s}),\\
& d(y_{2r}, y_{2s}), d(y_{2r}, y_{2s-1}), d(y_{2r}, y_{2r+1}), d(y_{2r}, y_{2s-1}),\\
& d(y_{2s-1}, y_{2r+1})\, d(y_{2r}, y_{2s-1}), d(y_{2s-1}, y_{2s}), d(y_{2r}, y_{2s-1}), d(y_{2r}, y_{2s}),\\
& d(y_{2r}, y_{2s-1}), d(y_{2r+1}, y_{2s}), d(y_{2r}, y_{2r+1}), d(y_{2r}, y_{2s}),\\
& d(y_{2s-1}, y_{2s}), d(y_{2r+1}, y_{2s}))\\
\leq\ & \varphi(d_{2r}\, d_{2s-1}, (K_m + d_{2s-1})(K_m + d_{2r}), d_{2r}(K_m + d_{2s-1}),\\
& (K_m + d_{2r})\, d_{2s-1}, K^2_m, d_{2r}(K_m + d_{2r}), d_{2s-1}(K_m + d_{2s-1}), K_m d_{2r},\\
& K_m(K_m + d_{2r}), K_m(K_m + d_{2s-1}), K_m(d_{2r} + K_m + d_{2s-1}), d_{2r}(K_m + d_{2s-1}),\\
& d_{2s-1}(d_{2r} + K_{m+} d_{2s-1}))
\end{aligned}$$

Since φ is upper semi continuous so letting $n \to \infty$ and using (ii) , we obtain,

$$\lim_{n\to\infty}[d(y_{2r+1}, y_{2s})]^2 \leq \varphi(0, \epsilon^2, 0, 0, \epsilon^2, 0, 0, 0, \epsilon^2, 0, \epsilon^2, \epsilon^2, 0, 0) < \epsilon^2$$

Therefore we obtain a contradiction and hence $\epsilon = 0$, similarly rest of other cases also give $\epsilon = 0$. Hence $\{y_n\}$ is a Cauchy sequence

Now, we prove our Main result by using this lemma.

Theorem 2.1 let P, Q, R, and S be four self maps of complete metric space (X,d) satisfying following conditions:

(1.5), (1.6), (1.12) one of P, Q, R and S is continuous at their coincidence point .

And (1.13) the pair P, R and Q , S are compatible of type (P).

Then P,Q,R and S have a unique common fixed point in X.

Proof: By lemma 2.1, there $\{y_n\}$ is a sequence in X defined by (1.7) such that the sequence $\{y_n\}$ is Cauchy. Sequence $\{y_n\}$ converges to a point u in X, there (X, d) is complete . Consequently, the subsequences $\{Px_{2n}\}$, $\{Rx_{2n}\}$, $\{Qx_{2n-1}\}$ and $\{Sx_{2n-1}\}$ converges to u.

Now, let us suppose that P is continuous, since P and R are compatible of type (P) ,it follows from proposition 1.4 that implies

$PRx_{2n} \to Pu$, $RRx_{2n} \to Pu$ as $n \to \infty$.

By using condition (1.6), we get

$[d(PRx_{2n}, Qx_{2n-1}]^2 \leq \varphi\, d(RRx_{2n} , PRx_{2n}) d(Sx_{2n-1} , Qx_{2n-1}),$
$d(RRx_{2n} , Qx_{2n-1}) d(Tx_{2n-1} , PRx_{2n})$,
$d(RRx_{2n} , PRx_{2n}) d(RRx_{2n} , Qx_{2n-1}), d(Sx_{2n-1} ,PRx_{2n}) d(Sx_{2n-1},Qx_{2n-1}),$
$[d(RRx_{2n} , Sx_{2n-1})]^2$, $d(RRx_{2n} , PRx_{2n}) d(Sx_{2n-1} , PRx_{2n}),$
$d(Sx_{2n-1} , Qx_{2n-1}) d(RRx_{2n} , Qx_{2n-1}), d(RRx_{2n} , Sx_{2n-1}) d(RRx_{2n} , PRx_{2n}),$ $d (RRx_{2n} , Sx_{2n-1}) d(Sx_{2n-1} , PRx_{2n}), d(Rx_{2n} , Sx_{2n-1}) d(Sx_{2n-1} , Qx_{2n-1}),$
$d(RRx_{2n} , Sx_{2n-1}) (RRx_{2n} , Qx_{2n-1}), d(RRx_{2n} , Sx_{2n-1}) d(PRx_{2n} , Qx_{2n-1}),$ $d(RRx_{2n} , PRx_{2n})d(PRx_{2n} , Qx_{2n-1}) ,d(Sx_{2n-1} , Qx_{2n-1})d(PRx_{2n} , Qx_{2n-1}))$

Since φ is upper semi- continuous , letting $n\to \infty$, we get,

$[d(Pu,u)]^2 \leq \varphi\ (0, [d(Pu,u)]^2 , 0, 0, [d(Pu,u)]^2 ,0, 0, 0, [d(Pu,u)]^2 ,0, [d(Pu,u)]^2 , [d(Pu,u)]^2 , 0, 0,)$

So, u = Pu . Since $P(X) \subseteq S(X)$, so there exist a point w in X such that u = Pu = Sw. Again by using (1.6) , we obtain ,

$[d(PRx_{2n}, Qw]^2 \leq \varphi\, d(RRx_{2n} , PRx_{2n}) d(Sw , Qw),$
$d(RRx_{2n} , Qw) d(Sw , PRx_{2n})$,
$d(RRx_{2n} , PRx_{2n}) d(RRx_{2n} , Qw), d(Sw ,PRx_{2n}) d(Sw, Qw),$
$[d(RRx_{2n} , Sw)]^2$, $d(RRx_{2n} , PRx_{2n}) d(Sw , PRx_{2n}),$
$d(Sw , Qw) d(RRx_{2n} , Qw), d(RRx_{2n} , Sw) d(RRx_{2n} , PRx_{2n})$,
$d(RRx_{2n} , Sw) d(Sw , PRx_{2n}), d(RRx_{2n} , Sw) d(Sw , Qw),$
$d(RRx_{2n} , Sw) (Sw , Qw), d(RRx_{2n} , Sw) d(RRx_{2n} , Qw),$
$d(RRx_{2n},Sw)d(PRx_{2n},Qw),$
$d(RRx_{2n} , PRx_{2n})d(PRx_{2n} , Qw) , d(Sw , Qw) d(Px_{2n} , Qw),)$

Letting $n \to \infty$, above inequality becomes

$[d(u, Qu)]^2 \leq \varphi\ (0, [d(u, Qu)]^2\ 0, 0, [d(u, Qu)]^2\ 0, 0, 0, [d(u, Qu)]^2 , 0, [d(u, Qu)]^2 ,$
$[d(u, Qu)]^2 , 0, 0)$

Hence Qu = u = Sw = Pu

Now Q and S are compatible mappings of type (A) and Sw = Qw = u.

So by proposition 1.3, we have SQw = QSw and thus Su = Qu. Now by using (1.6) again,we obtain

$[d(Px_{2n}, Qu]^2 \leq \varphi\, d(Rx_{2n} ,Px_{2n}) d(Su , Qu),$
$d(Rx_{2n} , Qu) d(Su , Px_{2n})$,

$d(Rx_{2n}, Px_{2n})\, d(Rx_{2n}, Qu), d(Su, Px_{2n})\, d(Su, Qu),$

$[d(Rx_{2n}, Su)]^2, d(Rx_{2n}, Px_{2n})\, d(Su, Px_{2n}),$

$d(Su, Qu)\, d(Rx_{2n}, Qu), d(Rx_{2n}, Su)\, d(Rx_{2n}, Px_{2n}),$

$d(Rx_{2n}, Su)\, d(Su, Px_{2n}), d(Rx_{2n}, Su)\, d(Su, Qu),$

$d(Rx_{2n}, Su)\, (Rx_{2n}, Qu), d(Rx_{2n}, Su)\, d(Px_{2n}, Qu),$

$d(Rx_{2n},Px_{2n})d(Px_{2n},Qu),$

$d(Su, Qu)d(Px_{2n}, Qu))$

letting $n \to \infty$, above inequality becomes

$[d(u, Qu)]^2 \leq \varphi(0, [d(u, Qu)]^2\ 0, 0, [d(u, Qu)]^2\ 0, 0, 0, [d(u, Qu)]^2\ 0, [d(u, Qu)]^2,$

$[d(u, Qu)]^2, 0, 0)$

Hence u = Qu = Sw = Pu.

Since $Q(X) \subseteq R(X)$, So there exist a point t in X such that u = Qu = Rt .

Again implies condition (1.6) that

$d^2(Pt, u) \leq \varphi(0, 0, 0, 0, 0, [d(Pt, u)]^2\ 0, 0, 0, 0, 0, 0, [d(Pt, u)]^2, 0)$

$< [d(Pt, u)]^2$

Hence Pt = u. Since P and R are compatible maps of type (R) and Pt = Rt = u. we get by using proposition 1.3 , RPt = PRt, and thus Ru = Pu = u. Hence u is a common fixed cpoint of P, Q, R and S.

Similarly, we can complete the proof when Q or R or S is continuous.

From condition (1.6), uniqueness follows easily.

Remark 2.1 If we put R = S in theorem (2.1) , such that R is continuous ,

PR = RA ; QR = RQ and $P(X) \subseteq R(X)$,$Q(X) \subseteq R(X)$ so that condition (1.6) becomes

$[d(Px, Ry)]^2 \leq \varphi(d(Rx, Px)d(Ry, Qy), d(Rx, Qy)d(Ry,Px),$

$d(Rx, Px)\, d(Rx, Qy), d(Ry, Px)\, d(Ry, Qy))$

then theorem (1.1) of Devi Prasad [5] is obtained by us.

Now we give the following example to show the validity of theorem (2.1).

Example 2.1 Let T = [0, 1] with usual metric in real line. P, Q, R and S are defined by

Pt = t/8 , Qt = t/6 , Rt = t/2 and St = t/3 for all $t \in [0, 1]$

And P(t) = [0, 8] $\subseteq$ [0, 2/3] = S(t)

Q(t)= [0, 1/6] $\subseteq$ [0, ½] = R(t)

Moreover, $|Pt - Rt| = |t/8 - t/2| = (3t)/8 \to 0$ iff $t \to 0$.

$|PPt - RRt| = |t/64 - t/4| = (15)t/64 \to 0$ iff $t \to 0$.

So that P and R are compatible of type (P) .Similarly,

$|SQt - QQt| = |2t^{18} - 1 - t^9| = |(t^9 - 1)(2t^9 + 1)| \to 0$ iff $t \to 0$.

Hence Q and S are also compatible of type (P).

Let the function φ is defined by us as

$\Phi(u_1, u_2, \ldots\ldots, u_{14}) = h \max\{u_1, u_2, u_3, \ldots\ldots, u_{14}\}$

Where $1/16 \leq h < ½$, $u_i \in R^+$ for $i = 1,2,3,\ldots\ldots.,14$.

Then φ satisfies (i) and (ii) . Also , we get

$$|Pt - Qv| = |t/8 - v/6| = 1/24|3t - 4v|;$$
$$|Rt - Sv| = |t/2 - 2t/3| = \frac{|3t-4v|}{6};$$
$$|Pt - Qv|^2 \leq 1/16\, |Rt - Sv|^2$$
$$\leq \varphi\,(d(Rt, Pt)d(Sv, Qv), \ldots, d[(Rt, Sv)]^2 \ldots\ldots)$$

Therefore condition (1.6) is satisfied and hence the hypothesis of theorem 2.1, is satisfied and clearly t = 0 is the unique common fixed point of P, Q, R and S .

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A REPLESNISHMENT POLICY FOR DETERIORATING ITEMS UNDER INFLATION WITH RAMP TYPE DEMAND RATE AND PARTIAL EXPONENTIAL TYPE BACKLOGGING

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ABSTRACT :
In this paper, we develop an order-level inventory system for deteriorating items under inflation with ramp type demand function and partial exponential type-backlogging function of time. Three costs are considered under inflation as significant: deterioration, holding, shortage. The backlogging rate is an exponentially decreasing, time-dependent function specified by a parameter. For this model we derive results, which ensure the existence of a unique optimal policy and provide the solution procedure for the problem. The method is illustrated by numerical example, and sensitivity analysis of the optimal solution with respect to the parameters of the system is carried out.

Keywords: Inventory, Deteriorating items, Inflation, Ramp type demand.

INTRODUCTION:

Most of the Literature available in the field of inventory management does not take into account the effects of inflation and time value of money. This has happened mostly because of the belief that inflation would not influence the policy variables to any significant degree. But during the last twenty years the monetary situation of most of the countries, affluent or otherwise, has changed to such an extent due to large scale inflation and consequent sharp decline in the purchasing power of money, that it has not been possible to ignore the effects of inflation and time value of money any further and so several efforts have been made by researchers to reformulate the optimal inventory management policies taking into account inflation and time value of money etc.

The first attempt in this direction was by Buzacott[2]. In this chapter, he dealt with an economic order quantity model with inflation subject in different types of pricing policies. After Buzacott [2]. Several other researchers have extended his approach to various interesting situations taking into consideration the time value of money, different inflation rates for the internal and external costs, infinite and finite replenishment rate, with or without shortage etc. In this connection, the works of Misra [5, 6], Aggarwal [1],Jeya Chandra and Bahner [3] , etc. are worth mentioning. But in all these studies, the market demand rate has been assumed to be a constant and unsatisfied demand is completely backlogged.

However, for fashionable commodities and high tech products with short product life cycle, he willingness for a customer to wait for backlogging during a shortage period is diminishing with the length of the waiting time. Hence, the longer the waiting time is the smaller the backlogging rate would be. To reflect this phenomenon, Papactristos and Skouri [7] established a partially backlogged inventory model in which the backlogging rate decreases exponentially as the waiting time increases.

Teng et al [8] extended the fraction of unsatisfied demand back ordered to any decreasing function of the waiting time up to the next replenishment. Mandal and Pal [4] and Sharma etal. (10) investigated an order level inventory model for deteriorating items, where the demand rate is a ramp type function of time. Kun-Shan-Wu and Liang-Yuh-Ouyang [9] extended the work to allow inventory starts with shortage. This type of demand pattern is generally seen in the case of any new brand of consumer goods coming to the market. The demand rate for such items increases with time up to a certain time and then ultimately stabilizes and becomes constant. It is believed that this type of demand rate is quite realistic. In this paper, we develop an inflationary modal for deteriorating items with ramp type demand rate and partial-exponential type backlogging. For precision, in the paper, we provide the exact solution for the problem, and method is illustrated by numerical example. Sensitivity analysis of the optimal solution with respect to the parameters of the system is also carried out.

ASSUMPTIONS AND NOTATIONS:

The mathematical model of the deterministic inventory replenishment problem with ramp type demand rate is based on the following assumptions:

(1) The replenishment rate is infinite, thus replenishments are instantaneous.

(2) The lead time is zero.(3) The on hand inventory deteriorates at a constant rate $\theta(0<\theta<1)$ per unit time. The deteriorated items are with drawn immediately from the warehouse and there is no provision for repair or replacement.

(4) The rate of demand R(t) be ramp type demand function of ***t***.

$$R(t) = D[t - (t-\mu)H(t-\mu)] \quad D_0 > t$$

Where H (t—μ) is Heaviside's function defined as Follows:

$$H(t-\mu) = \{^{1\ t\geq\mu}_{0\ t<\mu}$$

(5) Unsatisfied demand is backlogged at a rate $e^{-\alpha x}$ where x is the time up to the next replenishment and 'α' a parameter

$$0 < \alpha < \frac{1}{T}$$

(6) The unit price is subject to the same inflation rate as other related costs.

The inflation rate is the rate of decrease in the purchasing power of money.

The following notations are used throughout this chapter:

1. T - The fixed length of each ordering cycle.
2. S - The maximum inventory level for each ordering.
3. r - The inflation rate.
4. C_h - The inventory holding cost per unit per unit of time.
5. C_s - The shortage cost per unit per unit of time.
6. C_d - Deterioration cost per unit of deteriorated item.
7. I(t) - The on hand inventory at time t over [0,T).
8. CI - The amount of inventory carried during a cycle.

9. DI - The total number of items which deteriorate during a cycle.

10.SI - The amount of shortage during a cycle.

MATHEMATICAL MODELS AND SOLUTIONS:

The objective of the inventory problem here is todetermine the optimal order quantity so as to keep the total Relevant cost as low as possible under inflation. Based onwhether the inventory starts with shortages or not, there aretwo possible models under the assumptions described above.

1. The inventory model starts without shortages:

In this section, we shall discuss the inventory model for deteriorating items under inflation where the inventory starts without shortages.

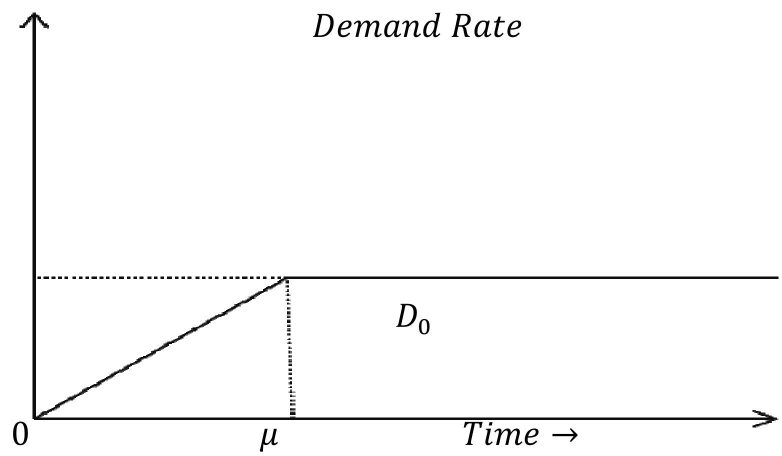

Fig. 1: A ramp type function of the demand rate [Adapted fromMandal and Pal [4]]

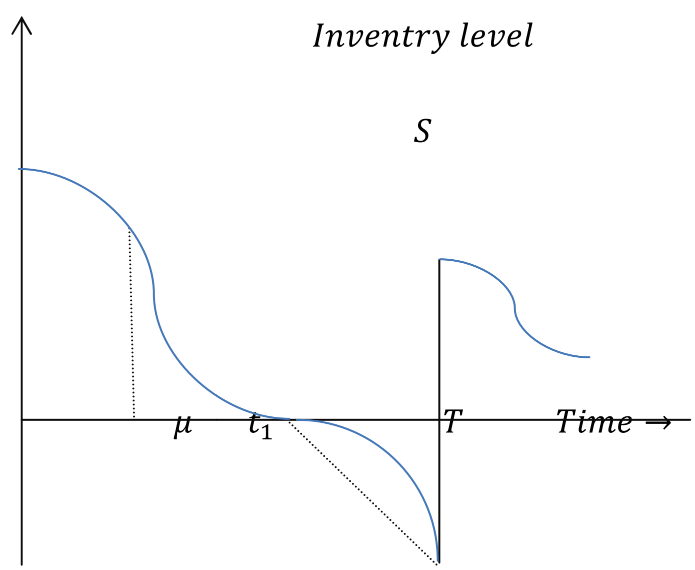

Fig.2: Graphical representation of inventory Model

The fluctuation of the inventory level in the system is given fig. 2. Replenishment is made at time t = 0 when the inventory level is at its maximumS. The inventory at t=0gradually reduces to 0 at t_1 time units. The depletion of inventory level during the interval [0, t_1] is due to the jointeffectof the demand and the deterioration of items. At t_1, the inventory level reaches zero, and thereafter, shortages areallowed to occur during the time interval [t_1, T), and thedemand during period [t_1, T), is partially backlogged. The total number of backlogged items is replaced by the next replenishment.

The inventory level of the system at the time t over period [0, T], can be described by the following equations:

$$\frac{dI(t)}{dt} = -R(t) - \theta I(t) \quad 0 \le t \le t_1 \qquad \text{........(1)}$$

and$$\frac{dI(t)}{dt} = -e^{-\alpha(T-t)}R(t) t_1 \le t < T \qquad \text{........(2)}$$

Mandal and pal [4] assumed that $\mu < t_1$ therefore, theabove equations become

$$\frac{dI(t)}{dt} + \theta I(t) = -D_0 t \qquad 0 \le t \le \mu \qquad \ldots\ldots\ldots..(3)$$

With boundary Condition$I(0) = S$

$$\frac{dI(t)}{dt} + \theta I(t) = -D_0 \mu \qquad \mu \le t \le t_1 \qquad \ldots\ldots\ldots..(4)$$

With boundary Condition $I(t_1) = 0$

and

$$\frac{dI(t)}{dt} = -e^{-\alpha(T-t)} D_0 \mu \qquad t_1 \le t < T \qquad \ldots\ldots\ldots..(5)$$

With boundary Condition $I(t_1) = 0$

The solution of equation (3) is

$$I(t) = Se^{-\theta t} - D_0\left[\frac{t}{\theta} - \frac{1}{\theta^2}\left(1 - e^{-\theta t}\right)\right] \quad 0 \le t \le \mu \qquad \text{........ (6)}$$

The solution of equation (4) is

$$I(t) = \frac{D_0\mu}{\theta}\left[e^{\theta(t_1 - t)} - 1\right] \quad \mu \le t < t_1 \qquad \text{........ (7)}$$

and the solution of equation (5)

$$I(t) = \frac{D_0\mu}{\alpha}\left[e^{-\alpha(T-t)} - e^{-\alpha(T-t_1)}\right] t_1 \le t < T \qquad \text{........ (8)}$$

From equation (6) and (7), the values of I (t) at t=μ should coincide, which implies that

$$S = \frac{D_0\mu}{\theta} e^{\theta t_1} - \frac{D_0}{\theta^2}\left(e^{\theta\mu} - 1\right) \qquad \text{........ (9)}$$

The amount of inventory carried during the period [0, t_1] is

$$CI = \int_0^{t_1} I(t)dt = \int_0^{\mu} I(t)dt + \int_{\mu}^{t_1} I(t)dt \qquad \text{..... (10)}$$

The total number of items which deteriorate during $[0, t_1]$ $is\ DI$

$$DI = S - \int_0^{t_1} R(t)dt = \theta \int_0^{t_1} I(t)dt \qquad \text{......... (11)}$$

The amount of shortage during the period [t_1,T)

$$SI = \int_{t_1}^{T} I(t)dt \qquad \text{..........(12)}$$

Following the approach of Misra[5,6],we obtain the present value of the inventory holding cost during the period [0,t_1] as

$$= C_h \int_0^{t_1} e^{-rt} I(t)dt = C_h \left[\int_0^{\mu} e^{-rt} I(t)dt + \int_{\mu}^{t_1} e^{-rt} I(t)dt \right]$$

$$= C_h \left[\frac{S\theta^2 - D_0}{\theta^2(\theta+r)} \left(1 - e^{-(\theta+r)\mu}\right) + \frac{D_0(r-\theta)}{\theta^2 r^2} \left(1 - e^{-r\mu}\right) + \frac{D_0 \mu e^{-rt_1}}{r(\theta+r)} + \frac{D_0 \mu}{\theta(\theta+r)} e^{\theta t_1 - (\theta+r)\mu} \right] \quad ....(13)$$

The Present value of the deterioration cost during the period [0, t_1] as

$$C_d \int_0^{t_1} e^{-rt_1} \theta I(t)dt = C_d \theta \left[\frac{S\theta^2 - D_0}{\theta^2(\theta+r)} \left(1 - e^{-(\theta+r)\mu}\right) + \frac{D_0(r-\theta)\mu}{\theta^2 r^2} \left(1 - e^{-r\mu}\right) + \frac{D_0 \mu e^{-rt_1}}{r(\theta+r)} + \frac{D_0 \mu}{\theta(\theta+r)} e^{\theta t_1 - (\theta+r)\mu} \right]$$

..... (14)

The present value of the shortage cost during the period [t_1,T) as

$$C_s \int_{t_1}^{T} e^{-rt} I(t)dt = C_s \int_{t_1}^{T} e^{-rt} \frac{D_0 \mu}{\alpha} \left\{ e^{-\alpha(T-t)} - e^{-\alpha(T-t_1)} \right\} dt$$

$$= C_s \left[\frac{D_0 \mu}{\alpha(\alpha - r)} e^{-rT} - \frac{D_0 \mu}{r(\alpha - r)} e^{-\alpha T + (\alpha - r)t_1} + \frac{D_0 \mu}{\alpha r} e^{(\alpha + r)T + \alpha t_1} \right] \quad ... ... (15)$$

The order quantity during the period [0, T) is-

$$\boldsymbol{Q} = S + I(T) = \frac{D_0 \mu}{\theta} e^{\theta t_1} - \frac{D_0}{\theta^2} \left(e^{\theta\mu} - 1\right) + \frac{D_0 \mu}{\alpha} (1 - e^{-\alpha(T-t_1)} \quad .... (16)$$

Thus the total relevant cost of the system during the time interval [0, T), which is

$$X_1 = (C_h + C_d\theta) \left[\frac{S\theta^2 - D_0}{\theta^2(\theta+r)} \left(1 - e^{-(\theta+r)\mu}\right) + \frac{D_0(r-\theta)}{\theta^2 r^2} \left(1 - e^{-r\mu}\right) + \frac{D_0 \mu e^{-rt_1}}{r(\theta+r)} + \frac{D_0 \mu e}{\theta(\theta+r)} e^{\theta t_1 - (\theta+r)\mu} \right] + C_s \left[\frac{D_0 \mu}{\alpha(\alpha - r)} e^{-rT} - \frac{D_0 \mu}{r(\alpha - r)} e^{-rT + (\alpha - r)t_1} + \frac{D_0 \mu}{ar} e^{-(\alpha + r)T + at_1} \right]$$

...... (17)

Thus , the average total cost per unit time is

$$TC_1(t_1) = \frac{X_1}{T} \quad .... (18)$$

To minimize the average total cost per unit of time, the optimal value of t_1 can be obtained by solving the following equation

$$\frac{dTC_1(t_1)}{dt_1} \quad .... (19)$$

Which also satisfies the conditions

$$\left. \frac{d^2 TC_1(t_1)}{dt_1^2} > 0 \right|_{t = t_1}$$

Equation (19) is equivalent to

$$(C_h + C_d\theta) \left[\frac{D_0 \mu e^{\theta t_1}}{(\theta+r)} \left(1 - e^{-(\theta+r)\mu}\right) - \frac{D_0 \mu}{(\theta+r)} e^{-rt_1} + \frac{D_0 \mu e^{\theta t_1 - (\theta+r)\mu}}{(\theta+r)} \right] + C_s \left[\frac{D_0 \mu}{r} e^{-(\alpha+r)T + at_1} - \frac{D_0 \mu}{r} e^{-\alpha T + (\alpha - r)t_1} \right]$$

.... (20)

Equation (20) is a non-Linear equation. Thus equation can be easily solved using any iterative method when the value of the parameters is presented.

By using the optimal value t^*_1, the optimal value of S*, the minimum average total cost per unit of time and the optimal order quantity can be obtained from equation(9) , equation(18) and equation(16) respectively .

THE INVENTORY MODEL STARTS WITH SHORTAGES:

In this section, we will discuss the inventory model for deteriorating items under inflations, where the inventory starts with shortages. The behavior of the inventory system at any time during a given cycle is depicted in Figure3. There are two different situations may arise due to time $t_1 (i) \mu < t_1,$ and $(ii)\ \mu > t_1.$

The inventory system starts with shortage at $t = 0$ and accumulate up to $t = t_1.$ The quantity received at $t_1$is used partly to meet the accumulated shortages in the previous cycle from time 0 to $t_1.$ The rest of procurement accounts for the demand and deterioration in [t_1, *T*). The inventory level gradually falls to zero at time *T*.

The inventory level of the system at time t over the period [0,T) can described by the following equations:

$$\frac{dI(t)}{dt} = -e^{-\alpha(t_1-t)}R(t)$$

$$0 \le t < t_1 \qquad \text{….. (21)}$$

and $$\frac{dI(t)}{dt} + \theta I(t) = R(t) \qquad t_1 \le t < T \qquad \text{…… (22)}$$

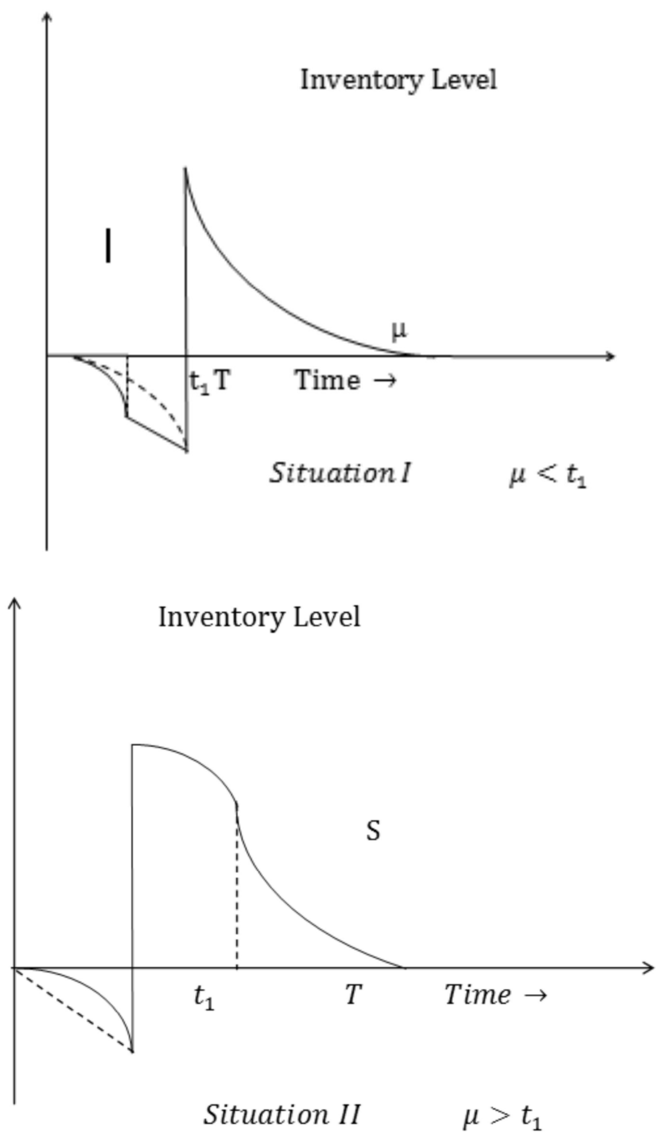

Figure 3: Graphical representation of Inventory models

Situation I : When $\mu < t_1$

In this situation the above two governing equation become

$$\frac{dI(t)}{dt} = -e^{-\alpha(t_1-t)}D_0 t \qquad 0 \le t \le \mu \qquad \text{..........(23)}$$

$$\frac{dI(t)}{dt} = -e^{-\alpha(t_1-t)}D_0 \mu \qquad \mu \le t \le t_1 \qquad \text{.......... (24)}$$

and

$$\frac{dI(t)}{dt} + \theta I(t) = -D_0\mu \qquad t_1 \le t < T \qquad \text{...... (25)}$$

The solutions of the differential equation (23)-(25) with the boundary conditions $I(0) = 0$ and $I(T) = 0$ are

$$I(t) = -\frac{D_0}{\alpha^2}\left[e^{-\alpha(t_1-t)}(\alpha t - 1) + e^{-\alpha t_1}\right] \qquad 0 \le t \le \mu \qquad \text{........ (26)}$$

$$I(t) = -\frac{D_0}{\alpha^2}\left[e^{-\alpha(t_1-t)}\mu\alpha - e^{-\alpha(t_1-\mu)} + e^{-\alpha t_1}\right] \quad \mu \le t < t_1 \qquad \text{.......... (27)}$$

$$I(t) = \frac{D_0\mu}{\theta}\left[e^{\theta(T-t)} - 1\right] t_1 \le t < T \qquad \text{..... (28)}$$

Following the approach of Misra [5, 6], we obtain the present value of the holding cost during $[t_1, T)$ as

$$C_h\int_{t_1}^{T} e^{-rt}I(t)dt = C_h\int_{t_1}^{T} e^{-rt}\frac{D_0\mu}{\theta}\left[e^{\theta(T-t)} - 1\right]dt$$

$$= C_h\left[\frac{D_0\mu}{r(\theta+r)}e^{-rT} + \frac{e^{\theta T-(\theta+r)t_1}}{\theta(\theta+r)}D_0\mu - \frac{D_0\mu e^{-rt_1}}{r\theta}\right] \qquad \text{... ... (29)}$$

The present value of the deterioration cost during the period $[t_1, T)$ as

$$C_d\int_{t_1}^{T} e^{-rt}\theta I(t)dt = C_d\int_{t_1}^{T} e^{-rt}D_0\mu\left[e^{\theta(T-t)} - 1\right]dt$$

$$= C_d\theta\left[\frac{D_0\mu}{r(\theta+r)}e^{-rT} + \frac{D_0\mu}{r(\theta+r)}e^{\theta T-(\theta+r)t_1} - \frac{D_0\mu e^{-rt_1}}{r\theta}\right]$$

......(30)

The present value of the shortage cost during the period $[0, t_1]$ as

$$C_s\int_0^{t_1} e^{-rt}I(t)dt = C_s\left[\int_0^{\mu} e^{-rt}I(t)dt + \int_{\mu}^{t_1} e^{-rt}I(t)dt\right]$$

$$= C_s\frac{D_0}{\alpha^2}\left[\frac{\alpha^2}{r(\alpha-r)^2}e^{-\alpha t_1}(1 - e^{(\alpha-r)\mu}) + \frac{e^{-(\alpha+r)t_1}}{r}(e^{\alpha\mu} - 1) + \frac{\mu\alpha e^{-rt_1}}{(\alpha-r)}\right]$$

.........(31)

The order quantity during the period [0, T) is

$$Q = \frac{D_0\mu}{\theta}\left[e^{\theta(T-t_1)} - 1\right] + \left[\frac{D_0\mu}{\alpha} + \frac{D_0 e^{-\alpha t_1}}{\alpha^2} - \frac{D_0}{\alpha^2}e^{-\alpha(t_1-\mu)}\right] \qquad \text{.........(32)}$$

Thus, the total relevant cost of the system during the time interval [0,T), which is

$$X_2 = (C_h + C_d\theta)\left[\frac{D_0\mu}{r(\theta+r)}e^{-rT} + \frac{D_0\mu}{\theta(\theta+r)}e^{\theta T-(\theta+r)t_1} - \frac{D_0\mu e^{-rt_1}}{r\theta}\right] +$$

$$C_s\frac{D_0}{\alpha^2}\left[\frac{\alpha^2}{r(\alpha-r)^2}e^{-\alpha t_1}\left(1 - e^{(\alpha-r)\mu}\right) + \frac{e^{-(\alpha+r)t_1}}{r}(e^{\alpha\mu} - 1) + \frac{\mu\alpha e^{-rt_1}}{(\alpha-r)}\right]$$

......(33)

Thus, the average total cost per unit time is

$$TC_2(t_1) = \frac{X_2}{T} \qquad \text{.........(34)}$$

The optimal value of t_1 for the minimum average total cost per unit of time is the solution

$\frac{dTC_2(t_1)}{dt_1} = 0$ ………… (35)

Provided that the value of t_1 satisfies the condition

$$\left. \frac{d^2TC_2(t_1)}{dt_1^2} > 0 \right|_{t = t_1^*}$$

Equation (35) is equivalent to

$$(C_h + C_d\theta)\frac{D_0\mu}{\theta}\left[e^{-rt_1} - e^{\theta T-(\theta+r)t_1}\right] + C_s\left[\frac{D_0\alpha}{r(\alpha-r)^2}\left(e^{(\alpha-r)\mu}-1\right)e^{-\alpha t_1} - \frac{D_0(\alpha+r)}{r\alpha^2}e^{-(\alpha+r)t_1}(e^{\alpha\mu}-1) - \frac{D_0\mu r e^{-rt_1}}{\alpha(\alpha-r)}\right] = 0$$

…….(36)

For different value of the various parameters, equation (36) can be solved and find the optimal value t_1^*.

By using the optimal value of t_1^*, the minimum average total cost per unit of time can be obtained from equation (34).

Situation II: $(\mu > t_1)$

In this situation, the above two governing equations, equation (21) and equation (22), becomes

$\frac{dI(t)}{dt} = -e^{-\alpha(t_1-t)}D_0 t \qquad 0 \le t \le t_1$ …………….. (37)

$\frac{dI(t)}{dt} + \theta I(t) = -D_0 t \qquad t_1 \le t \le \mu$ …………….. (38)

and

$\frac{dI(t)}{dt} + \theta I(t) = -D_0\mu \qquad \mu \le t < T$ …………….. (39)

The solutions of the differential equations (37)-(39) with the boundary conditions $I(0) = 0$ and $I(T) = 0$ are

$$I(t) = -\frac{D_0}{\alpha^2}\left[e^{-\alpha(t_1-t)}(\alpha t - 1) + e^{-\alpha t_1}\right] \quad 0 \le t \le t_1$$

…………….. (40)

$I(t) = Se^{-\theta(t-t_1)} - \frac{D_0}{\theta^2}\{(\theta t - 1) - (\theta t_1 - 1)e^{-\theta(t-t_1)}\} t_1 \le t \le \mu$

….. (41)

$I(t) = \frac{D_0\mu}{\theta}\left[e^{\theta(T-t)} - 1\right] \qquad \mu \le t < T$ ….. (42)

Now, in equations (41)-(42), the values of I(t)at $t = \mu$ should coincide, which implies that

$S = \frac{D_0\mu}{\theta}e^{\theta(T-t_1)} - \frac{D_0}{\theta^2}\left[e^{\theta(\mu-t_1)} + \theta t_1 - 1\right]$ …….. (43)

As discussed in situationI, the average total cost per unit of time is

$$TC_3(t_1) = \frac{1}{T}\left[(C_h + C_d\theta)\left\{\frac{D_0e^{-\mu r}}{r^2(\theta+r)} + \frac{D_0e^{-(\theta+r)t_1}}{\theta^2(\theta+r)}\left(\mu\theta e^{\theta T} - e^{\mu\theta}\right) + \frac{D_0\mu e^{-rT}}{r(\theta+r)} - \frac{D_0}{\theta^2 r^2}(\theta r t_1 - r + \theta)e^{-rt_1}\right\} + C_s\frac{D_0}{\alpha^2}\left\{\frac{(\alpha t_1 - 1)}{(\alpha-r)}e^{-rt_1} - \frac{\alpha}{(\alpha-r)^2}e^{-rt_1} - \frac{e^{-(\alpha+r)t_1}}{r} + \frac{\alpha^2}{r(\alpha-r)^2}e^{-\alpha t_1}\right\}\right]$$

……..(44)

The optimal value of $t_1$for the minimum average total cost per unit of time is the solution

$$\frac{dTC_3(t_1)}{dt_1} = 0$$ …..(45)

Provided that the value of t_1 satisfies the condition

$$\frac{d^2TC_3(t_1)}{dt_1^2} > 0$$
$$t = t_1^*$$

Equation (45) is equivalent to

$$(C_h + C_d\theta)\left\{\frac{D_0}{\theta}(\theta t_1 - 1)e^{-rt_1} - \frac{D_0}{\theta^2}e^{-(\theta+r)t_1}\left(\mu\theta e^{\theta T} - e^{\theta\mu}\right)\right\} + C_s\frac{D_0}{\alpha^2}\left\{\frac{(\alpha + r - \alpha r t_1}{(\alpha - r)}e^{-rt_1} + \frac{\alpha r e^{-rt_1}}{(\alpha - r)^2} + \frac{\alpha + r}{r}e^{-(\alpha+r)t_1} - \frac{\alpha^3 e^{-\alpha t_1}}{r(\alpha - r)^2}\right\} = 0 \quad \text{…….. (46)}$$

Equation (46) can be solved and find the optimal value of ${t_1}^*$. Using the optimal value${t_1}^*$, the optimal value $S^*$and the minimum average total cost per unit of time can be obtained from equation (44) and (45) respectively.

The optimal order quantity Q^* is

$$Q^* = S^* + \frac{D_0 {t_1}^*}{\alpha} - \frac{D_0}{\alpha^2}(1 - e^{-\alpha {t_1}^*})$$

NUMERICAL EXAMPLE:

The theory presented above can be illustrated using an numerical example adopted from Kun-Shan WU and Liang-Yuh-Ouyang [9] .the input parameters are as follows:

$C_h =$ \$ 3per unit per year, $C_d =$ \$ 5 per unit, $C_s =$ \$ 15 per unit per year,$D_0 =$ \$ 100 units, $\mu = .12$ year, $\theta = .001$, T= 1 Year, $\alpha = .08, r = 0.5$.

The optimal solutions for model I and model II are those given in Table 1. Notice that the exact solution for Model I and Model II are better than to the result of Kun& Shan Wu and Liang –Yuh- Ouyang[9] because the former has a smaller minimum average total cost per unit of time (14.6494 vs, 15.1349 per model I and 13.0835 vs 13.3968 per model-II)

Optimal Solution of the Proposed Inventory Systems.

Optimal Solution	Model I	Model II: Situation-I: $\mu < {t_1}^*$	Model II: Situation- I I: $\mu > {t_1}^*$
${t_1}^*$	0.8276	0.219	---
S^*	9.2153	9.3757	---
Q^*	11.2699	11.2710	---
TC^*	14.6494	13.0835	---

--- denotes the infeasible solution

SENSITIVITY ANALYSIS:

We will now study the sensitivity of the optimal solution to changes in the values of the different parameters associated with the inventory system in example. The result is shown in **Table-2** Sensitivity Analysis of Numerical Example for Model I

A careful study of Table-2 reveals the following points.

1. It can be seen that S* is insensitive to change in the value of the parameter θ, α, C_d It is moderately sensitive to change in the value of parameters $C_h, C_s,\ r$ and highly sensitive to change in the value of parameters $\mu$and D_0.

Table-2

Parameter	%Charge	% Change in		
		S*	Q*	TC*
r	+50% +25% -25% -50%	-2.46% -0.11% +0.117% +0.23%	-0.03% -0.001% +0.002% +0.003%	-1.02% -0.75% +0.045% +1.02%
a	+50% +25% -25% -50%	-0.13% -0.065% +0.065% +0.13%	-0.065% -0.032% +0.032% +0.064%	-0.253% -0.228% +0.223% +0.045%
C_h	+50% +25% -25% -50%	-8.68% -4.509% +4.89% +10.218%	-0.122% -0.059% +0.052% +0.096%	+37.85% +19.773% -21.64% -45.25%
C_d	+50% +25% -25% -50%	-0.026% -0.013% 0% +0.013%	0% 0% 0% 0%	-0.141% +0.04% -0.033% -0.073%
C_s	+50% +25% -25% -50%	+6.63% +3.884% -5.884% -16.133%	+0.068% +0.043% -0.079% -0.255%	+6.025% +3.503% -5.556% -14.852%
θ	+50% +25% -25% -50%	+0.01% 0% -0.011% -0.022%	+0.018% +0.01% -0.01% -0.018%	-0.136% -0.128% -0.064% -0.187%
M	+50% +25% -25% -50%	+44.14% +22.56% -23.54% -48.05%	+45.21% +23.00% -23.80% -48.40%	+48.72% +24.28% -24.93% -49.85%
D_0	+50% +25% -25% -50%	+50.001% +25% -25% -50%	+50% +25% -25% -50%	+49.69% +24.74% -25.16% -50.08%

2. It is seen that the optimal order quantity Q* is insensitive to changes in the value of the parameters $C_h, C_s, \theta, r, \alpha$. It is highly sensitive to changes in the value of parameters $\mu$and D_0.It is not affected by the change in the value of the parameterC_d.

3. It can be seen that the optimum total cost is insensitive to changes in the values and parameters C_d, θ, α It is moderately sensitive to change in the value of parameter C_s, r and highly sensitive to change in the values of parameters C_h, μ and D_0

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FUZZY DECISION TREES AS A DECISION MAKING FRAMEWORK IN THE PRIVATE SECTOR

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ABSTRACT :
Systematic approaches to making decisions in the private sector are becoming very common. Most often, these approaches concern expert decision models. The expansion of the idea of the development of e-participation and e-democracy was influenced by the development of technology. The solution presented in this papers concerns fuzzy decision making framework. This framework combines the advantages of the introduction of the decision making problem in a tree structure and the possibilities offered by the flexibility of the fuzzy approach. The possibilities of implementation of the framework in practice are introduced by case studies of investment projects appraisal in a community and assessment of efficiency and effectiveness of private sector.
Keywords : Decision tree, Appraisal tree, Fuzzy set, Decision making, privatesector.

1. INTRODUCTION

Making decision in the private sector is a common subject of research; however, using systematic approaches is not common when making decisions. The private sector is supposed to act in public interest and consider the interest of all stakeholders. It is obvious that a large number of diverse stakeholders have needs and wishes that must be considered when making decisions, which in the private sector can be clearly stated despite the different views of the definition of the term “The public interest”.

In general, the contribution of the research is the definition of the decision making framework for the private sector, which comprises suitable methods and approaches within the general framework. The core of the solution is decision trees, which represent a common base of qualitative multi-attribute decision models. The use of the fuzzy approach enables the decision makers to appraise the attributes of alternatives more easily and accurately[5]. Within the general definition, a comprehensive definition of the fuzzy appraisal tree is given. The main scientific contribution of the work is the definition of the fuzzy appraisal tree. Decision trees aswells as fuzzy decision trees supporting the appraisal have not been formalized to the stage of classification and comparative trees yet, thus the definition of the fuzzy appraisal tree is an important contribution to the decision trees theory. The solution of enables the use of any types of variable. The aggregation over the appraisal tree combines values of different types of variables without limitations. Furthermore, the solution exceeds the limitation of the number of vertices and their attributes of appraisal trees that use decision rules.

2. DECISION MAKING IN THE PRIVATE SECTOR

In the application of a systematic approach when making decision in the private sector it is important to consider the following points. Any negligence with respect to these points could possible cause difficulties to the systematic approach to making decisions in the private sector[4].

- A complex and less – transparent stakeholder network.
- Many diverse interests,
- Multipleproblem perceptionsand multiplepreferences,
- A large set of appraisal criteria.
- Aggregation of many and often divergent interests of society into such notations as "general welfare", which only makes the conflict.

The systematic approach to the decision making process is based on systems for decision - making support that include methods, models and tools, and offer help with the quality of decision – making. An approach such as this must suppress the causes for the slow application of this type of solution and must enable:

- The integration of numerous stakeholders and group formation,
- Insight into multiple problem perceptions and multiple preferences and coordination,
- The handling of large sets of appraisal Criteria,
- A simple and understandable introduction to the decision making problem and the decision,
- Analysis of difference in preference and the realization of an opinion reconciliation process and a stakeholder concordance search.

3. FUZZY SETS AND FUZZY LOGIC

Fuzzy logic and approximate reasoning are parts of the framework with the definition of the linguistic variables. The review of fuzzy methods is completed with an introduction to the transformations between crisp and fuzzy and linguistic and fuzzy variables (fuzzyfication, defuzzyfication, linguistic variable to fuzzy number mapping and approximation).

The concept of a characteristic function of a (cantorian or crisp) set was generalized by L.A. Zadeh[11]by replacing, in the co-domain, the two-element set {0, 1} by the unit interval [0, 1]. Logically speaking, this is supposed to work in logic with a continuum of truth values (fuzzy logic) rather than in classical Boolean logic with two values, true and false, only.

Definition: 3.1

Fuzzy set[11]

Given a (crisp) universe of discourse, x, the fuzzy set $\tilde{A}$ (more precisely, the fuzzy subset $\tilde{A}$ of x) is given by its membership function $\mu_{\tilde{A}}$(x) : x → [0, 1], and the value $\mu_{\tilde{A}}$(x) is interpreted as the degree of membership of x in the fuzzy set $\tilde{A}$. The group of all fuzzy subsets of x is denoted as F(x).

Definition 3.2

Fuzzy number[13]

A fuzzy number $\tilde{A}$ is a convex normalized (Sup px $\mu_{\tilde{A}}$ (x) =1) fuzzy set over the real numbers with a continuous membership function having only one mean value $x_0 \in \mathbf{R}\mu_{\tilde{A}}(x_0) = 1$.

If the mean value covers a subinterval [a, b] ⊆ [0, 1] then we are talking about a fuzzy interval. If the membership function of a fuzzy number of intervals is constructed of linear functions, theyare triangular fuzzy numbers and the later are trapezoidal fuzzy numbers.

Definition: 3.3

Trapezoidal fuzzy number

A trapezoidal fuzzy number is expressed as $\tilde{A}$ = (a, b,α,β) and defined by the linear membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \dfrac{a-x}{\alpha} & \text{if } a - \alpha \leq x \leq a \\ 1 & \text{if } a \leq x \leq b \\ 1 - \dfrac{x-b}{\beta} & \text{if } b \leq x \leq b + \beta \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

A triangular fuzzy number is a degenerated trapezoidal fuzzy number (a = b). For this reason, from this point the term fuzzy number will be used for fuzzy interval (trapezoidal fuzzy number), as well as for fuzzy number (triangular fuzzy number). As a short break, have a look at a graph of a fuzzy number (more precisely, a fuzzy interval or trapezoidal fuzzy number)

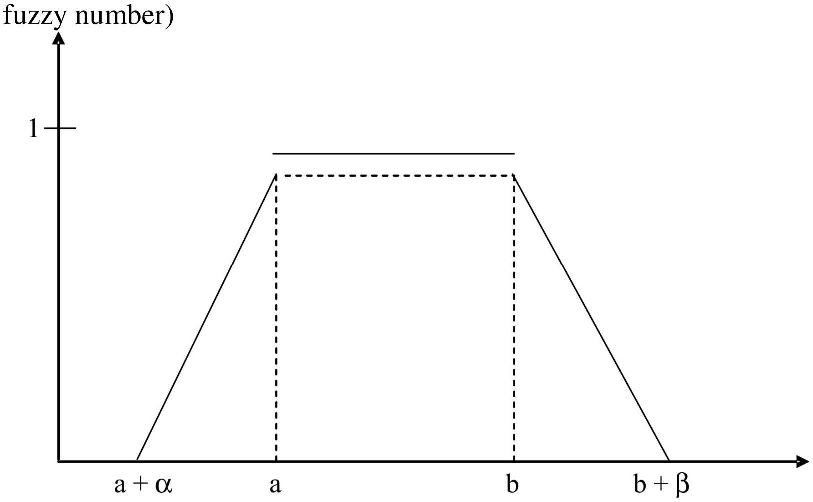

Figure 3.1 Graph of a fuzzy interval

For fuzzy numbers, the computation necessary for algebraic operations are considerably simplified. The calculations within the decision – making framework are only done with positive fuzzy number ($\mu_{\tilde{A}}(x) = 0, \forall x < 0$), and therefore only the arithmetic for positive fuzzy numbers is introduced. (The definitions comprise the fuzzy numbers $\tilde{A}$ = (a, b, α, β) and $\tilde{B}$ = (c, d, γ, δ))

TABLE 3.1 Arithmetic operations for trapezoidal fuzzy numbers[2].

Operations	Result	
$\frac{1}{\tilde{A}}$	$\left(\frac{1}{b},\frac{1}{a},\frac{\beta}{b(b+\beta)},\frac{\alpha}{a(a-\alpha)}\right)$	3.2
$\tilde{A}+\tilde{B}$	(a +c, b +d, α + y, β +δ)	3.3
$\tilde{A}-\tilde{B}$	(a – d, b – c, α +δ, β + y)	3.4
$\tilde{A}.\tilde{B}$	(ac, bd, aγ + ca - αγ, bδ + dB + Bδ)	3.5
$\frac{\tilde{A}}{\tilde{B}}$	$\left(\frac{a}{b},\frac{b}{c},\frac{a\delta+d\alpha}{d(d+\delta)},\frac{b\gamma+c\beta)}{c(c-\gamma)}\right)$	3.6

Zadeh introduced mapping between linguistic variables and fuzzy sets by the definition of a linguistic variables.

Definition: 3.4

Linguistic variable[12].

A linguistic variable is defined by a quintuple (K, T(k), U, G, $\tilde{M}$) in which k is the name of the variable, T(K) (or simply T) is the term set of k, that is, the set of names for linguistic values k, with each value being a fuzzy variable denoted generically be x and ranging over a universe of discourse U which is associated with the base variable u ; G is a syntactic rule (which usually has the form of grammar) for generating names x of values of k; and M is a semantic rule for associating each x with its meaning $\tilde{M}$ (x), which is a fuzzy subset of U. A particular X, that is a name generated by G is called a term. A term consisting of a word or words which function as a unit (i.e., always occur together) is called an atomic term. A concatenation of components of a composite term is a sub-term.

An example of a term set is:

T = {Reject, lowest, very low, Low, Middle, High, Very high, Highest, Must Be} (3.7)

Themodelling of linguistic variables with trapezoidal fuzzy numbers was proposed by Bonissone and Decker[2]. A choice of the cardinality of the term set depends on the characteristics of the problem in this case, and the same is true for the membership functions of the corresponding fuzzy numbers any kind of term set can be considered without any major changes, and in that respect the framework is flexible.

A metric of the fuzzy sets is required as a definition of all the mappings between crisp values (real numbers), fuzzy numbers and linguistic values. The Tran – Dickstein distance takes into account the fuzziness of the fuzzy sets and is confirmed in practice in an environmental-vulnerability assessment [9]. We have, therefore, decided to choose it for our framework. For trapezoidal fuzzy numbers the general definition is simplified as:

Definition: 3.5

Tran – Dickstein distance for trapezoidal fuzzy numbers (f (α) = α)[9].

$$D_T^2(\tilde{A},\tilde{B},\alpha)=\left(\frac{a+b}{2}-\frac{c+d}{2}\right)^2+\frac{1}{3}\left(\frac{a+b}{2}-\frac{c+d}{2}\right)$$

$$[\ \beta - \alpha - \delta, \lambda] + \frac{2}{3}\left(\frac{b-a}{2}\right)^2+\frac{1}{9}\left(\frac{b-a}{2}\right)$$

$$[\beta + \alpha] + \frac{2}{3}\left(\frac{d-c}{2}\right)^2 + \frac{1}{9}\left(\frac{d-c}{2}\right) \tag{3.8}$$

$$[\delta + \gamma] + \frac{1}{18}[\beta^2 + \alpha^2 + \delta^2 + \gamma^2] - \frac{1}{18}$$

$$[\alpha\beta + \gamma\delta] + \frac{1}{12}[\beta\gamma + \alpha\delta + \beta\delta + \alpha\gamma]$$

The proposed framework introduces parallel use of three types of variables, the real number (crisp value), the fuzzy number and the linguistic variable.

Definition: 3.6

Real number ←→ fuzzy number ←→ linguistic variable transformations

Fuzzyfication $V_F : l \to \tilde{L}$

Fuzzyfication makes the transformation from normalized real numbers $l \in \mathbf{R}$ to fuzzy sets $L \in F(x)$ (in our case, fuzzy numbers) using membership functions. It is carried out in two steps:

Mapping T_M : L $\to \tilde{L}$ of the real number $l \in \mathbf{R}$ to the fuzzy set $\tilde{L} \in F(x)$, where in the case of multiple corresponding fuzzy sets the weighted average operator is used.

$$\tilde{L}_F = \frac{1}{\sum_K \tilde{\mu}_k(l)} \sum_K \tilde{\mu}_k . \tilde{L}_F ; K = 1, N;$$

N is number of fuzzy sets tuched by l, $\tilde{L}_K$,are the fuzzy sets tuched by l and $\tilde{\mu}_k(x)$ are the membership functions of the fuzzy sets $\tilde{L}_F$

Translation T_T ; $\tilde{L}_F \to \tilde{L}_l$ of the fuzzy set $\tilde{L} \in F(x)$ so that the result of defuzzyfication of fuzzy set $\tilde{L}_l$, T_{DF} : $\tilde{L}_l \to x$ is equal to the input real number $l \in \mathbf{R}$.

DefuzzyficationT_{DF} : $\tilde{L} \to l$.

Defuzzyfication makes the transformation from fuzzy sets $\tilde{L} \in F(x)$ to real numbers $l \in \mathbf{R}$. A “centre of gravity” method was choosen for all the possible transformations of fuzzy sets into crisp values. The method is the most trivial weighted average and has a distinct geometrical meaning

$$x_{COG} = \frac{\int_x x.\mu(x)dx}{\int_x \mu(x)dx} \tag{3.9}$$

A simple calculation for a fuzzy number $\tilde{A}$ (a, b, α, β) gives the simple formula

$$x_{COG} = \frac{-a^2 + b^2 + a\alpha + b\beta - \frac{\alpha^2}{3} + \frac{\beta^2}{3}}{-2a + 2b + \alpha + \beta} \tag{3 .10}$$

linguistic variable $L \in T(k)$ to fuzzy variable $\tilde{L} \in F(x)$ mapping T_M : L $\to \tilde{L}$.

The mapping of linguistic values into fuzzy numbers is part of linguistic variable definition where suitable parameters are defined

- The name of the linguistic variable,
- The cardinality of the term set and the terms, the elements of the term set.
- For each term the corresponding fuzzy number (mapping functions).

The linguistic variable "Appraisal", with, nine values and names was used for this study:

TABLE 3.2
LINGUISTIC VARIABLES "APPRAISAL" MAPPING FUNCTION

Reject	Lowest	Very Low	Low	Medium	High	Very High	Highest	Must Be
0	.01	.10	.22	.41	.63	.78	.98	1
0	.02	.18	.36	.58	.80	.92	.99	1
0	.01	.06	.05	.09	.05	.06	.05	0
0	.05	.05	.06	.07	.06	.05	.01	0

Fuzzy set $\tilde{L} \in F(x)$ to linguistic value $L \in T(k)$ approximation T_A : $\tilde{L} \rightarrow L$.

The fuzzy number $\tilde{A}$ is approximated to a linguistic value $\tilde{L}_{approx}$so that the closet fuzzy number $\tilde{L}$, representative of the nearest linguistic value, is found:

$L_{approx} = L : D_T(\tilde{A}, \tilde{L}, \alpha) = \min D_T(\tilde{A}, \tilde{L}_i, \alpha)$; i =1, …., n (3.11)

For higher granularity of the ends results we introduced the approximation deviation. This is defined as the relative number of the difference in distance of the approximated fuzzy number and the fuzzy number image of the linguistic approximation and the difference between two adjacent linguistic values[1]:

$\tilde{A}$

The approximation with the deviation is then labelled as

$\leftarrow L_{approx}$, if Dev % L – 25 %

L_{approx}, if - 25% $\leq$Dev % $\leq$ 25 %

$L_{approx}\rightarrow$, if Dev %<25% (3.12)

At this point, we are well equipped with all that is needed to define the proposed model. We know that in order to perform the appraisal, an appraisal tree should be constructed and that in the private sector it is very suitable to perform an appraisal with the help of fuzzy variables and fuzzy aggregation. Therefore, in acomprehensive definition ofthe fuzzyappraisal framework and within it, the definition of the fuzzy appraisal tree is presented.

4. FUZZY APPRAISAL FRAMEWORK

The suggested appraisal framework resulted from the problem when solving the group multi-attribute decision making in the private sector. An investigation of the problem and the development of the solution lead to a general appraisal framework combining the advantages of the introduction of a tree structure and the use of a fuzzy approach for the appraisal of attributes or indicator as well as a comparison of the criteria and perspectives.

The entire fuzzy appraisal framework includes the definition of the fuzzy appraisal tree, averaging operators for the calculation of the average value of forests (groups of trees, with respect to groups of evaluators, group of alternatives, organization units of the same kind etc.) method for tree comparison, and tree classification (regarding the root, regarding the individual nodes, regarding the structure, etc.) methods for the analysis of tree variability (regarding the root, regarding the individual nodes, regarding the structure, etc.) and methods for tree optimization (efficiency, information, entropy, etc.)

Definition: 4.1

Fuzzy appraisal framework

Fuzzy appraisal frame constitute a forest with fuzzy appraisal tees over which the following is defined:

- Fuzzy appraisal trees
- Averaging operators O_{AVg} : $((\tilde{T}_1,....,\tilde{T}_n) \rightarrow \tilde{T}_{Avg}$

For the calculation of average tree values in chosen sub – forests.

- Methods for fuzzy tree comparison and fuzzy tree classification,
- Variability measures, and
- Methods for fuzzy tree optimization.

With a given framework, it is possible to use three types of variables – real, fuzzy and linguistic, the values of which represent an equivalent appraisal of an attribute, criteria, indicator of perspective represented by the nodes. Ingoing values (in the leaves) and calculated values (in the nodes) are recalculated from the ingoing type into the other two –real number, fuzzy number, fuzzy number, linguistic variable. All the necessary transformations are defined in each node and proceed during the recalculations. The values in the inner nodes are filled from the aggregation functions over fuzzy numbers. The aggregation functions overlinguistic values are not considered (simplicity, distinction from existing systems based on system rules). For special cases, the aggregation function is defined over real variables. Ingoing variables for aggregation operators are defined by the connections from the successors. Like the nodes, the connections, which also represent the weights are evaluated with all three types of variables and equipped by transformations to transform one into the other.

Definition: 4.2

Fuzzy appraisal tree

A fuzzy tree $\tilde{T} = (\tilde{V}, \tilde{E})$ consists of a finite, nonempty set of fuzzy nodes (or vertices) $\tilde{V}$ and a set of fuzzy edges $\tilde{E}$. A fuzzy vertex $\tilde{V}$ consists of: Three variables $l \in \mathbf{R}$, $\tilde{L} \in$ M (x), L∈L(k); (crisp variable l, fuzzy number $\tilde{L}$ and linguistic variable L), four transformations between them,
fuzzyfication T_F : $l \rightarrow \tilde{L}$, defuzzyfication T_{DF} : $\tilde{L} \rightarrow l$, approximation T_A: $\tilde{L} \rightarrow$l, and mapping T_M : L$\rightarrow \tilde{L}$ A fuzzy aggregation operator over the fuzzy variables of children (for internal nodes)

f : ($\tilde{L}_{i+1,j1}$, …. $\tilde{L}_{i+1,j,kij}$) $\rightarrow \tilde{L}_{ij}$ where i is the level of the node, j is the position of the node at the level i, and K_{ij} is the number of children of the node.

A fuzzy edge $\tilde{E}_{i,j}$ = ($\tilde{V}_{i,j}$, $\tilde{V}_{i+1,j,k}$) consists of a path from the parent to a child and of the weight $\tilde{W}_{i,j,k}$ which consists of three variables and four transformations between them

$\tilde{V}_{i,j}$,				
$l \in \mathbf{R}$	T_F: l←	$\tilde{L} \in$ F(x)	T_A: $\tilde{L} \leftarrow$	L∈L(K)
O_{Agg}: $\tilde{L}_{i+1, j,Kij} \rightarrow \tilde{L}_{ij}$				

Figure – 6.1 the structure of the fuzzy vertex $\tilde{V}$

For a function appraisal framework to work, averaging operators to drive aggregation functions and to calculate averages of fuzzy forests are needed. Because of the simplicity principle, we have opted, among the many averaging operators[13], for generalized operators of the weighted mean of fuzzy numbers expressed by the formula in Definition 4.3.

Definition: 4.3

[13]Generalised operators of the weighted mean of fuzzy numbers are:

$$h_a^w(a_1,....,a_n) = (\sum_{i=1}^{n} w_i a_i^a)^{\frac{1}{a}} a \in [0,1], i \in N_n, a \in \mathbf{R}\ (\alpha \neq 0) \quad (4.1)$$

Where for the vector $\tilde{W}$ = (W_1, …,W_n) it holds $\sum_{i=1}^{n}$ W_i=1, $W_i \geq 0 \forall_i \in \mathbf{N}_n$. Then vector $\tilde{W}$ is termed the weighted vector, and its components W_1 the weights. In the simplest version (equal weights $W_1 = \frac{1}{n}$ and α =1), it is simply the arithmetic mean.

Comparison and classification is based on the comparison of calculated average values approximated into linguistic values.

The proximity measure and consensus measure are chosen for the analysis of the variability in a forest of appraisal results.

5. APPLICATION EXAMPLES

The specific definition of a fuzzy appraisal framework is in general a choice of system elements according to the needs and possibilities of a specific problem. In this chapter the implementation of the fuzzy appraisal framework for two cases is presented. The first one, the optimal selection of community investment projects, was the environment where the idea of the fuzzy appraisal framework was born. The second one is the project balanced scorecard as an assessment and benchmarking tool is private sector running at Faculty of administration where the first implementation after definition of framework is going on.

5.1 Selection of investment project in a municipality for private sector

The case is focused on the question of the optimum choice of investment projects in a local community burned by various circumstances that could results in the municipality is inopportune investment orientation decision making in municipalities takes place successively with two groups of participants. Professional services asses the investments projects and merge them into investment options according to professional Criteria.

The proposals are then revised and approved by the mayor and forwarded to the municipal council, which then decides independently and autonomously. The decision makers are confronted with various difficulties resulting from un systematic approach political decision makers are reluctant to take professional arguments into consideration, while professional tend to disregard the political circumstances: however, an optimum decision is achieved only if all opinions and comments are dealt with in the decision making process.

We have therefore been seeking a solution to the issue of making optimal decisions on investment in local government, in the phase of preparing the investments as well as in the phase of initiating then realization and financing. The solution would have to establish a process that allow confrontation and coordination of diverse opinions and interests on the professional and political levels, in professional political as well as in professional – professional and political – political relations.

Based on the previous discussions, the fuzzy appraisal framework presented in section 4 represents an appropriate approach to the solution of the given problem. The decision tree contains knowledge of the structure of the values that determine to what extent an individual alternative is suitable for inclusion in the budget. We have determined the structure of the appraisal tree, taking in to account framework of deciding on capital investments in the private sector [3], legally prescribed definitions and the analysis of the method of decision making in local communities in Slovenia.

The appraisal model for investment projects in local communities was defined according to the needs and possibilities, based on the general definition of the fuzzy appraisal framework (Definition 4.2) with adaptations as follows.

Definition: 5.2

Fuzzy appraisal model for selection of investment projects in a municipality.

(1) Fuzzy appraisal model for selection of investment projects in a municipality is a fuzzy appraisal framework.

(2) The input values are linguistic variables (among the transformations in definition 4.1 point 1 fuzzyfication T_F: l $\rightarrow \tilde{L}$ is not needed).

(3) The fuzzy aggregation operators over the fuzzy variables of children (for internal nodes) O_{Agg}: ($\tilde{L}_{i+1,j,1},\ldots \tilde{L}_{i+1,j,kij}$) $\rightarrow \tilde{L}_{ij}$, is derived from (Def 4.3), where $\alpha = 1$ and equal weights $w_i = \frac{1}{n}$ for all edges are chosen:

$$\tilde{A}_{ij} = \frac{1}{K_{ij}} \sum_{K} \tilde{A}_{i+1,j,k}; i = I-1,\ldots i; j = 1,\ldots J \quad K = 1, \ldots K_{ij} \tag{5.1}$$

Where I is the number of levels of the tree, i is the current level of the tree, J_i is the branching of the tree, j is the position of the node at the i-th level, K_{ij} is the number of children of the parent in question at the level i+1, and K is the position of the child of the parent in question.

4) The averaging operator O_{Avg} : ($\tilde{T}_1,\ldots, \tilde{T}_n$) $\rightarrow \tilde{T}_{Avg}$ for the average tree value calculation in chosen sub – forests is derived ( ), where $\alpha = 1$ and equal weights $W_i = \frac{1}{n}$ for all edges are chosen:

$$\tilde{A}_{ij} = \frac{1}{|G|} \sum_{G} \tilde{A}_{i+j}; i = I-1,\ldots I; j = 1,\ldots j_i \tag{5.2}$$

Where G is the set of appraisers.

5) Variability measures are the proximity and consensus measure over the set of appraisers G (Definition 4.2)

The appraisal tree including three nodes (project contribution, feasibility and risk and cost / benefit appraisal), where the first two nodes each included three leaves and the third node included only two leaves[1]. The model was tested in three Slovenian municipalities. The set of appraisal projects include from seven to nine investment projects. Two types of appraisal were invited, representatives of municipal government and municipal councillors. Due to the reluctance of municipal councillors, the appraisal groups were rather small, comprising from nine to fifteen appraisers. We analysed the results represented with linguistic values and prepared a qualitative representation of results, where we considered the differences between projects and appraisal groups. The proposed solution attracted great interest, since the problem is of everyone's concern. It has been proven that the chosen method of appraisal is suitable for the chosen environments. An interview was performed after each case study concerning the usefulness and suitability of the suggested approach for decision making in a chosen environments. The results proved the approach to be suitable due to the evaluators having no problems during the appraisal. The content of the appraisal was a bigger problem due to the evaluators not being introduced to it and / or the importance of the project was underestimated, also financially. This is a matter of preparation and organization appraisals processes, in which case the fuzzy appraisal framework can contribute to but not solve the problem.

5.3 BALANCED SCORECARD AS AN ASSESSEMENT AND BENCH MAKING TOOL IN THE PRIVATE SECTOR

The fuzzy approach can be also effectively used when solving the problem of how to measure the successfulness of organizations with balanced scorecard. The balanced scorecard joins success indicators into four business perspectives. Customers, finance, process and learning and growth. In the profit sector final result is measured with the financial perspective. It is enabled by the other three perspectives, which indicate success fullness of the organization in the near future[6].

An organization is tree structured, where leaves are single employees or small departments and nodes combine subordinate units. The result of a unit is given by indicators defined for the unit, where some of them are calculated from equal indicators of subordinate units, and the others carry the results of the unit in question. The indicators of a unit are leaves of the appraisal tree of the unit. The nodes at the first level of the tree represent four perspectives of successfulness. The indicators of such an appraisal tree are defined over different variables which are hard to aggregate into joint value. The situation is the natural environment of the fuzzy appraisal framework which offers somehow simple solutions for quite difficult problems.

As a result of the research studying the problem of the implementation of the balanced scorecard into the private sector organizations we introduce the structure of the fuzzy appraisal framework for balanced scorecard follows:

- Each organizational unit is the carrier of indicators, which represent the result of the unit in question, and joint indicators of the result of subordinate units,
- The indicators which are measuring the same results are by definition equal indicators,
- The equal indicators of unit at the chosen level are aggregated into the equal indicators of the unit at the upper level of the organisational tree,
- The indicators of an organizational unit are linked into the appraisal tree of the unit, at the top of which four perspective nodes are defined, and the root of the tree represents the general appraisal of a unit,
- The root of the organizational tree is "the organization", which links all the indicators defined for the subunits into joint appraisal tree.

Definition of the appraisal model for the balanced scorecard is based on the general definition of the fuzzy appraisal framework with adaptations as follows.

Definition: 5.4

Fuzzy appraisal model of the balanced scorecard.

1) Fuzzy appraisal model for selection of investment projects in a municipality is a fuzzy appraisal framework.
2) Each node is evaluated with three variables (Crisp, fuzzy and linguistic), where one of them is the input variable.

The fuzzy aggregation operator over the fuzzy variables of children.

$$O_{Agg} : (\tilde{L}_{i+1,j,1},\ldots \tilde{L}_{i+1,j,kij}) \rightarrow \tilde{L}_{ij},$$

$$\tilde{A}_{ij} = \sum_{K} \tilde{W}_{i+1,j,k} \tilde{A}_{i+1,j,k}, \sum_{K} \tilde{W}_{i+1,j,k} = 1;\ i=I-1,\ 1;\ j=1,\ J_i,\ K=1,\ldots,K_{ij} \tag{5.3}$$

Where I is the number of levels of the tree, i is the current level of the tree, j_i is the branching of the tree, j is the position of the node at the i-th level, K_{ij} is the number of children of the parent in question at the level i+1, and K is the position of the child of the parent in question. The weights regulate the contribution of the children to the aggregation value of the parent.

3) The model of balanced scorecard comprises single organizational tree structure, so the averaging operator over forests of trees is not needed.

4) Variability is not a greater issues in the balanced scorecard model, but in any case the measure of proximity and consensus measure are available.

CONCLUSIONS

The structure of a fuzzy appraisal framework in the private sector is presented in this paper. The purpose of the framework is to develop solutions with properties adjusted specially for use in the public people and private sector. The methods and approaches that lead to the satisfactory conclusion were systematically combined in the framework.

The theory of decision trees and the theory of fuzzy sets and fuzzy logic. This led to incorporating the desired properties[8],[10]:

♦ Clarity and conciseness, context sensitivity, flexibility:

♦ Allow the representation of cognitive uncertainties in decision making, providing more information to the decision maker, using linguistic terms with soft boundaries to accommodate vagueness and ambiguous in human thinking and perception, into a framework.

The appraisal framework defined general elements of the system and gives guidelines to form concrete solutions. The approach was realized through its use in practice one case is observed.

The assessment of the performance of organizations with indicators balanced scorecard. Case studies proved the framework to be a suitable basis for implementing solutions of different decision making problems in private sector. However the accommodation of the framework to the specific environment of the private sector is not a restriction but a generalization.

It incorporates more flexibility in the appraisal, which makes the solution easier to use. The framework gives practitioners and researchers a change to broaden their research method and tools, designed to make their appraisal application better and more user friendly for all kinds of use, both public and non-public.

However, new research challenges and motivation are perhaps even more important than a contribution to solving the solving the specific problem in practice. The most important research for the future is:

❖ The definitions of suitable membership function of fuzzy sets and fuzzy numbers.

❖ The modelling of linguistic variables with fuzzy sets in accordance with the operators' comprehension and understanding.

❖ The definition of adequate functions and operators over a fuzzy set (aggregating and averaging operators, distance etc.),

❖ The discussions of data variability defined with fuzzy tree structures

❖ The methods for fuzzy tree optimisation.

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APPLICATION OF INCLINE MATRICES IN YOGA ON ARTHRITIS

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ABSTRACT :

Inclines are the additively idempotent semirings in which products are less than or equal to factors. Thus inclines generalize Boolean algebra, fuzzy algebra and distributive lattatice. And the Boolean Matrices, the fuzzy matrices and the lattice matrices are the prototypical examples of the incline matrices. That is the matrices over inclines. In this paper We extend sanchez's approach for medical diagnosis using incline matrix and exhibit the techniques. This paper presents a study protocol examining the feasibility and acceptability of providing yoga to an urban, minority population with arthritis.

Keywords : Fuzzy Matrix, Fuzzy incline Matrix, Arthritis, Osteo arthritis rheumatoid arthritis, Medical Knowledge.

I. INDRODUCTIONS

Incline is an algebraic structure and is a special type of a Semiring. In an incline $(J,+,\bullet)$ with the order relation "$\leq$" defined on L as $X \leq Y$ if and only if X+Y = Y for $X,Y \in J$, the incline axioms, that is X+XY = X and Y+XY=Y, imply that $XY \leq X$ and $XY \leq Y$. Thus inclies are additively idempotent semirings in which products are less than (or) equal to either factor.

The concept of slope was introduced by Cao and Cater Cao, Kim and Roush [1] renamed it as incline. The notation of inclines and their applications are described comprehensively in Cao, Kim and Roush [1] Kim and Roush [7] have surveyed and outlined algebraic properties of inclines and incline matrices.

The field of medicine is one of the most fruitful and interesting areas of applications for fuzzy set theory. In the discrimination analysis, The symptoms are ranked according to the grade, of discrimination of each disease by a particular symptom and is represented in the form of a Matrix called a frequency distribution Matrix F = (f_{ij}) where f_{ij} is the ratio of the patients with disease 'd_i'. The matrix model may not yield more exact diagnosis in such case where several diseases affect a single patient or when a single disease distinct quite differently in different patients and at different disease stage. Moreover, with the increased Volume of information available to medico's from new medical technologies, the process of classifying different sets of symptoms under a single name of disease and determining and appropriate therapeutic action becomes increasingly difficult. Recently, there are varieties of models of medical diagnosis under the general frame work of fuzzy sets theory involving fuzzy matrices to deal with different complicating side of medical diagnosis.

Arthritis is a term often used to mean any disorder that affects joints. Symptoms generally include joint pain and stiffness. Other symptoms may include redness, warmth, swelling, and decreased rang of motion of the affected joints in some types other organs are also affected. On get can be gradual or Sudden. There are over 100 types of arthritis. The most Common forms are Osteo arthritis (degenerative joint disease) and rheumatoid arthritis.

The result in the present paper include some arthritis patients datas which were obtained for the Boolean matrices,the fuzzy matrices and the lattice matrices among their special case.

II. Preliminary :

Here we recall some preliminary definition regarding the topic.

Definition 2.1

Fuzzy Matrix

Let F_{mn} denote the set of all m x n matrices over F, if m = n, in short, we write F_n are called as membership value matrices, binary fuzzy relation matrices (or) in short, fuzzy matrices.

Definition 2.2

Fuzzy incline Matrix :

A non empty set £ with two binary operations '+' and '.' Is called an incline if it satisfy the following conditions.

1. (£, +) is a semilattice.
2. (£, •) is Semigroup
3. X(Y+Z) = XY+YZ for all X,Y,Z ∈ £
4. X+XY = X and Y+XY = y for all X,Y, ∈ £

Definition 2.3

Arthritis :

The word 'arthritis' comes from the Greek words for "joint inflammation"

Sometimes called rheumatism, arthritis is really not one disease but a category of over 100 diseases affecting primarily the joint of the body. Certain types of arthritis, through are more widespread and can affect other body tissues such as the eyes, nerves, kidneys or skin.

Definition 2.4

Osteoarthritis

Sometimes called degenerative joint disease, osteoarthritis is a wearing away of the cartilage the protect adjacent bone ends and enables them to glide smoothy. Injury or obesity can hasten the damage caused by aging.

Definition 2.5

Rhematoid Arthritis :

An autoimmune disease characterized by chronic inflammation of joints. Rheumatoid disease can also involve inflammation of tissues in other areas of the body, such as the lungs, heart, and eyes. Because it can affect multiple organs of the body, rheumatoid arthritis is referred to as a systemic illness. Although rheumatoid arthritis is a chronic illness, patients may experience long periods without symptoms. Also known as rheumatoid arthritis.

III. Algorithm

Step (i) :

Input the incline matrix value over the set of patient P over diseases D and write the input value over the set of symptoms S over D denoted by the knowledge matrix R_1 and R_2 respectively.

Step (ii) :

Input the incline matrix over the set P of Patients over D and write its relation Q.

Step (iii) :

Calculate the relation matrices under the composition (+,⊔), where the '+' is Maximum '⊔' minimum.

(i) $T_1 = QR_1$

(ii) $T_2 = QR_2$

(iii) $T_3 = Q(J-R_1)$ Where J is the Matrix with all its entries.

1. Which is the greatest element of £. And

(iv) $T_4 = Q(J-R_2)$

Case (iv) :

Calculate the diagnostic scores ST_1 and ST_2

$$\mathbf{ST_1 = max\{\, T_1(p_i, d_j), T_3(p_i, d_j)\} \text{ for } i = 1, 2, 3, 4\ j = 2}$$

$$\mathbf{ST_2 = max\{\, T_2(p_i, d_j), T_4(p_i, d_j)\} \text{ for } i = 1, 2, 3, 4\ j = 2}$$

Step (v) :

Find $\mathbf{S_k = max\{ST_1(p_i, d_j) - ST_2(p_i, d_j)\}}$ then we complete the patient P_i is suffering from the disease d_k.

Step (vi):

If S_k has more than one value then go to step 1 and repeat the process by reassessing the symptoms for the patient.

IV. Case Study

Consider 4 Patients Balu, Selva, Sundar, Karuna are denoted by the set P ={ Balu, Selva, Sundar, Karuna} and the set of symptoms S ={Pain, Stiffness, Swelling,Redness}. Let the set of disease D = {Osteoarthritis, Rhemetoid arthritis}.

Step 1 :

Let us take for the Matrix

$$R_1 = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.52 & 0.64 \\ 0.46 & 0.29 \\ 0.34 & 0.60 \\ 0.26 & 0.57 \end{pmatrix} \end{array}$$

$$R_2 = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ S_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.27 & 0.36 \\ 0.18 & 0.48 \\ 0.42 & 0.38 \\ 0.61 & 0.28 \end{pmatrix} \end{array}$$

Step ii :

$$Q = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{pmatrix} 0.90 & 0.64 & 0.28 & 0.46 \\ 0.87 & 0.42 & 0.36 & 0.18 \\ 0.46 & 0.58 & 0.30 & 0.37 \\ 0.19 & 0.27 & 0.48 & 0.72 \end{pmatrix} \end{array}$$

Step iii :

$$T_1 = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.47 & 0.58 \\ 0.45 & 0.56 \\ 0.27 & 0.29 \\ 0.18 & 0.41 \end{pmatrix} \end{array}$$

$$T_2 = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.24 & 0.32 \\ 0.23 & 0.31 \\ 0.23 & 0.27 \\ 0.48 & 0.20 \end{pmatrix} \end{array}$$

Step iv:

$$ST_1 = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.47 & 0.58 \\ 0.45 & 0.56 \\ 0.31 & 0.41 \\ 0.32 & 0.41 \end{pmatrix} \end{array}$$

$$ST_2 = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} 0.66 & 0.58 \\ 0.64 & 0.57 \\ 0.48 & 0.29 \\ 0.48 & 0.30 \end{pmatrix} \end{array}$$

Step v :

$$\mathbf{S_k = max\,\{ST_1(p_i, d_j) - ST_2\,(p_i, d_j)\}}$$

$$S_K = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cc} d_1 & d_2 \end{array} \\ \begin{pmatrix} -0.19 & 0 \\ -0.19 & -0.01 \\ -0.17 & 0.12 \\ -0.16 & 0.11 \end{pmatrix} \end{array}$$

From the above matrix, if the medico’s agress, then Balu(P_1), Selva (P_2), Sundar (P_3) and Karuna (P_4) suffer from Rhematiod arthritis(d_2).

CONCLUSION :

The medical induction process describes the process of reason from patients symptoms, signs and test result to diagnosis by means of medical knowledge. As fuzzy decision making is a most important scientific social and economic work, there exist several major approaches within the theories of fuzzy decision making.

Arthritis is a major public problem and one of the approaches to address this problems is through yoga, yoga is a promising modality for arthritis. Finally we have explain medical diagnosis problem using fuzzy incline matrices.

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OPTIMIZATION OF MENTAL HELTH SERVICES DELIVERY THROUGH GOAL PROGRAMMING : A CASE STUDY

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ABSTRACT :

The main aim of this paper is to develop a goal programming model which optimize the services in health care organizations to provide better services in health care system.

Keywords : Health Care delivery system, optimization, goal Programming

1. INTRODUCTION

The delivery of mental health care to the public has grown complex over the last two decades. Health care organizations need to promote positive mental health practices through education and some heath camps. This procedure provides quality mental health services, which are both accessible to the general public and less disruptive to the patient's life led to the development of community-based mental health centers in India. [1-3,6] These centers offer outpatient, day-care, emergency, inpatient, consultation, and education services. The development of the community programs has required that the role of centralized state in-patient institutions, or state hospitals, be re-examined. This has resulted in a trend towards deinstitutionalization in order better to utilize resources, provide a broader range of and more efficient services and above all, treat the patient in the least restrictive manner [10,12].

2. DATA OF THE PROBLEM

The public mental health system in the state where the case example is located in composed of three district centers and six regional society Mental Health programs in Hyderabad. The programs have one to three community mental health centers, depending on population. The district centers offer specialized inpatient psychiatric care on a regional basis, while the society-based mental health centers offer out-patient, partial care and limited in-patient services, as well as consultation and education services. The community-based centers service a much smaller area/population than the regional centers. The region mental health delivery system consists of a regional centre and three community mental health centers. The Regional Centre (RC) primarily serves the needs of the region's residents. In this presentation planning decisions for the RC and the largest Community Health Centre (CMHC) will be considered. The relevant information is given in the following Table 1

TABLE 1 Mental Health Services Data

Total Funds for Patient Care, Counseling, and Education	: Rs. 3, 83, 80,500
Expenditure of education programs per 100 citizens	: Rs.71,280
Staff hours per 100 citizens reached	: 35
Psychiatrist time	
RC – Fourteen psychiatrists 75 hours/mo	: 13,500hrs./yr.
CMHC – Five psychiatrists 75 hours/mo	: 4,050hrs./yr.
Professional staff time	: 17,550hrs/yr
RC – 60 Prof. Staff 85 Hours/ mo.	: 64,260hrs./yr.
CMHC – 38 Prof. Staff 85 Hours / mo.	: 42,840hrs./yr.

Average Annual Cost Per Patient For Each Diagnosis

RC	18060	18060	18060	12045	12045	15045	6015	6015
CMHC	2295	2295	2295	1530	1530	1912	765	765
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**

Average Psychaitric Hours Per Patient For Each Diagnosis

RC	32	32	32	21	21	27	10	10
CMHC	9	9	9	6	6	7	3	3
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**

Average Professional Staff Hours Per Patient For Each Diagnosis

RC	180	180	180	120	120	150	60	60
CMHC	63	63	63	42	42	52	21	21
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**

3. GOAL PROGRAMMING MODEL

Using Goal programming model the decision maker can examine trade-offs among various objectives as well as to assess budgetary and staff ramifications of various policy decisions through sensitivity analysis. The goal-programming model has been shown to be superior to the linear programming model as it can be used to solve single as well as multiple linear objectives subject to linear constraint equations. With the goal-programming model, the fiscal impact of various levels of staff and patient reallocation from state to community programs and the change in demand patterns can be assessed.

3.1 PRIORITY STRUCTURE

The mental health services delivery problem of the region involves various social, political, and economic objectives, some of which are in conflict. The specific objectives are:

P_1 = Do not exceed patient care and counseling budget.

P_2 = Achieve additional deinstitutionalization by reducing patients receiving care in the regional centre by 30 percent and by providing interim care for released patients in the community-based mental health centers. Additionally, treatment of non-critical disorders at regional centers should be avoided.

P_3 = Meet projected demand increase of 10 percent for all mental health services.

P_4 = Provide psychiatric care to all patients who require it.

P_5 = Provide mental health education to all primary and secondary school children in the area served by the CMHC.

P_6 = Provide necessary professional staff care to all patients.

P_7 = Double the number of citizens reached by the mental health education service of the CMHC.

3.2 MODEL VARIABLES

The planning function of mental health services requires knowledge of the changing distribution of patient load between regional and community centers with regard to the constraints of the system. Therefore the decision variables can be represented as number of patients and number of citizens affected. The variables are:

€∝ijk = Number of patients at a given treatment centre with a given diagnosis at a certain position in the system.

Where:

i is treatment centre type

i = 1, 2 with 1 = regional centre in-patient care facility
2 = community-based mental health centre (CMHC)

j is diagnosis type

j = 1, 2, ….., 8 with 1 = mental retardation
2 = organic brain syndrome
3 = psychosis
4 = neurosis
5 = personality disorder
6 = alcohol and drug dependence
7 = transitional situational disturbance
8 = other

K is patient position in system

K = 1,2,3,4 with 1 = initial patient count
2 = additions to original patient count
3 = count of released patients
4 = final patient count

β = number of citizens reached by educational programs (in 100s).

3.3 GOAL CONSTRAINTS

G1 Budget Constraint

The total cost of educational programs and patient therapy should not exceed the budget allocated for this purpose.

$$\sum_{i=1}^{2} \sum_{j=1}^{8} a_{ij} (\propto_{ij1} + \propto_{ij2}) + b\beta + d^-_1 - d^+_1 = \text{Rs.}3{,}83{,}80{,}500$$

when d_1^- is minimized

a_{ij} is the estimated average annual cost per patient episode for institutional type *i* and diagnosis type j

b is the expenditure of educational programs per 100 citizens.

G2 Regional Centre Patient Bad Reduction

In order to achieve the deinstitutionalization goal and further reduce the amount of in-patient care at the regional centre by a target level of 30 % of the following constraint results:

$$\sum_{j=1}^{8} \propto_{ij} + \propto_{ij4} + d^-_2 - d^+_2 = 0.70$$

Where d_2^- is minimized

G3 Follow-Up treatment

Once a patient has been discharged from a regional centre, it is desirable to provide follow-up treatment for released regional centre patients at community-based mental health centers. The target 30 percent reduction in regional centre patients means that approximately 160 patients would be transferred to community care, thus increasing estimated community-based additions by approximately 15 percent.

$$\sum_{j=1}^{8} \propto_{2j2} + \propto_{ij3} + d^-_3 - d^+_3 = 0.15,$$ where d_3^- is minimized.

G4 Non-Critical Disorder

Less than 12 percent of the patients in the regional centre should have diagnosis of neurosis, personality disorder or alcohol and drug dependence. After reassignment of patients along with new admissions and releases

$$\sum_{j=4}^{6} \propto_{ij4} + d^-_4 - d^+_4 = 0.12 \sum_{i=1}^{2} \sum_{j=1}^{8} \propto_{1j4}$$

Where d_4^+ is minimized.

G5 Demand Increase

Demographic statistics from the India show that demand for mental health services can be estimated to increase at 10 percent per year for the next two years.

$$\sum_{i=}^{2} \sum_{j=1}^{8} \propto_{1j4} + d^-_5 - d^+_5 = 0.10 \sum_{i=1}^{2} \sum_{j=1}^{8} \propto_{1j1}$$

Where d_5^- is minimized.

G6 Psychiatric Time

Because of the limited number of qualified psychiatrists patient can go for check up in available counseling hours only

$$\sum^{2} \sum^{8} P_{ij}\propto_{ij4} + d^-_6 - d^+_6 = \text{Total psychiatrist hours available}$$

i=1 j=1

Pij is the average psychiatric hours spent per episode with patients of the particular diagnosis in each type of institution d_6^- is minimized.

G7 School Education

To provide mental health education, the CMHC wishes to attain at least 3,000 students through school education programs since β is in units of hundreds of persons reached, the following constraint results.

$\beta + d^-_7 - d^+_7 = 30$

Where minimize d_7^-

G8 Professional Staff Time

Professional staff consisting of psychologists, social workers and psychiatric nurses can give their services for finite number of hours. Depending on their availability patient and education load must be limited.

$$\sum_{i=1}^{2} \sum_{j=1}^{8} S_{ij} \propto_{ij4} + C\,y + d^-_8 - d^+_8 = \text{Total psychiatrist hours available}$$

Where d^-_8 is minimized

S_{ij} is the average number of professional staff hours spent per case for each diagnosis and institution type; and C is the number of professional hours utilized to provide educational programs per 100 citizens.

G9 Community Education

In addition to the 3,000 students reached by education programs, the CMHC would like to reach an additional 3,000 citizens of their service area with educational programs on positive mental health practices.

$\beta + d^-_9 - d^+_9 = 60$

Where d_9^- is minimized.

3.4. SYSTEM CONSTRAINTS

At the beginning of the planning year, the regional centre and community-based centre are utilized by 190 and 1,010 patients, respectively. The patient breakdown for each diagnosis at the regional centre during the initial period is as follows:

TABLE 2: Initial Patient Diagnoses

Number of persons	**Regional Center**	**Community-Based Center**
Mental retardation	12	25
Organic brain syndrome	15	22
Psychosis	83	78
Neurosis	04	103
Personality disorder	28	108
Alcohol and drug	05	162
Situational disturbance	09	220
Other	17	309
Total	202	1027

TABLE 3: Diagnosis As Percent of Total Mental Health Clients

	Total Patients	Percent
Mental retardation	332	2.2
Organic brain syndrome	373	2.5
Psychosis	1604	10.7
Neurosis	1382	9.2
Personality disorder	1472	9.8
Alcohol and drug	3395	22.7
Situational disturbance	2716	18.2
Other	3736	24.9
Total	**15010**	

$x_{111} = 39$ $x_{141} = 4$ $x_{171} = 9$

$x_{121} = 12$ $x_{151} = 27$ $x_{181} = 15$

$x_{131} = 78$ $x_{161} = 3$

The initial patient counts for each

$x_{211} = 25$ $x_{231} = 78$ $x_{261} = 162$

$x_{221} = 22$ $x_{241} = 103$ $x_{271} = 220$

$x_{251} = 108$ $x_{281} = 309$

The sum of patients being treated at the beginning of the analysis year, plus additions to the system, minus discharged patients, will equal the total patients treated at the end of the period.

$$\sum_{i=1}^{2}\sum_{j=1}^{8} \propto_{ij1} + \sum_{i=1}^{2}\sum_{j=1}^{8} \propto_{ij2} - \sum_{i=1}^{2}\sum_{j=1}^{8} \propto_{ij3} = \sum_{i=1}^{2}\sum_{j=1}^{8} \propto_{ij4}$$

$$0.60 \sum_{i=1}^{2}\sum_{j=1}^{8} \propto_{ij1} + \sum_{i=1}^{2}\sum_{j=1}^{8} \propto_{ij2} = \sum_{i=1}^{2}\sum_{j=1}^{8} \propto_{ij3}$$

In order to ensure that the proportion of diagnosis remains consistent with statewide averages, the following constraints are necessary.

$$\sum_{i=1}^{2} \propto_{ik2} = \gamma_k \sum_{i=1}^{2}\sum_{j=1}^{8} \propto_{ij2} \text{ for each diagnosis type } k = 1, 2 \ldots\ldots\ldots\ldots\ldots\ldots 8$$

$$\sum_{i=1}^{2} \propto_{ik3} = \gamma_k \sum_{i=1}^{2}\sum_{j=1}^{8} \propto_{ij3}$$

$$\sum_{i=1}^{2} \propto_{ik4} = \gamma_k \sum_{i=1}^{2}\sum_{j=1}^{8} \propto_{ij4}$$

Any γ_j is the ratio of patients having a certain diagnosis to total patients served. The γ_j values for each of the eight diagnoses are as follows:

$\propto_j$	1	2	3	4	5	6	7	8
γ_j	2.2	2.5	10.7	9.2	9.8	22.7	18.2	24.9

4. RESULT AND ANALYSIS

The solution will be obtained by using LINGO package may be interpreted as follows. The mental health services model was used to examine three different scenarios in order to estimate their impact on mental health delivery for the region. The first scenario examined effects of the current budget on the previously stated regional goals. The second examined the effect of expanding community education programs to contact an additional 3,000 citizens. Such a programs has a promotional effect, which is estimated to increase demand for mental health services by an additional 10 percent. A third scenario determined the impact of a 10 percent increase in patient treatment costs on the mental health delivery system. The three runs and the priority structure associated with each are shown in Table 4 presents a summary of the results of the three results.

TABLE 4: Priority Structure

Goals and Priorities	Result 1	Result 2	Result3
	Present Budget	Additional demand with citizen education	10 percent increase in patient and education costs
Do not exceed patient care and counseling budget	P_1	P_4	P_2
Achieve phase-I of deinstitutionalization	P_2	P_1	P_1
Meet projected demand increase	P_3	P_2	P_3
Provide necessary psychiatric care	P_5	P_5	P_5
Provide mental health education to schoolchildren	P_4	---	P_4
Provide necessary professional staff care	P_6	P_6	P_6
Increase number of citizens reached by education services	P_7	P_3	P_7

TABLE 5: Model Results

Goals and Priorities	Result 1	Result 2	Result 3
	Current Budget	Additional demand with citizen education	10 percent increase in patient and education costs
Do not exceed budget	Achieved	Achieved	Achieved
Achieve phase-I deinstitutionalization	Achieved	Achieved	Achieved
Meet projected demand	Achieved	Achieved	Not Achieved
Provide necessary psychiatric care	Achieved	Achieved	Achieved
Provide education to school children	Achieved	Not Achieved	Not Achieved
Provide necessary professional staff care	Achieved	Achieved	Achieved
Increase number of citizens reached by education	Achieved	Not Achieved	Not Achieved
Annual underutilization of psychiatrist hours (d_6^-)	6,693	*6,246*	7,049
Increased patient demand which is unmet (d_8^-)	35,468	33,762	39,647
Underachievement of school education contacts (d_7^-)	0	0	76
Underachievement of citizen education contacts (d_9^-) in 100s)	0	13	40
Final patient count in regional centre	146	146	146
Final patient count in CMHC	1,604	1,604	1,604

In the first result all the goals can be satisfied. However, analysis of these results shows that approximately 40 percent of psychiatrist and professional staff time is not being utilized for patient care. This is an artifact of the model, which stems in part from the lower average staff times typically spent with the CMHC patient. Achievement of deinstitutionalization goal will require that some patients who typically received more intense treatments will enter community-based programs creating unused staff time. These slack staff hours can be interpreted as hours available for these special needs patients as well as for upgrading community-based care

The second scenario demonstrates that a trade-off must be made between further education programs and the handling of additional demand. Financial resources are not sufficient to allow achievement of both goals. Again, it can be noted that there are sufficient staff hours available to allow increased services for patients with special needs in the CMHC.

The third run examines the effects of further inflation on the mental health system by assuming a 10 percent increase in all costs without an accompanying budget increase. In this case all education programs are discontinued and 72 patients needing mental health services cannot be treated. However, sufficient staff is available.

5. CONCLUSION

The main objective of this study is to minimize the cost in delivery of health care services through Goal Programming approach. The study can be used for delivery process in health care services in future to reduce the expenditure of the government.

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ITERATION METHOD FOR INITIAL VALUE PROBLEMS OF NONLINEAR SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT :

In this article we prove the existence and approximation of solutions of periodic boundary-value problem of second-order ordinary nonlinear hybrid differential equations. We relay our results on Dhage Iteration principle or method embodied in a recent hybrid fixed point theorem of Dhage in partially ordered normed linear spaces. Our results are proved under weaker continuity and Lipschitz conditions. An examples illustrates the theory developed in this article.

1. STATEMENT OF THE PROBLEM

Given the real numbers $r.>0$ and $T>0$, Consider the closed and bounded intervals $I_0=[-r,o]$ and $I=[0,T]$ in R and Let $J=[-r,T]$. By $C=C(I_0,R)$ we denote the space of continuous real-valued functions on I_0. We equip the space C with the norm $\|\ .\ \|_c$ defined by

$$\|x\|_c = \sup_{-r\le\theta\le 0}|x(\theta)| \tag{1}$$

Clearly, C is a Banach space with this supremum norm and it is called the history space of the functional differential equation in question.

For any continuous function $x:J\to R$ and for any $t\in I$, we denote by x_t the element of the space C defined by

$$x_t(\theta)=x(t+\theta), \qquad ,-r\le\theta\le 0 \tag{2}$$

Differential equations involving the history of the dynamic systems are called functional differential equations and it has been recognized long back the importance of such problems in the theory of differential equations. Since then, several classes of nonlinear functional differential equations have been discussed in the literature for different qualitative properties of the solutions. A special class of functional differential equations has been discussed in Dhage [8,9,12], Dhage and Dhage [14] and Dhage and Dhage [15] for the existence and approximation of solutions via a new Dhage iteration method. Very recently the Dhage iteration method is successfully applied to first order hybrid functional differential equation of delay type by Dhage[12,13]. Therefore, it is desirable to extend this method to other functional differential equations involving delay. The present paper is also an attempt in this direction.

In this paper, we consider the nonlinear second order functional differential equation(In short FDE)

$$\left.\begin{array}{ll} x''(t)=f(t, x_t)+g(t, x_t), \ \ t\in I & \\ x_0=\phi, & x'(0)=\eta \end{array}\right\} \qquad (3)$$

Where $\phi\in C$ and $f+g: I\times C\to R$ is a continuous function.

Definition 1.1: A function $x\in C^2(J, R)$ is said to be a solution of the FDE (3) on J if

(1) $x_0=\phi,\ x'(0)=\eta$

(II) $x_t\in C$ *for each* $t\in I$, *and*

(III) x is twice continuously differentiable on I and satisfies the equation in (3)

The FDE (3) is well-known and extensively discussed in the literature for different aspects of the solutions. See Hale [18] , Ntouyas [20,21] and the references therein. There is a vast literature on nonlinear functional differential equations for different aspects of the solutions via different approaches and methods. The method of upper and lower solution or monotone method is interesting and well known, however it requires the existence of both the lower as well as upper solutions as well as certain inequality involving monotonicity of the nonlinearity. In this paper we prove the existence of solution for FDE(3) via Dhage iteration method which does not require the existence of both upper and lower solution as well as the related monotonic inequality and also obtain the algorithm for the solutions. The novelty of the present paper lies in its method which is completely new in the field of functional differential equations and yields the monotone successive approximations for the solutions under some well-known natural conditions.

The rest of the paper is organized as follow. Section 2 deals with the preliminary definitions and auxiliary results that will be used in subsequent sections of the paper. The main results are given in Section (3) and (4). Illustrative examples are also furnished at the end of each section.

2. AUXILIARY RESULTS

Throughout this paper, unless otherwise mentioned , Let $(E, \le, \|.\|)$ denote a partially ordered normed linear space. Two elements x and y in E are said to be comparable if either the relation $x\le y$ or $y\le x$ holds. A non-empty subset C of E is called a chain or totally ordered if all the elements of C are comparable. It is known that E is regular if $\{x_n\}$ is a nondecreasing(resp. nonincreasing) sequence in E such that $x_n\to x^*$ as $n\to\infty$, then $x_n\le x^*$ $(resp.\ x_n\ge x^*)$ for all $n\in N$.

The conditions guaranteeing the regularity of E may be found in Guo and Lakshmikatham [17] and the references therein. Similarly a few details of a partially ordered normed linear space are given in Dhage[4] while orderings defined by different order cones are given in Deimling [1] , Guo and Lakshmikantham [17], and the references therein.

We need the following definition (see Dhage[4,5,6] and the references therein) in what follows.

Definition 2.1: A mapping $T: E \to E$ is called **isotone** or **nondecreasing** if it preserves the order relation $\leq$, that is , if $x \leq y$ implies $Tx \leq Ty$ for all $x, y \in E$, Similarly, T is called nonincreasing if $x \leq y$ implies $Tx \geq Ty$ for all $x, y \in E$. Finally, T is called monotonic or simply monotone if it is either nondecreasing or nonincreasing on E .

Definition 2.2: A mapping $T: E \to E$ is called **partially continuous** at a point $a \in E$ if for $\varepsilon.>0$ there exists a $\delta>0$ such that $\|Tx - Ta\|, < \varepsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. T called partially continuous on E if it is partially continuous at every point of it. It is clear that if T is partially continuous on E, then it is continuous on every chain C contained in E and vice-versa.

Definition 2.3: A non empty subset S of the partially ordered Banach space E is called **partially bounded** if every chain C in S is bounded. An operator T on a partially normed linear space E into itself is called partially bounded if $T(E)$ is a partially bounded subset of E. T is called uniformly partially bounded if all chains C in $T(E)$ are bounded by a unique constant.

Definition 2.4: A non empty subset S of the partially ordered Banach space E is called **partially compact** if every chain C in S is a relatively compact subset of E. A mapping $T: E \to E$ is called partially compact if $T(E)$ is a partially relatively compact subset of E. T is called partially compact if T is a **uniformly partially bounded** if for any bounded subset of E, $T(S)$ is a partially relatively compact subset of E. If T is partially continuous and partially totally bounded, then it is called **partially completely continuous on E.**

Remark 2.1: Suppose that T is a nondecrasing operator on E into itself , then T is a partially bounded or partially compact if $T(c)$ is a bounded or relatively compact subset of E for each chain c in E.

Definition 2.5: The order relation $\leq$ and the metric d on a nonempty set E are said to be D- comparable if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \leq, \|.\|)$, the order relation $\leq$ and the norm $\|.\|$ are said to be D- comparable if $\leq$
and the metric d dined through the norm $\|.\|$ are D- comparable. A subset S of E is called Janhavi if the order relation $\leq$ and the metric d or the norm $\|.\|$ are $D-$comparable in it. In particular, if $S = E$, then E is called a **Janhavi metric or Janhavi Banach space.**

Definition 2.6: An upper semi-continuous and monotone nondecreasing function $\psi: R_+ \to R_+$ is called a $D-$function provided $\psi(0) = 0$. An operator $T: E \to E$ is called partially nonlinear D- contraction if there exists a $D-$function ψ such that

$$\| Tx - Ty \| \leq \psi(\| x - y \|) \tag{4}$$

For all comparable elements $x, y \in E$, where $0 < \psi(r) < r$ for $r > 0$. In particular, if $\psi(r) = kr, k > 0, T$ is called a partial Lipschitz operator with a Lipschitz constant k and moreover, if $o < k < 1, T$ is called a partial linear contraction on E with a contraction constant k.

Remark 2.2: Note that every partial nonlinear contraction mapping T on a partially ordered normed linear space E into itself is partially continuous but the converse may not be true.

The **Dhage iteration method** embodied in the following applicable hybrid fixed point theorem of Dhage [5] in a partially ordered normed linear space is used as a key tool for work contained in this paper. The details of other hybrid fixed point theorems involving the **Dhage iteration principle** and method are given in Dhage [5,6,7], Dhage [16] and the references therein.

Theorem 2.1(Dhage [5,6]): Let $(E,,\le,\|.\|)$ be a regular partially ordered complete normed linear space such that every compact chain C in E is Janhavi. Let $T: E \to E$ be a partially continuous, nondecreasing and partially compact operator. If there exits an element $x_0 \in E$ such that $x_0 \le Tx_0$ *or* $Tx_0 \le x_0$, then the operator equation $Tx = x$ has a solution x^* *in* E and the sequence $\{T^n x_0\}$ of successive iterations converges monotonically to x^*.

Theorem 2.2(Dhage [5,6]) : Let $(E, \le, \|.\|)$ be a partially ordered Banach space and let $T: E \to E$ be nondecreasing and partially nonlinear D- contraction. Suppose that there exists an element $x_0 \in E$ such that $x_0 \le Tx_0$ or $x_0 \ge Tx_0$. If T is continuous or E is regular, then T has a fixed point to x^*. Moreover, the fixed point x^* is unique if every pair of elements in E has a lower and an upper bound.

Remark 2.3: The regularity of E in above theorem 2.1 may be replaced with a stronger continuity condition of the operator T on E which is a results proved in Dhage [4].

Remark 2.4: The condition that every compact chain of E is Janhavi holds if every partially compact subset of E possesses the compatibility property with respect to the order relation $\le$ and the norm $\|.\|$ in it. This simple fact is used to prove the main existence results of this paper.

3. MAIN RESULTS

In this section, we prove existence and approximation results for the FDE(3) on a closed and bounded interval $J = [-r, T]$ under mixed partial Lipschitz and partial compactness type conditions on the nonlinearities involved in it. We place the FDE (3) in the function space $C(J, R)$ of continuous real-valued functions defined on J. We define a norm $\|.\|$ and the order relation $\le$ in $C(J, IR)$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{5}$$

and

$$x \le y \Leftrightarrow x(t) \le y(t) \quad \textit{for all} \quad t \in J \tag{6}$$

Clearly, $C(J, R)$ is a Banach space with respect to above supremum norm and also partially ordered w.r.to the above partially order relation $\le$. It is known that the partially ordered Banach space $C(J,R)$ is regular and lattice so that every pair of elements of E has a lower and an upper bound in it. See Dhage [4,5,6] and references therein. The following useful lemma concerning the Janhavi subsets of $C(J, R)$ follows immediately from the Arzela-Ascoli theorem for compactness.

Lemma 3.1: Let $(C(J,R), \le, \|.\|)$ be a partially ordered Banach space with the norm $\|.\|$ and the order relation $\le$ defined by (5) and (6) respectively. Then every partially compact subset of $C(J, R)$ is Janhavi .

Proof: The proof of the lemma well-known and appears in the papers of Dhage[6], Dhage and Dhage [14] and so we omit the details.

We introduce an order relation $\leq_c$ in C induced by the order relation $\leq$ defined in $C(J,R)$.This will avoid the confusion of comparison between the elements of two Banach Spaces C and $C(J,R)$. Thus, for any $x, y \in C$, $x \leq_c y$ implies $x(\theta) \leq y(\theta)$ for all $\theta \in I_0$. Note that if $x,y \in C(J,R)$ and $x \leq y$, then $x_t \leq_c y_t$ for all $t \in I$.

We need the following in what follows.

Definition 3.1: A twice differentiable function $u \in C^2(J,R)$ is said to be a lower solution of the FDE (3) if u is twice continuously differentiable on I and satisfies the inqualities

$$\begin{aligned} u''(t) &\leq f(t,u_t) + g(t,u_t), \quad t \in I \\ u_0 &\leq_c \phi, \quad u'(0) \leq \eta \end{aligned} \tag{*}$$

Similarly, a twice differential function $v \in C^2(J, R)$ is called an upper solution of FDE(3) if the above inequalities are satisfied with reverse sign.

We consider the following set of assumptions in what follows

(H_1) There exists a constant $M_f > 0$ such that $|f(t,x) + g(t,x)| \leq M_f$ for all $t \in I$ and $x \in C$

(H_2) $f(t,x) + g(t,x)$ is nondecreasing in x for each $t \in I$

(H_3) There exits $D-$function $\varphi : R_+ \to R_+$ such that

$$0 \leq \{[f(t,x) + g(t,x)] - [f(t,y) + g(t,y)]\} \leq \varphi(\| x - y \|_c)$$

For all $t \in I$ and $x, y \in C$, $x \geq_c y$.

(H_4) FDE (3) has a lower solution $u \in C^2(J,R)$.

Lemma 3.2: A function $x \in C(J, R)$ is a solution of the FDE (3) if and only if it is solution of the nonlinear integral equation

$$x(t) = \begin{cases} \phi(0) + \eta t + \int_0^t (t-s)[f(s, x_s) + g(s, x_s)]\,ds, & \text{if } t \in I \\ \phi(t), & \text{if } t \in I_0 \end{cases} \tag{7}$$

Theorem 3.1: Suppose that hypotheses (H_1),(H_2) and (H_4) hold. Then the FDE (3) has a solution x^* defined on J and the sequence $\{x_n\}$ of successive approximations defined by

$$x_0 = u$$

$$x_{n+1}(t) = \begin{cases} \phi(0) + \eta t + \int_0^t (t-s)\,[f(s, x_s^n) + g(s, x_s^n)]\,ds, & \text{if } t \in I \\ \phi(t), & \text{if } t \in I_0 \end{cases} \tag{8}$$

Where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotonically to x^*.

Proof: Set $E = C(J,R)$, Then , in view of Lemma (3.1), every compact chain C in E possesses the compatibility property with respect to the norm $\| . \|$ and the order relation $\leq$ so that every compact chain C is Janhavi in E.

Define an operator T on E by

$$Tx(t) = \begin{cases} \phi(0)+\eta t+\int_0^t (t-s)[f(s, x_s)+g(s, x_s)]ds, & \text{if } t\in I \\ \phi(t), & \text{if } t\in I_0 \end{cases} \tag{9}$$

From the continuity of the integral, it follows that T defines the operator $T: E \to E$. Applying Lemma (3.2), the FDE (3) is equivalent to the operator equation

$$Tx(t) = x(t), \; t \in J$$

Now, we show that the operator T satisfies all the conditions of Theorem (2.1) in a series of the following steps

Step I: T is nondecreasing on E.

Let $x, y \in E$ be such that $x \geq y$. Then $x_t \geq_c y_t$ for all $t\in I$ and by hypothesis (H_2), we get

$$Tx(t) = \begin{cases} \phi(0)+t\eta+\int_0^t (t-s)\,[f(s, x_s)+g(s, x_s)]\,ds, & \text{if } t\in I \\ \phi(t), & \text{if } t\in I_0 \end{cases}$$

$$\geq \begin{cases} \phi(0)+t\eta+\int_0^t (t-s)[f(s, y_s)+g(s, y_s)]\,ds, & \text{if } t\in I \\ \phi(t). & \text{if } t\in I_0 \end{cases}$$

$$= Ty(t)$$

For all $t\in J$. This show that the operator T is also nondecreasing on E.

Step II: T is partially continuous on E.

Let $\{x_n\}_{n\in N}$ be a sequence in a chain C such that $x_n \to x$ as $n\to\infty$. Then $x_s^n \to x_s$ as $n\to\infty$. Since f is continuous, we have

$$\lim_{n\to\infty} T x_n(t) = \begin{cases} \phi(0)+\eta t+\int_0^t \left[\lim_{n\to\infty} (t-s)\,\left[f(s, x_s^n)+g(s, x_s^n)\right]\right] ds, & \text{if } t\in I \\ \phi(t), & \text{if } t\in I_0 \end{cases}$$

$$= \begin{cases} \phi(0)+\eta t+\int_0^t (t-s)[f(s, x_s)+g(s, x_s)]ds, & \text{if } t\in I \\ \phi(t), & \text{if } t\in I_0 \end{cases}$$

$$= Tx(t),$$

For all $t\in J$. This shows that Tx_n converges to Tx pointwise on J.

Now we show that $\{Tx_n\}_{n\in N}$ is an equicontinuous sequence in E.

There are three cases:

Case (a): Let $t_1, t_2 \in J$ with $t_1 > t_2 \geq 0$. Then we have

$$\left| Tx_n(t_2) - Tx_n(t_1) \right| = \left| (t_2 - t_1)\eta + \int_0^{t_2} (t_2 - s)\left[f(s, x_s^n) + g(s, x_s^n) \right] ds - \int_0^{t_1} (t_1 - s)\left[f(s, x_s^n) + g(s, x_s^n) \right] ds \right|$$

$$= \left| (t_2 - t_1)\eta + \int_0^{t_2} (t_2 - t_1)\left[f(s, x_s^n) + g(s, x_s^n) \right] ds - \int_{t_2}^{t_1} (t_1 - s)\left[f(s, x_s^n) + g(s, x_s^n) \right] ds \right|$$

$$\leq \left| t_2 - t_1 \right| \left| \eta \right| + M_f \int_0^{t_2} \left| t_2 - t_1 \right| ds + M_f \int_{t_2}^{t_1} \left| t_1 - s \right| ds$$

$$\to 0 \ \ as \ t_2 \to t_1$$

Uniformly for all $n \in N$.

Case (b) : Let $t_1, t_2 \in J$ with $t_2 < t_1 \leq 0$. Then we have

$$\left| Tx_n(t_2) - Tx_n(t_1) \right| = \left| \phi(t_2) - \phi(t_1) \right| \to 0 \ as \ t_2 \to t_1$$ uniformly for all $n \in N$.

Case (c): Let $t_1, t_2 \in J$ with $t_2 < 0 < t_1$. Then we have

$$\left| Tx_n(t_2) - Tx_n(t_1) \right| \to 0 \quad as \quad t_2 \to t_1$$

Thus in all three cases, we obtain

$$\left| Tx_n(t_2) - Tx_n(t_1) \right| \to 0 \ \ as \ \ t_2 \to t_1$$

Uniformly for all $n \in N$. This shows that the convergence $Tx_n \to Tx$ is uniform and that T is a partially continuous operator on E into itself in view of Remark (2.1).

Step III: T is partially compact operator on E.

Let C be an arbitrary chain in E. We show that $T(C)$ is uniformly bounded and equicontinuous set in E. First we show that $T(C)$ is uniformly bounded. Let $y \in T(C)$ be any element. Then there is an element $x \in C$ such that $y = Tx$. By hypothesis (H_1)

$$\left| y(t) \right| = \left| Tx(t) \right|$$

$$\leq \begin{cases} \left| \phi(0) \right| + T\left| \eta \right| + T\int_0^t \left| f(s, x_s) + g(s, x_s) \right| ds, & if \ \ t \in I \\ \left| \phi(t) \right|, & if \ t \in I_0 \end{cases}$$

$$\leq \left\| \phi \right\| + T\left| \eta \right| + M_f T^2 = r$$

For all $t \in J$, Taking the supremum over t we obtain $\| y \| \leq \| Tx \| \leq r$ for all $y \in T(C)$. Hence $T(C)$ is a uniformly bounded subset of E. Next we show that $T(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then proceeding with the arguments that given in step II it can be shown that

$$\left| y(t_2) - y(t_1) \right| = \left| Tx(t_2) - Tx(t_1) \right| \to 0 \qquad as \qquad t_1 \to t_2$$

Uniformly for all $y \in T(C)$. This shows that $T(C)$ is an equicontinuous subset of E.

Now, $T(C)$ is a uniformly bounded and equicontinuous subset of functions in E and Hence it is compact in view of Arzela-Ascoli Theorem. Consequently $T: E \to E$ is a partially compact operator on E into itself.

Step IV: u satisfies the inequality $u \leq Tu$.

By hypothesis (H_4), the FDE (3) has a lower solution u defined on J. Then we have

$$\begin{cases} u''(t) \leq f(t, u_t) + g(t, u_t), & t \in I \\ u_0 \leq_c \phi, & u'(0) \leq \eta \end{cases}$$

Integrating the above inequality from 0 to t, we get

$$u(t) \leq \begin{cases} \phi(0) + \eta t + \int_0^t (t-s)\left[f(s, u_s) + g(s, u_s)\right] ds, & \text{if } t \in I \\ \phi(t), & \text{if } t \in I_0 \end{cases}$$

$$= Tu(t)$$

For all $t \in J$. As a result we have that $u \leq Tu$.

Thus, T satisfies all the conditions of Theorem (2.1) and so the operator equation $Tx = x$ has a solution. Consequently the integral equation and the equation (3) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (9) converges monotonically to x^*. This completes the proof.

Remark 3.1: The conclusion of Theorems (3.1) also remains true if we replace the hypothesis (H_4) with the following ones:

(H_4) The FDE (3) has an upper solution $v \in C^2(J, R)$.

The proof of the Theorem (3.1) under this new hypothesis is similar and can be obtained by closely observing the same arguments with appropriate modifications.

Example 3.1: Given the closed and bounded intervals $I_0 = \left[-\frac{\pi}{2}, \ 0\right]$ and $I = [0, 1]$, consider the FDE

$$\begin{aligned} & x''(t) = f(t, x_t) + g(t, x_t), \ t \in I \\ & x_0 = \phi, \qquad x'(0) = 1 \end{aligned} \tag{10}$$

Where $\phi \in C$ and $f_1 + g_1 : I \times C \to R$ is a continuous functions given by

$$\phi(\theta) = \sin\theta, \qquad \theta \in \left[-\frac{\pi}{2}, 0\right]$$

And

$$f_1(t,x)+g_1(t,x) = \begin{cases} \tanh\left(\|x\|_c\right)+1, & , \text{ if } x \geq_c 0 \text{ , } x \neq 0 \\ 1, & \text{if } x \leq_c 0 \end{cases}$$

For all $t \in I$.

Clearly, f_1+g_1 is bounded on $I \times C$ with $M_{f_1}=2$. Again, Let $x, y \in C$ be such that $x \geq_c y \geq_c 0$ .Then $\|x\|_c \geq \|y\|_c \geq 0$ and therefore, we have $f_1(t,x)+g_1(t, x)= \tanh\left(\|x\|_c\right)+1 \geq \tanh\left(\|y\|_c\right)+1= f_1(t,y)+g_1(t,y)$

For all $t \in I$. Again, if $x, y \in C$ be such that $x \leq_c y \leq_c 0$, then we obtain

$$f_1(t, x)+g_1(t, x)=1 = f_1(t, y)+g_1(t, y)$$

For all $t \in I.$ This shows that the function $f_1(t,x)+g_1(t,x)$ is nondecreasing in x for each $t \in I.$

Finally

$$u(t)=\begin{cases} t(t+1), & \text{if } t \in I \\ \sin t, & \text{if } t \in I_0 \end{cases}$$

is a lower solution of the FDE (10) defined on J . Thus f_1+g_1 satisfies the hypotheses $(H_1),(H_2)$ *and* (H_4). Hence we apply theorem (3.1) and conclude that the FDE (10) has a solution x^* on J and the sequence $\{x_n\}$ of successive approximation defined by

$$x_0(t)=\begin{cases} t(t+1), & \text{if } t \in I \\ \sin t, & \text{if } t \in I_0 \end{cases}$$

$$x_{n+1} = \begin{cases} t+\int_0^t (t-s)\left[f_1\left(s, x_s^n\right)+g_1\left(s, x_s^n\right)\right]ds, & \text{if } t \in I \\ \sin t, & \text{if } t \in I_0 \end{cases}$$

Converges monotonically to x^*.

Remark 3.2: The conclusion in Example 3.1 is also true if we replace the lower solution u with the upper solution v given by

$$v(t)=\begin{cases} t(2t+1), & \text{if } t \in [0,1] \\ \sin t, & \text{if } t \in \left[-\frac{\pi}{2}, 0\right] \end{cases}$$

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DRUG DISTRIBUTION IN THE BODY INVOLVING I-FUNCTION OF TWO VARIABLES

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ABSTRACT :
The objective of this paper is to determine the drug distribution in the body involving I-function of two variables.

1. INTRODUCTION:

The I–function of two variables introduced by Sharma & Mishra [2], are defined and represented as follows:

$$I\begin{bmatrix}x\\y\end{bmatrix} = I^{0,n:m_1,n_1:m_2,n_2}_{p_i,q_i:r:p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\left[\begin{matrix}x\\y\end{matrix}\middle| \begin{matrix}[(a_j:\alpha_j,A_j)_{1,n}],[(a_{ji}:\alpha_{ji},A_{ji})_{n+1,p_i}]\\ [(b_{ji}:\beta_{ji},B_{ji})_{1,q_i}]\end{matrix}\right.$$

$$\left. \begin{matrix}:[(c_j;\gamma_j)_{1,n_1}],[(c_{ji'};\gamma_{ji'})_{n_1+1,p_{i'}}];[(e_j;E_j)_{1,n_2}],[(e_{ji''};E_{ji''})_{n_2+1,p_{i''}}]\\ :[(d_j;\delta_j)_{1,m_1}],[(d_{ji'};\delta\gamma_{ji'})_{m_1+1,q_{i'}}];[(f_j;F_j)_{1,m_2}],[(f_{ji''};F_{ji''})_{m_2+1,q_{i''}}]\end{matrix}\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)\,x^{\xi}y^{\eta}\,d\xi d\eta, \tag{1}$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\sum_{i=1}^{r}[\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi-A_{ji}\eta)\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi+B_{ji}\eta)]},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_1}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{n_1}\Gamma(1-c_j+\gamma_j\xi)}{\sum_{i'=1}^{r'}[\prod_{j=m_1+1}^{q_{i'}}\Gamma(1-d_{ji'}+\delta_{ji'}\xi)\prod_{j=n_1+1}^{p_{i'}}\Gamma(c_{ji'}-\gamma_{ji'}\xi)]},$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_2}\Gamma(f_j-F_j\eta)\prod_{j=1}^{n_2}\Gamma(1-e_j+E_j\eta)}{\sum_{i''=1}^{r''}[\prod_{j=m_2+1}^{q_{i''}}\Gamma(1-f_{ji''}+F_{ji''}\eta)\prod_{j=n_2+1}^{p_{i''}}\Gamma(e_{ji''}-E_{ji''}\eta)]},$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, $p_{i'}$, $p_{i''}$, q_i, $q_{i'}$, $q_{i''}$, n, n_1, n_2, n_j and m_k are non negative integers such that $p_i \geq n \geq 0$, $p_{i'} \geq n_1 \geq 0$, $p_{i''} \geq n_2 \geq 0$, $q_i > 0$, $q_{i'} \geq 0$, $q_{i''} \geq 0$, (i = 1, …, r; i′ = 1, …, r′; i′′ = 1, …, r′′; k = 1, 2) also all the A's, α's, B's, β's, γ's, δ's, E's and F's are assumed to be positive quantities for standardization purpose; the definition of I-function of two variables given above will however, have a meaning even if some of these quantities are zero. The contour L_1 is in the ξ–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j\xi)$ (j = 1, ..., m_1) lie to the right, and the poles of $\Gamma(1-c_j+\gamma_j\xi)$ (j = 1, ..., n_1), $\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$ (j = 1, ..., n) to the left of the contour.

The contour L_2 is in the η–plane and runs from $-\omega\infty$ to $+\omega\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - F_j\eta)$ $(j=1,\ldots, n_2)$ lie to the right, and the poles of $\Gamma(1 - e_j + E_j\eta)$ $(j = 1, \ldots, m_2)$, $\Gamma(1 - a_j + \alpha_j\xi + A_j\eta)$ $(j = 1, \ldots, n)$ to the left of the contour. Also

$$R' = \sum_{j=1}^{p_i} \alpha_{ji} + \sum_{j=1}^{p_{i'}} \gamma_{ji'} - \sum_{j=1}^{q_i} \beta_{ji} - \sum_{j=1}^{q_{i'}} \delta_{ji'} < 0,$$

$$S' = \sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{p_{i''}} E_{ji''} - \sum_{j=1}^{q_i} B_{ji} - \sum_{j=1}^{q_{i''}} F\delta_{ji'} < 0,$$

$$U = \sum_{j=n+1}^{p_i} \alpha_{ji} - \sum_{j=1}^{q_i} \beta_{ji} + \sum_{j=1}^{m_1} \delta_j - \sum_{j=m_1+1}^{q_{i'}} \delta_{ji'} + \sum_{j=1}^{n_1} \gamma_j - \sum_{j=n_1+1}^{p_{i'}} \gamma_{ji'} > 0, \tag{2}$$

$$V = -\sum_{j=n+1}^{p_i} A_{ji} - \sum_{j=1}^{q_i} B_{ji} - \sum_{j=1}^{m_2} F_j - \sum_{j=m_{21}+1}^{q_{i''}} F_{ji''} + \sum_{j=1}^{n_2} E_j - \sum_{j=n_2+1}^{p_{i''}} E_{ji''} > 0, \tag{3}$$

and $|\arg x| < \frac{1}{2} U\pi$, $|\arg y| < \frac{1}{2} V\pi$.

2. DRUG DISTRIBUTION IN THE BODY:

The study of 'dose-response' relationships plays an important role in pharmacology. We consider here the problem of determining the dosage required so that the body concentration of the drug tends towards a certain value.

<u>Main Result:</u>

The drug distribution in the body in terms of I-function of two variables to be represented is:

$$\int I^{0,\ n\ :m_1+1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_{i'}+1,q_{i'}+1:r':p_{i''},q_{i''}:r''}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| \begin{matrix} \ldots\ldots,\ldots\ldots:\ldots,\ldots,(1+y,y_1):\ldots\ldots,\ldots\ldots \\ \ldots\ldots,\ldots\ldots:(y,y_1),\ldots\ldots,:\ldots\ldots,\ldots\ldots \end{matrix}\right] dy$$

$$= -k I^{0,\ n\ :m_1,n_1+1\ :m_2,n_2}_{p_i,q_i:r\ :p_{i'}+1,q_{i'}+1:r':p_{i''},q_{i''}:r''}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\middle| \begin{matrix} \ldots\ldots,\ldots\ldots:(-t,t_1),\ldots\ldots:\ldots\ldots,\ldots\ldots \\ \ldots\ldots,\ldots\ldots:\ldots\ldots,(1-t,t_1):\ldots\ldots,\ldots\ldots \end{matrix}\right]$$

$$+c\, I^{0,\ n\ :m_1,n_1\ :m_2,n_2}_{p_i,q_i:r\ :p_{i'},q_{i'}:r':p_{i''},q_{i''}:r''}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{4}$$

valid for $y > y_1$, $t > t_1$ and $|\arg z_1| < \frac{1}{2} U\pi$, $|\arg z_2| < \frac{1}{2} V\pi$, where U and V is given in (2) and (3) respectively.

Proof of the formula:

Let $y = y(t)$ be the quantity of the drug present in the patient's body at time t. Initially, at time $t = 0$ say, the patient is given a dose, say y_0, of the drug, then the distribution of drug in the body mathematically denoted as follows [1, p.94(5.15)]:

$$dy/dt = -ky$$

or $$dy/y = -k\,dt \tag{5}$$

where k is constant and the value of constant k depending on the particular drug used (some forms of penicillin, for example Benzyl penicillin, roughly obey (5)).

Solving (5) gives

$$y = y_0\, e^{-kt}$$

After a set interval, T say, an equal amount of the drug, y_0, is added so that the concentration at time T is

$$y(T) = y_0 + y_0\, e^{-kt}$$

On integrating, (5) provides

$$\int dy/y = -k\int dt + c$$

or

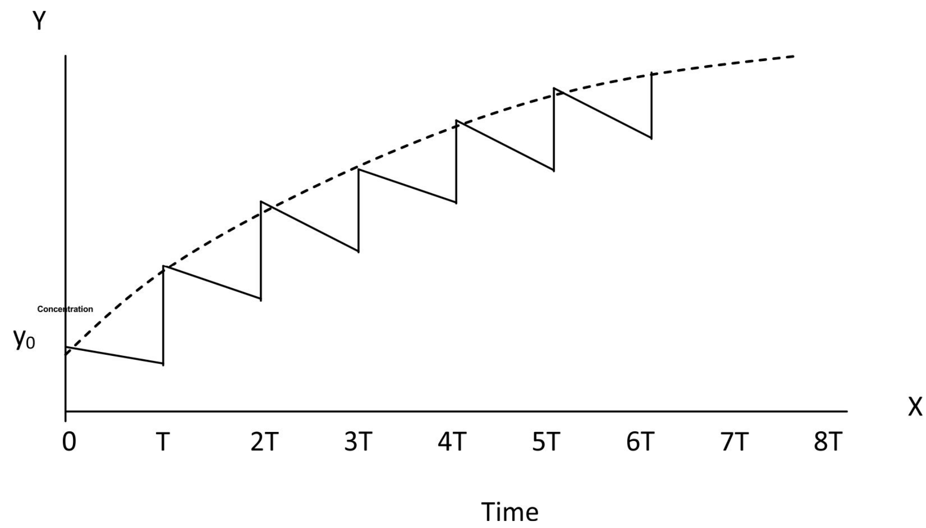
$$\int \frac{\Gamma(y)}{\Gamma(y+1)}dy = -k\,\frac{\Gamma(t+1)}{\Gamma(t)} + c \tag{6}$$

where c is constant.

Again put $t = t + t_1\xi$, $y = y - y_1\xi$ (since as time increase, the quantity of drug will be decreases) in (6) and multiply both side by $\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\,\theta_2(\xi)\theta_3(\eta)x^{\xi}y^{\eta}$, further integrate with respect to ξ,η in the direction of contour L_1, L_2 and use (1), we get (4).

Special Cases:

On specializing the parameters, I-function of two variables may be reduced to H-function, G-function, Lauricella's functions Legendre functions, Bessel functions, hypergeometric functions, Appell's functions, Kampe de Feriet's functions and several other higher transcendental functions. Therefore the result (4) is of general nature and may reduced to be in different forms, which will be useful in the literature on applied Mathematics and other branches.



3. CONCLUSION

Dose-response' relationships plays an important role in pharmacology. which describes the reversible transfer of a drug from one location to another within time in the body. It gives dosage required so that the body concentration of the drug tends towards a certain value .

This research will provide a good understanding of the key determinants of drug distribution as well as how data is obtained, interpreted and utilized to build a model with the objective of developing a more comprehensive understanding of a new entity

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SOFTWARE RELIABILITY GROWTH MODEL INVOLVING LOG LOGISTIC TEST EFFORT FUNCTION AND WIENER PROCESS

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ABSTRACT :

The paper presents continuous time based Software reliability growth model (SRGM). The model incorporates Wiener process which represent fluctuations in fault detection process. Model also considers time dependent fault detection rate which is approximated by log logistic test effort function(TEF).Weibull TEF based SRGM compared with proposed log logistic TEF based SRGM using statistical tools SSE , R2 and AIC. Results suggest that proposed model fits in real time scenario and predict faults detection data better.

Keywords : NHPP Models, Test Effort Function, Wiener process, Akaike's Information Criterion.

1. INTRODUCTION

Software reliability growth models (SRGMs) are a powerful tool for estimating the reliability of software. Basically, SRGMs are classified into two groups namely discrete time models(DTM) and continuous time models (CTM). DTM uses the number of test cases as a unit of fault detection period. In 1985, Yamada and Osaki[3] constructed discrete models. In year 1994 Kapur et al.[2] proposed a DTM with discrete Rayleigh testing effort curve. Kapur et al.[10] developed delayed S-shaped model based on DTM in year 1999. CTM uses machine execution time(CPU time) or calendar time as a unit of fault detection/removal period. Based on CTM, Goel and Okumoto [1] developed exponential model in 1979. Yamada et al.[6] formulated delayed S-shape model in 1983 . In year 2004 Lee et al[13] proposed several type of CTM based models such as exponential, delayed S-shaped, Inflection S-shaped.

In recent literature many researchers have developed DTM and CTM involving Test-effort functions. Testing effort such as the number of test-cases, testing coverage, CPU hours etc influence the reliability of software. In general Test-effort function represents time and resource constraint. Time constraint is the limited time available for testing as software has to be released in market. Resource constraint (human resource, CPU hours, etc.) is the limited resources available for testing.. Reliability of software depends on TEF and it is necessary to effectively consume TEF so as to achieve optimum reliability of software system. Many authors have developed SRGM incorporating TEF. In 1986 Ohtera, Narihisa and Yamada et al.[4] Saxena [20] proposed SRGM’s involving testing effort. Hishitani, Osaki and Yamada et al [5] developed model incorporating testing effort function given by Weibull distribution in year 1993. The blending of imperfect debugging with TEF done by Kapur and Younes [7] in 1994. In 1997 Shepperd and Schofield[8] estimated software project effort using analogies. Same year Logistic TEF was used by Huang, Kuo and Chen[9] and analysed reliability of software. In 1999 Huang, Kuo and Chen[11] incorporated TEF and efficiency while developing SRGM and estimated the cost. In 2002 Huang and Kuo [12] investigated a SRGM based on logistic TEF and predicted optimal software release policy involving cost-reliability criteria.

In 2008 Ahmad, Bokhari, Quadri and khan [14] proposed SRGM based on exponential weibull distribution. They incorporated various TEF and estimated optimal release policy. In 2011, log-logistic TEF with imperfect debugging used by Ahmad, Khan and Rafi [15]. They analyzed an inflection S- shaped SRGM. In 2012 Aggarwal, Kapur, Kaur and Kumar[16] incorporated various TEF in modular software. They categorized total faults as simple, hard and complex faults. These faults were considered as function of TEF described by Weibull type distribution. An optimization problem has been formulated of maximizing total faults removed subject to budgetary and reliability constrains. Genetic algorithm has been used to solve the problem. Reddy and Raveendrababu [17] in year 2012 incorporated generalized exponential TEF while developing SRGM.

In 2013 Shinji Inoue and Shigeru Yamada[18] proposed SRGM based on continuous time model such as lognormal process. They used Wiener process to represent fluctuations in fault detection and approximated fault detection rate with weibull based test effort function. ln this paper we propose SRGM based on continuous time model in which fault detection rate approximated by log logistic TEF . The Log-Logistic TEF captures increasing/decreasing failure occurrence phenomenon effectively. After introduction there are five section in this paper. Section 2 and 3 presents descriptions of various continuous time models and test effort functions respectively. Section 4 discussed development of model and solution. Section 5 provides estimation of the parameters and comparison using statistical tools. Finally conclusions have been highlighted in section 6.

Notations

E	Total testing effort consumed.
β	Scale parameter in Exponential, Rayleigh, Weibull and log logistic distributions.
γ	Shape parameter in Weibull, Generalized exponential and log logistic distribution.
α	Consumption rate of testing effort expenditures in the Logistic TEF.
λ	Constant parameter in the logistic TEF.
a	Total number of faults.
$m(t)$	Expected number of faults at time t.
b(t)	Fault detection rate at time t.
r	Fault detection rate constant.

2. CONTINUOUS TIME MODELS

2.1 Jelinsky- Moranda Model :

J-M model is one of the oldest models. In this model the failure intensity is the product of constant hazard rate ($\emptyset$) of an individual fault and the number of expected faults remaining in the software. The elapsed time between failures follows exponential distribution. The failure intensity at time t is given by $\frac{dm}{dt} = \emptyset[a - m(t)]$.

2.2 Goel – Okumoto Model:

G-O Model is the first Non Homogeneous Poisson Process model that takes the number of faults per unit time as independent Poisson random variables. It is similar to J-M model except that failure rate decreases with time. Parameters of model have physical interpretation and can be estimated by various statistical methods using experimental data.

The mean value function is $m(t) = a[1 - \exp(-rt)]$.

2.3 Yamada S-Shaped Model:

This NHPP model derived from G-O model. Mean value function of G-O model is of exponential nature. Yamada et al. reasoned that due to learning and skill improvements of programmer during debugging phase of software development cycle, the error detection curve is not exponential but rather S-Shaped.

The mean value function is $m(t) = a[1 - (1 + \mathrm{rt})\exp(-rt)]$.

2.4 Log-Logistic Model :

The NHPP models have constant, increasing or decreasing failure occurrence rates per fault. These models were inadequate to capture the failure processes underlying some of the failure data sets, which exhibit an increasing/decreasing failure occurrence rates per fault . The Log-Logistic model was proposed to capture increasing/decreasing failure occurrence rates per fault.

The mean value function is $m(t) = a\frac{(\gamma t)^k}{1+(\gamma t)^k}$ where γ and k are constant.

2.5 Log-Normal Model:

Mullen [27] showed that the distribution of failure rates for faults in software systems tends to be lognormal. The PDF of the lognormal distribution is given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} exp\left(\frac{1}{2}\frac{(ln(x) - \mu)^2}{\sigma^2}\right)$$

Where, x is the variate, μ and σ are mean and standard deviation respectively of the log of variate.

3. TEST EFFORT FUNCTIONS

3.1 Exponential TEF

It is non increasing function. The PDF (current testing effort function at any time t) is given by $\mathrm{f(t)} = \mathrm{E\beta exp}\,(-\beta t)$. CDF [cumulative testing effort function consumed in (0,t)] is given by $\mathrm{F(t)} = \int_0^t \mathrm{w(t)dt} \quad \mathrm{E}[1 - \exp(-\beta t)]$.

3.2 Rayleigh TEF

This TEF exhibits both increasing and decreasing phenomenon. PDF is represented by $\mathrm{f(t)} = 2\mathrm{E\beta t exp}\,(-\beta t^2)$ and CDF by $\mathrm{F(t)} = \mathrm{E}[1 - \exp(-\beta t^2)]$.

3.3 Weibull TEF

It is generalized case of Exponential and Rayleigh TEF. Also exhibits peak phenomenon initially increasing and then decreasing. Its PDF is $f(t) = \gamma E\beta t^{\gamma-1} exp\,(-\beta t^{\gamma})$ CDF is given by $F(t) = E[1 - exp(-\beta t^{\gamma})]$.

3.4 Logistic TEF

It is a smooth bell shaped function and represents TEF fairly accurate.PDF is given by $f(t) = \frac{E\,\lambda\,\alpha\,\exp\,(-\alpha t)}{[1+\lambda\,\exp\,(-\alpha t)]^2}$. CDF is given by $F(t) = \frac{E}{1+\lambda\,\exp\,(-\alpha t)}$.

3.5 Log Logistic TEF

It is similar to the log- normal distribution with elongated tails. Its PDF is

$f(t) = \frac{E\left(\frac{t}{\lambda}\right)^{-\beta}\beta}{\left[1+\left(\frac{t}{\lambda}\right)^{-\beta}\right]^2 t}$. CDF is given by $F(t) = \frac{E}{1+\left(\frac{t}{\lambda}\right)^{-\beta}}$.

4. MODEL DEVELOPMENT

The rate of fault detected at time t is proportional to remaining fault in software at time t.

$$\frac{\mathrm{dm}}{\mathrm{dt}} = \mathrm{b(t)}\big(\mathrm{a} - \mathrm{m(t)}\big) \quad (1)$$

Incorporating fluctuations in fault detection by introducing term $\sigma w(t)$. σ indicates a positive constant representing magnitude of the irregular fluctuations and $w(t)$ is standard Gaussian white noise.

$$\frac{dm}{dt} = (b(t) + \sigma w(t))\big(a - m(t)\big) \tag{2}$$

Assuming $b(t) = rf(t)$ where $f(t)$ is PDF of Test Effort Function. Equation becomes

$$\frac{dm}{dt} = (r\,f(t) + \sigma w(t))\big(a - m(t)\big) \tag{3}$$

Using $m(0) = 0$ Solution is

$$m(t) = a\left[1 - \exp\left\{-\left(\int_0^t (r\,f(t) + \sigma w(t))dt\right)\right\}\right] \tag{4}$$

As $\int_0^t f(t)dt = F(t)$ and $\int_0^t w(t)dt = W(t)$ where $W(t)$ is Wiener process. Equation becomes

$$m(t) = a[1 - \exp\{-(rF(t) + \sigma W(t))\}] \tag{5}$$

approximating $W(t)$ to Normal distribution, equation reduces to

$$m(t) = a\left[1 - \exp\left\{-\left(rF(t) + \frac{1}{2}\sigma^2 t\right)\right\}\right] \tag{6}$$

Applying Log Logistic TEF $F(t) = \frac{E}{1+\left(\frac{t}{\lambda}\right)^{-\beta}}$ Equation becomes

$$m(t) = a\left[1 - \exp\left\{-\left(\frac{rE}{1+\left(\frac{t}{\lambda}\right)^{-\beta}} + \frac{1}{2}\sigma^2 t\right)\right\}\right] \tag{7}$$

5. PARAMETER ESTIMATION AND MODEL COMPARISIONS

5.1 Parameter Estimation

Parameters of models are estimated by Maximum likelihood method using Musa's [19] SS1a failure dataset using Mat lab . The likelihood function for unknown parameters is given by

$$L = \prod_{i=0}^{n} \frac{[m(t_i) - m(t_{i-1})]^{(y_i - y_{i-1})}}{(y_i - y_{i-1})!} \exp\left[-(m(t_i) - m(t_{i-1}))\right]$$

There are n observed data pairs (t_i, y_i) where y_i is observed cumulative faults at time t_i. The parameters are estimated by maximizing likelihood function L.

5.2 Model Comparisons

The models are compared using various tools of Goodness of Fit (GoF).Some of the measures used to determine GoF are given below:

(i) Sum of Square Error (SSE).

(ii) Akaike's Information Criterion(AIC).

(iii) Coefficient of Determination (R^2).

Sum of Square Error (SSE)

SSE is the sum of squares of residuals between observed value and estimated value. It can be expressed as

$$SSE = \sum_{j=1}^{n} (y_j - H_J)^2$$

where y_j is observed cumulative faults at time j and H_J estimated cumulative faults at time j. Model with lower SSE fits better to given dataset.

Akaike's Information Criterion(AIC)

AIC is used to compare the models. It can be evaluated as AIC=-2*log(likelihood function at its maximum value)+2*N. Where N is the number of parameters. The model with lower AIC value is chosen as the best model to fit the data. In AIC, the compromise takes place between the maximized log likelihood and the number of free parameters estimated within the model (the penalty component) which is a measure of complexity or the compensation for the bias in the lack of fit when the maximum likelihood estimators are used.

5.3 Coefficient of Determination (R^2)

Coefficient of Determination is also known as multiple correlation coefficient. It measures the correlation between the dependent and independent variables. Value of R^2 vary from 0 to 1. $R^2 = 1$ is perfect fitting, $R^2 = 0$ no fitting and R^2 close to 1is good fitting.

R^2 is defined as : $$R^2 = \frac{\sum_{j=1}^{n} (H_J - \bar{y})^2}{\sum_{j=1}^{n} (y_j - \bar{y})^2} \qquad \bar{y} = \frac{1}{n} \sum_{j=1}^{n} y_j$$

where y_j is observed cumulative faults at time j and H_J estimated cumulative faults at time j. n is number of data points. Model fits better to given dataset if R^2 close to 1.

proposed model compared with respective previous model. Goodness of fit table is given below.

Table 1. Goodness of Fit for data set SS1a

Models Compared	SSE	R^2	AIC
Weibull TEF	3479.2	0.96	257.14
Log Logistic TEF (proposed)	3276.3	0.94	176.34

6. CONCLUSION

In this paper we have considered continuous time SRMG. Increasing/decreasing failure occurrence phenomenon represented by Wiener process and fault detection rate approximated by log logistic TEF. Goodness of Fit for data set SS1a shows that SSE and AIC values are lower for Log Logistic TEF. R^2 near to one which suggest that Log Logistic TEF based SRGM fits data better than Weibull TEF based SRGM.

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MATHEMATICAL ROUTING MODEL BASED ON NODES CONSTRAINTS IN A DISTRIBUTED SERVICE NETWORK

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ABSTRACT :
In Distributed Service Network, the main concern of the user is mostly focused on the immediate response time of a call for service. The user evaluates the efficiency of a service network in terms of how long he or she has been waiting until a service unit arrives at the scene of a call. In same manner after dispatching, routing is another important problem, the objective of which is how to select an optimal path for the server so that resource could meet the demand as soon as possible. In this paper the mathematical model is proposed to minimize the expected response time.
Keywords: Distributed Service Network, Routing problem, Response time, Mathematical model.

INTRODUCTION :

A distributed service network is a concept related to distribution and traveling: distribution of resources among facilities located at various locations and traveling of resources along a distributed network. There are many types of businesses and organizations that can fit into models of distributed service networks. In theory, almost every service provider can be modeled by means of a network even when one wanders through the long corridors of a mammoth bureaucratic organization while being transferred from one clerk to another, services are in fact, being received from a network.

To incorporate the term "Service" into this discussion, we must assume that the entity called a " distributed service network" provides something called 'service' . This must be in the form of labor such as maintenance or rescue; or in the form of equipment. Service is provided because there is a demand for it. Demand is materialized through service calls. Calls are presumed to be generated only on the nodes of the network.

Managing a distributed service network is not an easy task. It involves a variety of problems related to policy making in the long and in the short range. Here we discussed short term policies such as "Routing" depend upon the distance from origin to destination. In Distributed service network, the main concern of the users is mostly focused on the immediate response time of a call for service. The user evaluates the efficiency of a service network in terms of how long he or she has been waiting until a service unit arrives at the scene of a call. Therefore culmination of many planning efforts lies in the routing rule. A patrol routing problem refers to a server travelling from an origin to a destination under non-emergency circumstances. While the server is on the move, a call for service may arrive, and consequently the moving server will be assigned to that call. If there have no calls during that period the travel has terminated at destination node. Now the problem is how to select an optimal path for the server so that the expected response time to a call is minimized.

In short term decision problems pertaining to normal daily operations of a service network namely, dispatching and repositioning. A common assumption in these problems was that travel is performed along the shortest path to the assigned destination. However this assumption is adequate when the destination node is the scene of a service call, it is not necessarily appropriate when the travel has been initiated for other reasons, such as returning
to a home node after the termination of a service, or a task to patrol a certain area. Such assignment may cause casual travels through the network to a “no service” destination.

The obligatory starting point for the study aiming to describe the most important contributions to the understanding of the interpretations between location and allocation, is that of classical urban economics of 19th century and in particular that emanating from the German school. More precisely, we must go back Von Thunen (1826) and later Weber (1909). Hoover (1948) and Irshad (1956) who wrote specifically about the interrelationship of location and allocation; and to Christaller (1933), Palander (1935), Losch (1940) and Israd (1975) again Lefeber (1958)and Greenhut (1963) who have contributed to the question of a general location theory which implicitly takes into account transport costs as well.

A patrol routing problem is different from the regular routing problem. The objective function of regular routing problem is to minimize the total route length while the objective function of petrol routing problem is to minimize the expected response time to a random request for service. A regular routing problem is terminated only after the vehicle has serviced all the required links or nodes whereas the patrol is dispatched to it. Regular routing problems are usually deterministic in nature while patrol routing problems are probabilistic because of the stochastic nature of the demand for service .

Regular routing problems fall into two categories: The first one deals with a sequence of geographical points (nodes that designate pickup and/or delivery points) that must be traversed, in order, starting and ending at a depot or domicile, for example, the distribution of newspapers to newsstands and stores or the delivery of mail packages to addresses. The second category deals with a set of links (streets) in a network that must be traversed, in order, again starting and ending at a depot or domicile, for example, the delivery of mail to residences or the cleaning of streets.

In our routing problem, we are dealing with patrol routing problem since we have to minimize the expected response time.

MATHEMATICAL FORMULATION:

Let G (N, L) be an undirected network, where N is the sort of n nodes and L is the set of links. The fraction of service calls (demands) associated to each node i is h_i such that $\sum_{i=1}^{n} h_i = 1$.Demand for service over G are assumed to arrive according to Poisson distribution with total rate of λ calls per unit of time and at each node i independently with a mean rate of $\lambda_i = \lambda h_i$, the inter-arrival time distribution between consecutive calls for service is negative exponential with a cumulative distribution.

A single service unit on the network G is requested to perform a non-service task on its way from an origin node V towards a destination node W. The Service unit is equipped with a communication apparatus, so that it is capable of being dispatched to any call for service in the network at all times. The service unit starts moving from node V to node W ends when the next call for service arrives. When the new call for service occurs while the service unit is travelling, it is assumed that the unit is instantaneously informed and dispatched to the demanding node through a shortest path. It is also assumed that all travel in the network is performed at a constant speed

The objective to select a path from node V to node W such that the expected response time to any request for service is minimized. Intuitively, this path is close to the more congested nodes of the network. The two constraints are considered to make the problem practical. The first constraint is to limit the number of nodes of the travel path to vary only within some pre-specified range while the other constraint is on the travel time of the service unit.

We denote by S_m , a state of the system at stage m. Let Xm be the immediate destination from node S_m so that S_m and X_m are connected by a link. By definition, at state X_m there are m-1 more links to travel to node w. Given that the service unit at node S_m selects node X_m as the intermediate destination, the probability that the next call for service will occur before reaching state X_m is

$$1 - e^{\lambda - 1\,(S_m X_m)} \qquad \text{.................................(1)}$$

Now, we consider $F_m\,(S_m, X_m)$ to be the expected response time to the next call for service given that at stage m the service unit at node S_m selects X_m to be the immediate destination, and follows an optimal policy thereafter. The quantity $F_m(S_m, X_m)$ can be calculated according to the following recursion equation.

$$F_m(S_m, X_m) = e^{\lambda-1\,(S_m X_m)} g_m + e^{\lambda-1\,(S_m X_m)} \mathrm{F_{m-1}(X_m)} \qquad \text{.............................(2)}$$

The new request can occur only after the service unit has reached state X_m in the stage m-1 with probability $e^{\lambda-1\,(S_m X_m)}$. In this case, the resulting expected response time is the value of $\mathrm{F_{m-1}(X_m)}$,which has been determined according to an optimal policy. The g_m denotes the expected response time that is incurred when the new call for service arrives while the service unit is still travelling on link (S_m, X_m) . Further, we consider

$$M\left(1 - e^{\lambda-1\,(S_m X_m)}\right) g_m$$

This is calculated as the following manner. Given a request from service from node i while travelling on link (S_m, X_m), the service unit has two alternatives: (i) travel to node i via S_m and (ii) travel to node i via X_m. Let d(i,j) denotes the shortest distance between any nodes i and j, and

$$\theta_i = \left[1(S_m, X_m) + \mathrm{d}(X_m, \mathrm{i}) - \mathrm{d}(S_m, \mathrm{i})\ \right]/2 \qquad \text{................................ (3)}$$

be the time elapsed since the server unit has left node S_m and reached the point on link(S_m, X_m) for which the travel times to node i via either node $S_m$or X_m are equal. If $\theta_i < 0$ or $\theta_1 > 1\ (S_m, X_m)$, we set θ=1 or θ=1 (S_m, X_m). M can be expressed as

$$M = \sum_{i=1}^{n} h_i\left(1 - e^{-\lambda\theta}\right) \int_0^{\theta_1} [t + d(S_m, i)]\, \lambda e^{-\lambda t} dt / \left(1 - e^{-\lambda\theta_1}\right) - \left(e^{-\lambda\theta_1} - e^{-\lambda 1(S_m, X_m)}\right)$$
$$\times \int_{\theta_1}^{1(S_m, X_m)} \left[1(S_m, X_m) - \mathrm{t} + \mathrm{d}(X_m, \mathrm{i}) \lambda e^{-\lambda t} dt\right] / \left(e^{-\lambda\theta_1} - e^{-\lambda 1(S_m, X_m)}\right) \qquad \text{............ (4)}$$

We take into account that a call for service can come from any node i with probability h_i . Given a call from node i the service unit will be dispatched to node i via node S_m as long as the time elapsed since the unit has at S_m is less that θ_1; otherwise the service unit will be dispatched to node i via node X_m.Therefore, expression (4) can be rewritten as

$$M = \sum_{i=1}^{n} h_i\left[(1/\lambda)\left(1 - 2e^{-\lambda\theta_1} - e^{-\lambda 1(S_m, X_m)}\right) - d(X_m, i)\, e^{-\lambda 1(S_m, X_m)} + d(S_m, i)\right] \qquad \text{...................... (5)}$$

Now we use the definition of $F_m\,(S_m, X_m)$ to find the optimal path from node V to node W subject to a constraint on the number of nodes that can be visited. Let n_o be the number of nodes on the selected path including the destination node W. The constraint can be stated as a $\leq n_o \leq$ b

where a and b are lower and upper bounds on the value of n_o respectively. The problem is solved by a dynamic programming algorithm. The algorithm starts at stage zero and ends at stage b. at each stage m, we calculate for all possible states S_m , $F_m{}^*(S_m)$ can be calculated according to the recursive relationship.

$$F_m{}^*(S_m) = \min_{X_m}[F_m(S_m, X_m)] \qquad \text{......................} (6)$$

With the initial condition,

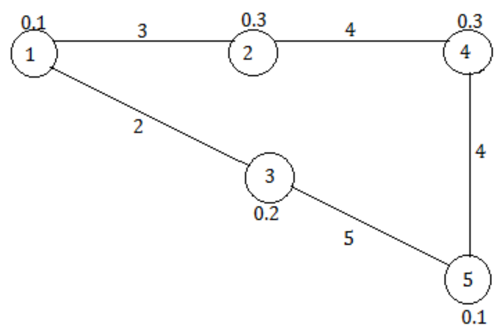
$$F_o{}^*(\mathrm{W}) = \sum_{i=1}^{n} h_i\, d(W, i) \qquad \text{.........................} (7)$$

where the expression (6) is the expected response time from node W whereas expression (7) gives an optimal path from S_m to W. If the maximum number of stages is equal to the upper bound b, the cost of the optimal path is not necessarily given by $F_b{}^*$(V) .The reason for this is that a path from V to W can be formed in an earlier stage than stage b. Therefore, the objective function can be expressed as :

$$\min_{a \leq m \leq b}[F_m{}^*(V)]$$

APPLICATION :

The sample network given below consists of five nodes and six links :



The travel time matrix with regard to the given sample network is tabulated as:

From / To	**1**	**2**	**3**	**4**	**5**
1	0	3	2	7	7
2	3	0	1	4	6
3	2	1	0	5	5
4	7	4	5	0	4
5	7	6	5	4	0

The objective is to find the optimal route in terms of ERT so that the server will pass through at least two nodes but no more than four nodes. Firstly, calculating the ERT for stage 0, i.e. node 5,

$$\mathrm{F_o}{}^*(5) = \sum_{\mathrm{i}=1}^{5} \mathrm{h_i}\, \mathrm{d}(5, \mathrm{i})$$
$$= \mathrm{h_1}\, \mathrm{d}(5,1) + \mathrm{h_2}\, \mathrm{d}(5,2) + \mathrm{h_3}\, \mathrm{d}(5,3) + \mathrm{h_4}\, \mathrm{d}(5,4) + \mathrm{h_5}\, \mathrm{d}(5,5)$$
$$= 0.1 \times 7 + 0.3 \times 6 + 0.2 \times 5 + 0.3 \times 4 + 0.1 \times 0 = 4.7$$

Node 5 can be reached from either node 4 or node 3, so we have to increase m by 1 setting m=1 and compute $\mathrm{F_1}(3)$ and $\mathrm{F_1}$ (4) as

$$\mathrm{F_1}{}^*(3) = M_1(3,5) + e^{-0.1\, 1(3,5)}\, \mathrm{F_o}{}^*(5) = M_1(3,5) + e^{-0.1 \times 5} \times 7 = 4.3746$$
$$\mathrm{F_1}{}^*(4) = M_1(4,5) + e^{-0.1 \times 5} \times 4.7 = 4.6240$$

This completes the calculation for stage 1. Now setting m=2, and examine all nodes that are connected to node 5 through two link. The [1, 2] is the set of two nodes so that node 1 can be connected to 5 via node 3. Therefore,

$$\mathrm{F_2}{}^*(1) = M_2(1,3) + e^{-0.1\, 1(1,3)}\, \mathrm{F_1}{}^*(3)$$

substituting $F_1^*(3)$ in the above equation, then

$$F_2^*(1) = 4.1846$$

Further, node 2 is connected to mode 5 by two alternative double link route; either via node 3 or via node 4. Therefore,

$$F_2^*(2) = \min\,[M_2(2,3) + e^{-0.1\times1}\,F_1^*(3)\,;\;M_2(2,2) + e^{-0.1\times4}\,F_1^*(4)$$

Calculating the above equation yields

$$F_2^*(2) = \min\,[4.1865\,; 4.0372]$$

This implies that the rout 2-3-5 can be discarded and the only candidate double link from node 2 to 5 via node 4, giving

$$F_2^*(2) = 4.0372$$

We have now to move stage 3, the following routes are candidates for stage 3:

(i) 2-1-3-5

(ii) 3-2-4-5

(iii) 3-1-3-5

(iv) 1-2-4-5

(v) 4-2-4-5

The routes containing the segment 2-3-5 have been ignored because it is inferior to 2-4-5 solution, i.e. the last candidate will lead to an infeasible solution.

The candidate route number 2, 3-2-4-5 is better than candidate number 3, and therefore,

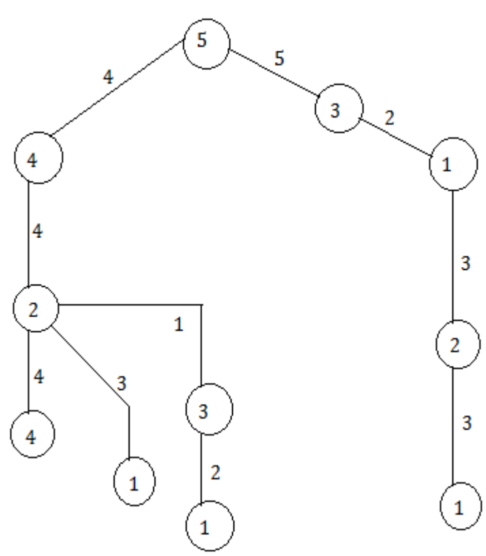
$$F_3^*(3) = M_3(3,2) + e^{-0.1\times1}\,F_2^*(2) = 3.8815$$

For the other route 1-2-4-5, the ERT is $F_3^*(1)$ 3.8845

The route 1-2-4-5 provides better performance in comparison with route 1-3-5. Lastly, there are two possibilities to proceed from node i.e. to node 2 or to node 3. It gives better result to proceed to node 3 (route 1-3-2-4-5)

$$F_4^*(1) = M_4(1,3) + e^{-0.1\times2}\,F_3^*(3) = 3.7809$$

Thus, the optimal path is 1-3-2-4-5 yielding ERT of 3.7809. the tree structure of entire solution is given as:



CONCLUSION:

In short term decision problems pertaining to normal daily operations of a service network namely, dispatching and repositioning. A common assumption in these problems was that travel is performed along the shortest path to the assigned destination. In this paper the importance of the optimal path of emergency management in distributive service network is discussed. The cost and response time of emergency management are considered in a mathematic programming model. Aiming at the successive emergencies with the probabilities of the potential emergencies, the mathematic model is proposed to reach the balance between the emergency that has happened and the potential demand to a certain degree.

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CONVERGENCE OF THREE-STEP ITERATIVE PROCESS FOR GENERALIZED NON-EXPANSIVE MAPPINGS IN CAT (0) SPACES

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ABSTRACT :

In this paper, we study three step iterative process for generalized non-expansive mappings, that is, the mappings which satisfy the condition (E) and establish strong convergence and delta-convergence results for the same in CAT(0) spaces. The results obtained in this paper extend and improve some results available in the literature.

Keywords: CAT(0) spaces, Generalized non-expansive mappings, Strong convergence.

AMS Mathematics Subject Classification (2010): 54E40, 54H25, 47H10.

1. INTRODUCTION:

One of the most important field in mathematics is the fixed point theory. This theory is used to solve a variety of problems in many areas such as economics, chemistry, computer science and engineering as well as many branches of mathematics. Fixed point theory for nonexpansive mappings has played a fundamental role in many aspects of functional analysis. A lot of iteration processes for various mappings in a CAT (0) space have been worked by many authors (see [3, 6−8]). In this paper, we study the new three-step iteration for generalized non-expansive mappings which are weaker than non-expansive mappings in a CAT (0) space.

In 2011, Falset et al. [4] introduced the following definition of generalized non-expansive mappings, that is, the mappings which satisfy the so-called condition (E).

Definition 1.1: Let C be a nonempty subset of a Banach space X and T: C→X be a single – valued mapping. Then T is said to satisfy condition (E_μ) on C, if there exists $\mu \geq 1$ such that $\|x - Ty\| \leq \mu \, \|Tx - x\| + \|x - y\|$ holds for all x, y ∈ C. T is said to satisfy condition (E) on C whenever T satisfies the condition (E_μ) for some $\mu \geq 1$.

Proposition 1.2: Every non-expansive mapping satisfies the condition (E), but the converse is not true.

Proposition 1.3 [10]: Let C be a bounded closed convex subset of a complete CAT(0) space X and T: C → X satisfies the condition (C). Then

d(x, Ty) ≤ 3d(Tx, x) + d(x, y) holds for all x, y ∈ C.

Remark 1.4 [10]: From the above Proposition 1.3, it follows that condition (C) is the special case for $\mu = 3$ in condition (E).

And here is an example which shows that there are mappings which satisfy condition (E) but not condition (C). Thus the mappings satisfying the condition (E) are more generalized than the mappings satisfying condition (C).

Example 1.5 [4]: In the space X = C[0, 1] under the supremum norm, consider a nonempty subset K of X defined as follows

$K = \{ f \in C[0, 1]: 0 = f(0) \leq f(x) \leq f(1) = 1\}$.

To any $g \in K$, associate a function $F_g : K \to K$ defined by

$F_g(h(t)) = (g \circ h)(t) = g(h(t))$.

It is easy to verify that F_g satisfies condition (E_1) but does not satisfy condition (C).

A metric space *X* is a *CAT*(0) *space* if it is geodesically connected and if every geodesic triangle in *X* is at least as 'thin' as its comparison triangle in the Euclidean plane.

For the definition of Δ-convergence, we collect some basic concepts.

Definition 1.6 [3]: Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{ r(x, \{x_n\}): x \in X\}$ and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X: r(x, \{x_n\}) = r(\{x_n\}) \}$.

Remark 1.7: In a CAT(0) space, $A(\{x_n\})$ consists of exactly one point as far as $\{x_n\}$ is a bounded sequence (see, e.g., [2], Proposition 7).

Definition 1.8 [3]: A sequence $\{x_n\}$ in a CAT(0) space X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write Δ-lim $x_n = x$ and call x the Δ-limit of $\{x_n\}$. Note that given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and given $y \in X$ with $y \neq x$, by uniqueness of the asymptotic center, we have

$$\limsup_{n\to\infty} d(x_n, x) = \limsup_{n\to\infty} d(x_n, y).$$

Thus every CAT(0) space satisfies the Opial property.

Lemma 1.9 [3]: Let X be a CAT(0) space. Then

$d((1-t)\, x \oplus t\, y, z) \leq (1-t)\, d(x, z) + t\, d(y, z)$ for all $x, y, z \in X$ and $t \in [0, 1]$.

Lemma 1.10 [3]: Let (X, d) be a CAT(0) space. Then

$d((1-t)\, x \oplus t\, y,\ z)^2 \leq (1-t)\, d(x, z)^2 + t\, d(y, z)^2 - t\,(1-t)\, d(x, y)^2$

for all $t \in [0, 1]$ and $x, y, z \in X$.

Lemma 1.11 [5]: Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.

Lemma 1.12 [1]: If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C then the asymptotic center of $\{x_n\}$ is in C.

In 2016, Thakur et al. [11] established a new three step iterative process in Banach spaces. Now we modify this iterative process into a CAT(0) space as follows:

Let C be a nonempty closed convex subset of a complete CAT(0) space X and T: C → C be a mapping. Then we define the sequence $\{x_n\}$ in C iteratively as:

$$\begin{Bmatrix} x_1 \in C \\ x_{n+1} = (1-\ \alpha_n)Tz_n \oplus \alpha_n Ty_n \\ y_n = (1-\ \beta_n)z_n \oplus \beta_n Tz_n \\ z_n =\ (1-\gamma_n)x_n \oplus \gamma_n Tx_n \end{Bmatrix} \tag{1.1}$$

where $\{\ \alpha_n\ \}_{n=1}^{\infty}$, $\{\ \beta_n\ \}_{n=1}^{\infty}$ and $\{\ \gamma_n\ \}_{n=1}^{\infty}$ are sequences of positive numbers in (0, 1).

2. MAIN RESULTS:

The following Lemma is a consequence of Lemma 2.9 of [6] which will be used to prove our main results.

Lemma 2.1 [6]**:** Let X be a complete CAT(0) space and x ∈ X. Suppose $\{t_n\}$ is a sequence in [b, c] for some b, c ∈ (0, 1) and $\{x_n\}$, $\{y_n\}$ are sequences in X such that $lim_{n\to\infty}\ sup\ \ d(x_n, x) \le$ r, $lim_{n\to\infty}\ sup\ \ d(y_n\,, x) \le$ r and $lim_{n\to\infty}\, d(t_n\, y_n \oplus (1-t_n)x_n\,, x)$ = r hold for some r ≥ 0. Then $lim_{n\to\infty}\, d(x_n, y_n)$ = 0.

Following result guarantees the existence of fixed point of the mappings satisfying condition (E) in CAT(0) spaces.

Lemma 2.2 [9]**:** Let C be a nonempty closed convex subset of a complete CAT(0) space X and T: C → C be a mapping satisfying condition (E). If $\{x_n\ \}$ is a sequence defined by (1.1), where $\{\ \alpha_n\ \}_{n=1}^{\infty}$, $\{\ \beta_n\ \}_{n=1}^{\infty}$ and $\{\ \gamma_n\ \}_{n=1}^{\infty}$ are in (0, 1). If $\{x_n\}$ is bounded and $lim_{n\to\infty}\, d(Tx_n\,, x_n)$ = 0, then F(T) is nonempty.

Lemma 2.3: Let C be a nonempty closed convex subset of a complete CAT(0) space X and T: C → C be a mapping satisfying condition (E). If $\{x_n\ \}$ is a sequence defined by (1.1), where $\{\ \alpha_n\ \}_{n=1}^{\infty}$, $\{\ \beta_n\ \}_{n=1}^{\infty}$ and $\{\ \gamma_n\ \}_{n=1}^{\infty}$ are in (0, 1). Then $lim_{n\to\infty}\, d(x_n\,, p)$ exists for all p ∈ F(T).

Proof: Let p ∈ F(T). Consider

$$\begin{aligned}
d(x_{n+1}, p) &= d((1-\ \alpha_n)Tz_n \oplus \alpha_n Ty_n, p) \\
&\le (1-\ \alpha_n)\, d(Tz_n, p) + \alpha_n\, d(Ty_n, p) \\
&\le (1-\ \alpha_n)\,[\ d(Tp, p) + d(z_n, p)] + \alpha_n\,[\ d(Tp, p) + d(y_n, p)] \\
&\le (1-\ \alpha_n)\, d(z_n, p) + \alpha_n\, d(y_n, p) \\
&= (1-\ \alpha_n)\, d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, p) + \\
&\quad + \alpha_n\, d((1-\ \beta_n)z_n \oplus \beta_n Tz_n\,, p) \\
&\le (1-\alpha_n)\,\{(1-\gamma_n)\, d(x_n\,, p) +\ \gamma_n\, d(Tx_n, p)\}\ + \\
&\quad + \alpha_n\,\{(1-\beta_n)\, d(z_n\,, p) +\ \beta_n\, d(Tz_n, p)\} \\
&\le (1-\alpha_n)\,\{(1-\gamma_n)\, d(x_n\,, p) + \gamma_n\,[\ d(Tp, p) + d(x_n, p)]\ \} + \\
&\quad + \alpha_n\{\ (1-\beta_n)\, d(z_n\,, p) +\ \beta_n\,[\,\mu\, d(Tp, p)\ +\ d(z_n, p)]\ \} \\
&= (1-\alpha_n)\,[\ (1-\gamma_n)\, d(x_n\,, p) + \gamma_n\, d(x_n\,, p)\ ] + \\
&\quad + \alpha_n\,[\ (1-\beta_n)\, d(z_n\,, p) +\ \beta_n\, d(z_n, p)\ ] \\
&= (1-\alpha_n)\, d(x_n\,, p) + \alpha_n\, d(z_n, p) \\
&= (1-\alpha_n)\, d(x_n\,, p) + \alpha_n\, d((1-\gamma_n)x_n \oplus \gamma_n Tx_n, p) \\
&\le (1-\alpha_n)\, d(x_n\,, p) + \alpha_n\,\{\ (1-\gamma_n)\, d(x_n\,, p) +\ \gamma_n\, d(Tx_n, p)\ \} \\
&\le (1-\alpha_n)\, d(x_n\,, p)
\end{aligned}$$

$+ \alpha_n \{ (1-\gamma_n) \, d(x_n, p) + \gamma_n [\mu \, d(Tp, p) + d(x_n, p)] \}$

$= (1-\alpha_n) \, d(x_n, p) + \alpha_n \{ (1-\gamma_n) \, d(x_n, p) + \gamma_n \, d(x_n, p) \}$

$= (1-\alpha_n) \, d(x_n, p) + \alpha_n \, d(x_n, p) = d(x_n, p)$

which shows that the sequence $\{ d(x_n, p) \}$ is decreasing and bounded below so that $\lim_{n\to\infty} d(x_n, p)$ exists.■

Lemma 2.4: Let C be a nonempty closed convex subset of a complete CAT(0) space X and T: C → C be a mapping satisfying condition (E). If $\{x_n\}$ is a sequence defined by (1.1), where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are in (0, 1). Then F(T) is nonempty if and only if $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Proof: Let p ∈ F(T) and by Lemma 2.3, $\lim_{n\to\infty} d(x_n, p)$ exists. Assume that $\lim_{n\to\infty} d(x_n, p) = c$. (2.1)

We first assume that $\lim_{n\to\infty} d(y_n, p) = c$.

Since $d(x_{n+1}, p) \leq d(y_n, p)$, therefore

$$\liminf_{n\to\infty} d(x_{n+1}, p) \leq \liminf_{n\to\infty} d(y_n, p)$$

$$\text{and so } c \leq \liminf_{n\to\infty} d(y_n, p). \quad (2.2)$$

On the other hand, $d(y_n, p) \leq d(x_n, p)$ implies that

$$\limsup_{n\to\infty} d(y_n, p) \leq c. \quad (2.3)$$

From (2.2) and (2.3), we get

$$\lim_{n\to\infty} d(y_n, p) = c. \quad (2.4)$$

Now, $d(Tx_n, p) \leq d(x_n, p)$ implies that $\limsup_{n\to\infty} d(Tx_n, p) \leq c$. (2.5)

By using (2.1), (2.4), (2.5) and Lemma 2.1, we get

$$\lim_{n\to\infty} d(Tx_n, x_n) = 0.$$ ■

Theorem 2.5: Let C be a nonempty closed convex subset of a complete CAT(0) space X and T: C → C be a mapping satisfying condition (E) with F(T) ≠ ∅. If $\{x_n\}$ and $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ are the sequences defined above in Lemma 2.4. Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.

Proof: By Lemma 2.4, we observe that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

We now let $\omega_w(x_n) = \cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

To show the Δ-convergence of $\{x_n\}$ to a fixed point of T, we claim that $\omega_w(x_n) \subset F(T)$ and is a singleton set. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 1.11, 1.12, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in C$.

Since $\lim_n d(v_n, T v_n) = 0$ and T satisfy the condition (E), there exists a $\mu \geq 1$ such that $d(v_n, Tv) \leq \mu \, d(v_n, Tv_n) + d(v_n, v)$.

By taking the lim sup of both sides, we have

$$\lim_n \sup d(v_n, Tv) \leq \lim_n \sup\{ \mu \, d(v_n, Tv_n) + d(v_n, v)\}$$

$$\leq \lim_n \sup d(v_n, v).$$

As $\Delta\text{-}\lim_n v_n = v$, by the Opial property

$$\lim_n \sup d(v_n, v) \leq \lim_n \sup d(v_n, Tv).$$

Hence Tv = v, that is, $v \in F(T)$.

We claim that u = v. Suppose not, by the uniqueness of asymptotic centers,

$$\begin{aligned}\lim_n \sup d(v_n, v) &< \lim_n \sup d(v_n, u) \\ &\leq \lim_n \sup d(u_n, u) \\ &< \lim_n \sup d(u_n, v) \\ &= \lim_n \sup d(x_n, v) \\ &= \lim_n \sup d(v_n, v),\end{aligned}$$

which is a contradiction and hence $u = v \in F(T)$. To show that $\{x_n\}$ Δ-converges to a fixed point of T, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemmas 1.11, 1.12, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_n v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that u = v and $v \in F(T)$. We can complete the proof by showing that x = v. If $x \neq v$, then in view of Lemma 2.3, $\{d(x_n, v)\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned}\lim_n \sup d(v_n, v) &< \lim_n \sup d(v_n, x) \\ &\leq \lim_n \sup d(x_n, x) \\ &< \lim_n \sup d(x_n, v) \\ &= \lim_n \sup d(v_n, v),\end{aligned}$$

which is a contradiction and hence the conclusion follows.■

Theorem 2.6: Let X, C, T and $\{x_n\}$ be as in Lemma 2.3, then $\{x_n\}$ converges strongly to a fixed point of T if and only if $lim_{n\to\infty}\inf d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf\{ d(x, p) : p \in F(T) \}$.

Proof: Necessity is obvious.

Conversely, suppose that $lim_{n\to\infty}\inf d(x_n, F(T)) = 0$. As proved in Lemma 2.3, we have $d(x_{n+1}, p) \leq d(x_n, p)$ for all $p \in F(T)$.

This implies that $d(x_{n+1}, F(T)) \leq d(x_n, F(T))$ so that $lim_{n\to\infty} d(x_n, F(T))$ exists. Thus by hypothesis $lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a positive integer n_0 such that

$d(x_n, F(T)) < \varepsilon/4$ for all $n \geq n_0$.

In particular, $\inf\{ d(x_{n_0}, p) : p \in F(T)\} < \varepsilon/4$. Thus there must exist $p^* \in F(T)$ such that $d(x_{n_0}, p^*) < \varepsilon/2$.

Now for all $m, n \geq n_0$, we have

$$\begin{aligned}d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(p^*, x_n) \\ &\leq 2\, d(x_{n_0}, p^*) \\ &< 2(\varepsilon/2) = \varepsilon.\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a complete CAT(0) space and so it must converge to a point q in C and $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(q, F(T)) = 0$ and closeness of F(T) forces q to be in F(T).■

Remark 2.7: The above delta – convergence and strong convergence results are also holds for mappings satisfying condition (C) as a special case for $\mu = 3$ in condition (E).

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TRACING OF POLYGONAL NUMBER FROM PYRAMIDAL NUMBER AND PENTATOPE NUMBER USING DIVISION ALGORITHM

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ABSTRACT :
In this communication, we find Polygonal number from Pyramidal number and Pentatope number by using division algorithm.
Keywords: Pyramidal number, Pentatope number, Polygonal number & Division Algorithm.

Introduction:

Number theory, a branch of pure mathematics is devoted primarily to the study of the integers. Number theorists study prime numbers as well as the properties of objects made out of integers or defined as generalizations of the integers. Integers can be considered either in themselves or as solutions to equations. One may also study real numbers in relation to rational numbers.

In [1-4], theory of numbers were discussed. In [5], a function $A: N \to N$ is defined by $A(n) = k$ where k is the smallest natural number such that n divides $k + \sum_{1}^{n} m^2$ & in [6], a function $A(n)$ is given by $A(n) = k$ where k is the smallest natural number such that n divides $k + \sum_{1}^{n} m$ and $k + \sum_{1}^{n} m^3$ were discussed. A pyramidal number is a figurate number that represents a pyramid with a polygonal base and a given number of triangular sides. A pentatope number is a number in the fifth cell of any row of **Pascal's triangle** starting with the 5-term row 1 4 6 4 1 either from left to right or from right to left. In [7], **Pyramidal numbers** and **pentatope** number were evaluated using z-transform & in [8 & 9], pentatope numbers were analyzed for its special dio-triples and dio-triples. A polygonal number is a number represented as dots or pebbles arranged in the shape of a regular polygon. In [10], centered polygonal numbers were evaluated using division algorithm.

In this communication, we obtain a function $A(n)$ given by $A(n) = k$ where k is the Polygonal number, such that $24PT_n$ divides $k + (\text{a bi-quadratic polynomial})$ & $6P_n^m$ divides $k + (\text{a cubic polynomial})$.

Notation:

$$PT_n = \frac{n(n+1)(n+2)(n+3)}{24} = \text{Pentatope number of rank 'n'}.$$

$$P_n^{\,m} = \frac{n\,(n+1)}{6}\left[(m-2)\,n+(5-m)\right] = \text{Pyramidal number of rank 'n' with sides 'm'}.$$

$$T_{m,n} = n\left(1+\frac{(n-1)(m-2)}{2}\right) = \text{Polygonal number of rank 'n' with sides 'm'}.$$

METHOD OF ANALYSIS:

SECTION A:

Tracing of Polygonal number from Pentatope number:

Let $A: N \to N$ be defined by $A(n)=k$ where k is the smallest natural number such that $24\,PT_n$ divides $k+\left(n^4+6n^3+(13-m)\,n^2+(m+2)n\right)$. If $24\,PT_n$ divides $\left(n^4+6n^3+(13-m)\,n^2+(m+2)n\right)$ then $A(n)=(m-2)\,n^2+(4-m)\,n$, otherwise $A(n)=\left((m-2)\,n^2+(4-m)\,n\right)-r$, where r is the least non-negative remainder when $\left(n^4+6n^3+(13-m)\,n^2+(m+2)n\right)$ is divided by $24\,PT_n$. Hence A is defined for all n. By division algorithm, such remainder is given by $\left(n^4+6n^3+(13-m)\,n^2+(m+2)n\right)-24\,q\,PT_n$ where q is the quotient when $\left(n^4+6n^3+(13-m)\,n^2+(m+2)n\right)$ is divided by $24\,PT_n$ & is given by the greatest integer function of $\dfrac{\left(n^4+6n^3+(13-m)\,n^2+(m+2)n\right)}{24\,PT_n}$,

i.e., $q = \dfrac{\left(n^4+6n^3+(13-m)\,n^2+(m+2)n\right)}{24\,PT_n}$

So that,

$$A(n)=(n^4+6n^3+11n^2+6n)-\left\{\left(n^4+6n^3+(13-m)n^2+(m+2)n\right)-\left[\frac{\left(n^4+6n^3+(13-m)n^2+(m+2)n\right)}{n^4+6n^3+11n^2+6n}\right]\times(n^4+6n^3+11n^2+6n)\right\}$$

$$=n^4+6n^3+11n^2+6n-n^4-6n^3-(13-m)n^2-(m+2)n$$

$$+\left[\left(\frac{n^4+6n^3+11n^2+6n}{n^4+6n^3+11n^2+6n}\right)-\left(\frac{(m-2)n^2+(4-m)n}{n^4+6n^3+11n^2+6n}\right)\right]\times(n^4+6n^3+11n^2+6n)$$

$$\therefore \quad A(n)=\left((m-2)\,n^2+(4-m)\,n\right)+\left[1-\left(\frac{(m-2)\,n^2+(4-m)\,n}{n^4+6n^3+11n^2+6n}\right)\right]\times(n^4+6n^3+11n^2+6n)$$

$$=\left((m-2)\,n^2+(4-m)\,n\right)+0 \qquad \left(\text{since}, 0<\left(\frac{(m-2)\,n^2+(4-m)\,n}{n^4+6n^3+11n^2+6n}\right)<1\right)$$

$$=\left((m-2)\,n^2+(4-m)\,n\right)$$

$$=2\,n\left[1+\frac{(n-1)\,(m-2)}{2}\right]$$

$$A(n)=2\,(\text{Polygonal number}).$$

SECTION B:

Tracing of Polygonal number from Pyramidal number:

Let $A: N \to N$ be defined by $A(n) = k$ where k is the smallest natural number such that $6\,P_n^m$ divides $k + \left((m-2)\ n^3 + (5-m)\,n^2 + n\right)$ where $m = 3, 4, ...20$. If $6\,P_n^m$ divides $\left((m-2)\ n^3 + (5-m)\,n^2 + n\right)$ then $A(n) = (m-2)\,n^2 + (4-m)\,n$, otherwise $A(n) = \left((m-2)\,n^2 + (4-m)\,n\right) - r$, where r is the least non-negative remainder when $\left((m-2)\ n^3 + (5-m)\,n^2 + n\right)$ is divided by $6\,P_n^m$. Hence A is defined for all n. By division algorithm, such remainder is given by $\left((m-2)\ n^3 + (5-m)\,n^2 + n\right) - 6\,q\,P_n^m$ where q is the quotient when $\left((m-2)\ n^3 + (5-m)\,n^2 + n\right)$ is divided by $6\,P_n^m$ & is given by the greatest integer function of $\dfrac{\left((m-2)\ n^3 + (5-m)\,n^2 + n\right)}{6\,P_n^m}$,

i.e., $q = \dfrac{\left((m-2)\ n^3 + (5-m)\,n^2 + n\right)}{6\,P_n^m}$

So that

$$A(n) = \left((m-2)\ n^3 + 3n^2 + (5-m)n\right) - \left\{\left((m-2)\ n^3 + (5-m)n^2 + n\right) - \left[\frac{\left((m-2)\ n^3 + (5-m)n^2 + n\right)}{(m-2)\ n^3 + 3n^2 + (5-m)n}\right] \times \left((m-2)\ n^3 + 3n^2 + (5-m)n\right)\right\}$$

$$= (m-2)\ n^3 + 3n^2 + (5-m)n - (m-2)\ n^3 - (5-m)n^2 - n$$
$$+ \left[\left(\frac{(m-2)\ n^3 + 3n^2 + (5-m)n}{(m-2)\ n^3 + 3n^2 + (5-m)n}\right) - \left(\frac{(m-2)n^2 + (4-m)n}{(m-2)\ n^3 + 3n^2 + (5-m)n}\right)\right] \times \left((m-2)\ n^3 + 3n^2 + (5-m)n\right)$$

$$\therefore \quad A(n) = \left((m-2)n^2 + (4-m)n\right) + \left[1 - \left(\frac{(m-2)n^2 + (4-m)n}{((m-2)\ n^3 + 3n^2 + (5-m)n)}\right)\right] \times \left((m-2)\ n^3 + 3n^2 + (5-m)n\right)$$

$$= \left((m-2)n^2 + (4-m)n\right) + 0 \qquad \left(\text{since}\ 0 < \left(\frac{(m-2)n^2 + (4-m)n}{((m-2)\ n^3 + 3n^2 + (5-m)n)}\right) < 1\right)$$

$$= \left((m-2)\,n^2 + (4-m)\,n\right)$$

$$= 2\,n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

$A(n) = 2\,(\text{Polygonal number}).$

CONCLUSION:

To conclude, one can find some special numbers through the division algorithm.

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LABELING ON DRAGON CURVE FRACTAL GRAPH

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ABSTRACT :

We discuss about the fractal which is a mathematical set of self similar patterns. The Dragon curves or Highway Dragon are also a fractal in the family of fractals. This fractal curve has been considered as a graph and also it has been tested in contrast with the labeling of cordial, edge cordial and total cordial labeling.

Keywords: Curve fractal graph, labeling of cordial, edge cordial and total cordial labeling

MSC Code: 05C78

1. INTRODUCTION

The Highway dragon (also known as the Harter–Highway dragon or the Jurassic Park dragon) was first investigated by NASA physicists John Highway, Bruce Banks, and William Harter. Many of its properties were published and discussed by a no. of scholars. Our study is to enable the Cordial, Total cordial, Edge cordial and Total edge Cordial for the above said dragon curve. So the dragon curve has been considered as a Graph with number of vertices and edges. The construction of the graph is also part of the study of properties of the said curve.

A Graph = $< V, E, \psi >$ consists of a non empty set V called the set of nodes of the graph, E is said to be the set of edges of the graph and ψ is the mapping from the set of edges E to a set of ordered or unordered pair of elements of V. It would be convenient to write a graph G as $<V, E>$ or simply as G.

A graph labeling is an assignment of integers to the vertices or edges, or both subject to certain conditions. Many types of labeling like harmonious, graceful, etc. are used by various researchers in practice. A graph G with q edges is harmonious if there is an injection f from the vertices of G to the group of integers modulo q such that when each edge 'xy' is assigned the label $|f(x) + f(y)|$ (mod q), the resulting edge labels are distinct.

A graph G with q edges is graceful if f is an injection from the vertices G of to the set $f: V \rightarrow \{0, 1, \dots, q\}$ such that, when each edge 'xy' is assigned the label $|f(x) + f(y)|$, the resulting edge labels are distinct. Eventually after the introduction of the concept of cordial labeling. Many researchers have investigated graph families or graphs which admit cordial labeling with minor variations in cordial theme like product cordial labeling, total product cordial labeling and prime cordial labeling.

Definition 1.1

If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

Definition 1.2

Let G be a graph. A mapping f: $E(G) \rightarrow \{0, 1\}$ is called binary edge labeling of G and $f(e)$ is called the label of the edge e of under f.

For an edge $e = uv$, the induced edge labeling $f^*: E(G) \rightarrow \{0, 1\}$ is given by $f^*(e) = |f(u) - f(v)|$. Let $vf(0)$, $vf(1)$ be the number of vertices of G having labels 0 and 1 respectively under f while $ef(0)$, $ef(1)$ be the number of edges having labels 0 and 1 respectively under f^*.

Definition 1.3

A binary vertex labeling of a graph G is called a cordial labeling if $|vf(0) - vf(1)| \leq 1$ and $|ef(0) - ef(1)| \leq 1$. A graph G is cordial if it admits cordial labeling

Definition 1.4

Let G be a graph with two or more vertices then the total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

Definition 1.5

A binary edge labeling of a graph G is called an edge cordial labeling if $|vf(0) - vf(1)| \leq 1$ and $|ef(0) - ef(1)| \leq 1$. A graph G is edge cordial if it admits cordial labeling.

Definition 1.6

Cahit [4] introduced edge-cordial labeling as a binary edge labeling f: $E(G) \rightarrow \{0,1\}$, with the induced vertex labeling given by $f(v) = \Sigma_{uv\in E} f(uv)(mod\ 2)$ for each $v \in V$ such that $|ef(0) - ef(1)| \leq 1$. And $|vf(0) - vf(1)| \leq 1$, where $ef(i)$ and $vf(i)(i = 0,1)$ denote the number of edges and vertices labeled with 0 and 1, respectively.

Definition 1.7

We define a total edge-cordial labeling of a graph G with vertex set V and edge set E as an edge-cordial labeling such that number of vertices and edges labeled with 0 and the number of vertices and edges labeled with 1 differ by at most 1, i.e., $|(v_f(0) + e_f(0)) - (v_f(1) + e_f(1))| \leq 1$. A graph with a total edge-cordial labeling is called a total edge-cordial graph.

2. MATHEMATICAL MODELS

The construction and its properties are well said already by many scholars. Here, we provide the pattern of the dragon diagram for the need of further study

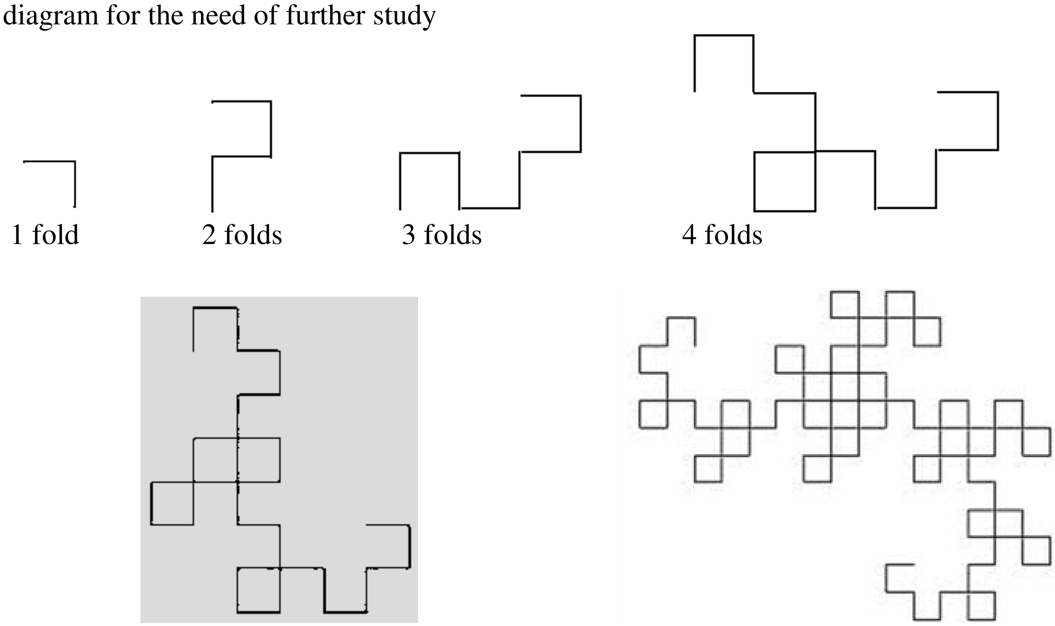
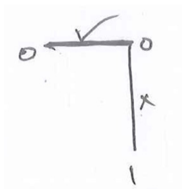


Fig.1: Construction of Highway dragon fractal

2.1 Cordial Labeling for Dragon curve

The pattern of labeling of vertices is easily understandable by seeing the figure (Fig.2.2). The dragon curve has twist in each iteration and it has more and more intersections developing further from iteration four. The vertices are numbered 0's and 1's to satisfy the condition of cordial and total cordial in each iteration. It is clearly observed that all the edges having label 0’s are denoted by a tick mark (✓) and correspondingly all edges are labeled with 1’s are denoted by a cross mark (×). Hence it satisfies existence of cordial labeling. In each iteration, the similar fashion of labeling is being tried to extend and is followed to check the existence of cordial labeling and total cordial labeling. Hence, the above fractal holds good for both cordial and total cordial labeling.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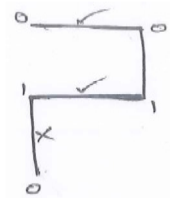
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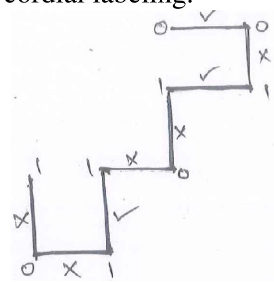
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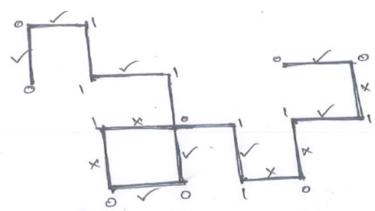
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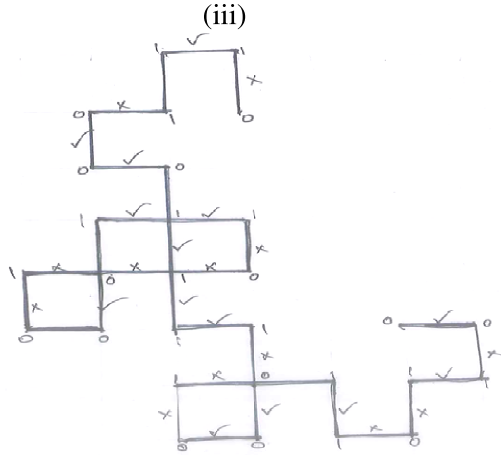
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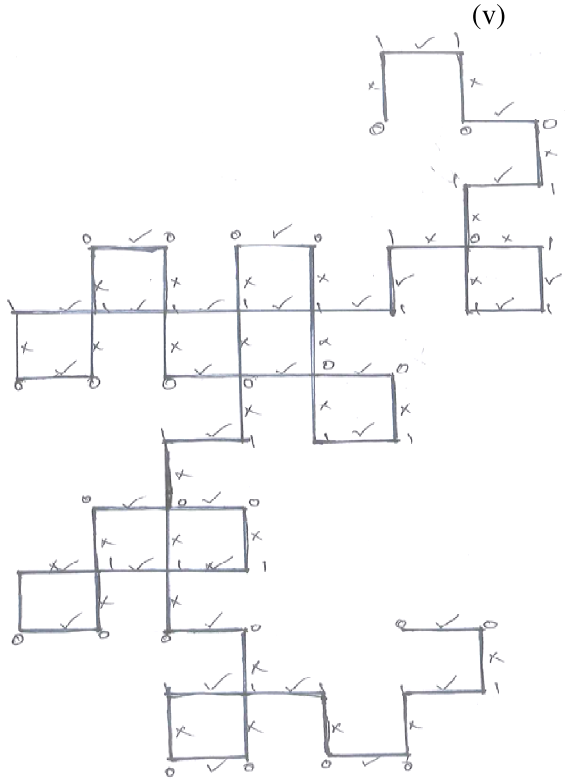
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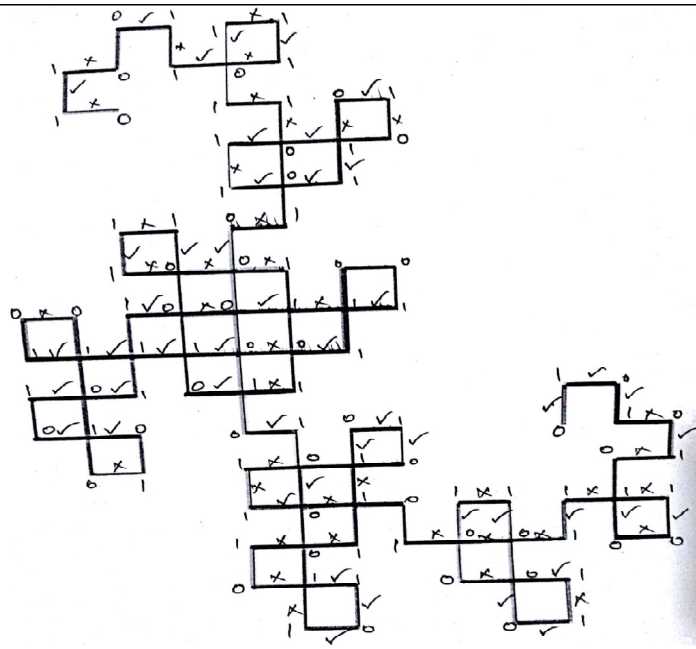
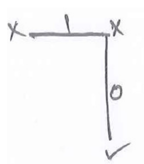


(vii)

Figure 2.2 Cordial labeling for first seven iterations

2.2 Edge Cordial Labeling for Dragon curve

Similar to the above, the pattern of labeling of edges are easily understandable by seeing the below figure (Fig 2.3). Initially iteration fractal graph is labeled with one zero and one adjacent to each other and it satisfies edge cordial labeling definition. Then for second iteration, to satisfy condition for vertices, edges are labeled with zeros and once as given in the figure. Further the pattern of labeling extended for the third and other iterations. Here, the vertices are labeled with label 0's denoted by a tick mark (✓) and correspondingly 1's denoted by a cross mark (✗). In each iteration, it satisfies the condition of edge cordial labeling and total edge cordial labeling. The results provided in the following table holds well for all iterations.



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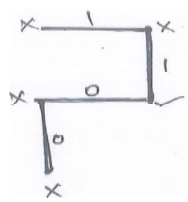

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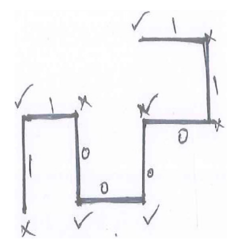

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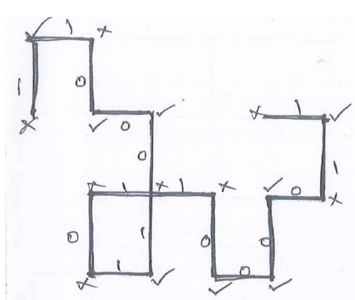

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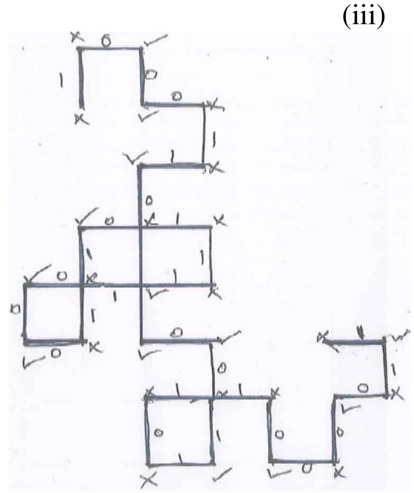

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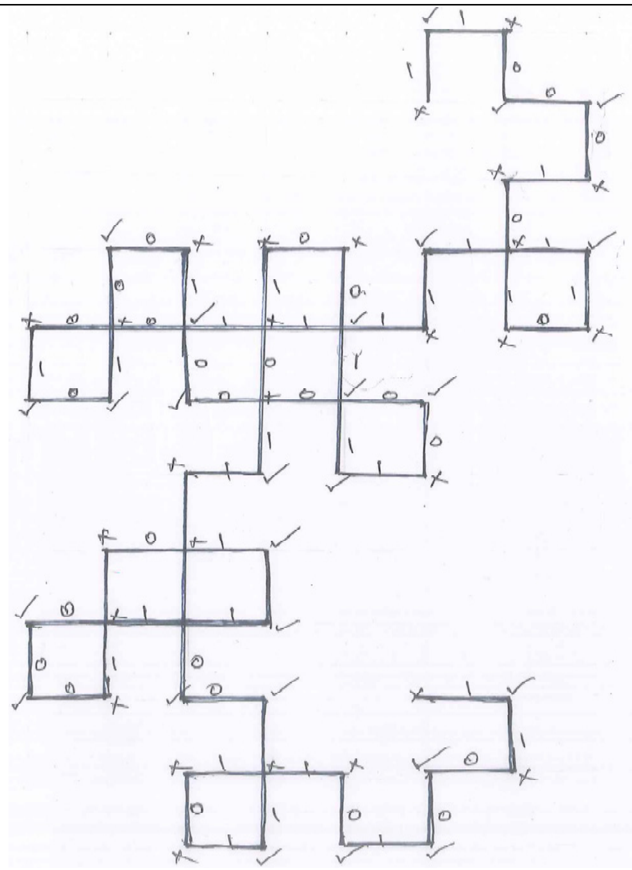

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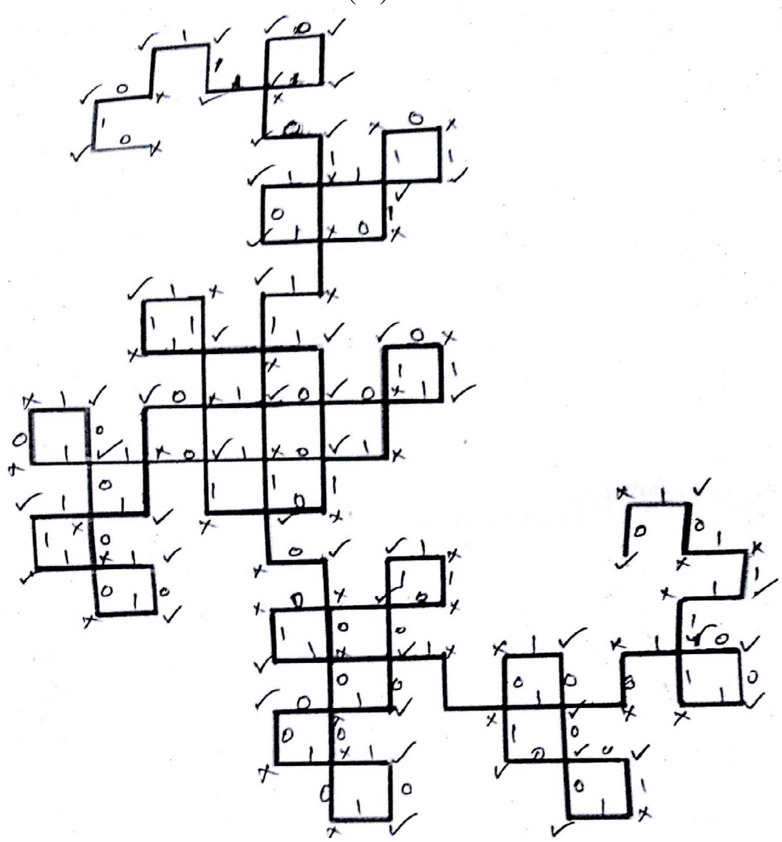

(vii)

Figure 2.3 Edge Cordial labeling for first seven iterations

The iteration wise no. of edges and vertices are provided in the Table 2.1.

Iterations	No.of Vertices	Vertices	No.of edges	Edges
1	3	$\lvert vf(0)\rvert = 2$ $\lvert vf(1)\rvert = 1$	2	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 1$
2	5	$\lvert vf(0)\rvert = 3$ $\lvert vf(1)\rvert = 2$	4	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 2$
3	9	$\lvert vf(0)\rvert = 5$ $\lvert vf(1)\rvert = 4$	8	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 4$
4	16	$\lvert vf(0)\rvert = 8$ $\lvert vf(1)\rvert = 8$	16	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 8$
5	29	$\lvert vf(0)\rvert = 14$ $\lvert vf(1)\rvert = 15$	32	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 16$
6	54	$\lvert vf(0)\rvert = 27$ $\lvert vf(1)\rvert = 27$	64	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 32$
7	103	$\lvert vf(0)\rvert = 51$ $\lvert vf(1)\rvert = 52$	128	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 64$
8	200	$\lvert vf(0)\rvert = 100$ $\lvert vf(1)\rvert = 100$	256	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 128$
9	393	$\lvert vf(0)\rvert = 196$ $\lvert vf(1)\rvert = 197$	512	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 256$
10	778	$\lvert vf(0)\rvert = 389$ $\lvert vf(1)\rvert = 389$	1024	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 512$
. . .	. . .	. . .	. . .	. . .
n	$2I_{n-1} - (n-2)$	$\lvert vf(0)\rvert = \lfloor ½\, 2I_{n-1} - n - 2\rfloor$ $\lvert vf(1)\rvert = \lvert vf(0)\rvert \pm 1$	2^n	$\lvert ef(0)\rvert = \lvert ef(1)\rvert = 2^{n-1}$

Where I_{n-1}, defines the number of vertices in the previous iteration. Since there are more intersections vertices using the induction method the number of vertices and corresponding number of 0's and 1's are calculated for this section.

3. CONCLUSION

The Dragon curve fractal graph leads to some application of the dragon curve in further emerging engineering and science fields. The existence of above said labeling are proved and the results are provided as detailed in table. However the Dragon curve fractal graph is cordial, total cordial, edge cordial and total edge cordial.

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MATHEMATICAL FORM OF IN-VIVO PRESSURE GRADIENT PROFILES – INPUT DATA FOR MATHEMATICAL MODELS OF HUMAN CVS

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ABSTRACT :

In this Paper, it is first shown why the input data used in the literature is too ideal or incorrect. Using Fourier series and numerical integration methods, for pulsatile flow of blood, in-vivo experimental data (pressure gradient profiles) for two different locations of human cardiovascular system (CVS) have been converted into a mathematical form that could be readily used as input data for such mathematical models. A comparison of experimental and mathematical profiles has been made and it is observed that these are in a good agreement (error less than 10%).

INTRODUCTION

Oscillatory motion of an incompressible viscous fluid in a long straight tube was studied by Grace (1928). Womersley (1955) considered physiological oscillating flows. He obtained the analytic expressions for axial velocity, flow rate, wall shear, etc. for a given oscillatory pressure gradient. Steady flow of an incompressible viscous fluid through a circular pipe was considered by Pai (1956), Schlichting (1960), and Bansal (1977). Velocity field, flow rate and wall shear were obtained in terms of pressure gradient. Many research papers (Skalak (1968), Fung et.al. (1972) etc.) have been presented with Womersley's model as base model. It is very interesting to note that the pressure gradient is required in these models as input data and the flow in these models is oscillatory.

Womersley (1955) considered blood flow through a circular pipe. He obtained exact solution of the equation of the motion of a viscous liquid in a circular tube under a pressure gradient of the form

$$\frac{p_1 - p_2}{l} = Ae^{i\omega t}, \qquad \ldots(1)$$

which is periodic in time and frequency is $f = \omega / 2\pi$.

Womersley (1955) considered only oscillatory part of the pressure gradient. Steady part of the pressure gradient profile was neglected by him. Obviously this is too ideal input form.

Sud and Sekhon (1985) considered a mathematical model of blood flow in artery subject to pulsating pressure gradient due to the normal heart action and periodic body acceleration. The artery was assumed as a rigid tube with infinite length. The governing equations were solved by using Laplace transforms technique [Sneddon (1974)].

The pressure gradient consists of two components, one of which is steady or non-fluctuating and the other fluctuating or oscillatory. Sud and Sekhon(1985) considered oscillatory as well as steady part of the pressure gradient profile i.e. pulsatile motion. For the human beings, following form of pressure gradient [Burton, (1966)] was taken

$$\frac{\partial p}{\partial z} = A_0 + A_1 \cos \omega_p t , \qquad \ldots(2)$$

where $\omega_p = 2\pi f_p$, f_p is pulse frequency, A_0 is a steady component and was calculated from known values of average velocity ($u_{z,avg}$) or average flow rate Q_{avg} as given below

$$A_0 = \frac{8\mu_f Q_{avg}}{\pi R^4} = \frac{4\mu_f u_{z,avg}}{R^2} \qquad \ldots(3)$$

R is the radius of the tube, $\mu_f = 4\times10^{-3}$ $kg\,m^{-1}\,s^{-1}$, $f_p = 1.2$ Hz.

Here A_1 is amplitude of fluctuating component of the pressure gradient. Sud and Sekhon (1985) calculated A_1 by using formula

$$A_1 = \frac{A_0}{5} \qquad \ldots(4)$$

Abdalla (1998) noticed that the forms of the actual pressure gradient in human CVS is quite different from the form of the pressure gradient considered by Sud and Sekhon (1985). Their argument for the computation of A_0 appears to be quite logical, but the argument for the computation of A_1 seems to be a little unconvincing. Sud and Sekhon (1985) considered $A_1 = A_0/5$ which is based on human blood pressure as 80-120 mm of Hg implying an average value of 100 and the amplitude of the oscillation as one fifth of the average value of blood pressure. But the aim is to compute the pressure gradient not the pressure, therefore $A_1 = A_0/5$ appears to be incorrect.

The results of the pressure gradient versus time derived by Sud and Sekhon (1985) and Abdalla (1998) are compared with the experimental results of Pedley(1980) and presented through Figure-1.

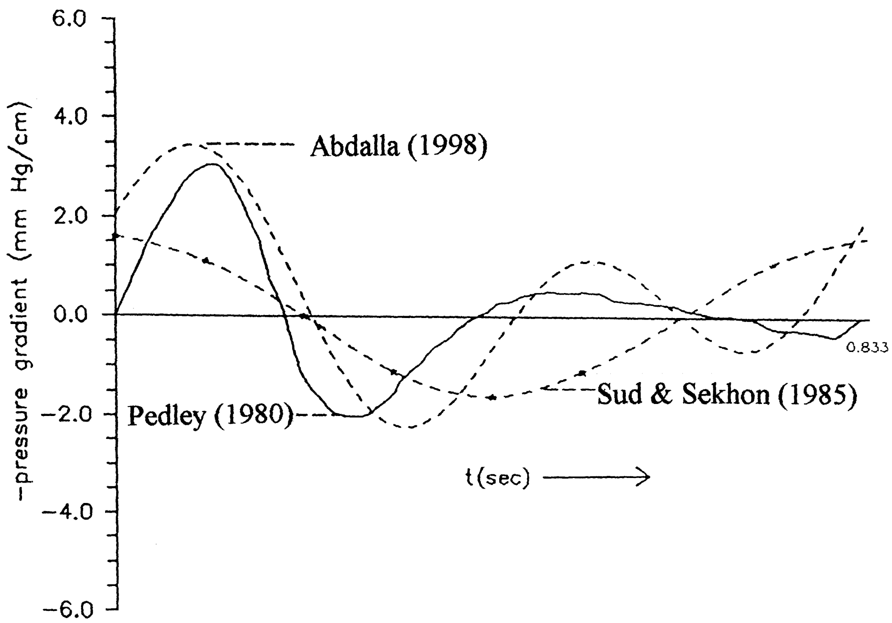

Fig. 1: Variation of Pressure gradient with time

The form of pressure gradient used by Abdalla (1998) is given by

$$-\frac{\partial p}{\partial z} = A_0 + \sum_{n=1}^{2} \left[A_n \cos n\omega_p t + B_n \sin n\omega_p t \right]. \qquad \ldots(5)$$

Toravi (2000) observed that Abdalla (1998) considered the time period as 0.833 sec for the mathematical forms of some experimental pressure gradient profiles. But, for calculation of the Fourier coefficients, Abdalla (1998) took the period T = 2π, which appears to be incorrect. It may further be noticed that the period does not have fixed values in all cases. It may change from person to person, with surroundings and due to health conditions etc.

Toravi (2000) obtained two in-vivo blood pressure gradient profiles in human CVS in thoracic aorta. Mathematical forms for these experimental profiles have been obtained in the present paper from the given geometrical forms.

MATHEMATICAL MODEL

Mathematical model for pressure gradient profiles is given by

$$-\frac{\partial p}{\partial z} = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos n\omega_p t + B_n \sin n\omega_p t \right], \qquad \ldots(6)$$

where

$$A_0 = \frac{1}{T} \int_0^T \left(-\partial p \middle/ \partial z \right) dt\,, \qquad \ldots(7)$$

$$A_n = \frac{2}{T}\int_0^T \left(-\partial p \big/ \partial z\right)\cos\left(n\omega_p t\right) dt\,, \qquad \ldots(8)$$

$$B_n = \frac{2}{T}\int_0^T \left(-\partial p \big/ \partial z\right)\sin\left(n\omega_p t\right) dt\,. \qquad \ldots(9)$$

$\omega_p = 2\pi f_p$, f_p is the frequency of the pressure gradient.

METHOD OF SOLUTION

Pressure gradient profiles are available in graphical form. Time scale is presented on the X-axis, which ranges from 0-T seconds and $\left(-\partial p \big/ \partial z\right)$ is presented on the Y-axis. The X-axis is divided into number of equal intervals and values of $\left(-\partial p \big/ \partial z\right)$ at these points are measured manually.

The values of Fourier coefficients A_0, $\mathrm{A_n}$ and $\mathrm{B_n}$ have been obtained by evaluating the integrals (7) to (9) numerically by using Simpson's 1/3 rule which is given by

$$\int_0^T f(t)\,dt = \frac{h}{3}\big[f(t_0) + f(t_{2m}) + 2\{f(t_2) + f(t_4) + \ldots\ldots\ldots + f(t_{2m-2})\} + 4\{f(t_1) + f(t_3) + \ldots\ldots\ldots + f(t_{2m-1})\}\big] \qquad \ldots(10)$$

where the function f (t) is defined in the interval [0, T]. The interval [0, T] is subdivided into an even number of equal subintervals; say 2m, of length $h = \frac{T}{2m}$ with end points $t_0 (= 0),\ t_1,\ t_2, \ldots\ldots\ldots,\ t_{2m-2},\ t_{2m-1},\ t_{2m} (= T)$.

EXPERIMENTAL PRESSURE GRADIENT PROFILES OF A HUMAN

List of available experimental pressure gradient profiles of human are as following:

1. Pressure gradient profile for main pulmonary artery of a human [Milnor (1989)],
2. Pressure gradient profile for thoracic aorta of a human [Milnor (1989)],
3. Pressure gradient profile for thoracic aorta of a subject 1 [Toravi (2000)]
4. Pressure gradient profile for thoracic aorta of a subject 2 [Toravi (2000)]

These pressure gradient profiles are converted in mathematical forms and presented through Figures 2 to 5. The values of the coefficients A_0, $\mathrm{A_n}$ and $\mathrm{B_n}$ of Fourier series for blood pressure gradient profiles in a human CVS are presented through Tables- 1 to 4.

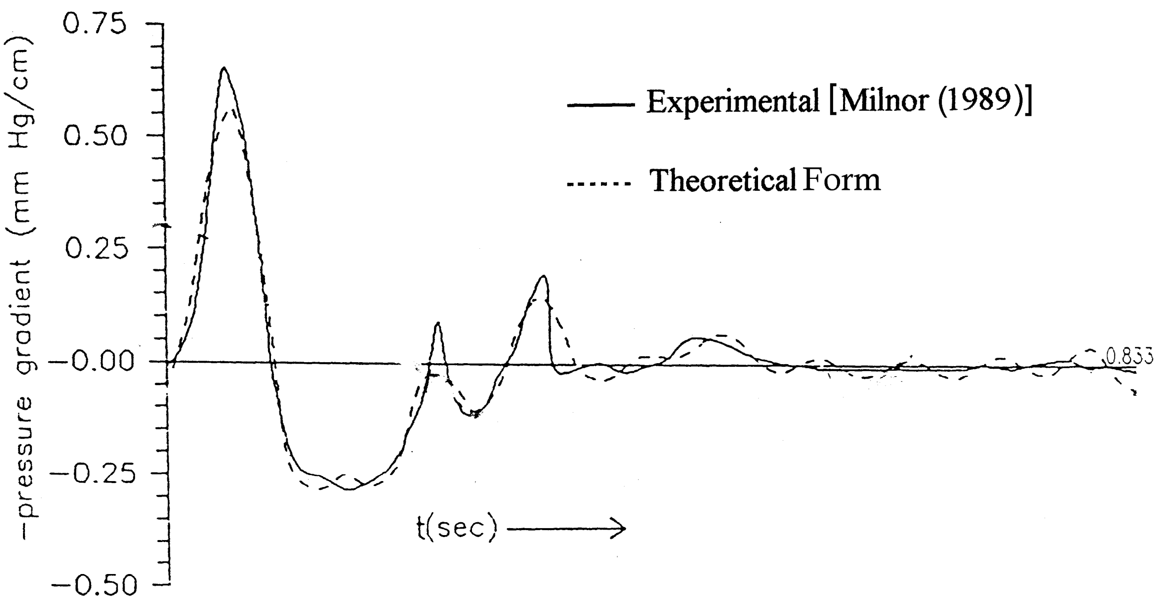

Fig.2. Pressure distribution versus time for main pulmonary artery R=1.35 cm [Milnor (1989)].

Table-1. The values of coefficients A_n and B_n ($\mathrm{kg\,m^{-2}s^{-2}}$) of Fourier series for the pressure gradient profile from direct data, for main pulmonary artery (R=1.35 cm) of a human [Milnor (1989)].

A_0	n	A_n	B_n
4.37	1	-283.11	492.89
	2	1326.80	44.80
	3	-856.82	-1049.19
	4	-201.67	1258.58
	5	445.49	-695.79
	6	-894.25	530.77
	7	621.14	195.01
	8	-392.64	9.33
	9	588.12	256.91
	10	-110.92	-629.61

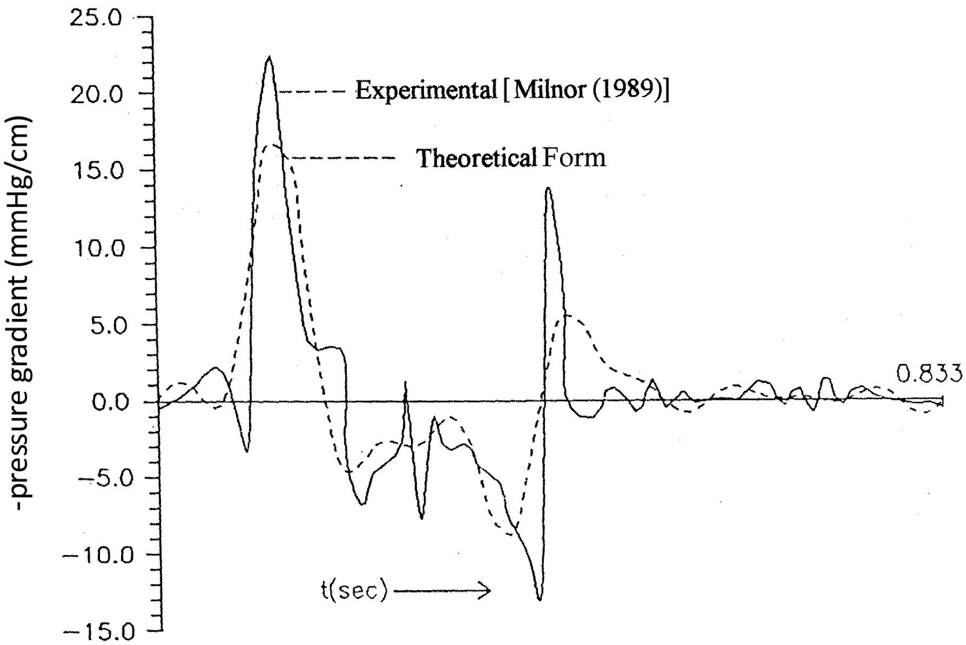

Fig. 3. Pressure gradient versus time for thoracic aorta R= 1.17 cm [Milnor(1989)].

Table-2. The values of coefficients A_n and B_n ($kg\,m^{-2}s^{-2}$) of Fourier series for the pressure gradient profile from direct data, for thoracic aorta (R=1.17 cm) of a human [Milnor (1989)].

A_0	n	A_n	B_n
863.95	1	-2560.78	-491.89
	2	-242.53	4963.95
	3	3538.34	384.78
	4	-1917.56	-88.66
	5	192.54	4314.25
	6	2623.37	520.70
	7	-593.51	-553.89
	8	838.30	1922.26
	9	1611.84	-443.50

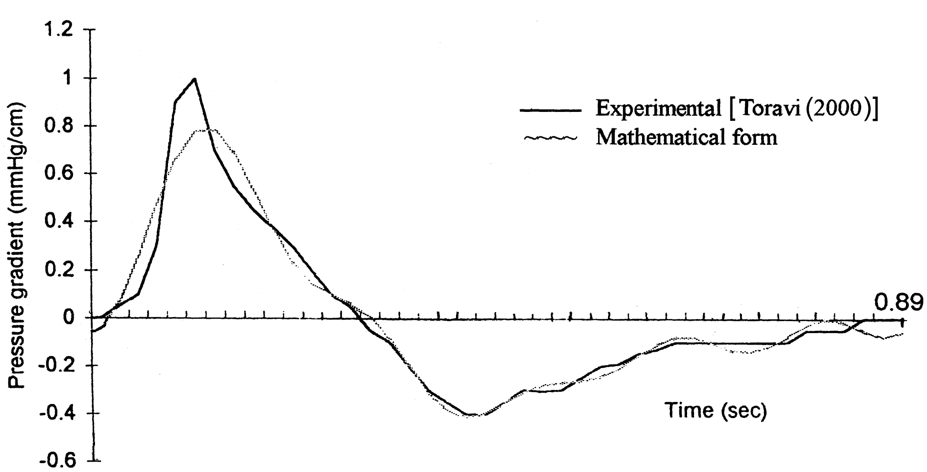

Fig. 4. Pressure gradient versus time for thoracic aorta when ω=1.12 Hz of healthy subject 1 [Toravi (2000)].

Table-3. The values of coefficients A_n and B_n $(kg\,m^{-2}s^{-2})$ when ω = 1.12 Hz in human CVS at thoracic aorta.

A_0	n	A_n	B_n
74.12	1	3777.86	2980.09
	2	-1587.12	2549.98
	3	-941.65	496.88
	4	-1093.21	58.24
	5	-816.37	-679.62

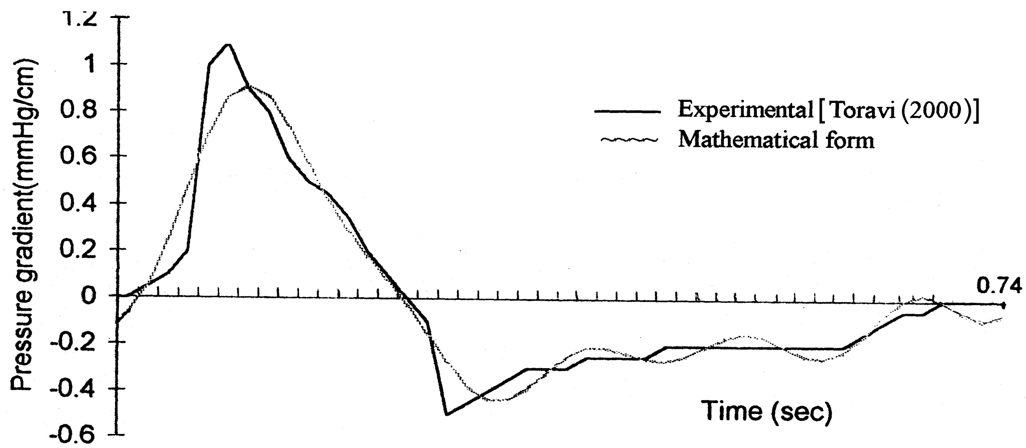

Fig. 5. Pressure gradient distribution versus time for thoracic aorta when ω=1.35 Hz of healthy subject 2 [Toravi(2000)].

Table-4. The values of coefficients A_n and B_n $(kg\,m^{-2}s^{-2})$ when ω = 1.35 Hz. in human CVS at thoracic aorta.

A_0	n	A_n	B_n
101.07	1	4052.95	3641.44
	2	-1304.54	3335.64
	3	-1716.40	-137.55
	4	-852.66	-239.88
	5	-1126.56	-520.86

RESULTS AND DISCUSSION

It is observed that Pressure gradient profiles are used as input data in mathematical models of pulsatile flow of blood. The input data used in literature for the mathematical models for the pulsatile flow of blood is either too ideal [Womersley (1955), Verma and Sharma (1983) etc.] or incorrect [Sud and Sekhon (1985), Sharma, Ariel and Chaturani (1995), Sharma and Mishra (2002) etc.]. Obviously the obtained results from such models will not be relevant and useful to the physical situations, which they correspond to.

Many mathematical models have been discussed in this paper. It is noticed that this form is incorrect due to confusion in blood pressure and blood pressure gradient. Difference between experimental profile [Pedley (1980)] and mathematical forms [Sud and Sekhon (1985) and Abdalla (1998)] is shown in Figure 1.

Pressure gradient profiles are considered in Fourier series form. Four experimental pressure gradient profiles of a human given by Milnor (1989) and Toravi (2000) are converted into mathematical form. Pressure gradient profiles [Milnor (1989)] for main pulmonary artery and thoracic aorta of a human are presented in mathematical form and shown through Figures 2 and 3 respectively. Values of Fourier coefficients are calculated and shown through Tables-1 and 2 respectively. Pressure gradient profiles [Toravi (2000)] in thoracic aorta of two subjects are converted in mathematical form and shown through Figures 4 and 5. Corresponding values of Fourier coefficients are given by Tables-3 and 4.

CONCLUSIONS

Looking at the importance of the input data for mathematical models for pulsatile flow of blood, mathematical forms of four pressure gradient profiles for humans have been obtained with a reasonable accuracy (error about 10%). Thus we have provided a better input data for mathematical models for pulsatile flow of blood.

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FACTORS RELATED TO STUDENTS PERFORMANCE IN STATISTICS COURSE : A CASE STUDY

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ABSTRACT :

The purpose of the present study is to investigate the structural relationships among mathematics achievement, Attitude Toward Statistics and statistics outcomes by testing a structural model. The Survey of Attitude Toward statistics (SATS)by Candace Schau is used in this study.

INTRODUCTION :

Regression analysis is a set of statistical processes for estimating the relationships among variables. It includes many techniques for modeling and analyzing several variables, when the focus is on the relationship between a dependent variable and one or more independent variables (or 'predictors'). More specifically, regression analysis helps one understand how the typical value of the dependent variable (or 'criterion variable') changes when any one of the independent variables is varied, while the other independent variables are held fixed.

Most commonly, regression analysis estimates the conditional expectation of the dependent variable given the independent variables – that is, the average value of the dependent variable when the independent variables are fixed. Less commonly, the focus is on a quantile, or other location parameter of the conditional distribution of the dependent variable given the independent variables. In all cases, a function of the independent variables called the **regression function** is to be estimated. In regression analysis, it is also of interest to characterize the variation of the dependent variable around the prediction of the regression function using a probability distribution. A related but distinct approach is necessary condition analysis (NCA), which estimates the maximum (rather than average) value of the dependent variable for a given value of the independent variable in order to identify what value of the independent variable is necessary but not sufficient for a given value of the dependent variable.

Regression analysis is widely used for prediction and forecasting, where its use has substantial overlap with the field of machine learning. Regression analysis is also used to understand which among the independent variables are related to the dependent variable, and to explore the forms of these relationships. Many techniques for carrying out regression analysis have been developed. Familiar methods such as linear regression and ordinary least squares regression are parametric, in that the regression function is defined in terms of a finite number of unknown parameters that are estimated from the data. Nonparametric regression refers to techniques that allow the regression function to lie in a specified set of functions, which may be infinite-dimensional.

The performance of regression analysis methods in practice depends on the form of the data generating process, and how it relates to the regression approach being used. Since the true form of the data-generating process is generally not known, regression analysis often depends to some extent on making assumptions about this process. These assumptions are sometimes testable if a sufficient quantity of data is available. Regression models for prediction are often useful even when the assumptions are moderately violated, although they may not perform optimally. However, in many applications, especially with small effects or questions of causality based on observational data, regression methods can give misleading results.

The earliest form of regression was the method of least square which was published by Legendre in 1805, and by Gauss in 1809. Legendre and Gauss both applied the method to the problem of determining, from astronomical observations, the orbits of bodies about the Sun (mostly comets, but also later the then newly discovered minor planets). Gauss published a further development of the theory of least squares in 1821, including a version of the Gauss–Markov theorem.

The term "regression" was coined by Francis Galton in the nineteenth century to describe a biological phenomenon. The phenomenon was that the heights of descendants of tall ancestors tend to regress down towards a normal average (a phenomenon also known as regression toward the mean). For Galton, regression had only this biological meaning, but his work was later extended by Udny Yule and Karl Pearson to a more general statistical context. In the work of Yule and Pearson, the joint distribution of the response and explanatory variables is assumed to be Gaussian. This assumption was weakened by R.A. Fisher in his works of 1922 and 1925. Fisher assumed that the conditional distribution of the response variable is Gaussian, but the joint distribution need not be. In this respect, Fisher's assumption is closer to Gauss's formulation of 1821.

In the 1950s and 1960s, economists used electromechanical desk calculators to calculate regressions. Before 1970, it sometimes took up to 24 hours to receive the result from one regression.

Regression methods continue to be an area of active research. In recent decades, new methods have been developed for robust regression, regression involving correlated responses such as time series and growth curves, regression in which the predictor (independent variable) or response variables are curves, images, graphs, or other complex data objects, regression methods accommodating various types of missing data, nonparametric regression, Bayesian methods for regression, regression in which the predictor variables are measured with error, regression with more predictor variables than observations, and causal inference with regression.

Multiple Linear Regression

Multiple regression is an extension of simple linear regression. It is used when we want to predict the value of a variable based on the value of two or more other variables. The variable we want to predict is called the dependent variable (or sometimes, the outcome, target or criterion variable). The variables we are using to predict the value of the dependent variable are called the independent variables (or sometimes, the predictor, explanatory or regressor variables).

The multiple regression model for a response variable, y with observed values $y_1, y_2 \ldots y_n$ (n is sample size) and q explanatory variables $x_1, x_2, \ldots x_q$ with the observed values, $x_{1i}\ x_{2i}, \ldots x_{qi}$ for i=1,2,3,. . .n is

$$Y_i = \beta_0 + \beta_{1x_{1i}} + \beta_2 x_{2i} + \beta_3 x_{3i} + \ldots + \beta_q x_{qi} + €i$$

The term €i is the residual or error for individual i and represents the deviation of the observed value of the response for this individual from that expected by the model. The error terms are assumed to have a normal distribution with variance---. The regression co-efficients β_1,β_2,. . . β_q are generally estimated by least squares. A measure of the fit of the model is provided by the multiple correlation co-efficient R defined as the correlation between the observed values of the response variable and the values predicted by the model. The value of R^2 gives the proportion of the variability of the response variable accounted for by the explanatory variables.

OBJECTIVE

The purpose of the study is to answer the following question- Do students' attitude toward statistics and mathematics have a relationship with students' final scores in statistics course?

Our null hypothesis is-

Ho: There is no significant relationship between students' scores on 1) the maths scores obtained on previous course 2) students' attitude toward statistics and 3) student scores on the math quiz.

Here Dependent variable is- Student' final score.

Independent variable are- 1) Math scores obtained on previous course 2) scores on math quiz 3)scores on the survey of attitude toward statistics.

Definition

Student performance and students achievement in statistics course means the final scores in the statistics course at the end of the semester.

Attitude toward statistics is defined as the attitude of students toward statistics materials taught in the statistics courses and measured by the instruments SATS-36(Schau).

Survey of Attitude Toward Statistics (SATS) :The SATS is one of the instruments to measure the students' attitude toward Statistics.SATS was created by Candace Schau in 1992. The SATS has two major uses. The first use is to help instructors evaluate their student's attitudes toward the statistician their statistics courses. The 2nd is for educational research into students' statistics attitude.

The original version of the SATS (called SAT – 28) contains 28 items that use a Likent-type scale to assess four components of student's attitudes toward statistics. The four components are : affect, cognitive competence, value and difficulty. In 1995, the SATS – 28 was revised to include two more attitude component – interest and effort. This version of SATS is called SATS – 36. SATS – 36 is a seven point (from strongly disagree to strongly agree) Likert-type questionnaire. SATS – 36 contains 36 items that assess six component. The six components are 1) Affect (six items) : Positive and negative feelings concerning statistics. Example : "I am scared by statistics".

2) Cognitive Competence (six items) : Attitudes about intellectual knowledge and skills when applied to statistics. Example : "I can learn statistics".

3) Value (nine item) : Attitudes about the usefulness, relevance and worth of statistics in personal and professional life. Example : "I use statistics in my everyday life".

4) Difficulty (seven items) : Attitudes about the difficulty of statistics as a subject. Example : "Most people have to learn a new way at thinking to do statistics".

5) Interest (four items) : Student's level of individual interest in statistics. Example :I am interested in learning statistics.

6) Effort (four items) : The amount of effort students spend on learning statistics. Example :I plan to attend every statistics class session.

Mills, Gal and Ginsburg also studied the attitudes of students towards the statistics subjects enrolled in an introductory statistics course as well as additional research regarding important variables related to student attitudes by using the instrument SATS.

Mathematics achievement means score on mathematics of previous course. Mathematics Subject is not compulsory for commerce, Economics and Education students. But mathematical knowledge or basic mathematical concepts is essential for Statistics. So we tested students' math skill by checking their scores obtained on previous mathematics exam.

Math quiz provide information about the students' mathematical knowledge on basic material being prepared. The quiz is designed to provide information on students mathematical knowledge without preparation and their knowledge of basic material. The particular math concept covered by the quiz are similar to those reviewed in math textbook of class-x and class-xii. In math quiz, 15 multiple-choice mathematics questions were examined by the students. The math quiz as well as the percentage of the survey respondents who answered each question incorrectly are presented in following table.

Table for percentage of incorrect answers.

Question	Percentage of students
Q1	12.6%
Q2	12.9%
Q3	10.2%
Q4	11.7%
Q5	18.4%
Q6	14.7%
Q7	33.5%
Q8	20.1%
Q9	19.6%
Q10	13.5%
Q11	13.85
Q12	31.3%
Q13	19.4%
Q14	14.1%
Q15	12.1%

While these mathematical skills may not be used directly in statistical calculations, the above result have seen that some students would likely have difficulty in not only performing statistical calculations but also understanding or interpreting statistical calculations. For example, a student who does not understand percentage or division may have difficulty in understanding means or standard deviations, and cannot interpret descriptive statistics. Also a student who cannot find the slope of a line will likely be unable to correctly interpret the slope in a linear regression.

METHODOLOGY

This study examined a number of factors that may affect students' achievement in statistics course. One factor is students' attitude toward statistics and other factor is Mathematical skills. This study was carried out on the undergraduate non-majoring statistics students of three subject viz Commerce, Economics and Education enrolled in Dibrugarh University. A sample of 345 under-graduate students of these three subjects participated in the study. This sample included 135 Commerce students, 121 Economics students and 89 Education students. To determine which factors are important for student success, we examined 1. The students' attitude toward statistics by using the most recent Survey of Attitude Toward Statistics(SATS)designed by Schau in 1995 and updated by her in 2003, and viewing the effect of each subscale on the final grades of the students 2. Students' math score on previous exam 3. The students' scores on a test of very basic mathematical concept.

ANALYSIS

The purpose of this research was to investigate the relationship between several factors and students' achievement in statistics courses. This relationship was tested using a multiple regression model to determine which factor improved students' achievement in statistics courses.

Table-1 provides regression statistics

Multiple R	.763007
R square	.58218
Adjusted R	.5722
Standard Error	5.115
Observations	345

From the above table, we see that the adjusted R square value is .5722. The adjusted R square gives more accurate information about the fitness of the model. In our study, the adjusted R square value .5722 tells us that the 8 factors account for 57.22% variance in the achievement on statistics.ie the independent variables can predict 57% of the variance in the dependent variable (statistics outcome). Therefore we may say that our regression model is good.

Table-2 Estimated Coefficients and significance of statistics outcomes

variable	Co-efficient (β)	p-value
Intercept	49.5520	9.77E-24
Math achievement	.27357	.3489
Effort	1.504	0.0245
Affect	.0374	.9588
Cognitive Competence	-1.008	0.05
Difficulty	-1.7426	0.0252
Value	-0.551	.26343
Interest	-1.3294	0.058
Math quiz	.1532	.02

Using the regression co-efficient for independent variables and the constant term given under the column labeled β, we can construct Least Square Equation for predicting the statistics outcomes ie regression model for variable on independent variables as-

Statistics outcomes= Constant+$\beta_1$math achievement+$\beta_2$effort+$\beta_3$affect+$\beta_4$cognitive competence+$\beta_5$difficulty+$\beta_6$value+$\beta_7$interest+$\beta_8$math quiz

Y=constant+$\beta_1 x_1$+$\beta_2 x_2$+$\beta_3 x_3$+$\beta_4 x_4$+$\beta_5 x_5$+$\beta_6 x_6$+$\beta_7 x_7$+$\beta_8 x_8$

Y= 49.5520+.273573x1+1.504x2+(-9.629)x3+(-1.008)x4+(-1.743)x5+(-0.551)x6+(-1.329)x7+.8508x8

Again our null hypothesis is- There is no significant relationship between statistics outcome on 1) math achievement 2) attitude toward statistics and3) scores on math quiz.

The p-value for effort, cognitive competence, difficulty, interest and mathematics quiz are .024, .05, .025, .05 and .02. These values are significant at 5% significance level and the values of the other factors are not significant. So we can claim that statistics outcomes is positively related to the factors effort, cognitive competence, difficulty, interest and mathematics quiz, but not to the other factors. Therefore we may say that students' basic mathematics skill, their effort to understand statistics, their attitude about intellectual knowledge and interest are related to their performance in the statistics course.

CONCLUSION

The purpose of our study was to determine the factors related to students' achievements in statistics courses. Schau's model was adopted to investigate the relationship between the independent variables and dependent variable. Furthermore, to predict higher achievement in students' performances, not only were students' attitude toward statistics measured, but also their beliefs evaluations in statistics were measured. Therefore, in this study we included other factors such as mathematics skills represented by their score on a quiz of simple mathematics and score on previous exam of mathematics. From the result we found that there is significant relationship between students' score on mathematics quiz and statistics achievement. We also found significant relationship between students' attitude toward statistics and statistics achievement. Thus the more the students know in statistics, the more positive their attitude will be toward statistics and consequently they will do better in the final exam.

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PROFIT ANALYSIS OF A SYSTEM WORKING IN HIGH TEMPERATURE ZONES WITH EFFECT OF DAY- NIGHT ON REPAIR

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ABSTRACT :
In the present paper, a reliability model is developed for a system which works in high- temperature zones like north India during summer. The system performance is affected by the temperature and the repair is affected by the time of failure. The system may get failed during day or night hours. If it fails during night hours, the repair may not be possible during that time and the system has to wait till day time when the repair facilities/ components are available. The working of the system depends upon the temperature also as the system may not work when temperature reaches beyond certain limit. The present study is based on the fabric manufacturing system which does not remain functional beyond 220c and in that case the system is put to down mode as otherwise the fabric thickness may get destroyed. Various measures, of system effectiveness are obtained by using regenerating point technique. Using these measures profit of the system is evaluated. The conclusions concerning the reliability and profit of the system are given on the basis of graphical study.
Key Words: Fabric Manufacturing System, High Temperature Zones, Regenerating Point Technique, Failure during Day and Night hours, Profit Analysis.

1. INTRODUCTION

Reliability modeling contains lot of studies on the reliability and cost-benefit analysis of various systems. Most of the researchers concentrated their interest on system reliability and analyzed various single or two-unit cold standby under different situations considering various concepts of failure and repair. These studies contributed by various researchers including Welke et al. (1995), Rizwan and Taneja (2000), Gupta et al. (2003), Tuteja et al. (2006), Kumar and Bhatia (2011) and Padmavathi et al. (2012) wherein the concepts/situations of operating and rest periods, centrifuge systems, instructions, ash water pump systems, desalination plant with online repair and emergency shutdown and different types of recovery taken up. Also, Singh and Taneja (2013) analyzed reliability analysis of a power generating system through gas and steam turbines with scheduled inspection, Rizwan et al. (2015), Sachdeva and Taneja (2016), Srinivasu and Sridharan (2017) and Adlakha and Taneja (2017) worked on different industrial systems under different assumptions.

But in none of these studies, the effect of high temperature on the working of the system has been taken into consideration. However, there are situations where such effects can be observed. Observing such a situation in a fabric manufacturing company, Sheetal et al. (2018) analyzed reliability and profit of a system with effect of temperature on operation. Later on, it was also observed by the authors that when such systems get failed in night hours, they have to wait for till day time due to non-availability of repair facilities/ components during night hours and hence the present paper.

The present paper deals with a reliability model for a system which works in high- temperature zones like north India during summer. As high temperature zones affects adversely the working of the system and/or the quality of the product the system is required to be put to down mode till the temperature is maintained to the acceptable limit. The present study is based on the fabric manufacturing system which does not remain functional beyond 22^0c and in that case the system is put to down mode as otherwise the fabric thickness may get destroyed. The system is made operative from down mode as soon as possible by using air conditioner. The system may get failed during day or night hours. If it fails during night hours, the repair is not possible during that time and the system has to wait till day time when the repair facilities/ components are available easily.

The system is analyzed by making use of regenerative point technique. Various measures of system effectiveness such as mean time to system failure (MTSF), availability, busy period analysis, expected down time and expected number of visits of the repairman are derived. The profit incurred to the system is also evaluated and graphical study is done. On the basis of the information gathered from a fabric manufacturing plant, the estimates of rates, costs and probabilities are obtained using method of maximum likelihood, interesting numerical results have been obtained with regard to MTSF, availability and profitability of the system.

Other Assumptions for the Model are:

Initial state is considered as the state of working in the high temperature zones.

All the random variables follow arbitrary distributions.

After every repair, the system becomes like a new one.

2. NOMENCLATURE

h_1 (t), H_1 (t)	h_1 (t), H_1 (t) p.d.f. and c.d.f. of time for increasing the temperature beyond certain limit
f (t), F(t)	p.d.f. and c.d.f. of failure time
g (t), G(t)	p.d.f. and c.d.f. of repair time
h_2 (t), H_2(t)	p.d.f and c.d.f. of time for maintaining the temperature to acceptable range
w(t), W(t)	p.d.f. and c.d.f . of waiting time till the morning hours when market gets opened.
p_1	probability that system fails during day hours.
q_1	probability that system fails during night hours.
(D)	Down state
(Op)	Operative state
(F_{wr})	Failed state waiting for repair
(F_{ur})	Failed state under repair
A_0	Availability
DT_0	Expected down time
B_0	Busy period analysis
V_0	Expected Number of visits by the repairman
TM_0	Expected Number of the temperature is maintained whenever it reaches beyond the acceptable limit
©	Symbols for Laplace Convolution

Ⓢ Symbols for Stieltjes Convolution

* Symbols for Laplace Transforms

** Symbols for Laplace Stieltjes Transforms

3. ANALYSIS OF MODEL

3.1 Transition Probabilities and Mean Sojourn Times

The transition diagram showing various states of transition of system are shown in **Fig. 1.** The epochs of entry into the states 0, 1, 2, and 3 are regenerative states. The possible transition probabilities are given below:

$q_{01}(t)=h_1(t)\overline{F(t)}$ $\qquad$ $q_{10}(t)=h_2(t)$

$q_{02}(t)=q_1 f(t)\overline{H_1(t)}$ $\qquad$ $q_{23}(t)=w(t)$

$q_{03}(t)=p_1 f(t)\overline{H_1(t)}$ $\qquad$ $q_{30}(t)=g(t)$

The non–zero elements p_{ij} can be obtained by $p_{ij}=\lim\limits_{s\to 0} q_{ij}^*(s)$ and it can be easily verified that

$$p_{01}+p_{02}+p_{03}=1\ ,\ p_{10}=p_{23}=p_{30}=1$$

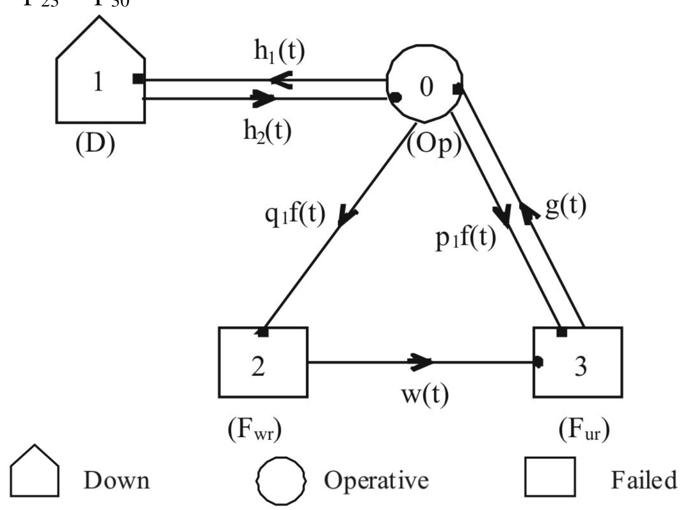

Fig.1: State Transition Diagram

Mean sojourn times (μ_i) in the regenerative state 'i' is defined as the same time of stay in that state before transition to any other state. If T denote the sojourn in the regenerative state 'i', then

$$\mu_i = E(T) = p_r(T>t)$$

Thus, $\mu_0=\int_0^\infty \overline{H_1(t)}\,\overline{F(t)}\,dt$, $\qquad \mu_1=\int_0^\infty \overline{H_2(t)}=\int_0^\infty t\,h_2(t)dt=-h_2^{*\prime}(0)$

$$\mu_2=\int_0^\infty \overline{W(t)}\,dt=-w^{*\prime}(0)\ ,\quad \mu_3=\int_0^\infty \overline{G(t)}\,dt=-g^{*\prime}(0)$$

The unconditional mean time taken by the system to transit to any regenerative state j when time is counted from the epoch of entrance into state 'i' is mathematically stated as:

$$m_{ij}=\int_0^\infty t\,q_{ij}(t)\,dt$$

$$m_{01}+m_{02}+m_{03}=\int_0^\infty t(h_1(t)\overline{F(t)}+q_1 f(t)\overline{H_1(t)}+p_1 f(t)\overline{H_1(t)})dt=K_0(\text{says})$$

$$m_{10}=\int_0^\infty th_2(t)\,dt=\mu_1\ ,\ m_{23}=\int_0^\infty t\,w(t)\,dt=\mu_2\ ,\ m_{30}=\int_0^\infty t\,g(t)\,dt=\mu_3$$

3.2 Reliability and Mean Time to System Failure (MTSF)

Let $\phi_i(t)$ be the c.d.f. of first passage of time from the regenerative state i to a failed state. Regarding the failed states as absorbing states, we have the following recursive relations for $\phi_i(t)$:

$$\phi_0(t) = Q_{01}(t) \circledS \phi_1(t)+ Q_{02}(t) + Q_{03}(t)$$

$$\phi_1(t) = Q_{10}(t) \circledS \phi_0(t)$$

Taking Laplace–Stieltjes Transforms (L.S.T.) of these relations and solving them by Cramer's rule for $\phi_0^{**}(s)$, we obtain

$$\phi_0^{**}(s)=\frac{N(s)}{D(s)}$$

where, $N(s)=Q_{02}^{**}(s)+Q_{03}^{**}(s)$ and $D(s)=1-Q_{01}^{**}(s)Q_{10}^{**}(s)$

Now, the mean–time to system failure (MTSF) when the system starts from the state '0' is

$$\text{MTSF}=\lim_{s\to 0}\frac{1-\phi_0^{**}(s)}{s}=\lim_{s\to 0}\frac{1-\frac{N(s)}{D(s)}}{S(D(s))}=\frac{N}{D}$$

where, $N = \mu_0 + p_{01}\mu_1$, $D = 1 - p_{01}\, p_{10}$

3.3 Availability Analysis

$$A_0(t) = M_0(t) + q_{01}(t) \copyright A_1(t) + q_{02}(t) \copyright A_2(t) + q_{03}(t) \copyright A_3(t)$$

$$A_1(t) = q_{10}(t) \copyright A_1(t)$$

$$A_2(t) = q_{23}(t) \copyright A_3(t)$$

$$A_3(t) = q_{30}(t) \copyright A_0(t)$$

where $M_0(t)=\int_0^\infty \overline{H_1(t)}\,\overline{F(t)}\,dt$

Taking Laplace transforms (L.T.) of these relations and solving them for $A_0^*(s)$, we obtain

$$A_0^*(s)=\frac{N_1(s)}{D_1(s)}$$

where, $N_1(s)=M_0^*(s)$, $D_1(s) = 1-q_{01}q_{10}-q_{02}q_{23}q_{30}-q_{03}q_{30}$

The Steady state availability of the system is given by

$$A_0=\lim_{s\to 0}(sA_0^*(s))=\lim_{s\to 0}\left(s\frac{N_1(s)}{D_1(s)}\right)=\frac{N_1(0)}{D_1'(0)}=\frac{N_1}{D_1}$$

where, $N_1 = M_0$, $D_1 = \mu_0 + p_{01}\,\mu_1 + p_{02}\,(\mu_2 + \mu_3)$

3.4 Other Measures of System Effectiveness

Using probabilistic arguments, the recursive relations for various measures of the system effectiveness of the system are obtained in the similar fashion as done in the preceding sections and there results have been shown below and derivations have been skipped to avoid repetition of similar derivation.

Expected Busy period (B_0) $=N_2/D_1$

Expected down time (DT_0) $= N_3/D_1$

Expected number of visits by the repairman (V_0) $=N_4/D_1$

Expected number of time the temperature is maintained whenever it reaches beyond the acceptable limit (TM_0) $=N_5/D_1$

where, $N_2 = W_1\, p_{01}$, $N_3 = W_2\, p_{02} + W_3$, $N_4 = p_{02} + p_{03}$ and $N_5 = p_{01}$

4. PROFIT ANALYSIS

Using the measures obtained as above, the expected profit per unit time incurred to the system, in steady state, is given by

$$\text{Profit}\,(P_0) = C_0A_0 - C_1(DT_0) - C_2B_0 - C_3V_0 - C_4(TM_0)$$

where

- C_0 Revenue per unit up time
- C_1 Goodwill loss per unit up time during which the system remains in down
- C_2 Cost per unit time during which the repairman is engaged.
- C_3 Cost per visit of the repair man
- C_4 Cost per Maintenance

5. NUMERICAL RESULTS AND INTERPRETATION:

Following particular case is taken up to find various numerical results:

$f(t) = \lambda e^{-\lambda t}$, $g(t) = \alpha e^{-\alpha t}$, $h_1(t) = \alpha_1 e^{-\alpha_1 t}$, $h_2(t) = \alpha_2 e^{-\alpha_2 t}$, $w(t) = \beta e^{-\beta t}$

Using the values estimated from the data collected i.e. $\lambda=0.04167$, $\alpha=1$, $\alpha_1= 0.0455$, $\alpha_2=0.75$, $\beta=0.93$, $C_0=5000$, $C_1=2500$, $C_2=2000$, $C_3=2000$ and $C_4=1000$; values of various measures of system effectiveness are obtained as:

- Mean Time to System Failure (MTSF) = 25.46966 hour
- Availability (A_0) =0.919729
- Expected Busy Period (B_0) per hour =0. 092996
- Expected Fraction of Down time (DT_0) =0.10751
- Expected number of Visits of repairman (V_0) per hour = 0. 038325
- Expected number of times per hour the temperature is maintained whenever it reaches beyond the acceptable limit (TM_0)=0.042308
- Profit incurred per hour to the system(P)=2756.454

Nature of MTSF, Reliability and Availability with regard to failure rate, repair rate, etc is shown in **Figs 2 to 4** which reveal that:

i) MTSF and Availability both get decreased with increase in the values of failure rate (λ). However, they have higher values for higher values of repair rate(α).

ii) Reliability decreases with increase in the value of time (t) as well as the failure rate (λ).

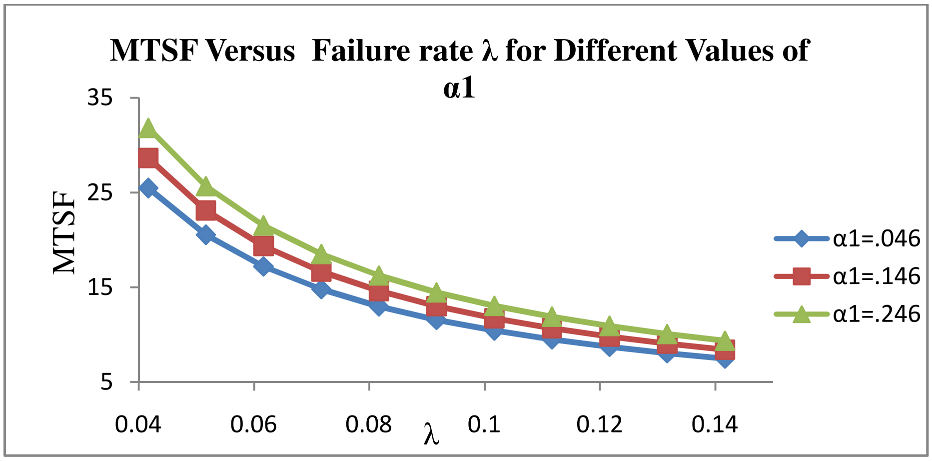

Fig. 2

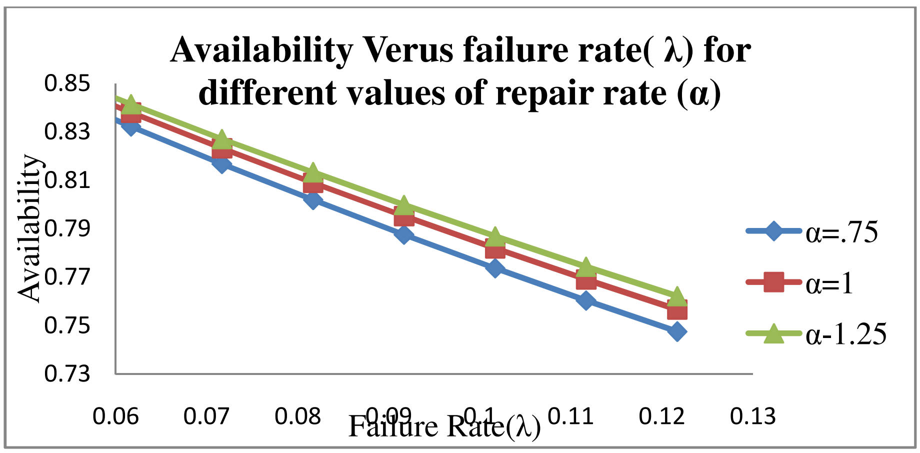

Fig. 3

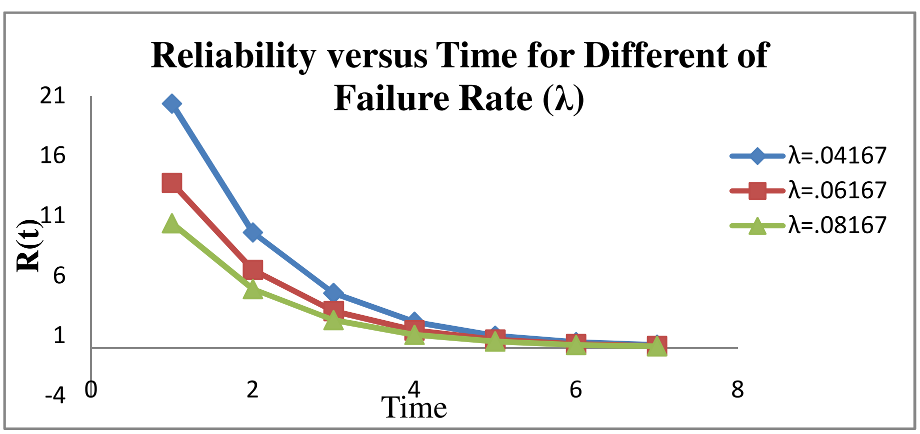

Fig. 4

The profitability aspect has been studied graphically with respect to various parameters and using the expressions for various measures of system effectiveness as shown in **Fig. 5 to 6.**

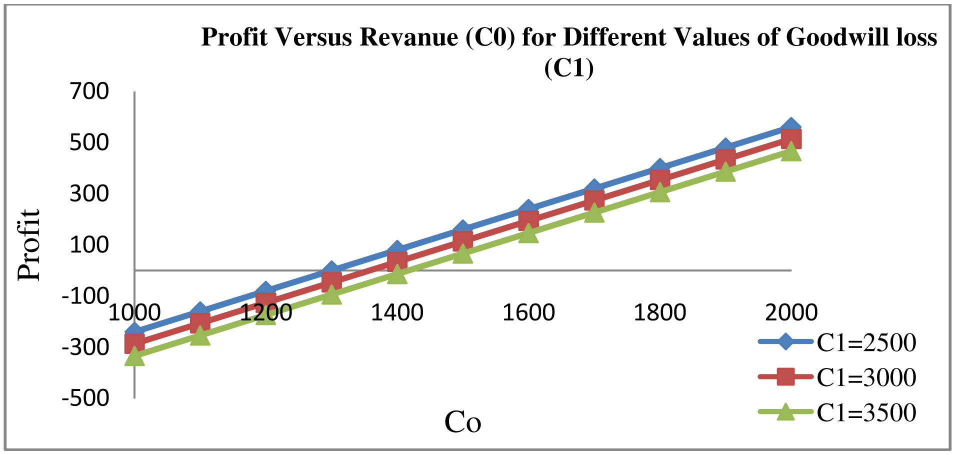

Fig. 5

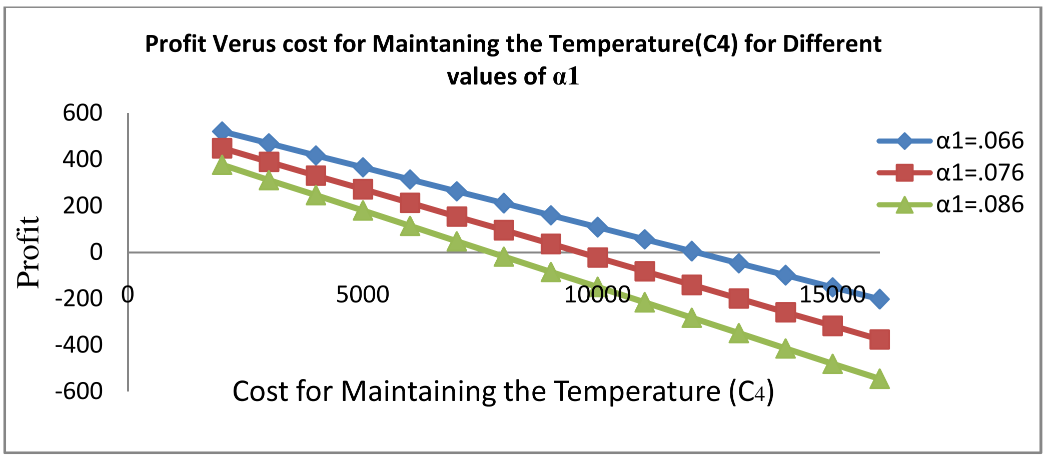

Fig. 6

The interpretations drawn from the above graphs are tabulated as follows:

Fig.	Assumed value of the Parameter for which the information by the plant was not provided	Profit		For	Profit ≥ 0 if
		Increses	Decreases		
5	C_{02}=5000, C_2=2000, C_3=1000, C_4=1000, λ= 0.04167, α=0.75	With increase in C_{01}	With increase in C_1	C_1=2500	$C_0 \geq 1298.901$
				C_1=3000	$C_0 \geq 1361.901$
				C_1=3500	$C_0 \geq 1421.901$
6	C_{01}=5000, C_2=1000, C_3=1000, C_4=1000, λ= 0.04167, α=0.75	With decrease in α_1	With increase in C_4	α_1=0.066	$C_4 \geq 736.386$
				α_1=0.076	$C_4 \geq 1078.492$
				α_1=0.086	$C_4 \geq 1121.901$

CONCLUSION

This paper develops a reliability model on a fabric manufacturing system and analysed the expressions for transition probabilities and system performance measures. The results obtained for a particular case highlight the importance of study as on the basis of cut-off points one can fix the lower/upper values for various parameters involved, e.g., the cut-off point for revenue per unit up time help in fixing the price of the product in such a way so that the system is profitable.

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PATH ANALYSIS FOR MTSF & AVAILABILITY OF A THREE UNIT STANDBY STOCHASTIC SYSTEM

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ABSTRACT :

To evaluate easily and quickly, the key parameters of a stochastic system for its profit/cost analysis, has always remained the need of the hour. With the increase in the number of the states in the state- space of a stochastic system, it becomes very cumbersome and more difficult to find the MTSF, availability and other key parameters of the stochastic system, using the various methods for reliability and availability analysis. Also while using the Regenerative Point Technique (RPT), many state equations ,has to be written and solved recursively after taking Laplace/Stieltjes transforms of these state equations and then applying the concepts of limits takes a lot of time. To overcome this difficulty, Gupta [3] introduced a new approach known as Regenerative Point Graphical Technique(RPGT), for finding quickly the key parameters of the system (under steady state conditions).In this paper, the path analysis of a three unit stand-by stochastic system (in which initially, two units are operative along with the third similar unit in cold stand-by mode and concept of two types of failures (minor/major) and two types of repairing facility using ordinary/expert servers) has been done to determine the MTSF and availability of the system using RPGT.

Keywords: Reachable state, Regenerative state, MTSF, Availability, Primary Circuit, Path, RPGT.

1. INTRODUCTION:

The researchers including Chander& Bansal [2], Kadyan et al [5], Shakeel &Vinod [7], Renu et al [8] and many others have used the Regenerative Point Technique and other methodologies [9], for doing the reliability and availability analysis of various stochastic systems. With the increase in the number of states in the state-space, to which a system can transit& increase in the number of transitions from any state to the others states, the Regenerative Point Technique and other methodologies, becomes very time consuming and cumbersome for doing the analysis for finding the key the parameters of the system. To overcome this difficulty, Gupta [3] introduced a new approach known as Regenerative Point Graphical Technique (RPGT). Gupta et al [4] used it for finding the key parameters of a system (under steady state conditions).

For all the simple paths from initial state to the other states and the various circuits along the different paths along with the level of circuits, which are located in totality, the Regenerative Point Graphical Technique (RPGT) proved more effective and efficient to find various key parameters. In this paper, Path analysis of the system, analyzed by Singh, D.V. et al [1], has been done to find the MTSF and availability of the system by using RPGT.

2. THE SYSTEM:

It is a three unit standby system in which initially, two units are operative along with the third similar unit in the cold stand-bymode. There can be two types of failures (minor/major) and two types of repairing facility using ordinary/expert servers. On the failure of a unit, the standby unit becomes operative instantaneously and the inspection is carried out of the failed unit, to detect the type of failure- whether it is minor or major. The unit with minor failure is repaired by the ordinary repairman while that with the major failure, is repaired only by the expert repairman.In case of further failure of another unit then the operating unit is stopped i.e. put in the down state. The system is in up-state/available only if two units are operative simultaneously.

3. ASSUMPTIONS, NOTATIONS & SYMBOLS:

The notation for pathsand algebra of paths Gupta [3] has been used. The various assumptions, notations and symbols used are given as under:

3.1 Assumptions:

i. The system starts from the good state '0' at time t = 0.

ii. All the three identical units and the operating units have the same failure rate.

iii. The inspection/repairs can start only if the server is available and the server cannot leave the system while repairing it.

iv. The expert repairman can do repairs of both types of failures.

v. The unit after repairs, works as a new one.

vi. Inspection of a failed unit finishes before the failure of any other unit.

vii. All random variables are independent and un-correlated.

viii. The distributions of the failure times are exponential and that of the inspection times and repair times may have general distributions which are different for minor/major repairs.

3.2 Notations:

pr/*	:	Probability/Laplace transformation.
$\underline{k}$	:	Non regenerative state '*k*'
$\{a_0,a_1,\ldots,a_{n-1},a_n\}$	:	A directed path from the state a_0 to a_n, through the states $a_1, \ldots, a_{n-1}$ to reach the state a_n.
$q_{i,j}(t)/q_{i,\underline{k},j}(t)$	:	Probability density function (p.d.f.) of the first passage time from a regenerative state *i* to a regenerative state *j* or to a failed state *j* without visiting any other regenerative state in (0, *t*]/ while visiting $\underline{k}$ only once in (0, *t*], given that the system entered regenerative state *i* at*t* = 0.
$(i, j)/\ p(i,j)$	:	Steady state transition probability from the regenerative state *i* to the regenerative state *j*, without visiting any other states. $(i,j) = p(i,j) = \lim_{s\to 0} q^{*}_{i,j}(s)$;$(i,j,k) = (i,j)\ (j,k) = p(i,j)\ .\ p(j,k)$
$(i,\underline{k},j)\ /p(i,\underline{k},j)$	:	Steady state transition probability from the regenerative state *i* to the regenerative state *j*, visiting non regenerative state $\underline{k}$. $(i,\underline{k},j) = p(i,\underline{k},j) = \lim_{s\to 0} q^{*}_{i,\underline{k},j}(s)$
$\overline{cycle}$		A circuit formed through un-failed states.
k-$\overline{cycle}$	:	A circuit formed through un-failed states, with terminals at the regenerative state *k*.
k-cycle	:	A circuit with terminals at the regenerative state *k*.
$V(\overline{k,k})$	:	Transition probability factor of the reachable state *k* of the k-$\overline{cycle}$ formed through un-failed states.
$V(k,k)$	:	Transition probability factor of the reachable state *k* of the *k-cycle*.
$V(i,j)$	:	Transition probability factor of the reachable state *j* from the*i*-state.
$(i \xrightarrow{sr} j)$	:	*r*-th directed simple path from *i*- state to *j*- state; *r* takes positive integral values for different paths from *i*- state to *j*- state.
$(i \xrightarrow{sff} j)$	:	a directed simple failure free path from *i*- state to *j*- state.

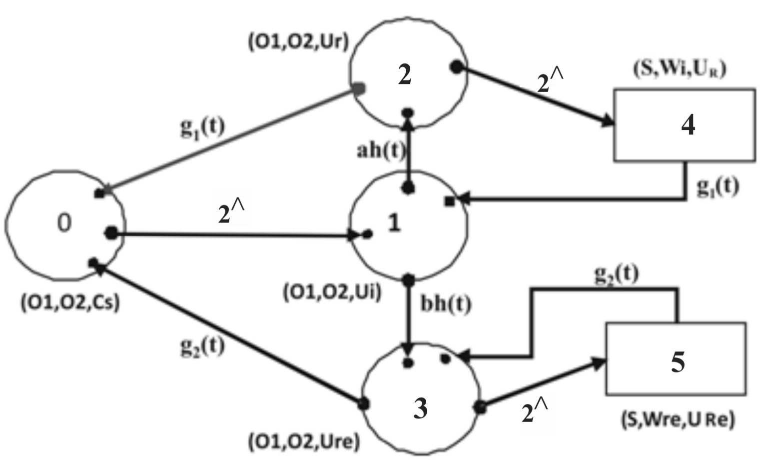
R_i (t)	:	Reliability of the system at time *t*, given the system is initially in the regenerative state '*i*'.
μ_i / μ_i'	:	Mean sojourn time of the state '*i*'/total un-conditional time spent before transiting to any other regenerative state(s), given that the system entered regenerative state '*i*' at $t = 0$.
f_i	:	Fuzziness measure of the*i*-state; $f_i = 0$,if '*i*' is a failed state; $f_i = 1$,if '*i*'is an up state.
CS	:	Unit is in cold standby mode.
O1/O2/S	:	Operating Units/ Unit is in the down state.
λ	:	Constant failure rate of a unit.
a/b	:	Probability of minor/major failure of a unit and a + b =1.
$g_1(t)/G_1(t)$	:	p.d.f. /c.d.f. of the repair timefor the ordinary repairman.
$g_2(t)/G_2(t)$	:	p.d.f. /c.d.f. of the repair timeforthe expert repairman.
h(t)/H(*t*)	:	p.d.f. /c.d.f. of the inspection time.
$\overline{G_1}(t)/\overline{G_2}(t)$ / $\overline{H}(t)$	:	$\overline{G_1}(t) = 1 - G_1(t)$ / $\overline{G_2(t)} = 1 - G_2(t)$ /$\overline{H}(t) = 1 - H(t)$
Ui/Ur/Wi	:	Failed unit is under inspection/ordinary repairs/ waiting for inspection.
Ure/Wre	:	Failed unit under repair/waiting repairs by an expert repairman.
UR/URe	:	Failed unit under repairs from previous state by ordinary repairman /expert repairman.

4. STATE TRANSITION DIAGRAM:

It is a directed graph in which the vertices (called nodes) are represented by the states belonging to the state-space of the stochastic system and the directed edges represent the various transitions from one state to the other reachable states. There are six states belonging to the *state-space* of the stochastic system, to which the system can transit under the given assumptions/conditions (Table-1). The epochs of entry into states 0, 1, 2 & 3 are the regenerative points and hence these states are the regenerative states. The states 4 & 5 are the failed states&non-regenerative states. Accordingly, the transition diagram of the system based upon the above assumptions/conditions is shown in Fig.1.

TABLE -1

TYPE OF STATE	SYMBOL	STATES
Regenerative State	•	0, 1, 2 & 3
Up-State	○	0, 1, 2 & 3
Failed State	□	4 &5



(Fig. 1)

5. PATH ANALYSIS OF THE SYSTEM:

There are 6 states (vertices) from0 to 5 and 9 transitions (edges)in the transition diagram (directed- graph) of the stochastic system.The terminal states of all the 9transitions (edges) are shown in Table -2 andthe various directed paths from each state (vertex) to the other reachable states are shown in Table - 3.

TABLE -2

Sr. No.	Terminal States of Transitions	
	From	To
1	0	1
2	1	2
3	1	3
4	2	0
5	2	4
6	3	0
7	3	5
8	4	1
9	5	3

TABLE- 3

Paths from State '*i*' to the Reachable State '*j*': P0						
i	*j* = 0	*j* =1	*j* =2	*j* =3	*j* =4	*j* =5
0	{0,1,2,0} {0,1,3,0}	{0,1}	{0,1,2}	{0,1,3}	{0,1,2,4}	{0,1,3,5}
1	{1,2,0} {1,3,0}	{1,2,0,1} {1,3,0,1} {1,2,4,1}	{1,2}	{1,3}	{1,2,4}	{1,3,5}
2	{2,0} {2,4,1,3,0}	{2,0,1} {2,4,1}	{2,0,1,2} {2,4,1,2}	{2,0,1,3} {2,4,1,3}	{2,4}	{2,0,1,3,5} {2,4,1,3,5
3	{3,0}	{3,0,1}	{3,0,1,2}	{3,0,1,3} {3,5,3}	{3,0,1,2,4}	{3,5}
4	{4,1,2,0} {4,1,3,0}	{4,1}	{4,1,2}	{4,1,3}	{4,1,2,4}	{4,1,3,5}
5	{5,3,0}	{5,3,0,1}	{5,3,0,1,2}	{5,3}	{5,3,0,1,2,4}	{5,3,5}

5.1 Path Levels :On taking initial state '0' at time t= 0, all the paths from the state '0' to other states are considered andthe primary, secondary circuits along the different paths from the initial state '0' to other reachable states are as given under. The number of primary circuits w.r.to the state '0', is 2 and there are no secondary circuits as shown in the Table – 4. The failure free paths from initial state to the un-failed states and circuits through un-failed states are shown in Table -5.

P_0: $0 \rightarrow 1$

P_1: $1 \to 2 \to 4 \to 1$

P_0: $0 \to 1 \to 2$

P_1: $1 \to 2 \to 4 \to 1$

P_0: $0 \to 1 \to 3$

P_1: $1 \to 2 \to 4 \to 1$;

P_1: $3 \to 5 \to 3$

P_0: $0 \to 1 \to 2 \to 0$

P_1: $1 \to 2 \to 4 \to 1$

P_0: $0 \to 1 \to 2 \to 4$

P_1: $1 \to 2 \to 4 \to 1$

P_0: $0 \to 1 \to 3 \to 0$

P_1: $1 \to 2 \to 4 \to 1$

P_1: $3 \to 5 \to 3$

P_0: $0 \to 1 \to 3 \to 5$

P_1: $1 \to 2 \to 4 \to 1$

P_1: $3 \to 5 \to 3$

TABLE-4

Vertex 'j'	Paths from initial state '0' to vertex 'j'		Circuits		
	$(\mathbf{0} \xrightarrow{s_r} \boldsymbol{j})$	P_0	P_1	P_2	Distinct Circuits
0	$(\mathbf{0} \xrightarrow{s_1} \mathbf{0})$	{0,1,2,0}	{1,2,4,1}	-	
	$(\mathbf{0} \xrightarrow{s_2} \mathbf{0})$	{0,1,3,0}	{1,2,4,1}		
			{3,5,3}		
1	$(\mathbf{0} \xrightarrow{s_1} \mathbf{1})$	{0,1}	{1,2,4,1}	-	
2	$(\mathbf{0} \xrightarrow{s_1} \mathbf{2})$	{0,1,2}	{1,2,4,1}	-	$N(P_1)=2$ $N(P_2)=0$
3	$(\mathbf{0} \xrightarrow{s_1} \mathbf{3})$	{0,1,3}	{1,2,4,1}	-	
			{3,5,3}		
4	$(\mathbf{0} \xrightarrow{s_1} \mathbf{4})$	{0,1,2,4}	{1,2,4,1}	-	
5	$(\mathbf{0} \xrightarrow{s_1} \mathbf{5})$	{0,1,3,5}	{1,2,4,1}	-	
			{3,5,3}		

Table - 5

Vertex 'j'	Failure Free Paths from initial state '0' to un-failed state 'j'		Failure Free Circuits
	$(0 \xrightarrow{s_r(sff)} j)$	P_0	P_1
0	$(0 \xrightarrow{s_1(sff)} 0)$	{0,1,2,0}	--
	$(0 \xrightarrow{s_2(sff)} 0)$	{0,1,3,0}	--
1	$(0 \xrightarrow{s_1(sff)} 1)$	{0,1}	--
2	$(0 \xrightarrow{s_1(sff)} 2)$	{0,1,2}	--
3	$(0 \xrightarrow{s_1(sff)} 3)$	{0,1,3}	--

6. TRANSITION PROBABILITIES AND MEAN SOJOURN TIMES:

The following transition probabilities which are determined from the p.d.f.'s $q_{i,j}(t)$ & $q_{i,\underline{k},j}(t)$ and the mean sojourn times(μ_i) and totalun-conditional times (μ'_i) are as under:

6.1 Transition Probabilities:

$p(0,1)=1; p(1,2) = a; p(1,3) = b;\ \ p(\underline{4},1) = 1;\ \ \ p(\underline{5},3)=1$

$p(2,0) = g_1^*(2\lambda), p(2,\underline{4},1) = 1 - g_1^*(2\lambda)$

$p(3,0) = g_2^*(2\lambda),\ p(3,\underline{5},3) = 1 - g_2^*(2\lambda)$

$p(1,2) + p(1,3) = 1;\ p(2,0) + p(2,\underline{4},1) = 1;\ p(3,0) + p(3,\underline{5},3) = 1$

$p(2,0) + p(2,\underline{4}) = 1;\ p(3,0) + p(3,\underline{5}) = 1$

6.2 Mean Sojourn Times:

$$\mu_0 = \frac{1}{2\lambda};\ \mu_1 = h^{*'}(0);\quad \mu_2 = \frac{1-g_1^*(2\lambda)}{2\lambda};\ \mu_3 = \frac{1-g_2^*(2\lambda)}{2\lambda}$$

6.3 Total Un-Conditional Times:

$\mu'_0 = \mu_0;\ \mu'_1 = \mu_1;\ \mu'_3 = \int_0^\infty t.\,d\{G_1(t)\} = -\ g_1^{*'}(0)$ and $\mu'_3 = \int_0^\infty t.d\{G_2(t)\} = -\ g_2^{*'}(0)$.

7. EVALUATION OF MTSF & AVAILABILITY OF THE SYSTEM:

The mean time to the system failure (*MTSF*) and availability of the system (under steady state conditions) are evaluated, by applying Regenerative Point Graphical Technique (RPGT) and using '0' as the initial state (at time t = 0) as under:

7.1 **Mean Time to System Failure:** From Fig.1, the regenerative un-failed states to which the system can transit (with initial state '0' at time 't' = 0), before transiting to any failed states are: i = 0,1, 2 &3. The mean time to system failure (MTSF) is obtained by applying RPGT as under:

$$MTSF = \left[\sum_{i,s_r}\left\{\frac{\{pr(0 \xrightarrow{s_r(sff)} i)\}.\mu_i}{\prod_{k_1 \neq 0}\{1-V_{\overline{k_1,k_1}}\}}\right\}\right] \div \left[1-\sum_{s_r}\left\{\frac{\{pr(0 \xrightarrow{s_r(sff)} 0)\}}{\prod_{k_2 \neq 0}\{1-V_{\overline{k_2,k_2}}\}}\right\}\right]$$

$$= \begin{matrix}(0,0).\mu_0+(0,1)\mu_1+\\(0,1,2)\mu_2+(0,1,3)\mu_3\end{matrix} \div \left[1-(0,1,2,0)-(0,1,3,0)\right] = \boldsymbol{N_0} \div \boldsymbol{D_0} \text{ where}$$

$\boldsymbol{N_0}$ = [$\boldsymbol{p(0,0)}\ \boldsymbol{\mu_0} + \boldsymbol{p(0,1)}\ \boldsymbol{\mu_1} + \boldsymbol{p(0,1).p(1,2)}\ \boldsymbol{\mu_2} + \boldsymbol{p(0,1).p(1,3)}\ \boldsymbol{\mu_3}$] and

$\boldsymbol{D_0}$ = [1- $\boldsymbol{p(0,1).p(1,2).p(2,0)}$ - $\boldsymbol{p(0,1).p(1,3).p(3,0)}$]

7.2 Availability of the System:

From Fig.1, the regenerative available un-failed/up-states to which the system can transit, are: j = 0, 1, 2, 3. Therefore, the total fraction of time for which the system remains available i.e. the availability of the system (with initial state '0'at time t = 0), using RPGT, is given by:

$$\boldsymbol{A_0} = \left[\sum_{j,s_r}\left\{\frac{\{pr(0 \xrightarrow{s_r} j)\}f_j.\mu_j}{\prod_{k_1 \neq 0}\{1-V_{k_1,k_1}\}}\right\}\right] \div \left[\sum_{i,s_r}\left\{\frac{\{pr(0 \xrightarrow{s_r} i)\}.\mu_i'}{\prod_{k_2 \neq 0}\{1-V_{k_2,k_2}\}}\right\}\right]$$

$$= \left[(0,0)\mu_0+\left\{(0,1)/(1\text{-}L_1)\right\}\mu_1+\left\{(0,1,2)/(1\text{-}L_1)\right\}\mu_2+\left\{(0,1,3)/(1\text{-}L_1)(1\text{-}L_3\right\}\mu_3\right]$$

$$\div \left[(0,0)\mu_0'+\left\{(0,1)/(1\text{-}L_1)\right\}\mu_1'+\left\{(0,1,2)/(1\text{-}L_1)\right\}\mu_2'+\left\{(0,1,3)/(1\text{-}L_1)(1\text{-}L_3\right\}\mu_3'\right]$$

Where $L_1 = 1-(1,2,\underline{4},1) = 1-(1,2)(2,\underline{4},1); L_3 = 1-(3,\underline{5},3) = (3,0)$

$= \boldsymbol{N_1} \div \boldsymbol{D_1}$ where

$$\boldsymbol{N_1} = \left[\boldsymbol{p(3,0)}\left\{\left(1-p(1,2).p(2,\underline{4},1)\right)f_0\mu_0 + f_1\mu_1 + p(1,2)f_2\mu_2\right\} + p(1,3)f_3\mu_3\right]$$

And $D_1 = p(3,0)\left\{\left(1-\boldsymbol{p(1,2).p(2,\underline{4},1)}\right)\boldsymbol{\mu_0} + \mu_1 + p(1,2)\mu_2'\right\} + p(1,3)\mu_3'$.

8. CONCLUSION:

On taking $\boldsymbol{f_i}$ = 1 for all the up-states namely 0, 1, 2 & 3, the mean time to system failure (MTSF) and the availability of the system, are the same as obtained by D.V. et al [1]. So, the path analysis of the transition of states to which the three unit stand-by stochastic system can transit, helps to findits MTSF and the availability, more quickly and without writing and solving any state equations and thus saves time & energy and is more economical

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THE EXPONENTIATED MARSHALL-OLKIN DISCRETE UNIFORM DISTRIBUTION WITH APPLICATION IN SURVIVAL AND HAZARD ANALYSIS ON PROGESTERONE, ESTROGEN AND OTHER VARIOUS HORMONES

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ABSTRACT :
The E-MO-U distribution can be a useful characterization of life time data analysis of a given system. In application part we analyze the study of effect of 3-week progesterone administration on sleep architecture and multiple hormonal profiles was done. The protocol allowed us to explore the effects of the treatment both on normal sleep and on sleep disturbed by an iv catheter. The Exponentiated Marshall-Olkin Discrete Uniform Distribution with application in Survival and Hazard analysis was fitted to study the behavior of plasma progesterone, androsanediol glucuronide, estrodiol under placebo and progesterone treatment. This mathematical model is capable of modeling the behavior of various hormones, and found that this discrete uniform distribution provides the best fit to all the data.
Keywords: Progesterone, Estradiol, TSH, Androsanediol Glucuronide, E-MO-U, Exponentiated reliability function Classification: 33C90, 62E99

1. INTRODUCTION:

Some times in real life it is difficult or inconvenient to get samples from a *continuous distribution. Almost always the observed values are actually discrete* because they are measured only finite number of decimal places and cannot really *constitute all the points in a continuum. Even if the measurements are taken on* continuous scale the observations seem to be making discrete model more appropriate. The discrete measurement is also taken to save the space. The continuous variables are measured by non-overlapping class frequencies whose union constitute the whole range of random variables and multinomial law is used to model such kind of situations.

In survival analysis the survival function may be a function of count random variable that is a discrete version of underlying continuous random variable. Also it is observed that the continuous failure timed at a generated from a complex system poses more derivational problem than that of discrete version of the underlying continuous one. These discrete life time distributions played only marginal role in reliability analysis. Therefore there is need to focus on more realistic discrete life time distributions. [8]. That is discretization of continuous lifetime model is an interesting and intuitively appealing approach to derive a discrete lifetime model corresponding to continuous one [4].

For a continuous distribution $R^+ = [0,\infty)$ with probability density function g(x) (pdf) and a cumulative distribution function G(x) (cdf), one can construct a discrete counterpart supported on the set of integers $N_0 = \{0,1,2\ldots\}$, whose probability mass function (PMF) is of the form

$$P_y = P(Y = y) = G(y + 1) - G(y), \quad y=0, 1, 2\ldots.$$

or

$$P_y = P(Y = y) = \bar{G}\,(y + 1) - \bar{G}\,(y) \tag{1.1}$$

Where $\bar{G}\,(y)$ is the survival function (sf) of the random variable X. The resulting PMF will be in a compact form if the continuous (sf) is in compact form.

Marshall and Olkin generalization introduced by Marshall and Olkin [6] using any discrete distribution. This generalization is generalized by adding an extra parameter θ >0 to the base distribution using

$$\overline{G}\,(y) = \frac{\theta.\bar{Q}(y)}{1-(1-\theta)\bar{Q}(y)}, \tag{1.2}$$

$$\bar{F}(y) = (\bar{G}(y))^{\gamma} = \left(\frac{\theta\bar{Q}(y)}{1-(1-\theta)\bar{Q}(y)}\right)^{\gamma}$$

There are many family of distributions discussed on continuous case, not much work is seen in discrete case. The reason behind is that it is difficult to obtain compact mathematical expressions for moments, reliability and estimation in the discrete set up. Using discretizing technique defined in (1.1), the probability mass function (PMF) corresponding to the Marshal-Olkin family is given by

$$\mathrm{P(x) = G(x) - G(x-1)} = \frac{\theta \mathrm{q(x)}}{[1-(1-\theta)\bar{\mathrm{Q}}(\mathrm{x})][1-(1-\theta)\bar{\mathrm{Q}}(\mathrm{x}-1)]}$$

For a discrete uniform distribution with PMF p(y) = 1/α; y=1,2,…α. Sandhya and Prashanth [9] introduced the Marshall-Olkin discrete uniform distribution with a survival function given by:

$$\bar{G}(y) = \frac{\theta(\alpha-y)}{\alpha\theta+(1-\theta)y}; y = 1,2, \ldots \alpha \qquad \alpha, \theta > 0,$$

And its corresponding PMF and CDF are respectively given by

$$g(y) = \frac{\alpha\theta}{[\alpha\theta+(1-\theta)y][\alpha\theta+(1-\theta)(y-1)]},$$

$$G(y) = 1 - \frac{\theta(\alpha-y)}{\alpha\theta+(1-\theta)y} = \frac{y}{\alpha\theta+(1-\theta)y} \tag{1.3}$$

2. EXPONENTIATED MARSHALL-OLKIN DISCRETE UNIFORM (E-MO-U) DISTRIBUTION

By inserting the above equation (1.3) into the resilience parameter family of distribution, the CDF of the resulting discrete distribution is the exponentiated Marshall-Olkin discrete uniform distribution and given by:

$$F(y) = \left(\frac{y}{\alpha\theta+(1-\theta)y}\right)^{\gamma}, y = 0,1,2, \ldots \alpha \tag{2.1}$$

In which $\gamma > 0$ is the resilience parameter, and the corresponding PMF is given by

$$f(y) = \left(\frac{y+1}{\alpha\theta+(1-\theta(y+1)}\right)^{\gamma} - \left(\frac{y}{\alpha\theta+(1-\theta)y}\right)^{\gamma} \tag{2.2}$$

3. THE SURVIVAL FUNCTION:

The E-MO-U distribution can be a useful characterization of life time data analysis of a given system. The survival function (SF), $\bar{F}\,(y)$, of the E-MO-U distribution is defined as:

$$\bar{F}(y) = 1 - \left(\frac{y}{\alpha\theta+(1-\theta)y}\right)^{\gamma}, y = 0,1,2 \ldots \alpha \tag{3.1}$$

3.1 The Hazard Rate and the cumulative Hazard Rate Functions:

The hazard rate function is given by:
$h(y) = \frac{Py}{\bar{F}(y)}$ and defined by:

$$h(y) = \frac{\left(\frac{y+1}{\alpha\theta+(1-\theta)(y+1)}\right)^{\gamma} - \left(\frac{y}{\alpha\theta+(1-\theta)y}\right)^{\gamma}}{1-\left(\frac{y}{\alpha\theta+(1-\theta)y}\right)^{\gamma}} \tag{3.2}$$

we note that h(x) might be a constant, increasing or decreasing depending or even bathtub on the values of the parameters involved.

For $\gamma = 1$, the hazard rate of the E-MO-U distribution reduced to hazard rate of the Marshall-Olkin uniform distribution. Sandhya and C.B. Prasanth [9] shows that, for the Marshall-Olkin uniform distribution the distribution is with increasing failure rate (IFR) when $\theta = \frac{\alpha-2x}{2\alpha-2x}$, decreasing failure rate (DFR) when $\theta < \frac{\alpha-2x}{2\alpha-2x}$ and constant failure rate at $\theta = \frac{\alpha-2x}{2\alpha-2x}$. This results are valid for any given value of $\gamma > 0$.

4. APPLICATION:

We used survival and hazard rate functions to investigate the effect of 3 week progesterone administration both on sleep architecture and on multiple hormone profiles in normal cycling women, and found that endogeneous progesterone could modulate the secretion of hormones primarily or partially regulated by the sleep-wake cycle (GH, prolatin, TSH).

Very few studies have objectively characterized the effect of progesterone administration on sleep [3,11], although preclinical studies have shown that certain neuroactive progesterone metabolites produce sedative-like-effects [13]. Eight postmenstrual women, aged 48-74 year (mean 57.4 year), were selected after a careful clinical and biological evaluation. Investigations were performed 2-17 year (mean 8.0 year) after natural menopause (seven samples) or bilateral ovariectomy (one sample). Mean age at menopause was 49.4 year. Out of eight three samples have never undergone hormonal replacement therapy, and five samples were off such treatments for 2-6 months at the time of enrollment.

5. RESULT:

Under placebo, (Figure 1) 24-h progesterone levels averaged $0.15 \pm 0.04 \mu g/liter$. Under oral nonmicronized progesterone treatment, 24-h levels averaged $8.1 \pm 1.2 \mu g/liter$. Minimal values were observed at 2200-2300 hr, just before progesterone administration. Therefore, levels increased rapidly to reach maximum values at 0100-0200 hr (2-3 hrs after drug ingestion).

Under placebo, (Figure 2) androstanediol glucuronide levels were stable over the 24-hr span. Under progesterone, levels were increased and 24-h profiles paralleled simultaneous progesterone patterns, with maximum values at 0100-0200h.

Estradiol were low stable and similar in both conditions (Figure 3). Individual DHEAS levels were stable and similar in both conditions but varied considerably among individuals.

6. MEDICAL FIGURES:

Figure 1:

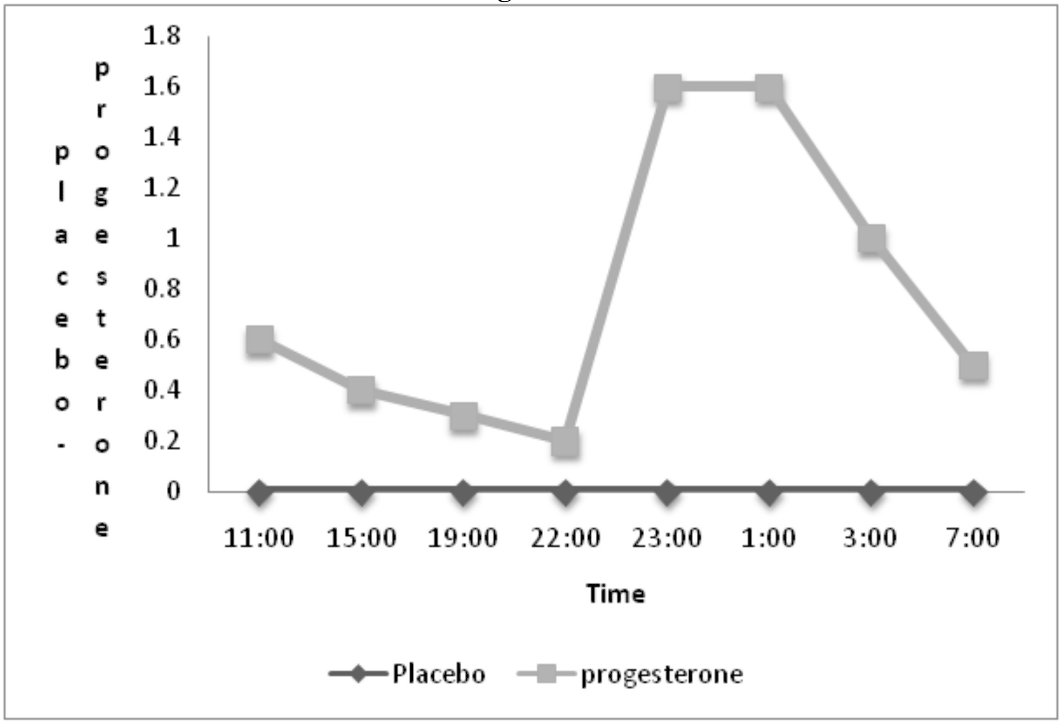


Figure 2:

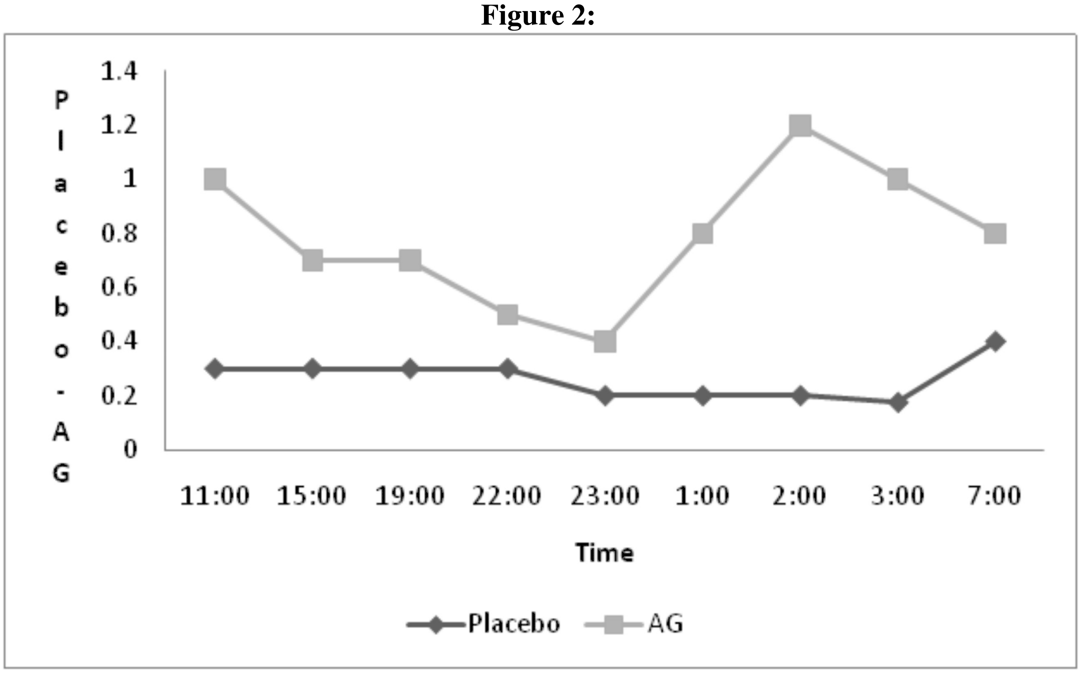
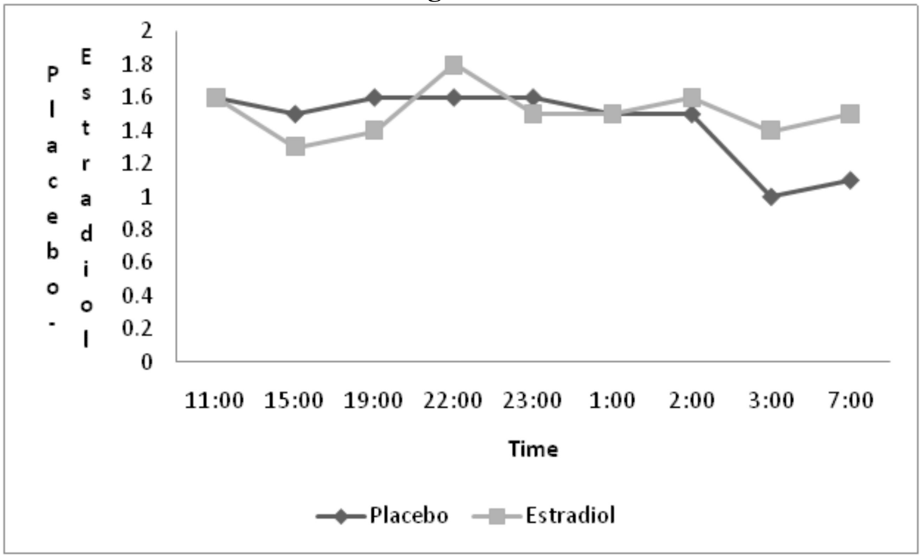


Figure 3:



7. MATHEMATICAL FIGURES:

Survival Figures:

Figure 4:

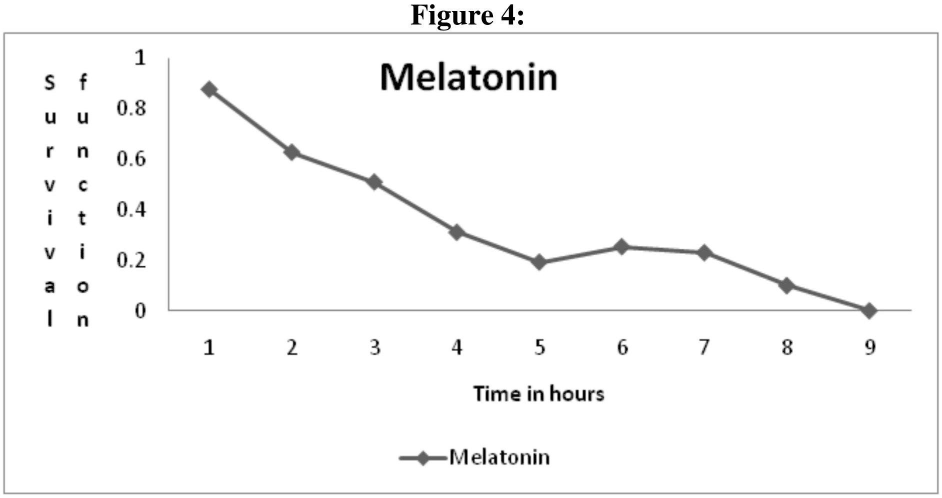


Figure 5:

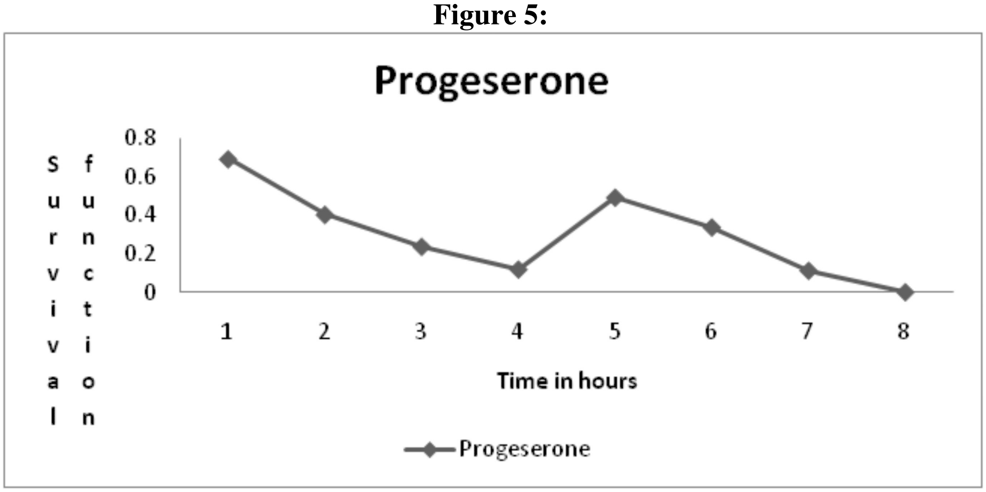
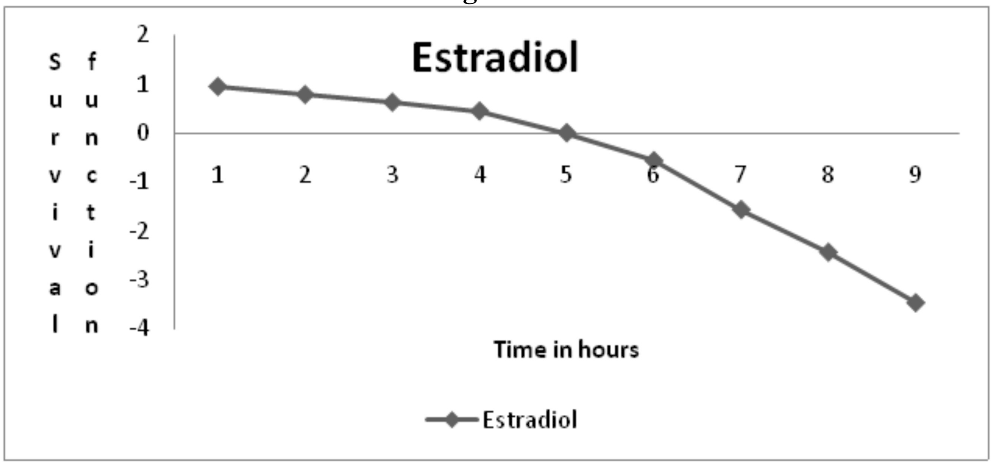


Figure 6:



Hazard Figures:

Figure 7:

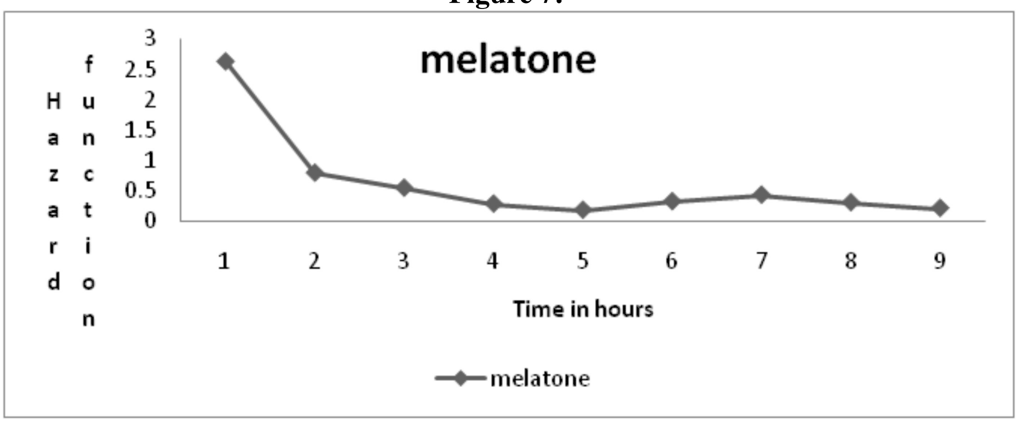


Figure 8:

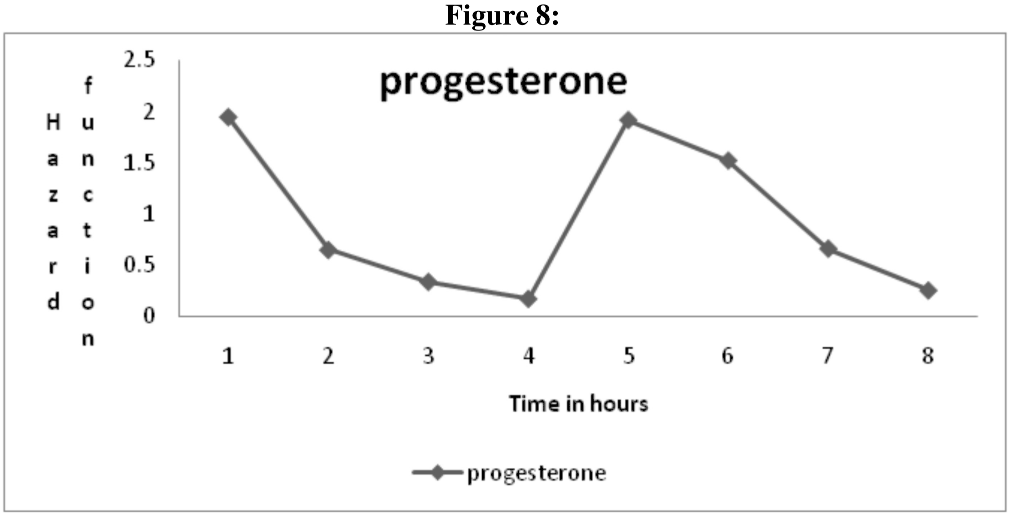
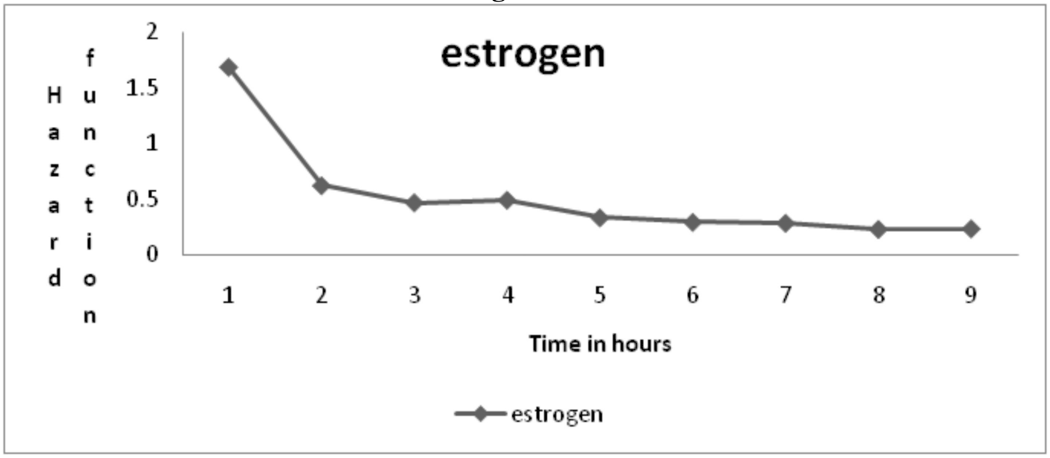


Figure 9:



8. DISCUSSION:

This study performed under well-controlled experimental conditions in a group of eight healthy postmenopausal women, is the first investigation where the effects of semi-chronic progesterone treatment on sleep architecture and on multiple 24-h hormonal profiles were simultaneously analyzed. Despite the modest sample size, the present results provide valuable indications about progesterone action on the neuroendocrine system.

None of the volunteers complained of sleep problems or of vasomotor symptoms. To be habituated to the study procedure, they were hospitalized in the sleep laboratory for two consecutive nights several weeks before the beginning of the investigation. During the laboratory session, sleep duration an architecture during night 1 (without catheter) were normal [5,10] and progesterone administration had no effect. Consistent with findings previously reported in healthy older women [5,14], sleep profiles under placebo were markedly disrupted during night 2 by the indwelling iv catheter and blood sampling procedure. In contrast, those sleep alteration were largely prevented under progesterone. Thus progesterone administration had no effect on normal sleep, but improved the sleep duration and sleep quality.

Very few studies have investigated the effects on sleep patterns of progesterone administration. They have yielded inconsistent results, probably because the protocol design varied widely from one study to another, in particular in view of the fact that progesterone could exert opposite effects at low or high concentration [1]. In the absence of nocturnal blood sampling, prolonged administration of progesterone alone [11] or of progesterone combined with estrogen [7] to healthy postmenopausal women without sleep disorders reduced wakefulness, but no other significant effect on sleep architecture was detected.

Typical postmenopausal LH, FSH, estradiol, estrone, DHEAS and androstanediol glucuronide profiles were observed under placebo. Progesterone administration had no effect on LH but was associated with a slight but significant decrease in FSH levels, consistent with negative feedback action at the hypothalamo-pituitary level [12,2]. Under placebo progesterone and androstanediol glucuronide levels were not related to each other. However, after progesterone administration, the acute nocturnal elevation in progesterone concentrations was associated with a proportional and parallel increase in androstanediol glucuronide, while estrodiol, estrone profiles were not altered.

9. CONCLUSION:

The medical results reveal that the progesterone may remit normal sleep in postmenopausal women when sleep is disturbed by environmental conditions. Progesterone would not act as a convectional somnolent (i.e, it would not induce artificial sleepiness), but it would rather act as a “physiologic” regulator. Moreover it would enhance sleep quality. Thus the use of progesterone might provide novel therapeutic approach for the exploration of sleep disturbances, in particular in the elderly.

The mathematical figures (4 to 9) show a difference in between fourth and fifth points of time (in hours) in both Survival functions curve as well as in the hazard functions curve. There is a decrease in hazard rate up to fourth time point (in hours), increases till fifth time point (in hours) and then decreases .The mathematical hazard functions curves give a clear idea compared with the corresponding medical zigzag curves.

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STRESS INTENSITY FACTOR OF GRIFFITH CRACK OPENED BY THERMAL STRESS IN ISOTROPIC INFINITE STRESS FREE STRIP

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ABSTRACT :
The closed forms expressions of stress-intensity factors and of crack shape are obtained by using Fourier transform method. The concern of the present paper is the problem of an interior Griffith crack opened by thermal stress in isotropic infinite stress free strip. Under plane strain condition the closed form expressions for the stress intensity factors and the crack shape are obtained by solving the Fredholm integral equations.
Keywords : Stress Intensity Factor, Thermal Stress, Fourier Transform .Isotropic Stress, Griffith crack.

INTRODUCTION

After Second World War there was a sudden and rapid development of thermo-elasticity, stimulated by various engineering sciences. The field aircraft and machine structures gas and steam turbines and new topics of chemical and nuclear engineering have given risk to numerous problems of thermal stress.
Oleziak and Sneddon [1] had solved the problem of penny-shaped crack opened by thermal stress by using Fourier transform method. Florence and Goodier [2] used complex variable to solve crack opening due to thermal stress.

The problem of stress-intensity factor in an infinite isotropic strip with rigidly lubricated strip had been solved by Kushwaha and Chandra [3] and Saraj [4] solved crack opening by heated wedge in isotropic rectangle while edges were insulated. Anil Kumar [5] had solved the above problem with constant temperature.

In the present research endeavor we are extending the method of Harendra and Kushwaha[6] to isotropic stress-free strip while there is general heat distribution in the stip.We shall divide the problem at a general point (x,y) into two problems namely : [A] Heat problem [B] Elasticity problem
Thus,

$$\sigma_{ij}(x,y)=\sigma_{ij}^{(H)}(x,y)+\sigma_{ij}^{(E)}(x,y), i,j=x,y \tag{1.1}$$

$$u_i(x,y)=u_i^{(H)}(x,y)+u_i^{(E)}(x,y), i=x,y \tag{1.2}$$

The physical problem is translated into a mathematical problem as crack occupies y =o, $o\le|x|\le b$ region. The boundary value problem is –

$$\sigma_{xy}(a,y)=\sigma_{xx}(\pm a,y)=o, o\le|y|<\infty, \tag{1.3}$$

$$\sigma_{xy}(x,o)=o, o\le|x|\le a, \tag{1.4}$$

$$u_y(x,o)=o, b\le|x|\le a \tag{1.5}$$

$$\sigma_{yy}(x,o)=o, o\le|x|<b \tag{1.6}$$

Thus, a Griffith crack of length 2b, being stress-free i.e. opened by thermal stress $\sigma_{yy}{}^{(H)}(x,o)$. The width of the strip is 2a. We assumed that the strip is under plane strain condition and all physical quantities vanish as $\sqrt{x^2+y^2}$.

We assumed that the crack axis is x-axis and y-axis passes through the middle of crack axis. The problem is symmetrical; therefore, the boundary conditions (1.3)-(1.6) are reduced to –

$$\sigma_{xy}(a,y)=\sigma_{xx}(a,y)=o,o,y<\infty, \tag{1.7}$$

$$\sigma_{xy}(x,o)=o,o\le x<a \tag{1.8}$$

$$u_y^{(E)}(x,o)=o,b\le x\le a \tag{1.9}$$

$$\sigma_{yy}^{(E)}(x,o)=-\sigma_{yy}^{(H)}(x,o),o\le x<b \tag{1.10}$$

It is also observed that, see Burniston [7] that –

$$u_y^{(E)}(x,o)>o,-b<x<b \tag{1.11}$$

Which means that the crack really opens out and the faces of the crack do not meet each other other than at crack tips. It is being assumed that –

$$u_y^{(H)}(x,o)=o \tag{1.12}$$

The plan of the paper is as follows:

In next section we shall formulate the heat problem. Section-3 will formulate the elasticity problem. Section-4 will reduce the problem to dual series relation whose solution will be given in terms of Fredholm integral equation. Section-5 will solve the Fredholm integral equation. The physical quantities of interest will be given in terms of solution of Fredholm integral equation in section-6. The special case of heat distribution will be discussed in section-7. Section-8 will discuss the formation of plastic zone length at crack tips.

2. FORMULATION OF HEAT PROBLEM

First we take finite Fourier transform with respect to x and integral Fourier transform with respect to other variable of equations of equilibrium.

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=o,\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=o, \text{ (Eqn. of equilibrium)} \tag{2.1}$$

and

$$\sigma_{ij}(x,y)=2\mu e_{ij}+\alpha_t(\gamma e_{kk}-T)\delta_{ij} \text{ (Stress-strain relation)} \tag{2.2}$$

$$e_{ij}=½\left(u_{ji}J+u_{ji}\right),i,j=x,y \text{ (strain component)} \tag{2.3}$$

$$\gamma=(3\lambda+2\mu)/\alpha_t e_{kk}=e_{xx}+e_{yy} \text{ (Elastic constant } \gamma\text{)} \tag{2.4}$$

With α_f as coefficient of linear expansion of medium, λ and μ are Lame's constants T is temperature and

$$\delta_{ij}=\left\{\begin{matrix} l, & i=j \\ o & i\pm j \end{matrix}\right\} \tag{2.5}$$

The symbol (,) in suffix of u is derivative with respect to variable succeeding it. Taking appropriate Fourier transforms of (2.1) and (2.2)-(2.4), then using the transforms of (2.2)-(2.4) into that of (2.1) and then eliminating $\bar{u}_{ycs}$, we get, where T, will satisfy Poisson's equation –

$$u_x^{(H)}(x,y)=\sum_{n=1}^{\infty}\alpha_n\sin(\alpha_n x)\int_o^{\infty}\cos(\varsigma y)\ \bar{Q}_{cc}w_o d\zeta, \tag{2.6}$$

(Q_{cc} comes from Poisson eqn.) It defines heat distribution.

$$\left(\text{where, } w_o=\frac{8(1+\eta)}{E\beta^2\left(\alpha_n^2+\varsigma^2\right)2},\beta^2=\frac{2(1-\eta)}{1-2\eta}\right) \tag{2.7}$$

$$u_y^{(H)}(x,y)=\frac{1}{2}v(o,y)+\sum_{n=1}^{\infty}v(\alpha_n,y)\cos(\alpha_n x) \tag{2.8}$$

$$\left(\text{where, } v(\alpha_n,y)=-(a\pi)^{-1}\alpha_t\int_o^{\infty}\varsigma\sin(\varsigma y)[w_o]Q_{cc}(\alpha_{nt}\varsigma)\ d\varsigma\right) \tag{2.9}$$

Where η is Poisson ratio and E is Young's modulus of the medium.

Thus, we get the displacement components given by (2.7) and (2.9). From displacement components we can easily evaluate the stress components through stress-strain relations (2.3).

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)T=Q(x_1y) \tag{2.10}$$

(Heat distribution is controlled by this poisson eqn.)

With Q (x,y) as known functions.

3. FORMULATION OF ELASTICITY PROBLEM

In this section we shall discuss the crack opening due to thermal stresses developed by temperature distribution in the medium.

The solution of equations of equilibrium (2.1) and stress-strain relations (2.3) with the absence of temperature term, T, there. For this problem we follow the method given by Sneddon [8]. We assume the solutions as :-

$$u_x^{(E)}(x,y)=\left[\sum_{n=1}^{\infty}\alpha_n^{-1}\sin(\alpha_n x)\left\langle(1-\eta)G,yy+\eta\alpha_n^2G\right\rangle+\right.$$

$$\left.\int_o^{\infty}\varsigma^{-1}\cos(\varsigma y)\left\langle(1-\eta)H,xx+\eta\varsigma^2H\right\rangle d\varsigma\right] \tag{3.1}$$

$$u_y^{(E)}(x,y)=½u_{yc}(o,y)+\sum_{n=1}^{\infty}u_{yc}(\alpha_n,y)co(\alpha_n x)$$

$$+\int_o^{\infty}\varsigma^{-2}\sin(\varsigma y)\left[(1-\eta)H,xxx+\eta\varsigma^2H,x\right]d\varsigma \tag{3.2}$$

$$u_{yc}(\alpha_n,y)=\alpha_n^{-2}\left[(1-\eta)G,yyy+\eta\alpha_n^2G,y\right], \tag{3.3}$$

and

$$G \equiv G(\alpha_n, y) = (A_n + yB_n)\bar{e}^{\alpha_n y} \tag{3.4}$$

$$H \equiv (x, \varsigma) = \langle C(\varsigma) + xD(\varsigma)\rangle \cosh(\varsigma x) \tag{3.5}$$

Where $An, Bn, C(\varsigma), D(\varsigma)$ are four arbitrary constants to be determined,

Evaluating stress components we get –

$$\sigma_{xy}^{(E)}(x,y) = \sum \alpha_n \sin(\alpha_n x)\left[-(A_n + yB_n)\alpha_n \bar{e}^{\alpha_n y} + B\bar{e}^{\alpha_n y}\right]$$
$$+\int_o^\infty \varsigma \sin(\varsigma y)\left[\varsigma(C + xD)\operatorname{coch}(\varsigma x) + D\sinh\varsigma x\right]d\varsigma \tag{3.6}$$

$$\sigma_{xx}^{(E)}(x,y) = \sum_{n=1}^{\infty} \alpha_n \cos(\alpha_n x)\left[(A_n + yB_n)\alpha_n \bar{e}^{\alpha_n y}\right]$$
$$+\int_o^\infty \varsigma \cos(\varsigma y)\left[\varsigma(C + xD)\cosh\varsigma x + 2D\sinh\varsigma x\right]d\varsigma \tag{3.7}$$

$$\sigma_{yy}^{(E)}(x,y) = \sum_{n=1}^{\infty} \alpha_n^{\,2} \cos(\alpha_n x)\left[(A_n + yB_n)\bar{e}^{\alpha_n y}\right]$$
$$+\int_o^\infty \varsigma^2 \cos(\varsigma y)\left[(c + xD)\cosh\varsigma x\right]d\varsigma \tag{3.8}$$

The boundary conditions (1.7)-(1.8), after using (3.6) and (3.7), give

$$D(\varsigma) = -\frac{\varsigma\, c(\varsigma)}{a\varsigma + \tanh(\varsigma a)} \tag{3.9}$$

$$\sum_{n=1}^{\infty} (-1)^n \alpha_n B_n \left\langle \frac{2\alpha_n + \alpha_n^{\,2} - \varsigma^2}{\left(\varsigma^2 + \alpha_n^{\,2}\right)^2} \right\rangle = F_1(\varsigma) + F(\varsigma) \tag{3.10}$$

With

$$F_1(\varsigma) = (a-1)\varsigma \frac{C(\varsigma)\cosh(\varsigma a)}{a\varsigma + \tanh a\varsigma} \tag{3.11}$$

With

$$F(\varsigma) = \int_o^\infty \sigma_{xx}^{(H)}(a,y)\cos(\varsigma y)\,dy, \tag{3.12}$$

and

$$B_n = \alpha_n A_n \tag{3.13}$$

Thus we get three equation involving four constants. We evaluate three constants in terms of fourth…… as Bu.

4. REDUCTION TO AND SOLUTION OF DUAL SERIES (REDUCTION)

In this section we shall reduce the boundary value problem to the solution of dual series by using the mixed boundary condition (1.9)-(1.10) and using the equations (3.9)-(3.13).

$$\sum_{n=1}^{\infty} B_n \cos(\alpha_n x) = o, b \le x \le a \tag{4.1}$$

$$\sum_{n=1}^{\infty}\alpha_n B_n \cos(\alpha_n x) = T_1(x) - T(x), o \le x < b, \tag{4.2}$$

$$T(x) == \sigma_{yy}^{(H)}(x,o), Bo = \frac{u_o}{4(1-\eta)}, \tag{4.3}$$

$$T_1(x) = \sum_{n=1}^{\infty}(-1)^n \alpha_n B_n \int_o^{\infty} \frac{T_2(\varsigma,o,x)\left(2\alpha_n + \alpha_n{}^2 - \varsigma^2\right)}{\left(\alpha_n{}^2 + \varsigma^2\right)} d\varsigma + T_3(x) \tag{4.4}$$

$$T_3(x) = \int_o^{\infty}\sigma_{xx}^{(H)}(a,y)dy\int_o^{\infty}T_2(\varsigma,x)\cos(\varsigma y)d\varsigma \tag{4.5}$$

$$T_2(\varsigma,x) = \frac{\left[\varsigma(a-x) + \tanh(a\varsigma)\right]\cosh(\varsigma x)}{\varsigma\cosh(a\varsigma)} \tag{4.6}$$

The equations (4.1)-(4.2) are called dual series equations.

4.1 Solution

We assume the trial solution, see Parihar [7], $\alpha_n B_n = 2\int_o^b g(t)\sin(\alpha_n t)\, dt, B_o = 2\int_o^b tg(t)dt,$ (4.7)

The substitution of (4.7) into (4.1) and using the relation

$$\frac{qt}{2} + \sum \frac{\sin(\alpha_n t)\cos(\alpha_n x)}{n} = \begin{Bmatrix} \frac{\pi}{2}, & t > x \\ \pi/4, & t = x \\ o, & t < x \end{Bmatrix}$$

The equation is satisfied identically. The substitution of first of (4.7) into (4.2) and using the relations.

$$\sum_{n=1}^{\infty}\frac{\sin(\alpha_n x)\sin(\alpha_n t)}{n} = ½\log\left|\frac{\sin q(x-t)}{\sin q(x+t)}\right| \tag{4.8}$$

And the method of Parihar, we get

$$g(t) = \frac{2}{a^2}\frac{\sin(qt/2)}{\sqrt{G(t,b)}}\left[\Delta_o(t) + \int_o^b g(\alpha)k(\alpha,t)d\alpha\right], o \le t < b \tag{4.9}$$

$$\Delta_o(t) = \int_o^b \frac{\cos(qx/2)\sqrt{G(x,b)}T_o(x)dx}{G(x,t)} \tag{4.10}$$

$$K(\alpha,t) = \int_o^b \frac{\cos(qx/2)\sqrt{G(x,b)}}{G(x,t)}T_4(x,\alpha) \tag{4.11}$$

$$T_o(x) = T(x) + T_{o3}(x) \tag{4.12}$$

$$T_4(x,y) = \sum_{n=1}^{\infty}(-1)^n \sin(\alpha_n y)\cos(\alpha_n x)M(\alpha_n, x) \tag{4.13}$$

The equation (4.9) is Fredholm integral equation of second kind. In next section we shall solve this equation by approximate expansion method.

5. SOLUTION OF FREDHOLM INTEGRAL EQUATION

Before we solve Fredholm integral equation, numerically. We evolve $T_4(x,y)$ as an approximate expansion.

$$M(\alpha_n, x) = (a-x)\left[\sum_{r=o}^{\infty}(-1)^2 2\alpha_n(1+\alpha_n)\left\langle \frac{p_1R_1(p_1)}{2\alpha_n{}^2} + R_3(p_2)\right\rangle\right]$$

$$+\left[2\sum_{r=o}^{\infty}(-1)^r R_4(p3) - 4\sum_{r=o}^{\infty}\sum_{i=o}^{\infty}(-1)^{ltr}R_4(p_4)\right.$$

$$\left. +2\sum_{r=o}^{\infty}(-1)^r r_4(p_5) - 4\sum_{r=o}^{\infty}\sum_{i=o}^{\infty}(-1)^{ltr} R_4(p_6)\right] 2\alpha_4(1+\alpha_n)$$

$$(a-x)\left[\sum_{r=o}^{\infty}(-1)^r \left\langle R_1(p_1) + R_1(p_2)\right\rangle\right]_{---}$$

$$\left[2\sum_{r=o}^{\infty}(-1)^r \left\langle\{R_2(p_3)+R_2(p_5)\} - 4\sum_{\partial=o}^{\infty}(-1)\{R_2(4)+R_2(p_6)\}\right\rangle\right] \tag{5.1}$$

$$\left.\begin{array}{ll} p_1 = a(2r+1) - x, & p_2 = p_1 + 2x \\ p_3 = a(2r+1), & p_4 = a\{2+2r+3\} - x \\ p_5 = p_3 + 2x, & p_6 = p_4 + 2x \end{array}\right\} \tag{5.2}$$

$$\left.\begin{array}{l} R_1(p) = \frac{\pi}{4}\left[\cos(\alpha_n p) + \sin(\alpha_u p)\right]/\alpha_n \\ R_2(p) = \frac{\pi}{4\alpha_n^2}\left[\cos(\alpha_n p) - \sin(\alpha_n p)\right], \\ R_3(p) = R_1(p)/(2\alpha_n 2), R_4(p) = \left[\frac{pR_1(p)}{\alpha_n} + 2R_3(p)\right] \end{array}\right\} \tag{5.3}$$

$$R_4(x,y) = \sum_{i=1}^{4} T_{4i}(x,y), \tag{5.4}$$

$$\left.\begin{array}{l} T_{41} = \frac{\pi}{4}(a-x)\sum_{r=o}^{\infty}(-1)^r(p_1+a)\psi_2(x,y) \\ T_{42} = \pi\sum_{r=o}^{\infty}(-1)^r\left[(p_3+a)\psi_2(x,y) - \sum_{i=o}^{\infty}(-1)(2a+p_4+p_o)\Psi_2(x,y)\right] \\ T_{43} = \frac{\pi}{4}(a-x)\sum_{r=o}^{\infty}(-1)^r\Psi_2(x,y) \\ T_{44} = -\frac{\pi}{2}\sum_{r=o}^{\infty}(-1)^r\left[\phi_2(x,y) + \Psi_2(x,y) - 4\sum_{i=o}^{\infty}(-1)\Psi_2(x,y)\right] \end{array}\right] \tag{5.5}$$

$$\phi_2(x,y) = \int\phi_1(x,y)dy, \Psi_2(x,y) = \Psi_1(x) + (\beta_y - 1)\phi_1(x,y)$$

$$\phi(x,y) = \log\left|\frac{\sin q(x-2y)}{\sin q(x+2y)}\right|$$

$B_y - I = \frac{d}{dy} - 1$ is an operator.

$\Psi_1(x) = \pi - q\frac{x}{2}$

Now the solution of Fredholm integral equation is obtained by the method of Fox [8]. We used the method of Lowengrub and Srivastava [9], Kushwaha [10] of approximate expansion of Kernel. The integrals involved are well behaved. The substitution is

$$\sin(qt/2) = p\sin(qb/2) \tag{5.7}$$

Makes the limit of integration as 0 to 1 from o to b with p as new variable. Then the length of interval is divided into m intervals.

The variable is defined as:-

$$\left.\begin{array}{l} p_i = p_{i-1} + \frac{1}{i} = 1,2,3....m \\ t_o = p_o = o \end{array}\right\} \tag{5.8}$$

Thus the equation (4.9) gives a system of m linear equations

$$A_{ij}g_j(p_i) = -e_{j,}j = 1,2,3....m \tag{5.9}$$

With

$$A_{ij} = \frac{2}{a^2}\frac{\sin(qt_i/2)}{\sqrt{G(t_i,b)}}\beta_i, i = 1,......m \tag{5.10}$$

$$A_{ij} = \frac{2}{a^2}\frac{\sin(qt_i/2)}{\sqrt{G(t_i,b)}}K(p_j,t_2)i = j \tag{5.11}$$

$$\beta_i = K(p_i,t_i) - I, \quad e_j = \frac{2}{a_2}\frac{\sin(qt_j/2)}{\sqrt{G(t_j,b)}}\Delta_o(t_j), \tag{5.12}$$

6. PHYSICAL QUANTITIES:

The normal stress components and the crack opening displacement play an important role in fracture-design parameters, stress-intensity factors also play important role.

NORMAL STRESS

The normal stress component, $\sigma_{yy}^{(E)}(x,o),$ for $b < x \le a$ is obtained as –

$$\sigma_{yy}^{(E)}(x,o) = \frac{-2}{ar_1}\left[\sum_{n=1}^{\infty}\alpha_n B_n \cos(\alpha_n x) - T_1(x) + T(x)\right] \tag{6.1}$$

Substituting from first of (4.7) into (6.1) and evaluating the series etc. we get –

$$\sigma_{yy}^{(E)}(x,o) = \frac{2}{ar_1}\left[\int_o^b \frac{g(t)\sin(qt)}{G_i(x,t)}dt\right.$$
$$\left.\int_o^b g(\alpha)m_1(\alpha,x)d\alpha + T_3^{(x)}\right], b < x \le a \tag{6.2}$$

Where,

$$M_1(\alpha,x)=\sum_{n=1}^{\infty}(-1)^n\sin(\alpha_n\alpha)\,M(\alpha_n,x), \tag{6.3}$$

And $T_3(x)$ is defined in (4.5)

STRESS-INTENSITY FACTORS

The stress-intensity factors are defined as –

$$K_b=\ell\lim_{x\to b}\sqrt{x-b}\;\sigma_{yy}(x,o)$$
$$=\lim_{x\to b}\sqrt{x-b}\;\sigma_{yy}^{(E)}(x,o) \tag{6.4}$$
$$N_b=\lim_{x\to b^+}\sqrt{x-b}\;\sigma_{xy}(x,o)=o \tag{6.5}$$
$$M_b=\lim_{x\to b^+}\sqrt{x-b}\;\sigma_{xx}^{(E)}(x,o) \tag{6.6}$$

Now using (6.2) in (6.4) and evaluating the integrals and then limit we get –

$$K_b=\frac{2}{a\pi}\Psi_o(b)\left[\Delta_o(b)+\int_o^b g(\alpha)K(\alpha,b)\,d\alpha\right]$$
$$\Psi_o(b)=\left[\frac{2}{q}\tan(qb/2)\right]$$

Where Δ_o and $K(\alpha,b)$ are given in (4.10) and (4.11) respectively. $g(\alpha)$ will be obtained as solution of Fredholm integral equation. The component of normal stress due to heat is not singular therefore in limit it gives zero.

CRACK SHAPE

The crack opening displacement is called crack shape which is evaluated through the value of series (4.1) for $o\le x<b$.

$$u_y^{(E)}(x,o)=2(1-\eta)\int_x^b g(t)\,dt,\quad o\le x<b, \tag{6.9}$$

While g(t) be obtained as solution of Fredholm integral equation.

7. SPECIAL HEAT DISTRIBUTION

We consider the case of heat distribution in stress-free strip through heat sources defined as –

$$Q(x,y)=\frac{Q_o}{2}\delta(x)\left[\delta(y-h)+\delta(y+h)\right], \tag{7.1}$$

Where Q_o is intensity of heat source, two heat sources of magnitude Q_o are acting at point $(o,\pm h)$, see figure 2. Take cosine Fourier integral transform with respect to y and finitie Fourier cosine transform with respect to x, we get –

$$Q_{cc}(\alpha_n,\varsigma)=Q_o\cos(\varsigma h) \tag{7.2}$$

Using (7.2) in (2.7) and (2.8), we get

$$u_x^{(H)}(x,y)=Q_1\sum_{n=1}^{\infty}\sin(\alpha_n x)M_3(\alpha_n,y,h)/\alpha_n^{\,2} \tag{7.3}$$

$$M_3(\alpha_n y,h)=\left[\left\langle(y+h)\alpha_n+1\right\rangle e^{-\alpha_n(y+h)}+\left\langle(y-h)\alpha_n+1\right\rangle e^{-\alpha_n(y-h)}\right] \tag{7.4}$$

$$u_y^{(H)}(x,y) = Q_1\left[\tfrac{1}{2}v(o,y) + \sum_{n=1}^{\infty}\cos(\alpha_{nx})\left\{e^{-\alpha_n(y+h)} + e^{-\alpha_n(y-h)}\right\}\right] \tag{7.5}$$

$$V(o,y) = -1 = \text{constant}, \quad Q_1 = \frac{2(1+\eta)}{aE\beta^2}Q_o \tag{7.6}$$

$$\sigma_{xx}(x,y) = \frac{EQ_1}{2(M^2)}\left[\sum_{n=1}^{\infty}\cos(\alpha_n x)\{M_3(\alpha_n,y,h)/\alpha_n\eta\alpha_n\right.$$

$$\left.\left\langle \overline{e}^{\alpha_n(y+h)}\overline{e}^{\alpha_n}(y-h)\right\rangle\}\right] \tag{7.7}$$

$$\sigma_{xx}^{(H)}(x,y) = Q_2\left[C^+(y,h,x) + C^-(y,h,x)\right)$$

$$\sigma_{yy}(x,y) = \frac{EQ_1}{2(1-\eta^2)}\sum_{n=1}^{\infty}\cos(\alpha_n x)$$

$$\left\{\eta\frac{M_3(\alpha_n,y,h)}{\alpha_n} - \alpha_n\left\langle \overline{e}^{\alpha_n(y+h)} + \overline{e}^{\alpha_n(y-h)}\right\rangle\right\} \tag{7.8}$$

$$+C_1^+(y,h,x) + C_1^-(y,h,x) - \eta\left\langle S_1^+ + S_1^-\right\rangle \tag{7.9}$$

$$Q_2 = \frac{EQ_1}{2(1-\eta^2)}, C^{\pm}(y,h,x) = -\frac{4\left\langle \cos qx - \overline{e}^{q(y\pm h)}\right\rangle}{R(y\pm h,x)} \tag{7.10}$$

$$\left.\begin{aligned} &R(y,x) = \cosh(qy) - \cos(qx),\\ &s_1^+(y,h,x) = \frac{2\sin(qx)\sinh q(y\pm h)}{R^2(y\pm h,x)}\end{aligned}\right\} \tag{7.11}$$

$$C_1^{\pm}(y,h,x) = \operatorname{cosec}(qx)\tan^{-1}\left\langle\frac{\tanh q(h\pm h)/2}{\tan(qx/2)}\right\rangle \tag{7.12}$$

$$\left.\begin{aligned}\sigma_{yy}^{(h)}(x,y) = Q_2\Big[\eta\{&C^+(y,h,x) + C(y,h,x) + C^+(y,h,x)\\ &+C_1^-(y,h,x)\} - \left\langle S_1^+ + S_1^-\right\rangle\Big]\end{aligned}\right\} \tag{7.13}$$

We can easily evaluate $\sigma_{xx}{}^{(H)}(x,o)$

$$\left.\begin{aligned} &C^{\pm}(y,h,a) = -\overline{e}^{q(y\pm h)/2}\operatorname{sech} q(y\pm h)/2\\ &C_1^{\pm}(y,h,a,) = \tanh q(y\pm h)/2\\ &S_1^{\pm}(y,h,a,) = o\end{aligned}\right\} \tag{7.14}$$

$$\left.\begin{aligned} &C^{\pm}(o,h,x,) = -4\frac{(1+\sinh\, qh)}{R(b,x)}\\ &C_1{}^{\pm}(o,h,x,) = \pm\operatorname{cosec}(qx)\tan^{-1}\left\langle\frac{\tanh(qh/2)}{\tan(qx/2)}\right\rangle\\ &S_1{}^{\pm}(o,h,x,) = \pm\frac{2\sin(qx)\sinh(qh)}{R^2(h,x)}\end{aligned}\right\} \tag{7.15}$$

8. PLASTIC ZONE LENGTH

The plastic deformation is explained through the criteria of crack destruction. Dugdale [11] assumed the presence of uniform stress in the zone. He used the condition of finiteness of stress in zone length. This brings the condition that the stress-intensity factor is zero at crack tip. Zone length will vary according to mode, theory used or plane strain & plane-stress conditions.

In this section we solve the crack opening by thermal stress $\sigma_{yy}^{(H)}(x,o)$. The crack occupies the region y=o, $o \le |x| \le d < b,$ to the right of the tip (b,o) is elastic region while the region $d \le |x| \le d < b$ is plastic. It is being taken that there is singularity in $\sigma_{yy}^{(H)}(x,o)$ at tip (b,o). Thus the problem of determining the plastic zone length is reduced to the following mixed boundary value problem.

$$\sigma_{xy}(\pm a, y) = \sigma_{xx}(\pm a, y) = o, o \le |y| < \infty \tag{8.1}$$

$$\sigma_{xy}^{(E)}(x,o) = o, \quad o \le |x| \le a \tag{8.2}$$

$$\sigma_{y}^{(E)}(x,o) = o, \quad b < |x| \le a \tag{8.3}$$

$$\sigma_{yy}^{(E)}(x,o) = \begin{cases} -\sigma_{yy}^{(H)}(x,o) & o \le x \le d \\ T & d < x \le b \end{cases} \tag{8.4}$$

Where (b-d) is plastic zone length determined through

$$\Delta_o(b) + \int_o^b g(x) K(\alpha, b)\, d\alpha = T \cos^{-1} \frac{G(b,d)}{G(b,c)}$$

While $\Delta_o(t)$ and $K(\alpha,t)$ are given by (4.10)-(4.11)

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A NEW TEST FOR TESTING EXPONENTIALITY AGAINST INCREASING FAILURE RATE AVERAGE ALTERNATIVES

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ABSTRACT :
The problem of testing exponentiality against positive ageing has received much attention of the researchers. A particular type of positive aging, namely, increasing failure rate average (IFRA) is an important concept in the theory of reliability, which arise as life distributions from shock models. In this paper, a test procedure based on U-statistics is proposed for the problem of testing exponentiality against increasing failure rate average alternatives. The distributional properties of the test statistic are studied . The performance of the proposed test is compared with the tests existed in the literature in terms of Pitman asymptotic relative efficiency. It is observed that the proposed test performs better.
Keywords: Asymptotic Relative Efficiency, U-Statistics, Increasing Failure Rate Average, Sub-sample.

1. INTRODUCTION

One of the important areas of research in reliability and survival analysis is the problem of testing exponentiality against positive aging. One of the important classes of life distributions exhibiting positive aging property is increasing failure rate average (IFRA) class of distributions. IFRA class of distributions is the smallest class of probability distributions which is closed under formation of coherent systems. In this article, the problem of testing exponentiality against IFRA alternatives is considered. The definition of IFRA is presented below:

Definition 1.1:

Let F be a absolutely distribution function, with $F(0)=0$. Then, F is an increasing failure rate average distribution if

$$\overline{F}(bx) \geq \{\overline{F}(x)\}^b \quad x>0,\ 0<b<1, \qquad \text{Where } \overline{F}=1-F \tag{1.1}$$

The equality in (1.1) holds if and only if F is an exponential distribution.

The problem of testing exponentiality against ageing received much attention by the researchers. Some of the works in this area include those of Proschan & Pyke (1967), Bickel & Doksum (1969), Ahmed (1975), Hollander & Proschan (1972, 1975) and Koul (1977, 1978). However, the tests that have been developed specifically for testing for increasing failure rate average alternative only is due to Deshpande (1983). Pandit, Shetty and Bhadri (2008) proposed a test statistic for this problem based on subsample minima.

In Section 2, a new class of statistics based on U – Statistics with kernel depending on subsample minima is proposed. Section 3 deals with the mean and asymptotic variance of the proposed statistic. The Pitman ARE comparisons and conclusions are presented in section 4.

2. PROPOSED CLASS OF TEST STATISTICS

Let $X_1, X_2, X_3, ..., X_n$ be a random sample from a continuous probability distribution with distribution function F such that F(0) = 0.

The problem is to test

H_0 : $\overline{F}(bx) = \overline{F}^b(x)$. $\forall\ x > 0$ and $0 < b < 1$

H_1 : $\overline{F}(bx) \geq \overline{F}^b(x)$. $\forall\ x > 0$ and $\overline{F}(bx) > \overline{F}^b(x)$ for some x

Let

$$h_b(x_1, x_2, x_3, x_4) = \begin{cases} 1, \text{if } \mathrm{Min}(x_1, x_2) > \mathrm{bMax}(x_3, x_4) \\ 0, otherwise \end{cases}$$

where b is a fixed number such that $0 \leq b \leq 1$.

Define $U(b)$ as U-statistic based on the kernel $h_b(x_1, x_2, x_3, x_4)$, which is given by

$$U(b) = \binom{n}{4}^{-1} \sum h^*{}_b(x_{i_1}, .x_{i_2}, x_{i_3}, x_{i_4})$$

where summation is taken over all combinations of integers (i_1, i_2, i_3, i_4) chosen out of integers $(1,2,...,n)$ and $h^*{}_b(x_1, x_2, x_3, x_4)$ is the symmetrized version of $h_b(x_1, x_2, x_3, x_4)$

3. MEAN AND ASYMPTOTIC DISTRIBUTION OF THE PROPOSED STATISTIC

The mean of the proposed statistic U(b) is given by

$$E_{H_0}(U(b)) = P[Min(x_1, x_2) > bMax(x_3, x_4)]$$
$$= 2\int_0^{\infty} F(y)\overline{F}^{2b}(y)dF(y)$$
$$= \frac{1}{(b+1)(2b+1)} \text{ under } H_0$$

The asymptotic distribution of the test statistic is given in theorem 1.

Theorem 1: Let $X_1, X_2, X_3, ..., X_n$ denote a random sample from population with absolutely continuous c.d.f $F(x)$.

Then, the asymptotic distribution of $\sqrt{n}[U - E(U)]$ is normal with mean zero and variance $16\xi_1$, where

$$\xi_1 = Cov\left[h_b^*(X_1, X_2, X_3, X_4)h_b^*(X_1, X_5, X_6, X_7)\right].$$
$$= E[h(X_1, X_2, X_3, X_4)(X_1, X_5, X_6, X_7) - \gamma^2(F)]$$
$$= \int_0^{\infty} E^2[h(x, X_2, X_3, X_4)]dF(x) - \left[\frac{1}{(b+1)(2b+1)}\right]^2 \text{ under } H_0$$

Proof: Proof is the direct consequence of Hoeffding(1948) one sample U-Statistic theorem.

Now, to evaluate ξ_1 under H_0, consider

$$E[h(x, X_2, X_3, X_4)] = P[Min(x, X_2) > bMax(X_3, X_4)] + P[Min(X_3, X_2) > bMax(x, X_4)]$$
$$= P[X_2 > x > bMax(X_3, X_4)] + P[x > X_2 > bMax(X_3, X_4)]$$

$$+P[Min(X_2,X_3)>bx>bX_4]+P[Min(X_2,X_3)>bX_4>bx]$$

$$=A_1+A_2+A_3+A$$

Now,

$$A_1=P[X_2>x>bMax(X_3,X_4)]$$

$$=\overline{F}(x)+\overline{F}(x)\overline{F}^{\frac{2}{b}}(x)-2\overline{F}(x)\overline{F}^{\frac{1}{b}}(x) \text{ under } H_0$$

$$A_2=P[x>X_2>bMax(X_3,X_4)]$$

$$=1-\overline{F}(x)+\frac{b}{2+b}-\frac{b\overline{F}^{\frac{2}{b}+1}(x)}{2+b}-\frac{2b}{1+b}+\frac{2\overline{F}^{\frac{1}{b}+1}(x)b}{1+b} \text{ under } H_0$$

$$A_3=P[Min(X_2,X_3)>bx>bX_4]$$

$$=\overline{F}^{2b}(x)-\overline{F}^{2b+1}(x) \text{ under } H_0$$

$$A_4=P[Min(X_2,X_3)>bX_4>bx]$$

$$=\frac{\overline{F}^{2b+1}(x)}{2b+1}, \text{ under } H_0$$

Thus we get

$$E[h(x,X_2,X_3,X_4)]=1+\frac{b}{2+b}-\frac{2b}{1+b}+\overline{F}^{2b}(x)-\overline{F}^{2b+1}(x)\left(\frac{2b}{2b+1}\right)+\overline{F}^{\frac{2}{b}+1}(x)\left(\frac{2}{2+b}\right)-2\overline{F}^{\frac{1}{b}+1}(x)\left(\frac{1}{b+1}\right)$$

Now ξ_1 is obtained as $\xi_1=\int_0^\infty E^2[h(x,X_2,X_3,X_4)]dF(x)-\left[\frac{1}{(b+1)(2b+1)}\right]^2$.

For different values of b, $B^*=16\xi_1$ are evaluated and is given in Table 3.1.

Table 3.1 Values of $B^*=16\xi_1$

b	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$B^*=16\xi$	0.16702	0.15039	0.10133	0.06007	0.03252	0.01601	0.00691	0.00236	0.00098

4. ASYMPTOTIC RELATIVE EFFICIENCY

For asymptotic relative efficiency(ARE) comparisons, we have considered three parametric families of distributions, namely, Weibull, Linear Failure Rate and Makeham distributions. These depend upon a real parameter θ in such a way that $\theta=\theta_0$ yields a distribution belonging to null hypothesis whereas $\theta>\theta_0$ yields distribution from the alternative.

The alternatives considered for evaluation of ARE along with efficacies of U are presented below:

(i) Weibull Distribution:

The survival function of X is

$$\overline{F_{1\theta}}(x)=\exp\left(-x^\theta\right), \qquad x>0,\ \theta\ge 1, \theta_0=1$$

and $(\gamma'(F_{1\theta}))^2=\dfrac{b^2(\ln(b))^2(4b+3)^2}{(b+1)^4(2b+1)^2}$

For different values of b we have calculated the $(Eff(U(b))^2$ which is given in Table 4.1

Table 4.1 Efficacies of U(b) for Weibull distribution

b	$B^* = 16\xi_1$	$(Eff(U(b))^2 = \frac{(\gamma'(F))^2}{B^*}$
0.1	0.16702	1.2087
0.2	0.15039	1.2489
0.3	0.10133	1.2137
0.4	0.06007	1.1734
0.5	0.03252	1.1399
0.6	0.01602	1.1144
0.7	0.00691	1.0952
0.8	0.00236	1.0822
0.9	0.00098	0.4989

(ii) Linear Failure Rate Distribution:

The survival function of X is

$$\overline{F}_{2\theta}(x) = \exp\left(-\left(x + \theta\frac{x^2}{2}\right)\right), \qquad x > 0,\ \theta \ge 0, \theta_0 = 0$$

$$(\gamma'(F_{2\theta}))^2 = \frac{1}{16}b^2 \frac{\left(12b^3 - 6b^2 - 29b - 14\right)^2}{\left[\left(3b + 3b^2 + 1 + b^3\right)\left(8b^3 + 12b^2 + 6b + 1\right)\right]^2}$$

For different values of b we have calculated the $(Eff(U(b))^2$ which is given in Table 4.2

Table 4.2 Efficacies of U(b) for Linear Failure Rate distribution

b	$B^* = 16\xi_1$	$(Eff(U(b))^2 = \frac{(\gamma'(F))^2}{B^*}$
0.1	0.16702	0.2029
0.2	0.15039	0.2940
0.3	0.10133	0.3601
0.4	0.06007	0.4324
0.5	0.03252	0.5353
0.6	0.01601	0.7086
0.7	0.00691	1.0535
0.8	0.00236	1.9644
0.9	0.00098	2.9878

(iii) Makeham Distribution:

The survival function of X is

$$\overline{F}_{\theta}(x) = \exp\left(-x + \theta\left(x + e^{-x} - 1\right)\right), \quad x > 0,\ \theta \ge 0, \theta_0 = 0$$

$$(\gamma'(F))^2 = \left[-\left[\frac{1}{2}b\frac{\left(-18b^3 - 121b^2 + 32b^4 - 93b - 20\right)}{\left[(2b+3)(1+b)^2(3b+2)(2b+1)^2(3b+1)\right]}\right]\right]^2$$

For different values of b we have calculated the $\left(Eff\,(U\,(b)\right)^2$ which is given in Table 4.3.

Table 4.3 Efficacies of U(b) for Makeham distribution

b	$B^* = 16\xi_1$	$\left(Eff\,(U\,(b)\right)^2 = \dfrac{(\gamma'(F))^2}{B^*}$
0.1	0.16702	0.05017
0.2	0.15039	0.0791
0.3	0.10133	0.1050
0.4	0.06007	0.1363
0.5	0.03252	0.1820
0.6	0.01601	0.2598
0.7	0.00691	0.4168
0.8	0.00236	0.8389
0.9	0.00098	1.3876

The Pitman ARE's of V(b,K) test of Pandit, Shetty and Bhadri (2008) for different values of b=0.1,0.5,0.9 with respect to Deshpande (1983) J_b test are given in the table 4.4 below

Table 4.4 AREs of V(b,k) with respect to J_b

b	0.1	0.5	0.9
Weibull	0.8703	0.4461	0.7648
LFR	0.6233	0.4582	0.3711
Makeham	1.0222	1.1937	1.4943

The Pitman ARE's of U(b) test for different values of b=0.1,0.5,0.9 with respect to Deshpande (1983) Jb test are given in the table 4.5 below

Table 4.5 AREs of U(b) with respect to J_b

b	0.1	0.5	0.9
Weibull	1.0230	0.717	0.6579
LFR	1.1797	1.7295	4.4076
Makeham	1.4293	2.7322	9.4343

CONCLUSION

In this paper a new test for testing exponentiality against IFRA alternatives is considered which is based on U statistic with kernel of degree four. The asymptotic comparison in Pitman sense is made with the tests due to Deshpande (1983) and Pandit et.al (2008). It is observed that

1. The new test is better when the alternative is LFR or Makeham.
2. However for Weibull alternative Deshpande (1983) test is better.
3. New test can also be recommended for Weibull distribution as AREs of the new test relative to Deshpande (1983) test for Weibull is near to one.

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COMPARATIVE ANALYSIS OF TWO UNIT COLD STANDBY SYSTEM WITH THREE TYPES OF REPAIR POLICY AND REPLACEMENT TO A MODEL WITH INSTRUCTION AND TWO OUT OF THESE THREE TYPES POLICY

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ABSTRACT :

The present paper deals with comparative analysis of two unit cold standby system with three types of repair policy and replacement with a model having instruction and two of the three types of repair policy. In the first model damage unit sometimes may or may not be repairable and therefore, it is repaired or replaced. The two types of repair policy used are : resume repair policy and policy adopted at the more degraded stage due to damage made by the ordinary repairman during try for repair. The purpose of taking the idea of instruction in the paper is to avoid the possibility of incorrect process of repair done by the ordinary repairman. It has been assumed that the expert repairman when comes for repairing a failed unit repairs all the unit which fail during his stay at the system.

1. INTRODUCTION

Concept of three types of repair policy has been discussed for one–unit/two–unit cold standby systems in the earlier papers with various possibilities including the one that the ordinary repairman may not be able to successfully complete the repair. Then expert repairman is called on to do the job. But engaging an expert repairman for repair may be costly. Such cost can be reduced by introducing the idea of instruction which are given by the expert to the ordinary repairman before the start of repair by the ordinary repairman. So, the stay and cost of expert repairman can be reduced. The idea of instruction was first introduced by Kumar et al. (1996). Later on, it was discussed by some other authors including Siwach et al. (2001), Gupta and Taneja (2001) together with concepts of accidents, two of repairmen, etc. Here also, the concept of instruction may be introduced in order to nullify the possibility of incorrect process of repair adopted by the ordinary repairman.

Thus, the present paper investigates two-unit cold standby system with three types of repair policy and replacement with a model having instruction and two of the three types of repair policy. In the first model damage unit sometimes may or may not be repairable and therefore, it is repaired or replaced and in the second model with instruction and two of the three types repair policy. On the failure of an operative unit, an expert repairman comes to give instruction to the ordinary repairman for repairing it. After getting instruction, the ordinary repairman repairs the unit by adopting the correct process. However during repair, he may damage the unit due to mishandling or some other causes and in this case repeat repair policy (type-II) is adopted by the expert repairman. Chances are there that even after getting instruction he may not be able to complete the repair successfully and hence resume repair policy is adopted by the expert. If at the time of imparting instruction for a failed unit other unit is operative, it is assumed that instruction time finishes before the failure of the latter.

In the model of reference 4 where damaged unit may or may not be repairable and hence it is repaired or replaced in this case the profit is denoted by P_4.

In the model of reference 5 if at the time of completion of the repair of a failed unit by the expert, the second unit is found in failed state, it is also repaired by the expert and its profit is denoted by P_{51}.

Pair-wise comparative analysis of models through graphs is carried out for some particular cases as generally we take. As we know that no model is best in every situation. Some models may be better in one situation but worse in another situation, so it is highly required to make their comparative analysis.

2. COMPARISON BETWEEN THE MODELS DISCUSSED IN REFERENCE 4 AND REFERENCE 5

Models of two-unit cold standby systems with discussion rate and instruction rate have been discussed in reference 4 and reference 5. The graphs have been plotted for showing the behaviour of the difference varying cost/instruction rate/probability.

(A) **Fig. 1** depicts the behaviour of the difference of profits ($P_{51} - P_4$) with respect to instruction cost (C_8) for different values of instruction rate (γ). Following can be observed from the graph :

(i) It is observed that at the initial stages of the values of instruction cost (C_8), i.e., upto 160, the difference is lower for higher value of instruction rate (γ). But the difference decreases more rapidly for lower values of γ, as a result of which the trend gets reversed for $C_8 > 175$.

(ii) For $\gamma = 1$, ($P_{51} - P_4$) > or = or < 0 according as C_8 < or = or > 209.2. So Model 1 of reference 5 is better than model of reference 4 if $C_8 < 209.2$, the models are equally good if $C_8 = 209.2$ and model of reference 4 is better than Model 1 of reference 5 if $C_8 > 209.2$.

(iii) For $\gamma = 2$, ($P_{51} - P_4$) > or = or < 0 according as C_8 < or = or > 243.35. So Model 1 of reference 5 is better than model of reference 4 if $C_8 < 243.35$, the models are equally good if $C_8 = 243.35$ and model of reference 4 is better than model of reference 5 if $C_8 > 243.35$.

(iv) For $\gamma = 10$, ($P_{51} - P_4$) > or = or < 0 according as C_8 < or = or > 548.2. So Model 1 of reference 5 is better than model of reference 4 if $C_8 < 548.2$, the models are equally good if $C_8 = 548.2$ and model of reference 4 is better than Model 1 of reference 5 if $C_8 > 548.2$.

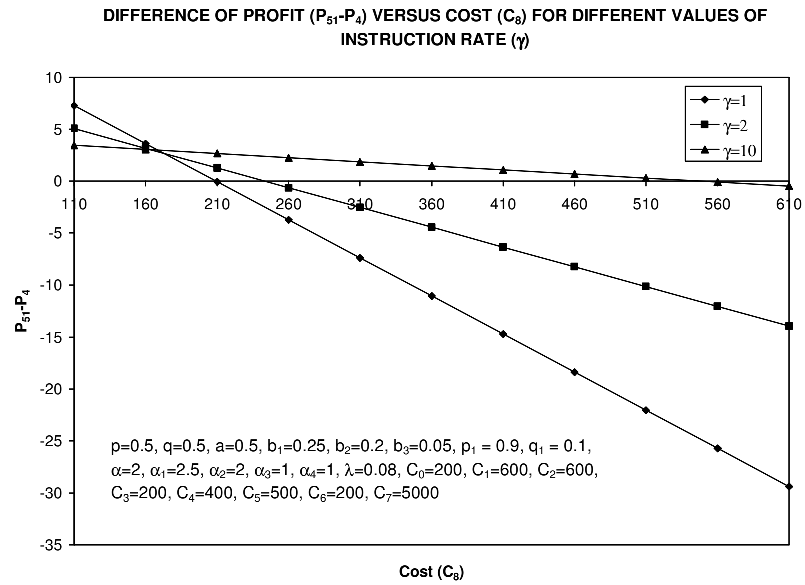

Fig. 1

(B) Fig. 2 depicts the behaviour of the difference of profits (P_{51} - P_4) with respect to probability (p_1) for different values of cost (C_7). Through this figure the following observations can be made:

(i) The difference increases with increase in the values of (p_1) and becomes higher for higher values of replacement cost (C_7).

(ii) For C_7 = 6000, ($P_{51} - P_4$) > or = or < 0 according as probability (p_1) > or = or < 0.968. So, Model 1 of reference 5 is better or worse than model of reference 4 if p_1 > or < 0.968. Both the models are equally good if p_1 = 0.968.

(iii)For C_7 = 8000, ($P_{51} - P_4$) > or = or < 0 according as probability (p_1) > or = or < 0.913. So, Model 1 of reference 5 is better or worse than model of reference 4 is p_1 > or < 0.913. Both the models are equally good if p_1 = 0.913.

(iv) For C_7 = 10000, ($P_{51} - P_4$) > or = or < 0 according as p_1 > or = or < 0.858. So Model 1 of reference 5 is better or worse than model of reference 4 if p_1 > or < 0.858. Both the models are equally good if p_1 = 0.858.

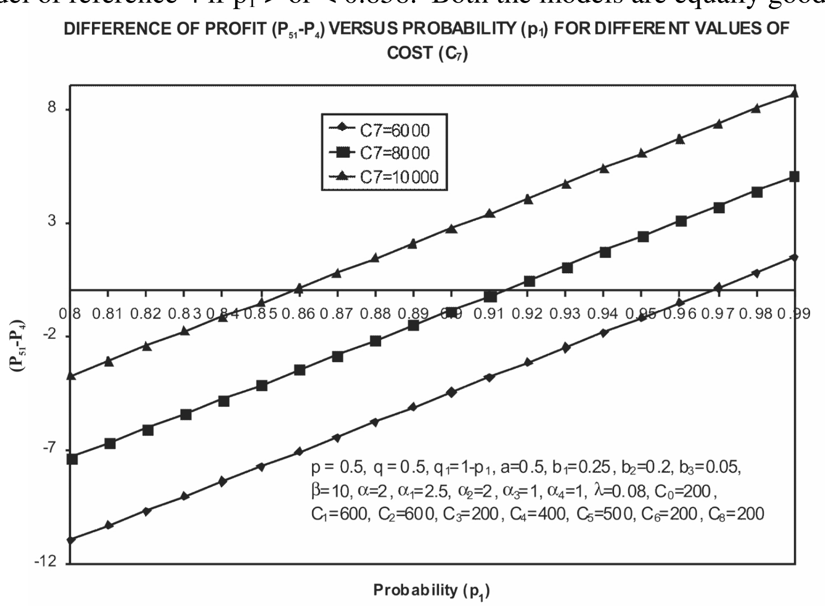

Fig. 2

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ON J_{α}^{β} - DIVERGENCE OF ORDER α AND TYPE β FOR INTUITIONISTIC FUZZY SET AND ITS PROPERTIES

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ABSTRACT :
In the present communication, we define $J_{\alpha}^{\beta}(A,B)$ $J_{\alpha}^{\beta}(A,B)$ -divergence of order α and type β for intuitionistic fuzzy set and discuss some properties. Section 2 presents the basic concepts for this study while section 3 presents the new intuitionistic fuzzy divergence J_{α}^{β} (A,B)) of order α and type β. Some particular cases have been discussed. Section 4 illustrates some properties.

Keywords : Kullback-Leibler Divergence, Intuitionistic Fuzzy Set, Fuzzy Divergence, Intuitionistic Fuzzy Divergence, $J_{\alpha}^{\beta}(A,B)$ -Divergence, Jensen's Inequality.

1.1 Introduction

The probabilistic information measure of divergence was introduced, first by Kullbackl and Leiobler [8], after the introduction of inaccuracy measure by Kerridge [7] and Shannon's [15] entropy viz.

$$I(P \| Q) = -\sum_{i=1}^{n} p_i \log q_i \tag{1}$$

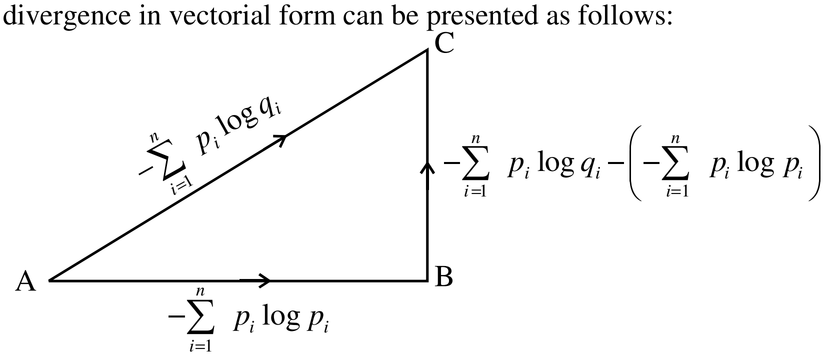
Kerridge inaccuracy [7]

and

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i \tag{2}$$

Shannon entropy [15]

Kullback-Leibler's [8] divergence in vectorial form can be presented as follows:



$$K(P\,||\,Q)=\sum_{i=1}^{n} p_i \log\frac{p_i}{q_i}$$

$$=-\sum_{i=1}^{n} p_i \log q_i-\left(-\sum_{i=1}^{n} p_i \log p_i\right)$$

= difference of Kerridge inaccuracy and Shanon entropy.

Kullback-Leibler divergence measure has found many applications in a variety of discipline. A series of probabilistic divergence measures can be viewed in Taneja [17]

Zadeh [24] introduced the fuzzy set theory just parallel to probabilistic information theory. Bhandari and Pal [2] introduced fuzzy divergence measure which has found many applications such as pattern recognition, clustering, signal theory and image processing etc.

Atanassov [1] generalized fuzzy set theory through intuitionistic fuzzy set theory after introducing 'hesitation degree of the function. Intuitionistic fuzzy set theoretic information measures, associated with them that are more appropriate on critical decision making [9, 10, 13, 19, 21, 25], medical diagnosis [15] and pattern recognition [11, 14, 18, 22, 24]. Wei and Ye [8] proposed an improved version of Vlachos and Sergiadis [18] intuitionistic fuzzy divergence and studied its application in pattern recognition. They used the mid-value of the membership, non-membership and hesitation values of two sets to propose a measure. Verma and Sharma [19, 20] proposed a generalized Wei and Ye [21] intuitionistic fuzzy divergence while Wei-Liang Hung and Min-Shien Yang [23] introduced the J-divergence of intuitionistic fuzzy sets with its application to pattern recognition.

In the present communication, we generalize the J-Divergence, $J_{\beta}^{\alpha}(A\,||\,B)$ of order α and type β for intuitionistic fuzzy sets and discuss its properties in section 3 and basic concepts have been reported in the section 2.

2. BASIC CONCEPTS FOR THE GENERALIZED $\mathbf{J}_{\beta}^{\alpha}(\mathbf{A}\,||\,\mathbf{B})$ FOR INTITONISTIC FUZZY SET.

2.1 Probabilistic Divergences Between Two

Probabilistic Divergence: The probabilistic divergence, between two distributions P and Q is defined as KL-Divergence, K(P||Q)

$$K(P\,||\,Q)=\sum_{i=1}^{n} p_i \log\frac{p_i}{q_i} \tag{3}$$

the vectorial form has been presented in section 1,

and more generalized form is

$$I(P\,||\,Q)=\sum_{i=1}^{n} p_i \log\left(\frac{p_i}{\frac{p_i+q_i}{2}}\right) \tag{4}$$

where $I(P\,||\,Q)=K\left(P,\ \frac{P+Q}{2}\right).$ (5)

For more detail *c.f.* Taneja [17].

Clearly both K and I are not symmetric. A symmetric divergence measure based on I can be defined as

$$S(P \| Q) = \frac{I(P \| Q) + I(Q \| P)}{2}. \tag{6}$$

S. Guiasu [6] introduced the Weighted Entropy, so considering the weighted distribution

$$W = (w_1, w_2, ..., w_n),$$

where $\sum_{i=1}^{n} w_i = 1$,

Jensen – Shannon divergence c.f. [10] was defined as

$$Js(P \| Q; W) = H(w_1 P + w_2 Q) - (w_1 H(P) - (w_w H(Q)) \tag{7}$$

2.2 FUZZY DIGERGENCE

Zadeh [24] introduced the fuzzy set theoretic concept in 1965. Later on Bhandari and Pal [2] in 1992 introduced the Fuzzy Divergence (FD) using the Fuzzy Entropy (F E) based on DeLuca and Termini [4] in 1972: Afterwards, using the idea of Lin, Shang and Liang [14] proposed the modified version of fuzzy divergence.

- DeLuca and Termini [4], Fuzzy Entropy:

$$H(A) = -\frac{1}{n} \sum \left[\mu_A(x) \log \mu_A(x) + (1 - \mu_A(x) \log(1 - \mu_A(x)) \right] \tag{8}$$

- Shang and Liang [14] fuzzy divergence:

$$J(A | B) = \sum_{i=1}^{n} \left[\mu_A(x) \log \frac{\mu_A(x)}{\left(\frac{\mu_A(x) + \mu_B(x)}{2} \right)} + (\mu_A(x)) \log \frac{(1 - \mu_A(x)}{1 - \left(\frac{\mu_A(x) + \mu_B(x)}{2} \right)} \right] \tag{9}$$

- Bhandari and Pal [2] : Divergence Measure :

$$D(A | B) = \frac{1}{n} \sum \left[\mu_A(x) \log \frac{\mu_A(x)}{\mu_B(x)} + (1 - \mu_A(x) \log \frac{1 - \mu_A(x)}{1 - \mu_B(x)} \right] \tag{10}$$

Later on Singh and Tomar [16] defined many symmetric and non-symmetric fuzzy divergences, and studied their properties and in equation.

2.3 INTUITIONISTIC FUZZY DIVERGENCE :

Let X be a set. An IFS A in X is defined as

$$A = \{x, \mu_A(x), \nu_A(x) \mid x \in X\}$$

where $\mu_A(x) : X \to [0, 1]$

$$\nu_A(x) : X \to [0, 1]$$

with the condition

$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \forall\, x \in X$, Atanassov [1]

For each IFS A in X, let the hesitation degree be

$$\pi_A(x) : X \to [0, 1]$$

with the condition.

$$\pi_A(X) = 1 - \mu_A(x) - \nu_A(x).$$

Wei and Ye [22] defined the intuitionistic fuzzy divergence as follows:

$$D(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\frac{\mu_A(x_i)+\mu_B(x_i)}{2}}+\nu_A(x_i)\log\frac{\nu_A(x_i)}{\frac{\nu_A(x_i)+\nu_B(x_i)}{2}}+\pi_A(x_i)\log\frac{\pi_A(x_i)}{\frac{\pi_A(x_i)+\pi_B(x_i)}{2}}\right] \tag{11}$$

Later on R. Verma and B.D. Sharma [19] modified [11] as follows :

$$D(A,B)=\frac{1}{n}\sum_{i=1}^{n}\left[\mu_A(x_i)\log\frac{\mu_A(x_i)}{\frac{\lambda\mu_A(x_i)+(1-\lambda)\mu_B(x_i)}{2}}+\nu_A(x_i)\log\frac{\nu_A(x_i)}{\frac{\lambda\nu_A(x_i)+(1-\lambda)\nu_B(x_i)}{2}}+\pi_A(x_i)\log\frac{\pi_A(x_i)}{\frac{\lambda\pi_A(x_i)+(1-\lambda)\pi_B(x_i)}{2}}\right] \tag{12}$$

where $0 \le \lambda \le 1$.

Considering intuitionistic fuzzy sets, A and B, Wen-Liang Hung and Mini-Shen Yang [23] defined the J_α-divergence between A and B as follows:

$$J_\alpha(A,B)=\begin{cases}\frac{-1}{\alpha-1}\left\{\left(\frac{\mu_A+\mu_B}{2}\right)^\alpha-\frac{1}{2}(\mu_A^\alpha+\mu_B^\alpha)+\left(\frac{\nu_A+\nu_B}{2}\right)^\alpha-\frac{1}{2}(\nu_A^\alpha+\nu_B^\alpha)\right.\\ \left.+\left(\frac{\pi_A+\pi_B}{2}\right)^\alpha-\frac{1}{2}(\pi_A^\alpha+\pi_B^\alpha)\right\}, \qquad \alpha\neq 1\ (\alpha>0)\end{cases} \tag{13}$$

In particular, when $\alpha = 1$,

$$J_1(A,B)=-\frac{1}{2}\begin{cases}(\mu_A+\mu_B)\log\left(\frac{\mu_A+\mu_B}{2}\right)-\mu_A\log\mu_A-\mu_B\log\mu_B\\ +(\nu_A+\nu_B)\log\left(\frac{\nu_A+\nu_B}{2}\right)-\nu_A\log\nu_A-\nu_B\log\nu_B\\ +(\pi_A+\pi_B)\log\left(\frac{\pi_A+\pi_B}{2}\right)-\pi_A\log\pi_A-\pi_B\log\pi_B\end{cases}, \tag{14}$$

The symmetric intuiti0onistic generalized Fuzzy divergence is as follows:

$$D(A\,||\,B)=D(A,B)+D(B,A) \tag{15}$$

We define $J_\beta^\alpha(A,B)$ in the next section 3, as a generalization of $J_\alpha(A,B)$.

3. GENERALIZED INTUITIONISTIC FUZZY DIVERGENCE $J_\alpha^\beta(A,B)$ OF ORDER α AND TYPE β

Let A and B be two IFS in X, We define $A\subseteq B$ and $A=B$ as follows

(i) $A\subseteq B$ if and only of $\forall\ x\in X$, $\mu_A(x)\le\mu_B(x)$ and $\nu_A(x)\ge\nu_B(x)$.

(ii) $A=B$ if and only if $\forall\ x\in X$, $\mu_A(x)=\mu_B(x)$ and $\nu_A(x)=\nu_B(x)$.

Definition: Let A and B be two intuitionistic fuzzy sets then

$$J_\alpha^\beta(A,B) = -\frac{1}{1-\alpha}\frac{\left[\begin{array}{l}\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2}\right)^{\alpha+\beta-1} - \frac{1}{2}\left(\mu_A^{\alpha+\beta-1}+\mu_\beta^{\alpha+\beta-1}\right)\\ +\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2}\right)^{\alpha+\beta-1} - \frac{1}{2}\left(\nu_A^{\alpha+\beta-1}+\nu_\beta^{\alpha+\beta-1}\right)\\ +\left(\frac{\pi_A(x_i)+\pi_B(x_i)}{2}\right)^{\alpha+\beta-1} - \frac{1}{2}\left(\pi_A^{\alpha+\beta-1}+\pi_\beta^{\alpha+\beta-1}\right)\end{array}\right]^\beta}{2^\beta-1}$$

where $\alpha \neq 1, \alpha > 0, \beta \neq 0, \beta > 0, \alpha+\beta-1 > 0$ (16)

- **Particular cases**

(i) when $\beta = 1$, (16) reduces to $J_\alpha(A,B)$ studied by Wen, L. Hung and Yang M.S. [23].

(ii) When $\beta = 1$ and $\alpha = 1$, (16) reduces to (14) i.e.

$$J_1^1(A,B) = -\frac{1}{2}\begin{cases}(\mu_A+\mu_B)\log\left(\frac{\mu_A+\mu_B}{2}\right) - \mu_A\log\mu_A - \mu_B\log\mu_B\\ +(\nu_A+\nu_B)\log\left(\frac{\nu_A+\nu_B}{2}\right) - \nu_A\log\nu_A - \nu_B\log\nu_B\\ +(\pi_A+\pi_B)\log\left(\frac{\pi_A+\pi_B}{2}\right) - \pi_A\log\pi_A - \pi_B\log\pi_B\end{cases} \tag{17}$$

Wen, L. Hung and Yang [23]

(iii) When $\beta = 1$, $\alpha = 2$.

$$J_2^1(A,B) = \frac{(\mu_A-\mu_B)^2+(\nu_A-\nu_B)^2+(\pi_A-\pi_B)}{4} = \left(e_{IFS}^1\ \ \frac{(A,B)}{2}\right)^2 \tag{18}$$

where e_{IFS}^1 (A,B) was defined as a distance between IFSs by Smidt and Kacprzyk [12, 13] so

$$\sqrt{2J_2^1(A,B)} = e_{IFS}^1(A,B) \tag{19}$$

4. SOME PROPERTIES OF $\mathbf{J}_\alpha^\beta(\mathbf{A},\mathbf{B})$

Theorem-1 : We have

(i) $0 \leq J_\alpha^\beta(A,B) \leq \frac{1}{\alpha-1}\left(1-\frac{1}{2^{\alpha-1}}\right)\frac{1}{(2^\beta-1)}$

and $J_\alpha^\beta(A,B) = 0$ if and only if $A = B$ (20)

(ii) $J_\alpha^\beta(A,B) = 0 = J_\alpha^\beta(B,A)$ (21)

(iii) $J_\alpha^\beta(A,B) \leq J_\alpha^\beta(A,C)$ and $J_\alpha^\beta(B,C) \leq J_\alpha^\beta(A,C)$ (22)

Proof : We have $J_\alpha^\beta(A,B) \geq 0$ and as A = B if and only if $J_\alpha^\beta(A,B) = 0$. Again we show that $J_\alpha^\beta(A,B) \leq L(\alpha,\beta)$, where

$$L(\alpha,\beta)=\frac{1}{\alpha-1}\left(1-\frac{1}{2^{\alpha-1}}\right)\frac{1}{(2^{\beta}-1)} \tag{23}$$

From theorem 1 and corollary 1 or Burbea and Rao [3] $J_{\alpha}^{\beta}(A,B)$ is convex if and only if $\alpha\in[1, 2]$ and $\beta= 1$. Thus for $\alpha \in [1, 2]$, $\beta = 1$, $J_{\alpha}^{\beta}(A,B)$ increases as $\| A - B \|$ increases, where $\| A-B\| = |\mu_A-\mu_B|+|\nu_A-\nu_B|+|\pi_A-\pi_B|$. Then $J_{\alpha}^{1}(A,B)$ for $\alpha \in [1, 2]$ attains its maximum at A = (1, 0, 0) and B = (0, 1, 0) or A = (1, 0, 0), B = (0, 0, 1) or A = (0, 1, 0), B = (0, 0, 1). It follows that $J_{\alpha}^{1}(A,B) \le L(\alpha,\beta)$. $L(\alpha,\beta)$ is undefined for $\alpha = 1$, $\beta = 1$. However if we define this case by the continuous limit $L(\alpha,\beta)$ as $\alpha \to 1$, $\beta = 1$, then we obtain $J_{1}^{1}(A,B)\le \log 2$

Obviously $J_{\alpha}^{\beta}(A,B)=0=J_{\alpha}^{\beta}(B,A)$, by Jensen's inequality.

Property (iii) Let A, B, C be three IFSs such that $A\subseteq B\subseteq C$ then $J_{\alpha}^{\beta}(A,B)\le J_{\alpha}^{\beta}(A,C)$ and $J_{\alpha}^{\beta}(B,C)\le J_{\alpha}^{\beta}(A,C)$. By the same reason, we obtain that

$$\| A-B\| \le \| A-C\|$$

and $\| B-C\|\le\| A-C\|$, if $A\subseteq B\subseteq C$

Thus, $J_{\alpha}^{\beta}(A,B)\le J_{\alpha}^{\beta}(A,C)$, $\forall\ \alpha\in[1,\ 2]$

Corollary 1. $J_{\alpha}^{\beta}(A,B)\ge 0$ with equality if and only if A = B.

2. $0\le J_{\alpha}^{\beta}(A,B)\le 1$.

Theorem 2: Upper Bound for $J_{\alpha}^{\beta}(B,A)$ i.e.

$$0\le J_{\alpha}^{\beta}(B,A)\le \tfrac{1}{\alpha-1}\left(1-\tfrac{1}{2^{\alpha-1}}\right)\left(\frac{1}{2^{\beta}-1}\right) \tag{24}$$

Proof: For the upper bound, we need the convexity of $J_{\alpha}^{\beta}(A,B)$. But the convexity of $J_{\alpha}^{\beta}(A,B)$ holds only for $\alpha = [1, 2]$ and $\beta \in [0, 1]$. Therefore by theorem 1, $J_{\alpha}^{\beta}(A,B)$ holds only for $\alpha \in [1, 2]$, $\beta \in [0, 1]$. However, the concavity of J_{α}^{β} can hold for $\alpha < 0$, $\beta > 0$. This implies when $\alpha>0,\ \beta>0,\ J_{\alpha}^{\beta}(A,B)\ge 0$ and as $J_{\alpha}^{\beta}(A,B)=0$ if and only if A = B. Moreover when $\alpha>0,\ \beta>0$.

$J_{\alpha}^{\beta}(A,B)=J_{\alpha}^{\beta}(B,A)$ also holds.

Hence $0\le J_{\alpha}^{\beta}(A,B)\le\frac{1}{\alpha-1}\left(1-\frac{1}{2^{\alpha}-1}\right)\left(\frac{1}{2^{\beta}-1}\right)$

The results directly follows from Jensen's inequality.

Theorem 3 : Let A, B $\in$ IFS (X)

(i) $J_{\alpha}^{\beta}(A\cup B,\ A\cap B)=J_{\alpha}^{\beta}(A,B)$ (25)

(ii) $J_{\alpha}^{\beta}(A\cap B,\ A\cup B)=J_{\alpha}^{\beta}(B,A)$ (26)

Proof : By definition $J_{\alpha}^{\beta}(A,B)$ is

$$J_\alpha^\beta(A,B) = -\frac{1}{1-\alpha}\frac{\begin{bmatrix}\left(\frac{\mu_A+\mu_B}{2}\right)^{\alpha+\beta-1} - \frac{1}{2}\left(\mu_A^{\alpha+\beta-1}+\mu_\beta^{\alpha+\beta-1}\right)\\ +\left(\frac{\nu_A+\nu_B}{2}\right)^{\alpha+\beta-1} - \frac{1}{2}\left(\nu_A^{\alpha+\beta-1}+\nu_\beta^{\alpha+\beta-1}\right)\\ +\left(\frac{\pi_A+\pi_B}{2}\right)^{\alpha+\beta-1} - \frac{1}{2}\left(\pi_A^{\alpha+\beta-1}+\pi_\beta^{\alpha+\beta-1}\right)\end{bmatrix}^\beta}{2^\beta-1}$$

Hence, $$J_\alpha^\beta(A\cup B,\ A\cap B) = -\frac{1}{n(1-\alpha)}\sum\frac{\begin{bmatrix}\left(\frac{\mu_{A\cup B}+\mu_{A\cap B}}{2}\right)^{\alpha+\beta-1} + \left(\frac{\nu_{A\cup B}+\nu_{A\cap B}}{2}\right)^{\alpha+\beta-1}\\ +\left(\frac{\pi_{A\cup B}+\pi_{A\cap B}}{2}\right)^{\alpha+\beta-1} - \frac{1}{2}\left(\mu_{A\cup B}^{\alpha+\beta-1}+\mu_{A\cap B}^{\alpha+\beta-1}\right)\\ -\frac{1}{2}\left(\nu_{A\cup B}^{\alpha+\beta-1}+\nu_{A\cap B}^{\alpha+\beta-1}\right) - \frac{1}{2}\left(\pi_{A\cup B}^{\alpha+\beta-1}+\pi_{A\cap B}^{\alpha+\beta-1}\right)\end{bmatrix}^\beta}{2^\beta-1} \tag{27}$$

$$= \frac{1}{n(1-\alpha)}\frac{\left[\begin{array}{l}\sum\left\{\begin{array}{l}\left(\frac{\mu_B+\mu_A}{2}\right)^{\alpha+\beta-1} + \left(\frac{\nu_B+\nu_A}{2}\right)^{\alpha+\beta-1} + \left(\frac{\pi_B+\pi_A}{2}\right)^{\alpha+\beta-1}\\ -\frac{1}{2}\left\{(\mu_B^{\alpha+\beta-1}+\mu_A^{\alpha+\beta-1}) + (\nu_B^{\alpha+\beta-1}+\nu_A^{\alpha+\beta-1}) + (\pi_B^{\alpha+\beta-1}+\pi_A^{\alpha+\beta-1})\right\}\end{array}\right\}\\ +\sum\left\{\begin{array}{l}\left(\frac{\mu_B+\mu_A}{2}\right)^{\alpha+\beta-1} + \left(\frac{\nu_B+\nu_A}{2}\right)^{\alpha+\beta-1} + \left(\frac{\pi_B+\pi_A}{2}\right)^{\alpha+\beta-1}\\ -\frac{1}{2}\left\{(\mu_A^{\alpha+\beta-1}+\nu_A^{\alpha+\beta-1}+\pi_A^{\alpha+\beta-1}) + (\mu_B^{\alpha+\beta-1}+\nu_B^{\alpha+\beta-1}+\pi_B^{\alpha+\beta-1})\right\}\end{array}\right\}\end{array}\right]^\beta}{2^\beta-1} \tag{28}$$

$$= J_\alpha^\beta(A,B)$$

(iii) By similar reasons, obviously $J_\alpha^\beta(A\cap B,\ A\cup B) = J_\alpha^\beta(A,B)$. (29)

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